UNSTABLE MODULES AND UNSTABLE ALGEBRAS. APPLICATIONS TO HOMOTOPY THEORY

by Lionel Schwartz

In this set of notes we present part of the work that started at the beginning of the 80's around the Sullivan and the Segal conjecture. It has lead to major advances in homotopy theory and the work is going on. The works of G. Carlsson, J. Lannes and H. Miller must be recalled here. It has something to do with the classification problem, for example it has lead to :

- *(i)* classifying maps from classifying spaces up to homotopy by algebraic invariants in a wide range of situation (Lannes, Jackowski, McClure, Oliver,...),
- *(ii)* classifying finite loop spaces (Dwyer, Wilkerson,...),
- *(iii)* related is the question of classification of delooping of finite loop spaces (Dwyer, Miller, Wilkerson, Notbohm,...),
- *(iv)* Steenrod's problem : existence and classification of spaces with polynomial cohomology (Aguade, Broto, Notbohm,...).

We first present the algebraic background, then present a sample of applications.

The mod 2 Steenrod algebra $\mathcal{A}_2$
Unstable modules and unstable algebras

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1.1 The mod 2 Steenrod algebra

The mod 2 Steenrod algebra $\mathcal{A}_2$ is the associative unital graded $\mathbf{F}_2$-algebra generated by elements Sq^i of degree i, $i > 0$, subject to the following relations, known as the Adem relations

$$\mathrm{Sq}^i\mathrm{Sq}^j = \sum_0^{[i/2]} C_{j-k-1}^{i-2k}\mathrm{Sq}^{i+j-k}\mathrm{Sq}^k$$

for all $i, j > 0$ such that $i < 2j$.

In these formulas Sq^0 is understood to be the unit. We will denote in the sequel the singular mod 2 cohomology of a space or of a spectrum X by H^*X, the reduced cohomology of X will be denoted by $\bar{H}^*X$. One has :

Theorem : For any space or spectrum X, H^*X is in a natural way a graded $\mathcal{A}_2$-module.

In particular it means that for any continuous map $f : X \to Y$ the induced map $f^* : H^*Y \to H^*X$ is $\mathcal{A}_2$-linear.

Classically, Sq^1 acts as the Bockstein homomorphism associated to the sequence

$$0 \to \mathbf{Z}/2 \to \mathbf{Z}/4 \to \mathbf{Z}/2 \to 0 \quad .$$

N. E. Steenrod constructed the operations on the cohomology of spaces, then J. Adem showed that these operations satisfy the aboved mentioned relations. The next theorem is a consequence of the work by H. Cartan and J.-P. Serre on the cohomology of the Eilenberg-Mac Lane spaces.

Theorem : The Steenrod algebra is the algebra of all natural stable transformations of mod 2 cohomology.

Let us recall that a cohomology operation τ of degree $d \geq 0$ is a family of natural transformations τ_i, $i \geq 0$ from the functor H^i (i-th cohomology group) to the functor H^{i+d} ($i + d$-th cohomology group). A cohomology operation is said to be stable if it commutes with the suspension isomorphism $\sigma_i \ : \bar{H}^i X \to \bar{H}^{i+1}\Sigma X$, *i.e.* if $\sigma_i \tau_i = \tau_{i+1}\sigma_i$. Here Σ denotes as usual the reduced suspension of the pointed space X. It is easy to prove that a stable cohomology operation is necessarily additive. The proof consists in the following commutative diagram :

$$\begin{tikzcd}
\bar{H}^i X \arrow[r, "\tau_i"] \arrow[d, "\sigma_i"'] & \bar{H}^{i+d} X \arrow[d, "\sigma_i"'] \\
\bar{H}^{i+1}\Sigma X \arrow[r, "\tau_{i+1}"] & \bar{H}^{i+d+1}\Sigma X \\
\bar{H}^{i+1}\Sigma X \vee \Sigma X \arrow[r, "\tau_{i+1}"] \arrow[u, "\delta^*"] & \bar{H}^{i+d+1}\Sigma X \vee \Sigma X \arrow[u, "\delta^*"]
\end{tikzcd}$$

where $\delta : \Sigma X \to \Sigma X \vee \Sigma X$ is the codiagonal.

