# Topological constructors

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## 1 Introduction

*Algebraic topology* is concerned by any kind of topological space, but using the *algebraic* tool soon leads to favour the constructors producing spaces which can be conveniently so analyzed. Two important constructors are considered here, the first one giving the *CW-complexes*, the second one the *simplicial sets*.

The CW-complexes were invented by J.H.C. Whitehead [19] and numerous well written texts about them are available. On one hand the references [11] and [7] give a reasonably complete study of the basic facts about CW-complexes; furthermore their didactical quality is high. On the other hand, most textbooks about algebraic topology, more precisely about homotopy theory, contain a section devoted to this subject; see for example [16, Section 7.6] and [18, Chapter II]. This situation allows us to be satisfied with a quick survey.

The situation is a little less comfortable with simplicial sets; several texts are in competition, each one favouring some point of view or other. The textbook [11] also deals with simplicial sets; they were then called semi-simplicial complexes but the modern framework of contravariant functors with respect to the category $\Delta$ is not considered in [11]; in the more recent textbook [7], this framework is systematically used, but the exposition does not go very far. The most classical reference is [12], a little book entirely devoted to this subject, a wonderful tool; it gives quite complete demonstrations for all the interesting basic theorems, in particular it contains numerous explicit formulas difficult to find elsewhere; it is also relatively exhaustive for an introduction to this theory; the bibliographical notes at the end of each chapter give a precise and useful idea of the birth of every important notion[1]. Maybe the unique difficulty is in the writing style; it is not at all didactical and without some experience previously acquired, the reader finds the subject rather esoteric[2]. We will try to organize the present lecture notes as a little complementary introduction to Peter May's book. The readers who can read German are advised to use also the book [10], significantly more detailed, provided

[1] Those of [7] are also quite interesting.

[2] To avoid a possible reader wastes some time, let us signal a curious error p.130: the map $\xi$ in Lemma 29.1 in general is not a *linear* homomorphism; but it is easy to enrich the hypotheses to obtain the correct demonstration.

with many interesting examples, containing also a few subjects not treated in [12], for example the Steenrod operations. It is really a pity this book has not been translated into English.

# 2 CW-complexes.

## 2.1 The definition.

**Definition 1** — A *CW-complex structure* on a topological space $X$ is defined by the following data:

1. An increasing sequence of subspaces $\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \ldots \subset X_n \subset X_{n+1} \subset \ldots \subset X$ is given; the whole space $X$ has the topology inherited from this filtration, that is, $U \subset X$ is open if $U \cap X_n$ is an open set of $X_n$ for every $n$; the subspace $X_n$ is called the *$n$-skeleton* of $X$;

2. For every integer $n \in \mathbb{N}$, an index set $A_n$ is given, indexing the $n$-disks $\{D^n_\alpha\}_{\alpha \in A_n}$ which are to be attached to $X_{n-1}$ to construct $X_n$;

3. For every $n \in \mathbb{N}$, an attaching map $\phi_n : B_n \to X_{n-1}$ is given; its source $B_n$ is the boundary of the $n$-th disk collection:

$$B_n = \partial(\coprod_{\alpha \in A_n} D^n_\alpha) = \coprod_{\alpha \in A_n} S^{n-1}_\alpha$$

and the $n$-th stage $X_n$ is the *pushout* of the diagram:

$$\coprod_{\alpha \in A_n} D^n_\alpha \hookleftarrow \coprod_{\alpha \in A_n} S^{n-1}_\alpha \stackrel{\phi_n}{\to} X_{n-1},$$

that is, the disjoint union of the left-hand and right-hand terms where $x$ is identified with $\phi_n(x)$ if $x \in B_n$.

In particular $B_0$, the boundary of a discrete point set, is empty; $\phi_0$ is the "empty map" with empty source and target. The image of a $n$-disk in $X_n$ and in $X$ is traditionally called a $n$-cell or simply a *cell*; the *cellular* homology, studied in another lecture series of this Summer School, is defined according to the structure of the cell sets.

A CW-complex is *reduced* if its 0-skeleton has only one point, which is then its *base-point*.

A relative definition can be given: A CW-complex structure for the pair $(X, X_{-1})$ of topological spaces, $X_{-1}$ being a subspace of $X$, is analogous, but you must replace the initial space $X_{-1} = \emptyset$ by the given possibly non empty $X_{-1}$.

The *homotopy type* of a CW-complex depends only on the homotopy classes of its attaching maps. This is a consequence of the homotopy extension property, stated later.

## 2.2 Examples.

### 2.2.1 The suspension tool.

We will have to use the *suspension functor* $S$; if $X = (X, *)$ is a pointed space, the *reduced* suspension is the space $SX = (X \times I)/((X \times \partial I) \cup (* \times I))$; if $f : X \to Y$ is a continuous (pointed) map between pointed spaces, there is a natural map $Sf : SX \to SY$. The $d$-sphere $S^d$ can be considered as the $d$-th suspension of the 0-sphere $S^0 = \partial I = \partial D^1$.

### 2.2.2 The spheres.

The $d$-sphere can also be considered as a CW-complex where the disk sets $A_n$ are empty except $A_0$ and $A_d$ which have one element; in other words $S^d = D^d/S^{d-1}$.

The canonical map of degree $k$ denoted by $z^k : S^1 \to S^1$ is the map defined on the unit circle of the complex plane mapping a complex number to its $k$-th power. The same notation $z^k : S^d \to S^d$ is used in dimension $d$; the latter map is the $(d-1)$-th suspension of the initial map $z^k : S^1 \to S^1$; the map so obtained is the canonical map of degree $k$ defined on the $d$-sphere.

### 2.2.3 The Moore spaces.

The $d$-sphere $S^d$ is also the Moore space $\text{Moore}(\mathbb{Z}, d)$, that is a simply connected space where every reduced homology group is null except $\widetilde{H}_d(S^d) = \mathbb{Z}$; the relation $d \geq 1$ is assumed. A version of the Moore space $X = \text{Moore}(\mathbb{Z}_k, d)$ is obtained as a CW-complex with three cells in dimensions 0, $d$ and $d+1$: only one way to construct $X_d$ $(= S^d)$ and the last attaching map $\phi_{d+1} : S^d \to S^d$ is the map $z^k$ defining the last stage $X_{d+1} = X$. This construction where an attaching map is the suspension of another one shows that, more generally, the (reduced) suspension of a reduced CW-complex is also a reduced CW-complex.

### 2.2.4 The Grassman manifolds.

The *Grassman* manifold $G^{2,2} = G$ of planes in $\mathbb{R}^4$ is a topological space. Let us describe a CW-complex structure for it. A plane can be defined by two independent elements, but many choices are possible. The following set of matrices contains such choices:

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},\ M_2(\alpha) = \begin{bmatrix} 1 & 0 \\ 0 & \alpha \\ 0 & \beta \\ 0 & 0 \end{bmatrix},\ M_3(\alpha,\gamma) = \begin{bmatrix} \alpha & \gamma \\ \beta & 0 \\ 0 & \delta \\ 0 & 0 \end{bmatrix},$$

$$M_4(\alpha,\beta) = \begin{bmatrix} 1 & 0 \\ 0 & \alpha \\ 0 & \beta \\ 0 & \gamma \end{bmatrix},\ M_5(\alpha,\gamma,\delta) = \begin{bmatrix} \alpha & \gamma \\ \beta & 0 \\ 0 & \delta \\ 0 & \varepsilon \end{bmatrix},\ M_6(\alpha,\beta,\delta,\varepsilon) = \begin{bmatrix} \alpha & \delta \\ \beta & \varepsilon \\ \gamma & 0 \\ 0 & \eta \end{bmatrix}.$$

If we add the conditions that each column vector is normalized ($\alpha^2 + \ldots = 1$), and that the last variable coefficient of each column is strictly positive, then every plane is represented exactly by one of these matrices. For example a matrix $M_2(\alpha)$ is associated to every plane containing the first axis, contained in the sum of the three first axes, but not in the sum of the two first ones; the inequality $|\alpha| < 1$ must be satisfied and $\alpha$ defines $\beta = +\sqrt{1-\alpha^2}$.

Our reference textbook [11] claims (pp. 13-14) these matrix sets define a CW-complex structure on $G$, but this is wrong. The 0-skeleton $G_0$ is the matrix $M_1$, that is, the plane spanned by the axes 1 and 2. The matrices $\{M_2(\alpha)\}_{\alpha\in[-1,+1]}$, an interval of planes, must be attached to $M_1$, identifying $M_2(1)$ and $M_2(-1)$ to $M_1$; we then have the 1-skeleton $G_1$, a circle of planes. We could hope to continue in the same way; the set of $M_3$-matrices should be parametrized by $\{(\alpha,\gamma) \in [-1,+1]^2\}$; it is natural to identify $M_3(\pm1,\gamma)$ with $M_2(0)$, at least if $|\gamma| < 1$, and also to identify $M_3(\alpha,\pm1)$ with $M_1$ if $|\alpha| < 1$; but you are unable to define such an identification for $M_3(\pm1,\pm1)$: the attaching map described in [11, p. 14] does not work in general in the corners.

A correct solution for this problem is given in [15, Section 6] and consists in choosing *orthonormal* bases for our planes, starting from the simplest vectors to the longest ones. For example the two columns of $M_3(\alpha,\gamma)$ are in general not orthogonal, so that it is better to use the matrices:

$$N_3(\alpha,\gamma) = \begin{bmatrix} \alpha & \gamma\beta \\ \beta & -\gamma\alpha \\ 0 & \delta \\ 0 & 0 \end{bmatrix}$$

where columns are orthogonal; there is a canonical bijection between the sets of matrices $M_3(\alpha,\gamma)$ and $N_3(\alpha,\gamma)$ for $|\alpha|, |\gamma| < 1$, $\beta = +\sqrt{1-\alpha^2}$ and $\delta = +\sqrt{1-\gamma^2}$, the bijection identifying two matrices when the associated planes are the same; but this bijection cannot be extended to the square boundaries. Our (right) 2-cell is the set $\{N_3(\alpha,\gamma); \alpha,\gamma \in [-1,+1]\}$; the attaching map is defined by $\phi_2(N_3(\pm1,\gamma)) = M_2(\mp\gamma)$ and $\phi_2(N_3(\alpha,\pm1)) = M_1$; in particular the four corners of the square are coherently mapped to $M_1$; the attaching map $\phi_2 : \partial([-1,+1]^2) \to G_1$ is of degree 2 and this subspace of planes in $G$ is the real projective plane.

Modifying in an analogous way the definitions of the matrix sets $M_5$ and $M_6$, you can prove the Grassman space $G = G^{2,2}$ admits a CW-complex structure with one 0-cell, one 1-cell, two 2-cells, one 3-cell and one 4-cell; its Euler characteristic is therefore 2.

More generally the Grassman space $G^{m,n}$ of $m$-subspaces in $\mathbb{R}^{m+n}$ has a CW-complex structure where any sequence $(p_m, \ldots, p_1)$ satisfying $\sum p_i = n$ produces a cell of dimension $\sum ip_i$; for example the set of $M_5$-matrices in our description of $G^{2,2}$ corresponds to the index $(1,1)$ giving a cell of dimension $(2 \times 1) + 1 = 3$.