1.2 Multiplicative and additive generators for the Steenrod algebra

We describe multiplicative generators for the Steenrod algebra.

Theorem : The operations Sq^{2^h}, $h \geq 0$ constitute a minimal system of multiplicative generators for $\mathcal{A}_2$.

Proof : This is a consequence of the Adem relations. More precisely one has :

Lemma : The operation Sq^m belongs to the right ideal of $\mathcal{A}_2$ generated by those Sq^{2^t} such that $t \geq 0$ and 2^t divides m.

Proof :

Write m as $2^\ell + 2^{\ell+1}a$. If $a \neq 0$ one has :

$$\mathrm{Sq}^{2^\ell}\mathrm{Sq}^{2^{\ell+1}a} = \sum_{0}^{2^{\ell-1}} C^{2^\ell-2t}_{(2^{\ell+1}a-t)-1}\mathrm{Sq}^{m-t}\mathrm{Sq}^t \quad .$$

The binomial coefficient $C^{2^\ell}_{(2^\ell a-t)-1}$ is easily checked to be non-zero mod 2. Indeed 2^ℓ appears in the 2-adic expansion of $2^{\ell+1}a - t$, because $t \leq 2^{\ell-1}$. Recall that a binomial coefficient C^i_n is non zero mod 2 if and only all the powers of 2 occurring in the 2-adic expansion of i occur in the 2-adic expansion of n.

Therefore,

$$\mathrm{Sq}^{2^\ell}\mathrm{Sq}^{2^{\ell+1}a} = \mathrm{Sq}^m + \sum_{1}^{2^{\ell-1}} \varepsilon_t\mathrm{Sq}^{m-t}\mathrm{Sq}^t \quad .$$

For t between 1 and $2^{\ell-1}$, the integer $m - t$ is not divisible by 2^ℓ. We can apply an induction hypothesis and assume that these operations Sq^{m-t} are in the right ideal generated by the operations Sq^{2^ℓ} for $t \leq \ell - 1$.

The demonstration of the theorem follows.

This system of generators is a minimal one. This is proved by looking at the action of the Steenrod algebra on $H^*B\mathbf{Z}/2$. It will be done later.

We now describe an additive basis for the Steenrod algebra. In order to do that we need to describe certain monomial elements in $\mathcal{A}_2$. For a sequence of integers $I = (i_1,\ldots,i_n)$, let Sq^I denote the operation $\mathrm{Sq}^{i_1}\ldots\mathrm{Sq}^{i_n}$. The sequence I is said to be *admissible* if $i_h \geq 2i_{h+1}$ for all $h \geq 1$, $(i_{n+1} = 0)$. The operation Sq^I with I admissible is called an admissible monomials if I is admissible.

Theorem : The admissible monomials Sq^I form a basis for $\mathcal{A}_2$.

Again it is a consequence of the Adem relations. We prove here just that the operations Sq^I I admissible span the graded vector space $\mathcal{A}_2$. Let $I = (i_1,\ldots,i_n)$ be an admissible sequence. Its moment is defined to be the integer $i_1 + 2i_2 + \ldots + ni_n$. If I is not admissible there exists h, $1 \leq h \leq n-1$ such that $i_h < 2i_{h+1}$. Using the Adem relations one gets:

$$\mathrm{Sq}^I = \sum_{0}^{[i_h/2]} \varepsilon_t \mathrm{Sq}^{I'} \mathrm{Sq}^{i_h+i_{h+1}-t} \mathrm{Sq}^t \mathrm{Sq}^{I''} \quad ,$$

where $\varepsilon_t \in \mathbf{F}_2$, $0 \leq t \leq [i_h/2]$,

$I' = (i_1, \ldots, i_{h-1})$ and $I'' = (i_{h+2}, \ldots, i_n)$.