The right reference about these questions is therefore [15] where it is explained how the cellular structure so defined for Grassman manifolds leads to *characteristic classes* of vector bundles.

#### 2.2.5 Projective spaces.

The Grassman space $G^{1,n}$ is nothing but the real projective space $P^n(\mathbb{R})$ which therefore admits a CW-complex structure with one $d$-cell for every $0 \leq d \leq n$. These considerations can be generalized in the same way to other $\mathbb{R}$-algebras $\mathbb{C}$ and $\mathbb{H}$, giving a structure of CW-complex to $P^n(\mathbb{C})$ and $P^n(\mathbb{H})$, and still more generally to the Grassman space $G^{m,n}(K)$, the base field $K$ being $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

#### 2.2.6 Morse functions and CW-complexes.

If a (differentiable) manifold $M$ is provided with a *Morse function*, that is, a $C^\infty$ function $f : M \to \mathbb{R}$ with only non-degenerate singularities (in a local chart near a singularity $x_0$, the function can be described $f(x) = f(x_0) + Q(x - x_0)$, $Q$ a non-degenerate quadratic form), and if the critical values $f(x_0)$ are different from each other, then a CW-complex structure is given through $f$. For example the ordinary torus $T$ described as the surface of the points of $\mathbb{R}^3$ at the distance 1 from the circle $\{(x, y, z); x^2 + y^2 = 1, z = 0\}$ can be provided with the Morse function $f(x, y, z) = x$; four critical values $-3$, $-1$, $+1$ and $+3$ with the respective *indices* 0, 1, 1 and 2, defining a CW-complex structure of the torus with one 0-cell, two 1-cells and one 2-cell. This point of view about manifold descriptions is the root of Milnor's version of the $h$-cobordism theorem [13, 14].

#### 2.2.7 Product of CW-complexes.

If $X$ and $Y$ are CW-complexes, they naturally define another product CW-complex which will be denoted by $Z = X \widetilde{\times} Y$. The underlying point sets behave in an ordinary way, no surprise, but some precautions are to be applied from the topological point of view.

The $n$-skeleton of $Z$ is $Z_n = \cup_{i=0}^n X_i \widetilde{\times} Y_{n-i}$ to be considered as recursively defined, so that we must describe the $(n+1)$-cells of $Z_{n+1}$. Every $(n+1)$-cell of $Z_{n+1}$ is more or less the product of a $p$-cell of $X$ by a $q$-cell of $Y$, the relation $p + q = n + 1$ being satisfied; this $(n+1)$-cell is understood as $D^p \times D^q$ and there is a canonical way to attach it to $Z_n$. Processing in such a way all the pairs of $p$-cells and $q$-cells satisfying $p+q=n+1$ gives the wished $(n+1)$-skeleton $Z_{n+1}$. No surprise yet.

The possible pitfall is about topology. We must decide whether $X\tilde{\times}Y$ and $X \times Y$ (the product as topological spaces) are homeomorphic; not always. There is a canonical bijective continuous map $\phi : X\tilde{\times}Y \to X \times Y$ but the inverse map is not necessarily continuous; it is also continuous if one of the factors $X$ or $Y$ is a finite CW-complex (the number of cells is finite) or when both have a countable set of cells, see [11]. The first standard counter-example is the following. Let us define $X$ as the wedge of an infinite number of intervals $[0,1]$ indexed by $\mathbb{N}$, the same for $Y$ but the index set is $\mathbb{R}$:

$$X = \vee_{\alpha\in\mathbb{N}} I_\alpha \quad ; \quad Y = \vee_{\beta\in\mathbb{R}} I_\beta.$$

It is a pretty exercise to understand the reason why in this case the inverse map $\phi^{-1}$ is not continuous at the base point, see [4], or [7, p. 59][3]. This is not really important: in fact the map $\phi$ is a homotopy equivalence and anyway the right point of view is to consider the product in the category of CW-complexes which is the $\tilde{\times}$-product, see [7, Section 2.2]; by the way, what about the morphisms?

## 2.3 Cellular maps.

We could imagine the good morphisms between CW-complexes should more or less be compatible with the cellular structure, but such a condition is a little too strong. For example let $X$ be the circle $S^1$ and $Y$ the wedge $S^1 \vee S^1$; then a map $f : X \to Y$ such that $\pi_1(f)$ sends the generator of $\pi_1(X)$ to the product of both generators of $\pi_1(Y)$ is not homotopic to $g$ mapping the 1-cell in one 1-cell. So that the appropriate definition is the following.

**Definition 2** — If $X$ and $Y$ are CW-complexes, a *cellular map* $f : X \to Y$ is a continuous map satisfying the relation $f(X_n) \subset Y_n$ for every $n$.

In this way the map $f : X \to Y$ just above is cellular if the base point is mapped to the base point. The same difficulty is met with the simplicial sets but frequently overcome thanks to the elegant *Kan condition*.

Up to homotopy, any map is cellular:

**Theorem 3** — *If $f : X \to Y$ is a continuous map between CW-complexes, $f$ is homotopic to a cellular map $g$.*

A relative version can also be stated and proved. This is the main tool allowing to build demonstrations by *climbing over the skeleton*.

[3] The analogy between the necessary negative argument and the incompleteness Gödel theorem is striking.

### 2.4 Extending homotopies.

Two main ingredients are used to prove the previous theorem; the most elementary is the homotopy extension property satisfied by the relative CW-complexes.

**Theorem 4** — *A relative CW-complex $(X, A)$ satisfies the* homotopy extension property*; that is, if $f : (X \times 0) \cup (A \times I) \to Y$ is a continuous given map, then a continuous extension $F : (X \times I) \to Y$ of $f$ can be constructed.*

PROOF. It is sufficient to recursively construct an extension $F_n : (X \times 0) \cup (X_n \times I) \to Y$; this is a direct consequence of the same property for the pair $(X, A) = (D^n, S^{n-1})$, which in turn results from a strong retracting deformation $(D^n \times I) \Rightarrow ((S^{n-1} \times I) \cup (D^n \times 0))$ consisting in using a radial process centered at $(0 \times 2)$. ■

The other main ingredient, significantly more interesting, is a dimension property; a simple case where the essential argument begins to be visible is the following proposition.

**Proposition 5** — *Let $f : S^1 \to S^2$ be a continuous map. Then $f$ is homotopic to a constant map.*

PROOF. Consider $S^2$ as the boundary of the 3-simplex $\Delta^3$, covered by 4 open sets $\{U_i\}_{0 \leq i \leq 3}$, the interiors of the unions of every combination of three faces, each interior being “centered" at a vertex $v_i$. The $f$-preimage of this covering is a covering of $S^1$ and a compactness argument gives $0 = a_0 < \ldots < a_n = 1$ ($S^1$ is parametrized by $[0, 1]$) such that $f([a_{j-1}, a_{j+1}]) \subset U_{i_j}$. We decide to define $f'(a_j) = v_j$ and to affinely extend $f'$ between $a_j$ and $a_{j+1}$, with values on the edge between $v_{i_j}$ and $v_{i_{j+1}}$. It is easy to see there is a canonical homotopy between $f$ and $f'$ and we have succeeded in deform $f$ into a map $f'$ which runs only along the edges of $\Delta_3$. In particular $v_{i_0} = v_{i_n}$, that is, $i_0 = i_n$. We are mainly interested by the *cellular* approximation $f'$ of $f$: the map $f$ is defined on a 1-dimensional object and the obtained deformation runs along the 1-skeleton of $\Delta^3$. Finally because this 1-skeleton can be deformed in $\Delta^3$ to a vertex, the homotopy of $f'$ and also $f$ to a constant map is obtained. ■

## 3 The category $\Delta$.

Some strongly structured sets of indices are necessary to define the notion of *simplicial object*; they are conveniently organized as the category $\Delta$. An object of $\Delta$ is a set $\underline{\mathbf{m}}$, namely the set of integers $\underline{\mathbf{m}} = \{0, 1, \ldots, m-1, m\}$; this set is canonically *ordered* with the usual order between integers.

A $\Delta$-morphism $\alpha : \underline{\mathbf{m}} \to \underline{\mathbf{n}}$ is an *increasing* map. Equal values are permitted; for example a $\Delta$-morphism $\alpha : \underline{\mathbf{2}} \to \underline{\mathbf{3}}$ could be defined by $\alpha(0) = \alpha(1) = 1$ and $\alpha(2) = 3$. The set of $\Delta$-morphisms from $\underline{\mathbf{m}}$ to $\underline{\mathbf{n}}$ is denoted by $\Delta(\underline{\mathbf{m}}, \underline{\mathbf{n}})$; the subset of injective (resp. surjective) morphisms is denoted by $\Delta^{\mathrm{inj}}(\underline{\mathbf{m}},\underline{\mathbf{n}})$ (resp. $\Delta^{\mathrm{srj}}(\underline{\mathbf{m}},\underline{\mathbf{n}})$).

Some *elementary* morphisms are important, namely the simplest non-surjective and non-injective morphisms. For geometric reasons explained later, the first ones are the *face morphisms*, the second ones are the *degeneracy morphisms*.

**Definition 6** — The *face morphism* $\partial_i^m : \underline{\mathbf{m-1}} \to \underline{\mathbf{m}}$ is defined for $m \geq 1$ and $0 \leq i \leq m$ by:

$$\begin{aligned} \partial_i^m(j) = & \; j & \text{if } j < i, \\ \partial_i^m(j) = & \; j+1 & \text{if } j \geq i. \end{aligned}$$

The face morphism $\partial_i^m$ is the unique injective morphism from $\underline{\mathbf{m-1}}$ to $\underline{\mathbf{m}}$ such that the integer $i$ is not in the image. The face morphisms generate the injective morphisms, in fact in a unique way if a growth condition is required.

**Proposition 7** — *Any injective $\Delta$-morphism $\alpha \in \Delta^{\mathrm{inj}}(\underline{\mathbf{m}},\underline{\mathbf{n}})$ has a unique expression:*

$$\alpha = \partial_{i_n}^n \circ \ldots \circ \partial_{i_{m+1}}^{m+1}$$

*satisfying the relation $i_n > i_{n-1} > \ldots > i_{m+1}$.*

Proof. The index set $\{i_{m+1}, \ldots, i_n\}$ is exactly the difference set $\underline{\mathbf{n}} - \alpha(\underline{\mathbf{m}})$, that is, the set of the integers where surjectivity fails. ■

Frequently the upper index $m$ of $\partial_i^m$ is omitted because clearly deduced from the context. For example the unique injective morphism $\alpha : \underline{\mathbf{2}} \to \underline{\mathbf{5}}$ the image of which is $\{0,2,4\}$ can be written $\alpha = \partial_5\partial_3\partial_1$.

If two face morphisms are composed in the wrong order, they can be exchanged: $\partial_i \circ \partial_j = \partial_{j+1} \circ \partial_i$ if $j \geq i$. Iterating this process allows you to quickly compute for example $\partial_0\partial_2\partial_4\partial_6 = \partial_9\partial_6\partial_3\partial_0$.