The moment of any sequence occurring on the right *i.e.* $(I', i_h + i_{h+1} - t, t, I'')$, $0 \leq t \leq [i_h/2]$) is strictly lower than the moment of I. The result follows by induction on the moment (note that it is defined for all sequences).

We will come back in the next paragraph to what we have not yet proved : minimality of the system of multiplicative generators, freeness of the system consisting of the Sq^I, I admissible.

1.3. Instability

The mod 2 cohomology

of a space X has, as an $\mathcal{A}_2$-module, a certain property called

instability :

- if $x \in H^*X$ and

$i > |x|$, then $\mathrm{Sq}^i x = 0$.

Here $|x|$ denotes

the degree of x.

Definition : An $\mathcal{A}_2$-module M unstable if it satisfies the preceding property.

In particular, it implies that an unstable $\mathcal{A}_2$-module M is trivial in negative degrees, indeed Sq^0 is identified with the unit, thus $\mathrm{Sq}^0 x = x$ is equal to 0 if $|x| < 0$. The cohomology of a spectrum, which need not be non trivial in negative degrees, is not in general an unstable $\mathcal{A}_2$-module. We will abbreviate "unstable $\mathcal{A}$-module" to "unstable module" most of the time. The category of $\mathbf{Z}$-graded $\mathcal{A}$-modules with morphisms $\mathcal{A}_2$-linear map of degree zero will be denoted by $\mathcal{M}$. The full subcategory whose objects are unstable modules. will be denoted by $\mathcal{U}$.

It is clear that the quotient of an unstable module is unstable, it is also clear that a sub-module of an unstable module is unstable. However it is not true that if one has a short exact sequence :

$$0 \to M' \to M \to M'' \to 0$$

with M' and M'' both unstable that M is unstable.

The mod-2 cohomology of a space has some more structure. It supports a product compatible with the unstable module structure, it is an unstable $\mathcal{A}_2$-algebra : the cup product determines in a natural way on the mod 2 cohomology of a space X the structure of an $\mathbf{F}_2$-graded commutative algebra. This algebra structure is related to the unstable module structure by two properties, the first one is known has the Cartan formula :

$$\mathrm{Sq}^i(xy) = \sum_{k+\ell=i} \mathrm{Sq}^k x\, \mathrm{Sq}^\ell y \quad \text{for all } x \quad \text{and any } y \in H^*X \quad ,$$

$$\mathrm{Sq}^{|x|} x = x^2 \quad \text{for} \quad \text{any} \quad x \in H^*X \quad .$$

Definition : An unstable $\mathcal{A}_2$-algebra K is an unstable module provided with maps $\mu : K \bigotimes K \to K$ and $\eta : \mathbf{F}_2 \to K$ which determine a commutative, unital, $\mathbf{F}_2$-algebra structure on K and such that the two properties stated above hold.

We will denote by $\mathcal{K}$ the category of unstable $\mathcal{A}_2$- algebras, morphisms being $\mathcal{A}_2$-linear algebra maps of degree zero. For convenience, we will abbreviate “unstable $\mathcal{A}_2$-algebra” to “unstable algebra”.

The Cartan formula can be reformulated as follows :

Proposition : There is an algebra map δ (diagonal) from $\mathcal{A}_2$ in $\mathcal{A}_2 \bigotimes \mathcal{A}_2$ which is determined by the following formula :

$$\delta(\mathrm{Sq}^i) = \sum_{k+\ell=i} \mathrm{Sq}^k \otimes \mathrm{Sq}^\ell \quad .$$

We refer to [SE] for the proof (we just indicate the basic idea).

In fact, this map determines a cocommutative Hopf algebra structure on $\mathcal{A}_2$. It can be used to provide the tensor product $M \bigotimes N$ of two unstable modules M and N with an unstable module structure. This structure being determined by the formula :

$$(\theta \otimes \theta')(m \otimes n) = \theta m \otimes \theta' m'$$

for all $\theta, \theta' \in \mathcal{A}_2$, $m \in M$ and $n \in N$. Then $M \bigotimes N$ becomes an unstable module by restriction via the diagonal map δ. If $M \cong H^*X$ and $N \cong H^*Y$ the structure determined on $M \otimes N \cong H^*(X \times Y)$ agrees with the natural one.