**Definition 8** — The *degeneracy morphism* $\eta_i^m : \underline{\mathbf{m+1}} \to \underline{\mathbf{m}}$ is defined for $m \geq 0$ and $0 \leq i \leq m$ by:

$$\begin{aligned} \eta_i^m(j) = & \; j & \text{if } j \leq i, \\ \eta_i^m(j) = & \; j-1 & \text{if } j > i. \end{aligned}$$

The degeneracy morphism $\eta_i^m$ is the unique surjective morphism from $\underline{\mathbf{m+1}}$ to $\underline{\mathbf{m}}$ such that the integer $i$ has two pre-images. The degeneracy morphisms generate the surjective morphisms, in fact in a unique way if a growth condition is required.

**Proposition 9** — *Any surjective $\Delta$-morphism $\alpha \in \Delta^{\mathrm{srj}}(\underline{\mathbf{m}},\underline{\mathbf{n}})$ has a unique expression:*

$$\alpha = \eta_{i_n}^n \circ \ldots \circ \eta_{i_{m-1}}^{m-1}$$

*satisfying the relation $i_n < i_{n+1} < \ldots < i_{m-1}$.*

PROOF. The index set $\{i_n, \ldots, i_{m-1}\}$ is exactly the set of integers $j$ such that $\alpha(j) = \alpha(j+1)$, that is, the integers where injectivity fails. ■

Frequently the upper index $m$ of $\eta_i^m$ is omitted because clearly deduced from the context. For example the unique surjective morphism $\alpha : \underline{\mathbf{5}} \to \underline{\mathbf{2}}$ such that $\alpha(0) = \alpha(1)$ and $\alpha(2) = \alpha(3) = \alpha(4)$ can be expressed $\alpha = \eta_0\eta_2\eta_3$.

If two face morphisms are composed in the wrong order, they can be exchanged: $\eta_i \circ \eta_j = \eta_j \circ \eta_{i+1}$ if $i \geq j$. Iterating this process allows you to quickly compute for example $\eta_3\eta_3\eta_2\eta_2 = \eta_2\eta_3\eta_5\eta_6$.

**Proposition 10** — *Any $\Delta$-morphism $\alpha$ can be $\Delta$-decomposed in a unique way:*

$$\alpha = \beta \circ \gamma$$

*with $\beta$ injective and $\gamma$ surjective.*

PROOF. The intermediate $\Delta$-object $\underline{\mathbf{k}}$ necessarily satisfies $k + 1 = \mathrm{Card}(\mathbf{im}(\alpha))$. The growth condition then gives a unique choice for $\beta$ and $\gamma$. ■

**Corollary 11** — *Any $\Delta$-morphism $\alpha : \underline{\mathbf{m}} \to \underline{\mathbf{n}}$ has a unique expression:*

$$\alpha = \partial_{i_n} \circ \ldots \circ \partial_{i_{k+1}} \circ \eta_{j_k} \circ \ldots \circ \eta_{j_{m-1}}$$

*satisfying the conditions $i_n > \ldots > i_{k+1}$ and $j_k < \ldots < j_{m-1}$.* ■

Finally if face and degeneracy morphisms are composed in the wrong order, they can be exchanged:

$$\begin{array}{rcll} \eta_i \circ \partial_j & = & \mathrm{id} & \text{if } j = i \text{ or } j = i+1; \\ & = & \partial_{j-1} \circ \eta_i & \text{if } j \geq i+2; \\ & = & \partial_j \circ \eta_{i-1} & \text{if } j < i. \end{array}$$

All these commuting relations can be used to convert an arbitrary composition of faces and degeneracies into the canonical expression:

$$\alpha = \eta_9\partial_6\eta_3\partial_7\eta_9\partial_8\eta_6\partial_2\eta_4\partial_9 = \partial_7\partial_6\partial_2\eta_2\eta_4\eta_6.$$

This relation means the image of $\alpha$ does not contain the integers 2, 6 and 7, and the relations $\alpha(2) = \alpha(3)$, $\alpha(4) = \alpha(5)$ and $\alpha(6) = \alpha(7)$ are satisfied.

**Corollary 12** — *A* contravariant *functor $X : \Delta \to \mathrm{CAT}$ is nothing but a collection $\{X_m\}_{m\in\mathbb{N}}$ of objects of the target category* CAT, *and collections of* CAT*-morphisms $\{X(\partial_i^m) : X_m \to X_{m-1}\}_{m\geq 1\,,\,0\leq i\leq m}$ and $\{X(\eta_i^m) : X_m \to X_{m+1}\}_{m\geq 0\,,\,0\leq i\leq m}$ satisfying the commuting relations:*

$$\begin{array}{rcll} X(\partial_i) \circ X(\partial_j) & = & X(\partial_j) \circ X(\partial_{i+1}) & \text{if } i \geq j, \\ X(\eta_i) \circ X(\eta_j) & = & X(\eta_{j+1}) \circ X(\eta_i) & \text{if } j \geq i, \\ X(\partial_i) \circ X(\eta_j) & = & \mathrm{id} & \text{if } i = j, j+1, \\ X(\partial_i) \circ X(\eta_j) & = & X(\eta_{j-1}) \circ X(\partial_i) & \text{if } j > i, \\ X(\partial_i) \circ X(\eta_j) & = & X(\eta_j) \circ X(\partial_{i-1}) & \text{if } i > j+1. \end{array}$$

In the five last relations, the upper indices have been omitted. Such a contravariant functor is a *simplicial object* in the category CAT. If $\alpha$ is an arbitrary $\Delta$-morphism, it is then sufficient to express $\alpha$ as a composition of face and degeneracy morphisms; the image $X(\alpha)$ is necessarily the composition of the images of the corresponding $X(\partial_i)$'s and $X(\eta_i)$'s; the above relations assure the definition is coherent.

# 4 Simplicial sets.

## 4.1 Terminology and notations.

**Definition 13** — A *simplicial set* is a simplicial object in the category of sets.

A simplicial set $X$ is given by a collection of sets $\{X(\underline{\mathbf{m}})\}_{m\in\mathbb{N}}$ and collections of maps $\{X_\alpha\}$, the index $\alpha$ running the $\Delta$-morphisms; the usual coherence properties must be satisfied. As explained at the end of the previous section, it is sufficient to define the $X(\partial_i^m)$'s and the $X(\eta_i^m)$'s with the corresponding commuting relations.

The set $X(\underline{\mathbf{m}})$ is usually denoted by $X_m$ and is called the set of $m$-simplices of $X$; such a simplex has the *dimension $m$*. To be a little more precise, these simplices are sometimes called *abstract* simplices, to avoid possible confusions with the *geometric* simplices defined a little later. An (abstract) $m$-simplex is only *one* element of $X_m$.

If $\alpha \in \Delta(\underline{\mathbf{n}}, \underline{\mathbf{m}})$, the corresponding morphism $X(\alpha) : X_m \to X_n$ is most often simply denoted by $\alpha^* : X_m \to X_n$ or still more simply $\alpha : X_m \to X_n$. In particular the faces and degeneracy operators are maps $\partial_i : X_m \to X_{m-1}$ and $\eta_i : X_m \to X_{m+1}$. If $\sigma$ is an $m$-simplex, the (abstract) simplex $\partial_i\sigma$ is its $i$-th face, and the simplex $\eta_i\sigma$ is its $i$-th degeneracy; we will see the last one is "particularly" abstract.

## 4.2 The structure of simplex sets.

**Definition 14** — An $m$-simplex $\sigma$ of the simplicial set $X$ is *degenerate* if there exist an integer $n < m$, an $n$-simplex $\tau \in X_n$ and a $\Delta$-morphism $\alpha \in \Delta(\underline{\mathbf{m}}, \underline{\mathbf{n}})$ such that $\sigma = \alpha(\tau)$. The set of non-degenerate simplices of dimension $m$ in $X$ is denoted by $X_m^{ND}$.

Decomposing the morphism $\alpha = \beta \circ \gamma$ with $\gamma$ surjective, we see that $\sigma = \gamma(\beta(\tau))$, with the dimension of $\beta(\tau)$ less or equal to $n$; so that in the definition of degenaracy, the connecting $\Delta$-morphism $\alpha$ can be required to be surjective. The relation $\sigma = \alpha(\tau)$ with $\alpha$ surjective is shortly expressed by saying the $m$-simplex $\sigma$ *comes from* the $n$-simplex $\tau$.

Eilenberg's lemma explains each degenerate simplex comes from a canonical non-degenerate one.

**Lemma 15** — **(Eilenberg's lemma)** *If $X$ is a simplicial set and $\sigma$ is an $m$-simplex of $X$, there exists a unique triple $T_\sigma = (n, \tau, \alpha)$ satisfying the following conditions:*

1. *The first component $n$ is a natural number $n \leq m$;*
2. *The second component $\tau$ is a non-degenerate $n$-simplex $\tau \in X_n^{ND}$;*
3. *The third component $\alpha$ is a $\Delta$-morphism $\tau \in \Delta^{\mathrm{srj}}(\underline{\mathbf{m}}, \underline{\mathbf{n}})$;*
4. *The relation $\sigma = \alpha(\tau)$ is satisfied.*

**Definition 16** — This triple $T_\sigma$ is called the *Eilenberg triple* of $\sigma$.

Proof. Let $\mathcal{T}$ be the set of triples $T = (n, \tau, \alpha)$ such that $n \leq m$, $\tau \in X_n$ and $\alpha \in \Delta(\underline{\mathbf{m}}, \underline{\mathbf{n}})$ satisfy $\sigma = \alpha(\tau)$. The set $\mathcal{T}$ certainly contains the triple $(m, \sigma, \mathrm{id})$ and therefore is non empty. Let $(n_0, \tau_0, \alpha_0)$ be an element of $\mathcal{T}$ where the first component, the integer $n_0$, is minimal. We claim $(n_0, \tau_0, \alpha_0)$ is the Eilenberg triple.

Certainly $n_0 \leq m$. The $n_0$-simplex $\tau_0$ is non-degenerate; otherwise $\tau_0 = \beta(\tau_1)$ with the dimension $n_1$ of $\tau_1$ less than $n_0$, but then $(n_1, \tau_1, \beta\alpha_0)$ would be a triple with $n_1 < n_0$. Finally $\alpha_0$ is surjective, otherwise $\alpha_0 = \beta\gamma$ with $\gamma \in \Delta^{\mathrm{srj}}(m, n_1)$ and $n_1 < n_0$; but again the triple $(n_1, \beta(\tau_0), \gamma)$ would be a triple denying the required property of $n_0$. The existence of an Eilenberg triple is proved and uniqueness remains to be proved.

Let $(n_1, \tau_1, \alpha_1)$ be another Eilenberg triple. The morphisms $\alpha_0$ and $\alpha_1$ are surjective and respective sections $\beta_0 \in \Delta^{\mathrm{inj}}(\underline{\mathbf{n_0}}, \underline{\mathbf{m}})$ and $\beta_1 \in \Delta^{\mathrm{inj}}(\underline{\mathbf{n_1}}, \underline{\mathbf{m}})$ can be constructed: $\alpha_0\beta_0 = \mathrm{id}$ and $\alpha_1\beta_1 = \mathrm{id}$. Then $\tau_0 = (\alpha_0\beta_0)(\tau_0) = \beta_0(\alpha_0(\tau_0)) = \beta_0(\sigma) = \beta_0(\alpha_1(\tau_1)) = (\alpha_1\beta_0)(\tau_1)$; but $\tau_0$ is non-degenerate, so that $n_1 = \dim(\tau_1) \geq n_0 = \dim(\tau_0)$; the analogous relation holds when $\tau_0$ and $\tau_1$ are exchanged, so that $n_1 \leq n_0$ and the equality $n_0 = n_1$ is proved.

The relation $\tau_0 = \beta_0(\alpha_1(\tau_1))$ with $\tau_0$ non-degenerate implies $\alpha_1\beta_0 = \mathrm{id}$, otherwise $\alpha_1\beta_0 = \gamma\delta$ with $\delta \in \Delta^{\mathrm{srj}}(\underline{\mathbf{n_1}}, \underline{\mathbf{n_2}})$ and $n_2 < n_1 = n_0$, but this implies $\tau_0$ comes from $\gamma(\tau_1)$ of dimension $n_2$ again contradicting the non-degeneracy property of $\tau_0$; therefore $\alpha_1\beta_0 = \mathrm{id}$ but this equality implies $\tau_0 = \tau_1$.

If $\alpha_0 \neq \alpha_1$, let $i$ be an integer such that $\alpha_0(i) = j \neq \alpha_1(i)$; then the section $\beta_0$ can be chosen with $\beta_0(j) = i$; but this implies $(\alpha_1\beta_0)(j) \neq j$, so that the relation $\alpha_1\beta_0 = \mathrm{id}$ would not hold. The last required equality $\alpha_0 = \alpha_1$ is also proved. ∎

Each simplex comes from a unique non-degenerate simplex, and conversely, for any non-degenerate $m$-simplex $\sigma \in X_m^{ND}$, the collection $\{\alpha(\sigma)\,;\, \alpha \in \Delta^{\mathrm{srj}}(\underline{\mathbf{n}}, \underline{\mathbf{m}})\,;\, n \geq m\}$ is a perfect description of all simplices coming from $\sigma$, that is, of all degenerate simplices *above* $\sigma$. This is also expressed in the following formula, describing the structure of the simplex set of any simplicial set $X$:

$$\coprod_{m \in \mathbb{N}} X_m = \coprod_{m \in \mathbb{N}} \coprod_{\sigma \in X_m^{ND}} \coprod_{n \geq m} \Delta^{\mathrm{srj}}(\underline{n}, \underline{m})(\sigma).$$

## 4.3 Examples.

### 4.3.1 Discrete simplicial sets.

**Definition 17** — A simplicial set $X$ is *discrete* if $X_m = X_0$ for every $m \geq 1$, and if for every $\alpha \in \Delta(\underline{\mathbf{m}}, \underline{\mathbf{n}})$, the induced map $\alpha^* : X_n \to X_m$ is the identity.

The reason of this definition is that the *realization* (see Section 4.4) of such a simplicial set is the discrete point set $X_0$; the Eilenberg triple of any simplex $\sigma \in X_m = X_0$ is $(0, \sigma, \alpha)$ where the map $\alpha$ is the unique element of $\Delta(\underline{\mathbf{m}}, \underline{\mathbf{0}})$.

### 4.3.2 The simplicial complexes.

A *simplicial complex* $K = (V, S)$ is a pair where $V$ is the *vertex set* (an arbitrary set, finite or not), and $S \subset \mathcal{P}_F(V)$ is a set of finite sets of vertices satisfying the properties:

1. For any $v \in V$, the one element subset $\{v\}$ of $V$ is an element of $S$;
2. For any $\tau \subset \sigma \in S$, then $\tau \in S$.

The *simplex* $\sigma \in S$ *spans* its elements. If $S = \mathcal{P}_F(V)$, then $K$ is the *simplex* freely generated by $V$, or more simply the simplex spanned by $V$.

The terminology is a little incoherent because a simplicial *set* is an object more sophisticated than a simplicial *complex*, but this terminology is so well established that it is probably too late to modify it.

The simplicial complex $K = (V, S)$ is *ordered* if the vertex set $V$ is provided with a *total* order[4]. Then a simplicial set again denoted by $K$ is canonically associated; the simplex set $K_m$ is the set of *increasing* maps $\sigma : \underline{\mathbf{m}} \to K$ such that the image of $\underline{\mathbf{m}}$ is an element of $S$; note that such a map $\sigma$ is not necessarily injective. If $\alpha$ is a $\Delta$-morphism $\alpha \in \Delta(\underline{\mathbf{n}}, \underline{\mathbf{m}})$ and $\sigma$ is an $m$-simplex $\sigma \in K_m$, then $\alpha(\sigma)$ is naturally defined as $\alpha(\sigma) = \sigma \circ \alpha$. A simplex $\sigma \in K_m$ is non-degenerate if and only if $\sigma \in \Delta^{\text{inj}}(\underline{\mathbf{m}}, V)$; if $\sigma \in K_m = \Delta(\underline{\mathbf{m}}, V)$, the Eilenberg triple $(n, \tau, \alpha)$ satisfies $\sigma = \tau \circ \alpha$ with $\alpha$ surjective and $\tau$ injective.

This in particular works for $K = (\underline{\mathbf{d}}, \mathcal{P}(\underline{\mathbf{d}}))$ the simplex freely generated by $\underline{\mathbf{d}}$ provided with the canonical vertex order. We obtain in this way the canonical structure of simplicial set for the *standard d-simplex* $\Delta^d$.

### 4.3.3 The spheres.

Let $d$ be a natural number. The simplest simplicial version $S = S^d$ of the $d$-sphere is defined as follows: the set of $m$-simplices $S_m$ is $S_m = \{*_m\} \coprod \Delta^{\text{srj}}(\underline{\mathbf{m}}, \underline{\mathbf{d}})$; if $\alpha \in \Delta(\underline{\mathbf{n}}, \underline{\mathbf{m}})$ and $\sigma$ is an $m$-simplex $\sigma \in S_m$, then $\alpha(\sigma)$ depends on the nature of $\sigma$:

1. If $\sigma = *_m$, then $\alpha(\sigma) = *_n$;

[4] Other situations where the order is not total are also interesting but will be considered later.

2. Otherwise $\sigma \in \Delta^{\text{srj}}(\underline{\mathbf{m}}, \underline{\mathbf{d}})$ and if $\sigma \circ \alpha$ is surjective, then $\alpha(\sigma) = \sigma \circ \alpha$, else $\alpha(\sigma) = *_n$ (the emergency solution when the natural solution does not work).

This is nothing but the canonical quotient $S^d = \Delta^d / \partial \Delta^d$, at least if $d > 0$; more generally the notion of simplicial subset is naturally defined and a quotient then appears. In the case of the construction of $S^d = \Delta^d / \partial \Delta^d$, the subcomplex $\partial \Delta^d$ is made of the simplices $\alpha \in \Delta(\underline{\mathbf{m}}, \underline{\mathbf{d}})$ that are not surjective.

The Eilenberg triple of $*_m$ is $(0, *_0, \alpha)$ where *alpha* is the unique element of $\Delta(\underline{\mathbf{m}}, \underline{\mathbf{0}})$. The Eilenberg triple of $\sigma \in \Delta^{\text{srj}}(\underline{\mathbf{m}}, \underline{\mathbf{d}}) \subset S_m$ is $(d, \text{id}, \sigma)$. There are only two non-degenerate simplices, namely $*_0 \in S_0$ and $\text{id}(\underline{\mathbf{d}}) \in S_d$, even if $d = 0$.

### 4.3.4 Classifying spaces of discrete groups.

Let $G$ be a (discrete) group. Then a simplicial version of its classifying space $BG$ can be given. The set of $m$-simplices $BG_m$ is the set of "$m$-bars" $\sigma = [g_1| \ldots |g_m]$ where every $g_i$ is an element of $G$. It is simpler in this situation to define the structure morphisms only for the face and degeneracy operators:

1. $\partial_0[g_1| \ldots |g_m] = [g_2| \ldots |g_m]$;
2. $\partial_m[g_1| \ldots |g_m] = [g_1| \ldots |g_{m-1}]$;
3. $\partial_i[g_1| \ldots |g_m] = [\ldots |g_{i-1}|g_i g_{i+1}|g_{i+2}| \ldots]$ if $0 < i < m$;
4. $\eta_i[g_1| \ldots |g_m] = [\ldots |g_i|e_G|g_{i+1}| \ldots]$, where $e_G$ is the unit element of $G$.

In particular $BG_0 = \{[\,]\}$ has only one element.

The $m$-simplex $[g_1| \ldots |g_m]$ is degenerate if and only if one of the $G$-components is the unit element.

The various commuting relations must be verified; this works but does not give obvious indications on the very nature of this construction; in fact there is a more conceptual description. Let us define the simplicial set $EG$ by $EG_m = \text{SET}(\underline{\mathbf{m}}, G)$, that is, the maps of $\underline{\mathbf{m}}$ to $G$ without taking account of the ordered structure of $\underline{\mathbf{m}}$ (the group $G$ is not ordered); if $\alpha \in \Delta(\underline{\mathbf{n}}, \underline{\mathbf{m}})$ there is a canonical way to define $\alpha : EG_m \to EG_n$; it would be more or less coherent to write $EG = G^\Delta$.

There is a canonical left action of the group $G$ on $EG$, and $BG$ is the natural quotient of $EG$ by this action. A simplex $\sigma \in EG_m$ is nothing but a $(m+1)$-tuple $(g_0, \ldots, g_m)$ and the action of $g$ gives the simplex $(gg_0, \ldots, gg_m)$. If two simplices are $G$-equivalent, the products $g_{i-1}^{-1} g_i$ are the same; the quotient $BG$-simplex $[g_1, \ldots, g_m]$ denotes the equivalence class of all the $EG$-simplices $(g, gg_1, gg_1g_2, \ldots)$, which can be imagined as a simplex where the *edge* between the vertices $i-1$ and $i$ $(i > 0)$ is labeled by $g_i$ to be considered as a (right) operator between the adjacent vertices. Then the boundary and degeneracy operators are clearly explained and it is even not necessary to prove the commuting relations, they can be deduced of the coherent structure of $EG$.

### 4.3.5 The Eilenberg-MacLane spaces.

The previous example constructs an *Eilenberg-MacLane* space, that is, a space with only one non-zero homotopy group. The *realization* process (see later) applied to the simplicial set $BG$ produces a model for $K(G,1)$: all the homotopy groups are null except $\pi_1$ canonically isomorphic to $G$. The construction can be generalized to construct $K(\pi,d)$, $d>1$, when $\pi$ is an *abelian* group. This requires the simplicial definition of homology groups, explained in another lecture series. See also [12, Chapter V] where these questions are carefully detailed.