Yet another equivalent way to express the Cartan formula is to say that the map $\mu : K \bigotimes K \to K$ is $\mathcal{A}_2$-linear.

1.4 Basic examples and applications

The basic example is the mod 2 cohomology of the space $B\mathbf{Z}/2$, $H^* B\mathbf{Z}/2$ is the polynomial algebra $\mathbf{F}_2[u]$ on one generator u of degree 1. The action of $\mathcal{A}_2$ on $\mathbf{F}_2[u]$ is completely determined by the fact it is an unstable algebra and one proves easily by induction on n that

$$\mathrm{Sq}^i u^n = C_n^i u^{n+i} \quad .$$

It is also worth to note the following easy consequences :

- $\mathrm{Sq}^i u^{2^h} \neq 0$ if and only if $i = 0$ or $i = 2^h$.
- $\mathrm{Sq}^I u = 0$ unless I is of the form $\Delta_n = (2^{n-1}, 2^{n-1}, \ldots, 1)$,
- $\mathrm{Sq}^{\Delta_n} = u^{2^n}$.

This is nothing else than the description of the unstable module $F(1)$ which is done in the next paragraph.

This computation allows us to prove that the elements Sq^{2^h}, $h \geq 0$, form a minimal set of multiplicative generators of $\mathcal{A}_2$. We need to show that the operation Sq^{2^t} is not a sum of products of elements of strictly lower degree). But it acts non-trivially on $H^{2^t} B\mathbf{Z}/2$ whereas decomposable elements act trivially.

Let V will be an elementary abelian p-group , in other words a finite dimensional $\mathbf{F}_2$-vector space. The mod 2 cohomology of BV will be denoted by $H^* V$, the reduced mod 2 cohomology of BV will be denoted by $\bar{H}^* V$. As a functor of V the cohomology $H^* V$ identifies with the graded symmetric algebra on the dual vector space $V^* : S^*(V^*)$, the elements of V^* having degree 1. Recall that the symmetric algebra over V^* is the smallest quotient of the tensor algebra over V^* which is commutative. If we denote by d the dimension of the vector space V and by $\{x_1, \ldots, x_d\}$ a basis of the dual V^* it identifies with the polynomial algebra $\mathbf{F}_2[x_1, \ldots, x_d]$.

A map φ of unstable $\mathcal{A}$-algebras from H^*W in H^*V is uniquely determined by its behavior in degree 1. Indeed it is completely determined by its action on the multiplicative generators that are in degree 1. So it is determined by the dual map of its restriction in degree 1 which nothing than a linear map $\varphi : V \to W$. It follows easily that the map

$$\mathrm{Hom}(V, W) \to \mathrm{Hom}_{\mathcal{K}}(H^*W, H^*V) \quad \varphi \mapsto \varphi^*$$

is a bijection.

1.5 Free objects in the category $\mathcal{U}$

The category $\mathcal{U}$ is abelian. It has enough projective objects. This is implied by :

Proposition : There is, up to isomorphism, a unique unstable module $F(n)$ with a class ι_n of degree n such that the natural transformation :

$$\mathrm{Hom}_{\mathcal{U}}(F(n), M) \to M^n, \quad f \mapsto f(\iota_n)$$

is an equivalence of functors.

As the functor $M \mapsto M^n$ from $\mathcal{U}$ to the category $\mathcal{F}$ of $\mathbf{F}_2$ vector spaces is right exact, the unstable module $F(n)$ is projective. The unstable module $F(n)$ is called the free unstable module on one generator of degree n.

Let us define the *excess* of an admissible sequence I to be

$$(i_1 - 2i_2) + (i_2 - 2i_3) + \ldots + (i_{n-1} - 2i_n) + i_n$$

and denote it by $e(I)$.

The suspension functor $\Sigma : \mathcal{M} \to \mathcal{M}$ is defined as follows : given an unstable module M, ΣM is given by $(\Sigma M)^n \cong M^{n-1}$. The $\mathcal{A}_2$-action is given by $\theta(\Sigma m) = \Sigma\theta m$ for all $m \in M$, $\theta \in \mathcal{A}_2$.