Let $\pi$ be a fixed abelian group, and $d$ a natural number. The simplicial set $E(\pi,d)$ is defined as follows. The set of $m$-simplices $E(\pi,d)_m$, shortly denoted by $E_m$, is $E_m = C^d(\Delta^m,\pi)$, the group of *normalized* $d$-cochains on the standard $m$-simplex with values in $\pi$. Such a cochain $\sigma$ is nothing but a map $\sigma : \Delta^m_d \to \pi$, defined on the (degenerate or not) $d$-simplices of $\Delta^m$, null for the degenerate simplices. If $\alpha$ is a $\Delta$-morphism $\alpha : \underline{\mathbf{n}} \to \underline{\mathbf{m}}$, this map defines a simplicial map $\alpha_* : \Delta^n \to \Delta^m$ which in turns defines a pullback map $\alpha^* : C^d(\Delta^m,\pi) \to C^d(\Delta^n,\pi)$ between $m$-simplices and $n$-simplices of $E_m$.

The simplicial set $E(\pi,d)$ so defined contains the simplicial subset $K(\pi,d)$, made only of the *cocycles*, those cochains the coboundary of which ($d : C^d(\Delta^m,\pi) \to C^{d+1}(\Delta^m,\pi)$ is null. In fact $E(\pi,d)$ is a *simplicial group*, that is, a simplicial object in the category of groups, and $K(\pi,d)$ is a simplicial subgroup. The quotient simplicial group $E(\pi,d)/K(\pi,d)$ is canonically isomorphic to $K(\pi,d+1)$ and this structure defines the Eilenberg-MacLane fibration:

$$K(\pi,d) \hookrightarrow E(\pi,d) \to K(\pi,d+1)$$

See later the section about *simplicial fibrations* for some details.

### 4.3.6 Simplicial loop spaces.

Let $X$ be a simplicial set. We can construct a new simplicial set $DT(X)$ (the acronym $DT$ meaning Dold-Thom) from X, where $DT(X)_m$ is the free $\mathbb{Z}$-module generated by the $m$-simplices $X_m$; the operators of $DT(X)$ are also "generated" by the operators of $X$. This is a simplicial version of the Dold-Thom construction, producing a new simplicial set $DT(X)$, the homotopy groups of which being the homology groups of the initial $X$. The simplicial set $DT(X)$ is also of *simplicial group*; its simplex *sets* are nothing but the chain groups at the origin of the simplicial homology of $X$, but in $DT(X)$, each simplicial "chain" of $X$ is *one* (abstract) simplex. See [12, Section 22].

The same construction can be undertaken, but instead of using the abelian group generated by the simplex sets $X_m$, we could consider the free *non-commutative* group generated by $X_m$. This also works, but then the obtained space is a simplicial model for the *James construction* of $\Omega\Sigma X$, the loop space of the (reduced) suspension of $X$. See [2] for the James construction in general and [3] for the simplicial case.

It is even possible to construct the "pure" loop space $\Omega X$, without any suspension. This is due to Daniel Kan [9] and works as follows. It is necessary to assume $X$ is reduced, that is with only one vertex: the cardinality of $X_0$ is 1. Let $X_m^*$ the set of all $m$-simplices, except those that are 0-degenerate: $X_m^* = X_m - \eta_0(X_{m-1})$; this makes sense for $m \geq 1$. Then let $GX_m$ be the free *non-commutative* group generated by $X_{m+1}^*$; to avoid possible confusions, if $\sigma \in X_{m+1}^*$, let us denote by $\tau(\sigma)$ the corresponding *generator* of $GX_m$. The simplicial object $GX$ to be defined is a simplicial *group*, so that it is sufficient to define face and degeneracy operators for the generators:

$$\begin{array}{rcll} \partial_i \tau(\sigma) & = & \tau(\partial_{i+1}\sigma), & \text{if } 1 \leq i \leq m; \\ \partial_0 \tau(\sigma) & = & \tau(\partial_1 \sigma)\tau(\partial_0 \sigma)^{-1}; & \\ \eta_i \tau(\sigma) & = & \tau(\eta_{i+1}\sigma), & \text{if } 0 \leq i \leq m. \end{array}$$

These definitions are coherent, and the simplicial set $GX$ so obtained is a simplicial version of the loop space construction. See [12, Chapter VI] for details and related questions, mainly the *twisted Eilenberg-Zilber theorem*, at the origin of the general solution described in [17] for the computability problem in algebraic topology.

### 4.3.7 The singular simplicial set.

Let $X$ be an arbitrary topological space. Then the *singular simplicial set* associated to $X$ is constructed as follows. The set of $m$-simplices $SX_m$ is made of the continuous maps $\sigma : \Delta^m \to X$; *one* (abstract) simplex is *one* continuous map but no topology is installed on $SX_m$; in particular when $SX$ will be *realized* in the following section, the *discrete* topology must be used. The source of the abstract $m$-simplex $\sigma$ is the geometric $m$-simplex $\Delta^m \subset \mathbb{R}^{\underline{m}}$ provided with the traditional topology. If $\alpha \in \Delta(\underline{\mathbf{n}}, \underline{\mathbf{m}})$ is a $\Delta$-morphism, this $\alpha$ defines a natural continuous map $\alpha_* : \Delta^n \to \Delta^m$ between geometric simplices, and this allows us to naturally define $\alpha^*(\sigma) = \sigma \circ \alpha_*$. An enormous simplicial set is so defined if $X$ is an arbitrary topological space; it is at the origin of the *singular homology* theory.

## 4.4 Realization.

If $K = (V, S)$ is a simplicial set, the realization $|K|$ is a subset of $\mathbb{R}^{(V)}$, the $\mathbb{R}$-vector space generated by the vertices $v \in V$; a point $x \in \mathbb{R}^{(V)}$ is a function $x : V \to \mathbb{R}$ almost everywhere null, that is, the set of $v$'s where $x$ is non-null is finite. Such a function can also be denoted by $x = \{x_v\}_{v \in V}$, the set of indexed values, or also the linear notations $x = \sum x_v.e_v$ or $x = \sum x_v.v$ can be used. Then $|K|$ is the set of $x$'s in $\mathbb{R}^{(V)}$ satisfying the following conditions:

1. For every $v \in V$, the inequality $0 \leq x_v \leq 1$ holds;
2. The relation $\sum_{v \in V} x_v = 1$ is satisfied;
3. The set $\{v \in V \textbf{ st } x_v \neq 0\}$ is a simplex $\sigma \in S$.

The right topology to install on $|K|$ is induced by all the finite dimensional spaces $\mathbb{R}^\sigma$ for $\sigma \in S$. In this way the realization $|K|$ is a CW-complex. In particular, if $\Delta^m$ is the simplex freely generated by $\underline{\mathbf{m}}$, the realization is the standard geometric $m$-simplex again denoted by $\Delta^m$, provided with its ordinary topology. In general the topology of $|K|$ is induced by its (geometric) simplices.

If $\alpha : \underline{\mathbf{m}} \to \underline{\mathbf{n}}$ is a $\Delta$-morphism, then $\alpha$ defines a covariant induced map $\alpha_* : \Delta^m \to \Delta^n$ (between the "simplicial" simplices or the geometric realizations, as you like) and for any simplicial set $X$ a contravariant induced map $\alpha^* : X_n \to X_m$.From now on, unless otherwise stated, $\Delta^m$ denotes the *geometric* standard simplex, that is, the convex hull of the canonical basis of $\mathbb{R}^{\underline{m}}$.

If $X$ is a simplicial set, the (*expensive*) realization $|X|$ of $X$ is:

$$|X| = \coprod_{m \in \mathbb{N}} X_m \times \Delta^m \ / \ \approx .$$

Each component of the coproduct is the product of the discrete set of $m$-simplices by the geometric $m$-simplex; in other words, each abstract simplex $\sigma$ in $X_m$ gives birth to a geometric simplex $\{\sigma\} \times \Delta^m$, and they are attached to each other following the instructions of the equivalence relation $\approx$, to be defined. Let $\alpha \in \Delta(\underline{\mathbf{m}}, \underline{\mathbf{n}})$ be some $\Delta$-morphism, and let $\sigma$ be an $n$-simplex $\sigma \in X_n$ and $t \in \Delta^m \subset \mathbb{R}^{\underline{m}}$. Then the pairs $(\alpha^*\sigma, t)$ and $(\sigma, \alpha_* t)$ are declared equivalent.

It is not obvious to understand what is the topological space so obtained. A description a little more explicit but also a little more complicated explains more satisfactorily what should be understood.

The *cheap* realization $\|X\|$ of the simplicial set $X$ is:

$$\|X\| = \coprod_{m \in \mathbb{N}} X_m^{ND} \times \Delta^m \ / \ \approx$$

where the equivalence relation $\approx$ is defined as follows. Let $\sigma$ be a non-degenerate $m$-simplex and $i$ an integer $0 \leq i \leq m$; let also $t \in \Delta^{m-1}$; the abstract $(m-1)$-simplex $\partial_i^* \sigma$ has a well defined Eilenberg triple $(n, \tau, \alpha)$; then we decide to declare equivalent the pairs $(\sigma, \partial_{i*}(t)) \approx (\tau, \alpha_*(t))$.

For example let $S = S^d$ be the claimed simplicial version of the $d$-sphere described in Section 4.3.3. This simplical set $S$ has only two non-degenerate simplices, one in dimension 0, the other one in dimension $d$. The cheap realization needs a point $\Delta^0$ and a geometric $d$-simplex $\Delta^d$ corresponding to the abstract simplex $\mathrm{id} \in \Delta(\underline{\mathbf{d}}, \underline{\mathbf{d}})$; then if $t \in \Delta^{d-1}$ and $0 \leq i \leq d$, the equivalence relation asks for the Eilenberg triple of $\partial_i(\mathrm{id}) = *_{d-1}$ which is $(0, *_0, \eta)$, the map $\eta$ being the unique element of $\Delta(\underline{\mathbf{d-1}}, \underline{\mathbf{0}})$. Finally the initial pair $(\mathrm{id}, \partial_{i*}t)$ in the realization process must be identified with the pair $(*_0, \Delta^0)$; in other words $\|S\| = \Delta^d / \partial\Delta^d$, homeomorphic to the unit $d$-ball with the boundary collapsed to one point.

**Proposition 18** — *Both realizations, the expensive one and the cheap one, of a simplicial set $X$ are canonically homeomorphic.*

PROOF. The homeomorphism $f : |X| \to \|X\|$ to be constructed maps the equivalence class of the pair $(\sigma, t) \in X_m \times \Delta^m$ to the (equivalence class of the) pair $(\tau, \alpha_*(t)) \in X_n \times \Delta^n$ if the Eilenberg triple of $\sigma$ is $(n, \tau, \alpha)$. The inverse homeomorphism $g$ is induced by the canonical inclusion $\coprod X_m^{ND} \times \Delta^m \hookrightarrow \coprod X_m \times \Delta^m$. These maps must be proved coherent with the defining equivalence relations and inverse to each other.

If $\alpha = \beta\gamma$ is a $\Delta$-morphism expressed as the composition of two other $\Delta$-morphisms, then an equivalence $(\sigma, \beta_*\gamma_* t) \approx (\gamma^*\beta^*\sigma, t)$ can be considered as a consequence of the relations $(\sigma, \beta_*\gamma_* t) \approx (\beta^*\sigma, \gamma_* t)$ and $(\beta^*\sigma, \gamma_* t) \approx (\gamma^*\beta^*\sigma, t)$, so that it is sufficient to prove the coherence of the definition of $f$ with respect to the *elementary* $\Delta$-operators, that is, the face and degeneracy operators.

Let us assume the Eilenberg triple of $\sigma \in X_m$ is $(n, \tau, \alpha)$, so that $f(\sigma, t) = (\tau, \alpha_* t)$. We must in particular prove that $f(\eta_i^*\sigma, t)$ and $f(\sigma, \eta_{i*} t)$ are coherently defined. The second image is the equivalence class of $(\tau, \alpha_*\eta_{i*} t)$; the Eilenberg triple of $\eta_i^*\sigma$ is $(n, \tau, \alpha\eta_i)$ so that the first image is the equivalence class of $(\tau, (\alpha\eta_i)_* t)$ and both image representants are even equal.

Let us do now the analogous work with the face operator $\partial_i$ instead of the degeneracy operator $\eta_i$. Two cases must be considered. If ever the composition $\alpha\partial_i \in \Delta(\underline{\mathbf{m-1}}, \underline{\mathbf{n}})$ is surjective, the proof is the same. The interesting case happens if $\alpha\partial_i$ is not surjective; but its image then forgets exactly one element $j$ $(0 \leq j \leq n)$ and there exists a unique surjection $\beta \in \Delta(\underline{\mathbf{m-1}}, \underline{\mathbf{n-1}})$ such that $\alpha\partial_i = \partial_j\beta$. The abstract simplex $\partial_j^*\tau$ gives an Eilenberg triple $(n', \tau', \alpha')$ and the unique possible Eilenberg triple for $\partial_i^*\sigma$ is $(n', \tau', \beta\alpha')$. Then, on one hand, the $f$-image of $(\sigma, \partial_{i*} t)$ is $(\tau, \alpha_*\partial_{i*} t) = (\tau, \partial_{j*}\beta_* t)$; on the other hand the $f$-image of $(\partial_i^*\sigma, t)$ is $(\tau', \alpha_*\beta_* t)$; but according to the definition of the equivalence relation $\approx$ for $\|X\|$, both $f$-images are equivalent. The coherence of $f$ is proved.

Let $\sigma \in X_m^{ND}$, $0 \leq i \leq m$, $t \in \Delta^{m-1}$ and $(n, \tau, \alpha)$ (the Eilenberg triple of $\partial_i^*\sigma$) be the ingredients in the definition of the equivalence relation for $\|X\|$; the pairs $(\sigma, \partial_{i*} t)$ and $(\tau, \alpha_* t)$ are declared equivalent in $\|X\|$; the map $g$ is induced by the canonical inclusion of coproducts, so that we must prove the same pairs are also equivalent in $|X|$. But this is a transitive consequence of $(\sigma, \partial_{i*} t) \approx (\partial_i^*\sigma, t) = (\alpha^*\tau, t) \approx (\tau, \alpha_* t)$. We see here we had only described the binary relations *generating* the equivalence relation $\approx$; the defining relation is not necessarily stable under transitivity. The coherence of $g$ is proved.

The relation $fg = \mathrm{id}$ is obvious. The other relation $gf = \mathrm{id}$ is a consequence of the equivalence in $|X|$ of $(\sigma, t) \approx (\tau, \alpha_* t)$ if the Eilenberg triple of $\sigma$ is $(n, \tau, \alpha)$. ■

### 4.4.1 Examples.

Let us consider the construction of the classifying space of the group $G = \mathbb{Z}_2$ described in Section 4.3.4. The universal "total space" EG has for every $m \in \mathbb{N}$ exactly two non-degenerate $m$-simplices $(0, 1, 0, 1, \ldots)$ and $(1, 0, 1, 0, \ldots)$. The only non degenerate faces are the 0-face and the $m$-face. For example the faces of $(0, 1, 0, 1)$ are $(1, 0, 1) \in EG_2^{ND}$, $(0, 0, 1) = \eta_0(0, 1)$, $(0, 1, 1) = \eta_1(0, 1)$ and

$(0, 1, 0) \in EG_2^{ND}$. Each non-degenerate $m$-simplex is attached to the $(m-1)$-skeleton of $EG$ like each hemisphere of $S^m$ is attached to the equator $S^{m-1}$ and $EG$ is nothing but the infinite sphere $S^\infty$. The details are not so easy; the key point consists in proving the geometric $m$-simplex corresponding for example to $\sigma = (0, 1, 0, 1, \ldots)$ with a few identification relations on the boundary, following the instructions read from the various iterated faces of $\sigma$, is again homeomorphic to the $m$-ball, its boundary to the $(m-1)$-sphere; the simplest case is $\Delta^2/\partial_1\Delta^2 \cong D^2$, for $\partial_1\Delta^2 = \Delta^1$ is contractible, and this can be extended to the higher dimensions.

The classifying space $BG$ is the quotient space of $EG$ by the canonical action of $\mathbb{Z}_2$, that is, the quotient space of $S^\infty$ by the corresponding action; so that $BG$ is homeomorphic to the infinite real projective space $P^\infty\mathbb{R}$; the $m$-skeleton (throw away all the non-degenerate simplices of dimension $> m$ and also *their* degeneracies) is a combinatorial description of $P^m\mathbb{R}$. If $\sigma_m = [1|1|\ldots|1|1]$ denotes the unique non-degenerate simplex of $BG$; then $\partial_0\sigma_m = \sigma_{m-1}$, $\partial_1\sigma_m = \eta_0\sigma_{m-2}$, $\ldots$, $\partial_{m-1}\sigma_m = \eta_{m-2}\sigma_{m-2}$ and $\partial_m\sigma_m = \sigma_{m-1}$.

Let us also consider the case of the singular simplicial set of a topological space $X$ (see Section 4.3.7). There is a canonical continuous map $f : |SX| \to X$ defined as follows; if $(\sigma, t)$ represents an element of $|SX|$, this means the (abstract) simplex $\sigma$ is a continuous map $\sigma : \Delta^m \to X$, but $t$ is an element of the geometric simplex $\Delta^m$, so that it is tempting to define $f(\sigma, t) = \sigma(t)$; it is easy to verify this definition is coherent with the equivalence relation defining $|SX|$. This map is always a weak homotopy equivalence, and is an ordinary homotopy equivalence if and only if $X$ has the homotopy type of a CW-complex.

### 4.4.2 Simplicial maps.

A natural notion of *simplicial map* $f : X \to Y$ between simplicial sets can be defined. The map $f$ must be a system $\{f_m : X_m \to Y_m\}_{m\in\mathbb{N}}$ satisfying the commuting relations $\alpha_X^* \circ f_m = f_n \circ \alpha_Y^*$ if $\alpha$ is a $\Delta$-morphism $\alpha \in \Delta(\underline{\mathbf{m}}, \underline{\mathbf{n}})$. If $f : X \to Y$ is such a simplicial map, a realization $|f| : |X| \to |Y|$, a continuous map, is canonically defined.

## 4.5 Products of simplicial sets.

**Definition 19** — If $X$ and $Y$ are two simplicial sets, the *simplicial product* $Z = X \times Y$ is defined by $Z_m = X_m \times Y_m$ for every natural number $m$, and $\alpha_Z^* = \alpha_X^* \times \alpha_Y^*$ if $\alpha$ is a $\Delta$-morphism.

The definition of the product of two simplicial sets is perfectly trivial and is however at the origin of several landmark problems in algebraic topology, for example the deep structure of the twisted Eilenberg-Zilber theorem, still quite mysterious, and also the enormous field around the Steenrod algebras.

Every simplex of the product $Z = X \times Y$ is a *pair* $(\sigma, \tau)$ made of one simplex in $X$ and one simplex in $Y$; both simplices must have the *same dimension*. It is tempting at this point, because of the "product" ambience, to denote by $\sigma \times \tau$ such a simplex in the product but *this would be a terrible error!* This is not at all the right point of view; the pair $(\sigma, \tau) \in Z_m$ is the unique simplex in $Z$ whose respective *projections* in $X$ and $Y$ are $\sigma$ and $\tau$ and this is the reason why the pair notation $(\sigma, \tau)$ is the only one which is possible. For example the diagonal of a square is a 1-simplex, the unique simplex the projections of which are both factors of the square; on the contrary, the "product" of the factors is simply the square, which does not have the dimension 1 and which is even not a simplex.

**Theorem 20** — *If $X$ and $Y$ are two simplicial sets and $Z = X \times Y$ is their simplicial product, then there exists a canonical homeomorphism between $|Z|$ and $|X| \times |Y|$, the last product being the product of CW-complexes (or also of $k$-spaces).*

If you consider the product $|X| \times |Y|$ as the product of topological spaces, the same accident as for CW-complexes (see Section 2.2.7) can happen.

PROOF. There are natural simplicial projections $X \times Y \to X$ and $Y$ which define a canonical continuous map $\phi : |X \times Y| \to |X| \times |Y|$. The interesting question is to define its inverse $\psi : |X| \times |Y| \to |X \times Y|$.

First of all, let us detail the case of $X = \Delta^2$ and $Y = \Delta^1$ where the essential points are visible. The first factor $X$ has dimension 2, and the second one $Y$ has dimension 1 so that the product $Z$ should have dimension 3. What about the 3-simplices of $Z$? There are 3 such *non-degenerate* 3-simplices, namely $\rho_0 = (\eta_0\sigma, \eta_2\eta_1\tau)$, $\rho_1 = (\eta_1\sigma, \eta_2\eta_0\tau)$ and $\rho_2 = (\eta_2\sigma, \eta_1\eta_0\tau)$, if $\sigma$ (resp. $\tau$) is the unique non-degenerate 2-simplex (resp. 1-simplex) of $\Delta^2$ (resp. $\Delta^1$). This is nothing but the decomposition of a prism $\Delta^2 \times \Delta^1$ in three tetrahedrons.

Note no non-degenerate 3-simplex is present in $X$ and $Y$ and however some 3-simplices must be produced for $Z$; this is possible thanks to the *degenerate* simplices of $X$ and $Y$ where they are again playing a quite tricky role in our workspace; in particular a pair of *degenerate* simplices in the factors can produce a *non-degenerate* simplex in the product! This happens when there is no common degeneracy in the factors.

For example the tetrahedron $\rho_0 = (\eta_0\sigma, \eta_2\eta_1\tau)$ inside $Z$ is *the* unique 3-simplex the first projection of which is $\eta_0\sigma$, and the second projection is $\eta_2\eta_1\tau$; the first projection is a tetrahedron collapsed on the triangle $\sigma$, identifying two points when the sum of barycentric coordinates of index 0 and 1 (the indices where injectivity fails in $\eta_0$) are equal; the second projection is a tetrahedron collapsed on an interval, identifying two points when the sum of barycentric coordinates of index 1, 2 and 3 are equal.