We now describe the unstable module $F(n)$:

Theorem : The unstable module $F(n)$ is isomorphic to $\Sigma^n(\mathcal{A}_2/(\mathrm{Sq}^I$, I admissible and $e(I) > n)$.

Hence this module has as an $\mathbf{F}_2$-basis the elements $\Sigma^n\mathrm{Sq}^I$, with I admissible and $e(I) \leq n$. We will denote $\Sigma^n\mathrm{Sq}^I$ by $\mathrm{Sq}^I\iota_n$.

Proof : We are not going to give it in detail and refer to [SE] or [Sc] for that. We indicate what one has to do. Consider the n-th suspension of $\mathcal{A}_2$. Let $\iota_n = \Sigma^n 1$ be the generator. By an abuse of notation we will keep the same name for the generator of $F(n)$.

A priori, all the elements $Sq^I \iota_n$ with $e(I) > n$ are mapped to zero in an unstable module. We have to show that the $\mathbf{F}_2$-span of the elements Sq^I ith I admissible and $e(I) > n$ is a sub-module, denoted $\mathcal{B}(n)$ of $\mathcal{A}_2$. This implies that any map from the n-th suspension of $\mathcal{A}_2$ to an unstable module factors the quotient by $\mathcal{B}(n)$, that we denote $F(n)$. This means that the span of a class of degree n in an unstable module is always a quotient of $F(n)$ *via* the map sending ι_n to x.

The unstable module $F(1)$ can be identified with the sub-module of $H^*\mathbf{Z}/2$ generated by the class u. The set $\{u, u^2, u^4, \dots\}$ is a basis for $F(1)$. Indeed note that the admissible monomial of excess less than one are just the Sq^{Δ_n}.

If $n = 2$ the admissible monomial of excess less than 2 are outside the Sq^{Δ_n} the monomials of the form $Sq^{\Delta_{i,j}}$ with $\Delta_{i,j} = (2^{i+j} + 2^j, \dots, 2^i + 1, 2^{i-1}, \dots, 1)$ with $i \geq 1$ and $j \geq 0$, or $Sq^{\Delta_{0,j}}$ with $\Delta_{0,j} = (2^{j+1}, \dots, 2)$ and $j \geq 0$.

It is easy to check that if M and N are unstable, $M \otimes N$ equipped with the diagonal action is unstable. Consider the unstable module $F(1)^{\otimes n}$; it is of dimension 1 in degree n. Therefore, there is just one non trivial map $\omega_n : F(n) \to F(1)^{\otimes n}$. The symmetric group $\mathcal{S}_n$ acts on $F(1)^{\otimes n}$ by permutation of the coordinates. If $p = 2$ the non trivial class of degree n of $F(1)^{\otimes n}$ is $\mathcal{S}_n$-invariant. Therefore, ω_n takes values in the invariants under the action of $\mathcal{S}_n$ which are denoted $(F(1)^{\otimes n})^{\mathcal{S}_n}$.

As an application we now prove that the Sq^I, I admissible form a linearly independent set.

To do that consider the map

$$Sq^I \mapsto Sq^I(x_1 \dots x_n) \in H^*\mathbf{Z}/2^{\oplus n} \cong \mathbf{F}_2[x_1, \dots, x_n] \quad .$$

We claim that this map is an injection when restricted to the subspace generated by elements such that $e(I) \leq n$. Let Δ_k be the sequence $(2^{k-1}, \dots, 1)$. If I is admissible it writes uniquely as $I = \Delta_{k_1} + \dots + \Delta_{k_s}$, with $k_1 \geq k_2 \geq \dots \geq k_s$. If the excess is less than n the ineger s has to be less than n. The element $Sq^I(x_1 \dots x_n)$ contains the monomial $x_1^{2^k_1} \dots x_s^{2^k_s} x_{s+1} \dots x_n$. It is the largest in the reverse lexichographic order that appears. The result follows easily.

Here is another description of $F(n)$ which is a consequence of what precedes :

Theorem : The map ω_n is an isomorphism from $F(n)$ onto $(F(1)^{\otimes n})^{\mathcal{S}_n}$.

Proof :

In fact the preceding argument shows that the map is a monomorphism, and an inductive argument along the same kind of ideas shows that it is surjective.

1.6 Instability and the Adem relations

The Adem relations are delicate to use. However some of them simplify in an unstable module M. Let us introduce an operation Sq_0. On an element $x \in M$, M unstable, it is given by :

$$\mathrm{Sq}_0 x = \mathrm{Sq}^{|x|} x \qquad .$$

Proposition : One has :

- $\mathrm{Sq}^i \mathrm{Sq}_0 x = \mathrm{Sq}_0 \mathrm{Sq}^{i/2} x$ if i is even,
- $\mathrm{Sq}^i \mathrm{Sq}_0 x = 0$ if i is odd.

Proof : Indeed, one applies the Adem relations and one gets:

$$\mathrm{Sq}^i \mathrm{Sq}^{|x|} x = \sum_0^{[i/2]} C_{|x|-t-1}^{i-2t} \mathrm{Sq}^{|x|+i-t} \mathrm{Sq}^t x$$

But $\mathrm{Sq}^{|x|+i-t}(\mathrm{Sq}^t x)$ is trivial as soon as $|x| + i - t > t + |x|$, *i.e.* $i > 2t$.

Therefore the term on the right reduces to zero if i is odd and to $\mathrm{Sq}^{|x|+i/2} \mathrm{Sq}^{i/2} x$ if i is even.

This leads us to introduce the following functor Φ from $\mathcal{U}$ to itself. Given $M \in \mathcal{U}$, define ΦM as a graded vector space by

$$(\Phi M)^n = M^{n/2}$$

if n is even,

$$(\Phi M)^n = \{0\}$$

if n is odd.

If x is an element of M we denote by Φx the corresponding element in ΦM.

The following formulas define an action of the Steenrod algebra on ΦM :

$\mathrm{Sq}^i \Phi x = \Phi \mathrm{Sq}^{i/2} x$, if i is even, and

$\mathrm{Sq}^i \Phi x = 0$ if i is odd.

The above relations express the fact that the map

$$\lambda_M \ : \Phi M \to M, \quad \Phi x \mapsto \mathrm{Sq}_0 x$$

is $\mathcal{A}_2$-linear.

The following result is particularly useful. Let $\Sigma\iota_{n-1} : F(n) \to \Sigma F(n-1)$ be the map which sends ι_n to $\Sigma\iota_{n-1}$:

Proposition : The following is an exact sequence of unstable modules

$$0 \longrightarrow \Phi F(n) \xrightarrow{\lambda_{F(n)}} F(n) \xrightarrow{\Sigma\iota_{n-1}} \Sigma F(n-1) \longrightarrow 0 \quad .$$

This is proved by using the canonical basis of $F(n)$. One checks that $\Phi F(n)$ identifies as a submodule of $F(n)$ with the $\mathbf{F}_2$-span of those $\mathrm{Sq}^I\iota_n$ where $e(I) = n$, *i.e.* all classes of the form $\mathrm{Sq}_0 x$. Therefore $\lambda_{F(n)}$ is injective. Next a class of the form $\mathrm{Sq}_0 x$ is always trivial in a suspension, by the instability condition. Thus the composite

map $\Sigma\iota_{n-1} \circ \lambda_{F(n)}$ is trivial. Exactness at $F(n)$ and $\Sigma F(n-1)$is proved using the canonical basis ot these modules.