Let us take a point of coordinates $r = (r_0, r_1, r_2, r_3)$ in the simplex $\rho_0$. Its first projection is the point of $X = \Delta^2$ of barycentric coordinates $s = (s_0 = r_0 + r_1, s_1 = r_2, s_2 = r_3)$; in the same way its second projection is the point of $Y = \Delta^1$ of barycentric coordinates $t = (t_0 = r_0, t_1 = r_1 + r_2 + r_3)$. So that:

$$\phi(\rho_0, (r_0, r_1, r_2, r_3)) = ((\sigma, (r_0 + r_1, r_2, r_3)), (\tau, (r_0, r_1 + r_2 + r_3)))$$

In the same way:

$$\begin{aligned}\phi(\rho_1,(r_0,r_1,r_2,r_3)) &= ((\sigma,(r_0,r_1+r_2,r_3)),(\tau,(r_0+r_1,r_2+r_3)))\\ \phi(\rho_2,(r_0,r_1,r_2,r_3)) &= ((\sigma,(r_0,r_1,r_2+r_3)),(\tau,(r_0+r_1+r_2,r_3)))\end{aligned}$$

The challenge then consists in deciding for an arbitrary point $((\sigma,(s_0,s_1,s_2)),(\tau,(t_0,t_1))) \in |X| \times |Y|$ what simplex $\rho_i$ it comes from and what a good $\phi$-preimage $(\rho_i, r)$ could be. You obtain the solution in comparing the sums $u_0 = s_0$, $u_1 = s_0 + s_1$, $u_2 = t_0$ ; the sums $s_0 + s_1 + s_2$ and $t_0 + t_1$ are necessarily equal to 1 and do not play any role. You see in the three cases, the values of $u_i$'s are:

$$\begin{aligned}((\eta_0\sigma,\eta_2\eta_1\tau),r) \Rightarrow\ & u_0 = r_0+r_1, u_1 = r_0+r_1+r_2, u_2 = r_0,\\ ((\eta_1\sigma,\eta_2\eta_0\tau),r) \Rightarrow\ & u_0 = r_0, u_1 = r_0+r_1+r_2, u_2 = r_0+r_1,\\ ((\eta_2\sigma,\eta_1\eta_0\tau),r) \Rightarrow\ & u_0 = r_0, u_1 = r_0+r_1, u_2 = r_0+r_1+r_2,\end{aligned}$$

so that you can guess the degeneracy operators to be applied to the factors $\sigma$ and $\tau$ from the order of the $u_i$'s; more precisely, sorting the $u_i$'s puts the $u_2$ value in position 0, 1 or 2, and this gives the index for the degeneracy to be applied to $\sigma$; in the same way the $u_0$ and $u_1$ values must be installed in positions "1 and 2", or "0 and 2", or "0 and 1" and this gives the two indices (in reverse order) for the degeneracies to be applied to $\tau$. It's a question of *shuffle*. Furthermore you can find the components $r_i$ from the differences between successive $u_i$'s. Now we can construct the map $\psi$:

$$\begin{aligned}\phi((\sigma,s)(\tau,t)) &= (\rho_0,(u_2,u_0-u_2,u_1-u_0,1-u_1)) \ \text{ if } u_2 \le u_0 \le u_1,\\ &= (\rho_1,(u_0,u_2-u_0,u_1-u_2,1-u_1)) \ \text{ if } u_0 \le u_2 \le u_1,\\ &= (\rho_2,(u_0,u_1-u_0,u_2-u_1,1-u_2)) \ \text{ if } u_0 \le u_1 \le u_2.\end{aligned}$$

There seems an ambiguity occurs when there is an equality between $u_2$ and $u_0$ or $u_1$, but it is easy to see both possible preimages are in fact the same in $|Z|$.

Now this can be extended to the general case, according to the following recipe. Let $\sigma \in X_m$ and $\tau \in Y_n$ be two simplices, $s \in \Delta_m$ and $t \in \Delta^n$ two geometric points. We must define $\psi((\sigma,s),(\tau,t)) \in |Z| = |X \times Y|$. We set $u_0 = s_0$, $u_1 = s_0 + s_1$, ..., $u_{m-1} = s_0 + \ldots + s_{m-1}$, $u_m = t_0$, $u_{m+1} = t_0 + t_1$, ..., $u_{m+n-1} = t_0 + \ldots + t_{n-1}$. Then we sort the $u_i$'s according to the increasing order to obtain a sorted list $(v_0 \le \ldots \le v_{m+n-1})$. In particular $u_m = v_{i_0}, \ldots, u_{m+n-1} = v_{i_{n-1}}$ with $i_0 < \ldots < i_{n-1}$, and $u_0 = v_{j_0}, \ldots, u_{m-1} = v_{j_{m-1}}$ with $j_0 < \ldots < j_{m-1}$. Then:

$$\begin{aligned}&\psi((\sigma,s),(\tau,t)) =\\ &((\eta_{i_{n-1}}\ldots\eta_{i_0}\sigma,\eta_{j_{m-1}}\ldots\eta_{j_0}\tau),(v_0,v_1-v_0,\ldots,v_{m+n-1}-v_{m+n-2},1-v_{m+n-1})).\end{aligned}$$

Now it is easy to prove $\psi \circ \phi = \mathrm{id}_{|Z|}$ and $\phi \circ \psi = \mathrm{id}_{|X|\times|Y|}$, following the proof structure clearly visible in the case of $X = \Delta^2$ and $Y = \Delta^1$.

It is also necessary to prove the maps $\phi$ and $\psi$ are continuous. But $\phi$ is the product of the realization of two simplicial maps and is therefore continuous. The map $\psi$ is defined in a coherent way for each *cell* $\sigma \times \tau$ (this time it is really the *product* $|\sigma| \times |\tau| \subset |X| \times |Y|$) and is clearly continuous on each cell; because of the definition of the CW-topology, the map $\psi$ is continuous. ■

If three simplicial sets $X$, $Y$ and $Z$ are given, there is only one natural map $|X \times Y \times Z| \to |X| \times |Y| \times |Z|$ so that "both" inverses you construct by applying twice the previous construction of $\psi$, the first one going through $|X \times Y| \times |Z|$, the second one through $|X| \times |Y \times Z|$ are necessarily the same: the $\psi$-construction is *associative*, which is interesting to prove directly; it is essentially the associativity of the Eilenberg-MacLane formula.

### 4.5.1 The case of simplicial groups.

Let $G$ be a *simplicial group*. The object $G$ is a simplicial object in the group category; in other words each simplex set $G_m$ is provided with a group structure and the $\Delta$-operators $\alpha^* : G_m \to G_n$ are group homomorphisms.

This gives in particular a continuous canonical map $|G \times G| \to |G|$; then identifying $|G \times G|$ and $|G| \times |G|$, we obtain a "continuous" group structure for $|G|$; the word *continuous* is put between quotes, because this does not work in general in the topological sense: this works always only in the category of "CW-groups" where the group structure is a map $|G| \times |G| \to |G|$, the source of which being evaluated in the CW-category; because of this definition of product, it is then true that $|G| \times |G| = |G \times G|$. The composition rule so defined on $|G|$ satisfies the group axioms; in particular the associativity property comes from the considerations about the associativity of the $\psi$-construction in the previous section.

## 4.6 Kan extension condition.

Let us consider the standard simplicial model $S^1$ of the circle, with one vertex and one non-degenerate 1-simplex $\sigma$. This unique 1-simplex clearly represents a generator of $\pi_1(S^1)$, but its double cannot be so represented. This has many disadvantages and correcting this defect was elegantly solved by Kan.

**Definition 21** — A *Kan* $(m, i)$*-hat* (Kan hat in short) in a simplicial set $X$ is a $(m+1)$-tuple $(\sigma_0, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_{m+1})$ satisfying the relations $\partial_j \sigma_k = \partial_{k-1} \sigma_j$ if $j < k$, $j \neq i \neq k$.

For example the pair $(\partial_0 \mathbf{id}, \partial_1 \mathbf{id}, \partial_2 \mathbf{id},)$ is a Kan $(3,3)$-hat in the standard 3-simplex $\Delta^3$ if $\mathbf{id}$ is the unique non-degenerate 3-simplex. Also the pair $(\sigma, \sigma)$ is a Kan $(2,1)$-hat of the above $S^1$.

**Definition 22** — If $(\sigma_0, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_{m+1})$ is a Kan $(m, i)$-hat in the simplicial set $X$, a *filling* of this hat is a simplex $\sigma \in X_{m+1}$ such that $\partial_j \tau = \sigma_j$ for $j \neq i$.

The 3-simplex **id** of $\Delta^3$ is a filling of the example Kan hat in $\Delta^3$. The example Kan hat of $S^1$ has no filling. A Kan $(m,i)$-hat is a system of $m$-simplices arranged like all the faces except the $i$-th one of a *hypothetical* $(m+1)$-simplex.

**Definition 23** — A simplicial set $X$ satisfies the *Kan extension condition* if any Kan hat has a filling.

The standard simplex $\Delta^d$ satisfies the Kan condition. The other elementary simplicial sets in general do not.

The simplicial sets satisfying the Kan extension condition have numerous interesting properties; for example their homotopy groups can be combinatorially defined [12, Chapter 1], a canonical *minimal* version is included, also satisfying the extension condition [12, Section 9], a simple decomposition process produces a Postnikov tower [12, Section 8].

The simplicial groups are important from this point of view: in fact a simplicial group always satisfies the Kan extension condition [12, Theorem 17.1]. For example the simplicial description of $P^\infty\mathbb{R}$ (see Section 4.4.1) is a simplicial group and therefore satisfies the Kan condition, which is not so obvious; it is even minimal. The singular complex $SX$ of a topological space $X$ also satisfies the Kan condition but in general is not minimal. These simplicial sets satisfying the Kan condition are so interesting that it is often useful to know how to *complete* an arbitrary given simplicial set $X$ and produce a new simplicial set $X'$ with the same homotopy type satisfying the Kan condition. The Kan-completed $X'$ can be constructed as follows.

Let us define first an elementary completion $\chi(X)$ for $X$. For each Kan $(m,i)$-hat of $X$, we decide to add the hypothetical $(m+1)$-simplex (even if a "solution" preexists), and the "missing" $i$-th face; such a completion operation does not change the homotopy type of $X$. Doing this completion construction for every Kan hat of $X$, we obtain the first completion $\chi(X)$. Then we can define $X_0 = X$, $X_{i+1} = \chi(X_i)$ and $X' = \lim_{\rightarrow} X_i$ is the desired Kan completion. You can also run this process in considering only the failing hats.

## 4.7 Simplicial fibrations.

A *fibration* is a map $p : E \rightarrow B$ between a *total space* $E$ and a *base space* $B$ satisfying a few properties describing more or less the total space $E$ as a *twisted product* $F \times_\tau B$. In the simplicial context, several definitions are possible. The notion of *Kan* fibration corresponds to a situation where a simplicial homotopy lifting property is satisfied; to state this property, the elementary datum is a Kan hat in the total space and a given filling of its projection in the base space; the Kan fibration property is satisfied if it is possible to fill the Kan hat in the total space in a coherent way with respect to the given filling in the base space. This notion is the simplicial version of the notion of *Serre fibration*, a projection where the homotopy lifting property is satisfied for the maps defined on polyhedra. The reference [12] contains a detailed study of the basic facts around Kan fibrations, see [12, Chapters I and II].