More generally for any integer k one has the following exact sequences :

$$0 \longrightarrow \Phi^{k+1} F(n) \xrightarrow{\lambda_{\Phi^k F(n)}} \Phi^k F(n) \longrightarrow \Sigma^{2^k} \Phi^k F(n-1) \longrightarrow 0 \quad .$$

They are obtained by applying the functor Φ to the preceding one. One has only to take care of exactness and to observe that for any unstable module M one has

$$\Phi\Sigma M \cong \Sigma^2 \Phi M \quad .$$

To conclude observe that $\mathrm{Ker}\lambda_M$ and $\mathrm{Coker}\lambda_M$ are, by the very definition, suspensions. One defines functors Ω and Ω_1 from $\mathcal{U}$ to itself by

$$\mathrm{Ker}\lambda_M = \Sigma\Omega_1 M \quad ,$$

$$\mathrm{Coker}\lambda_M = \Sigma\Omega M \quad .$$

In other words we have an exact sequence

$$0 \to \Sigma\Omega_1 M \to \Phi M \to M \to \Sigma\Omega M \to 0 \quad .$$

Theorem : The functor Ω is the left adjoint of the suspension functor Σ. The functor Ω_1 is the first left derived functor of Ω.

1.7 The category $\mathcal{U}$ is locally noetherian

Theorem : Let M be an unstable module which has finitely many generators as an $\mathcal{A}_2$-module. Then any sub $\mathcal{A}_2$-module of M has also finitely many generators as $\mathcal{A}_2$-module.

This is due to W. Massey and F. Peterson. It implies that the category $\mathcal{U}$ is locally noetherian, let us recall the definition.

Definition : An object in an abelian category is noetherian if it has the ascending chain property, *i.e.* if any ascending sequence of subobjects stabilizes.

Definition : A category is locally noetherian if it has a set of noetherian generators.

Recall that a set of objects $\{M_\alpha\}$, $\alpha \in A$, in an abelian category is called a set of generators, if for any object M in the category, there is an epimorphism from a certain direct sum of the M_α's onto M. We assume moreover, as part of the definition, that the category has exact filtered colimits.

The unstable modules $F(n)$ are obviously a set of generators for $\mathcal{U}$.

Proof :

We show that an unstable module which is finitely generated as module over the Steenrod algebra is noetherian. To do that it is enough to show it for the $F(n)$'s because, as we have seen, a finitely generated unstable module is the quotient of a finite direct sum of $F(n)$'s.

We note that this result would be false in the category of all modules over the Steenrod algebra, as shown by the following example : consider the Steenrod algebra $\mathcal{A}_2$ itself, the submodule of elements of strictly positive degree is not finitely generated as an $\mathcal{A}_2$ module.

Let us show that a sub-module M of $F(n)$ is finitely generated. We will work by induction on n and assume that $n > 0$. Let us suppose that the result is known for $n-1$. Let k be a given integer. Define M_k to be the sub $\mathcal{A}_2$-module of $F(n)$ of those x such that $\mathrm{Sq}_0^k x$ is in M. It is clear that $M_k \subset M_{k+1}$. Let $\Sigma M'_k$ be the image of M_k into $\Sigma F(n-1)$ under the map $\Sigma\iota_{n-1}$. By the induction hypothesis one can assume that for $k \geq k_0$ one has $M'_k = M'_{k+1}$. It implies that, if $k \geq k_0$, one has also $M_k = M_{k+1}$. One concludes that

M_k, $k \geq k_0$, is finitely generated. Indeed one checks that the lift of a generating system for $\Sigma M'_k$ (which can be supposed to be finite by induction) is a generating system for M_k.

Then to finish the proof one uses the exact sequences :

$$0 \to \Phi M_k \to M_{k-1} \to \Sigma M'_k \to 0$$

and the facts that M'_k is finitely generated by induction hypothesis and if M is finitely generated so is ΦM.

Theorem : The unstable module $F(p) \otimes F(q)$ is finitely generated for all p and q. Therefore, the tensor product of two finitely generated unstable modules is finitely generated.

We will not give the proof, just say that one can reduce to show that $F(1)^{\otimes n}$ is finitely generated (see [Sc]).

1.8 Nilpotent and locally finite unstable modules

Definition : An unstable module is locally finite if it is a direct limit of *finite modules.* An unstable module is nilpotent if it is the direct limit of unstable which have a finite filtration whose quotients are suspensions.