We will examine with a little more details the notion of *twisted cartesian product*, corresponding to the topological notion of fibre bundle. It is a key notion in topology, and the simplicial framework is particularly favourable for several reasons. In particular the Serre spectral sequence becomes well structured in this environment, allowing us to extend it up to a *constructive* version, one of the main subjects of another lecture series of this Summer School. We give here the basic necessary definitions for the notion of twisted cartesian product.

A reasonably general situation consists in considering the case where a simplicial group $G$ acts on the fiber space, a simplicial set $F$, the fiber space. As usual this means a map $\phi : F \times G \to F$ is given; source and target are simplicial sets, the first one being the product of $F$ by the simplicial set $G$, underlying the simplicial group; the map $\phi$ is a simplicial map; furthermore each component $\phi_m : (F \times G)_m = F_m \times G_m \to F_m$ must satisfy the traditional properties of the right actions of a group on a set. We will use the shorter notation $f.\, g$ instead of $\phi(f, g)$. Let also $B$ be our base space, some simplicial set.

**Definition 24** — A *twisting operator* $\tau : B \to G$ is a family of maps $\{\tau_m : B_m \to G_{m-1}\}_{m\geq 1}$ satisfying the following properties.

1. $\partial_0 \tau(b) = \tau(\partial_1 b)\tau(\partial_0 b)^{-1}$;
2. $\partial_i \tau(b) = \tau(\partial_{i+1}(b))$ if $i \leq 1$;
3. $\eta_i \tau(b) = \tau(\eta_{i+1} b)$;
4. $\tau(\eta_0 b) = e_m$ if $b \in G_{m+1}$, the unit element of $G_m$ being $e_m$.

In particular it is not required $\tau$ is a *simplicial map*, and in fact, because of the degree -1 between source and target dimensions, this does not make sense.

**Definition 25** — If a twisting operator $\tau : B \to G$ is given, the corresponding *twisted cartesian product* $E = F \times_\tau B$ is the simplicial set defined as follows. Its set of $m$-simplices $E_m$ is the same as for the non-twisted product $E_m = F_m \times B_m$; the face and degeneracy operators are also the same as for the non-twisted product with only one exception: $\partial_0(f, b) = (\partial_0 f.\, \tau(b), \partial_0 b)$.

The twisting operator $\tau$, the unique ingredient at the origin of a difference between the non-twisted product and the $\tau$-twisted one, acts in the following way: the twisted product is constructed in a recursive way with respect to the base dimension. Let $B^{(k)}$ be the $k$-skeleton of $B$ and let us suppose $F \times_\tau B^{(k)}$ is already constructed. Let $\sigma$ be a $(k+1)$-simplex of $B$; we must describe how the product $F \times \sigma$ is to be attached to $F \times B^{(k)}$; what is above the faces $\partial_i \sigma$ for $i \geq 1$ is naturally attached; but what is above the 0-face is shifted by the translation defined by the operation of $\tau(b)$. It is not obvious such an attachment is coherent, but the various formulas of Definition 25 are exactly the relations which must be satisfied by $\tau$ for consistency. It was not obvious, starting from scratch, to guess this is a good framework for working simplicially about fibrations; this was invented (discovered ?) by Daniel Kan [9]; the previous work by Eilenberg and MacLane [5, 6] in the particular case of the fibrations relating the elements of the Eilenberg-MacLane spectra was probably determining.

### 4.7.1 The simplest example.

Let us describe in this way the exponential fibration $\mathbf{exp} : \mathbb{R} \to S^1 : t \to e^{2\pi i t}$. We take for $S^1$ the model with one vertex $*_0$ and one non-degenerate edge $\mathrm{id}(\underline{\mathbf{1}}) = \sigma$ (see Section 2.2.2). For $\mathbb{R}$, we choose $\mathbb{R}_0 = \mathbb{Z}$ and $\mathbb{R}_1^{ND} = \mathbb{Z}$, that is one vertex $k_0$ and one non-degenerate edge $k_1$ for each integer $k \in \mathbb{Z}$; the faces are defined by $\partial_i(k_1) = (k+i)_0$ ($i = 0$ or 1). The discrete (see Section 4.3.1) simplicial group $\mathbb{Z}$ acts on the fiber; for any dimension $d$, the simplex group $\mathbb{Z}_d$ is $\mathbb{Z}$ with the natural structure, and $k_i \,.\, g = (k+g)_i$ for $i = 0$ or 1. It is then clear that the right twisting operator for the exponential fibration is $\tau(g) = 1$ for $g \in \mathbb{R}_1^{ND}$.

### 4.7.2 Fibrations between $K(\pi, n)$'s.

Let us recall (see Section 4.3.5) $E(\pi, d)$ is the simplicial set defined by $E(\pi, d)_m = C^d(\Delta^m, \pi)$ (only *normalized* cochains) and $K(\pi, n)$ is the simplicial subset made of the *cocycles*. The maps between simplex sets to be associated to $\Delta$-morphisms are naturally defined. A simplicial projection $p : E(\pi, d) \to K(\pi, d+1)$ associating to an $m$-cochain $c$ its coboundary $\delta c$, necessarily a cocycle, is also defined. The simplicial set $\Delta^m$ is contractible, its cochain complex is acyclic and the kernel of $p$, the potential *fibre space*, is therefore the simplicial set $K(\pi, d)$. The base space is clearly the quotient of the total space by the fibre space (*principal* fibration), and a systematic examination of such a situation (see [12, Section 18]) shows $E(\pi, d)$ is necessarily a twisted cartesian product of the base space $K(\pi, d+1)$ by the fiber space $K(\pi, d)$.

It is not so easy to guess a corresponding twisting operator. A solution is described as follows; let $z \in Z^{d+1}(\Delta^m, \pi)$ a base $m$-simplex; the result $\tau(z) \in Z^d(\Delta^{m-1}, \pi)$ must be a $d$-cocycle of $\Delta^{m-1}$, that is a function defined on every $(d+1)$-tuple $(i_0, \ldots, i_d)$, with values in $\pi$, and satisfying the cocycle condition; the solution $\tau(z)(i_0, \ldots, i_d) = z(0, i_0 + 1, \ldots, i_d + 1) - z(1, i_0 + 1, \ldots, i_d + 1)$ works, but seems a little mysterious. The good point of view consists in considering the notion of *pseudo-section* for the studied fibration; an actual section cannot exist if the fibration is not trivial, but a pseudo-section is essentially as good as possible; see the definition of pseudo-section in [12, Section 18]. When a pseudo-section is found, a simple process produces a twisting operator; in our example, the twisting operator comes from the pseudo-section $\rho(z)(i_0, \ldots, i_d) = z(0, i_0 + 1, \ldots, i_d + 1)$, quite natural.

The fibrations between Eilenberg-MacLane spaces are a particular case of universal fibrations associated to simplicial groups. See [12, Section 21].

### 4.7.3 Simplicial loop spaces.

A simplicial set $X$ is *reduced* if its 0-simplex set $X_0$ has only one element. We have given in Section 4.3.6 the Kan combinatorial version $GX$ of the loop space of $X$. This loop space is the fiber space of a *co-universal* fibration:

$$GX \hookrightarrow GX \times_\tau X \to X.$$

Only the twisting operator $\tau$ remains to be defined. The definition is simply... $\tau(\sigma) := \tau(\sigma)$ for both possible meanings of $\tau(\sigma)$; the first one is the value of the twisting operator to be defined for some simplex $\sigma \in X_{m+1}$ and the second one is the generator of $GX_m$ corresponding to $\sigma \in X_{m+1}$, the unit element of $GX_m$ if ever $\sigma$ is 0-degenerate (see Section 4.3.6). The definition of the face operators for $GX$ are exactly those which are required so that the twisting operator so defined is coherent.

It is again an example of *principal fibration*, that is the fiber space is equal to the structural group and the action $GX \times GX \to GX$ is equal to the group multiplication. This fibration is co-universal, with respect to $X$; in fact, let $H \hookrightarrow H \times_{\tau'} X \xrightarrow{p} X$ another *principal* fibration above $X$ for another twisting operator $\tau' : X \to H$. Then the free group structure of $GX$ gives you a unique group homomorphism $GX \to H$ inducing a canonical morphism between both fibrations.

If the simplicial space $X$ is 1-reduced (only one vertex, no non-degenerate 1-simplex), then an important result by John Adams [1] allows one to compute the homology groups of $GX$ if the initial simplicial set $X$ is of finite type; an intermediate ingredient, the *Cobar construction*, is the key point. One of the main problems in Algebraic Topology consists in solving the analogous problem for the iterated loop spaces $G^nX$ when $X$ is $n$-reduced; it is the problem of *iterating the Cobar construction*; one of the lecture series of this Summer School is devoted to this subject, organized around a *constructive* version of Algebraic Topology.

# References


[1] J. Frank Adams. *On the Cobar construction.* Proceedings of the National Academy of Science of the U.S.A., 1956, vol. 42, pp 409-412.

[2] Gunnar Carlsson and R. James Milgram. *Stable homotopy and iterated loop spaces.* in [8], pp 505-583.

[3] Dominique Dancète. *An effective version of the James construction.* To appear.

[4] C.H. Dowker. *Topology of metric complexes.* American Journal of Mathematics, 1952, vol. 74, pp. 555-577.

[5] Samuel Eilenberg, Saunders MacLane. *On the groups $H(\pi, n)$, I.* Annals of Mathematics, 1953, vol. 58, pp 55-106.

[6] Samuel Eilenberg, Saunders MacLane. *On the groups $H(\pi, n)$, II.* Annals of Mathematics, 1954, vol. 60, pp 49-139.

[7] Rudolf Fritsch and Renzo A. Piccinini. *Cellular structures in topology.* Cambridge University Press, 1990.

[8] *Handbook of Algebraic Topology* (Edited by I.M. James). North-Holland, 1995.

[9] Daniel M. Kan. *A combinatorial definition of homotopy groups*. Commentarii Mathematici Helvetici, 1958, vol. 67, pp 282-312.

[10] Klaus Lamotke. *Semisimpliziale algebraische topologie*. Springer, 1968.

[11] Albert T. Lundell and Stephen Weingram. *The topology of CW complexes*. Van Nostrand, 1969.

[12] J. Peter May. *Simplicial objects in algebraic topology*. Van Nostrand, 1967.

[13] John Milnor. *Morse theory*. Princeton University Press, 1963.

[14] John Milnor. *Lectures on the h-cobordism theorem*. Princeton University Press, 1965.

[15] *Characteristic classes*. Annals of Mathematical Studies #86, Princeton University Press, 1974.

[16] Edwin H. Spanier. *Algebraic Topology*. McGraw Hill, 1966.

[17] Francis Sergeraert. *Constructive algebraic topology*. `www-fourier.ujf-grenoble.fr/~sergerar/`

[18] George W. Whitehead. *Elements of homotopy theory*. Springer, 1978.

[19] J.H.C. Whitehead. *Combinatorial homotopy I*. Bulletin of the American Society, 1949, vol. 55, pp. 1133-1145.