Proposition : The full sub-category of $\mathcal{U}$ whose objects are locally finite modules (resp. nilpotent modules) is a thick sub-category.

A full sub-category is thick whenever given a short exact sequence :

$$0 \to M' \to M \to M" \to 0$$

as soon as two objects are in the sub-category so is the third.

Proof :

In the case of nilpotent modules it is obvious by definition. In the case of locally finite modules it is not. Indeed, the fact that M is locally finite if M' and $M"$ are deserves a proof [Sc], it can also be done as an application of Brown-Gitler modules (exercice!). This is false in the category $\mathcal{M}$ as shown by the following where $\bar{\mathcal{A}}_2$ denotes the suset of elements of non-zero degree of $\mathcal{A}_2$:

$$0 \to \bar{\mathcal{A}}_2/\bar{\mathcal{A}}_2.\bar{\mathcal{A}}_2 \to \mathcal{A}_2/\bar{\mathcal{A}}_2.\bar{\mathcal{A}}_2 \to \mathbf{F}_2 \to 0 \quad .$$

In this example the action on M' which generated over $\mathbf{F}_2$ by Sq^{2^h}, $h \geq 0$, is trivial. Whereas M is cyclic over $\mathcal{A}_2$, generated by 1 and infinite.

Here is another characterization of the nilpotent modules :

Proposition : An unstable module is nilpotent iff the following hold : for any $x \in M$, $\mathrm{Sq}_0^n x = 0$ as soon as n is sufficiently large.

The proof is left to the reader. Note that a non-trivial nilpotent module always contains a non-trivial suspension.

Proposition : The full sub-category of $\mathcal{U}$ whose objects are locally finite modules (resp. nilpotent modules) is stable under direct limits and tensor product.

Definition : An unstable module is reduced its largest nilpotent submodule is trivial.

As an example $H^*\mathbf{Z}/2$ is reduced as is more generally H^*V.

Injective objects in $\mathcal{U}$
Lannes' T functor

In this chapter we will describe objects in the category $\mathcal{U}$ and discuss Lannes' T functors and give examples of computations.

Let us first recall that $\{C_\alpha\}$, $\alpha \in A$, is a set of cogenerators for a category $\mathcal{C}$ if any object of the category embeds in a product of C_α's.

There are three basic types of injective objects in the category $\mathcal{U}$.

- The Brown-Gitler modules which are the obvious cogenerators for $\mathcal{U}$;
- The Carlsson modules which are cogenerators for the subcategory of reduced objects in $\mathcal{U}$;
- The Lannes-Zarati modules that are tensor product of Brown-Gitler modules by Carlsson modules.

2.1 Generalities and representability

Let Ab be the category of abelian groups. Recall that :

Definition : An object I in an abelian category $\mathcal{C}$ is injective if and only if the functor

$$\mathcal{C} \to \mathcal{A}b, \quad M \mapsto \mathrm{Hom}_{\mathcal{C}}(M, I)$$

is exact.

From the definition it is clear that any product of injective objects is again injective. For convenience, we will often abbreviate "injective unstable module" to "$\mathcal{U}$-injective".

The following statement is true in any locally noetherian abelian category which has exact filtered direct limits.

Proposition : Any filtered direct limit, in particular any direct sum, of $\mathcal{U}$-injectives is $\mathcal{U}$-injective.

The converse is also true, if any filtered colimit of injective objects is an injective object the category is locally noetherian.

An $\mathcal{A}_2$-module which is finite dimensional as an $\mathbf{F}_2$-vector space in every degree will be said to be of *finite type*

In order to construct $\mathcal{U}$-injectives, we construct contravariant functors from $\mathcal{U}$ to the category $\mathcal{E}$ of $\mathbf{F}_p$-vector spaces. The functors that we will consider will be representable and left exact. Thus the representing modules will be $\mathcal{U}$-injective.

The following lemma tell us when a contravariant functor $R : \mathcal{U} \to \mathcal{A}b$ is representable. It is classical and true in a much more general context. One essentially needs the existence of a set of projective generators.

Lemma : The functor R is representable if and only if it is right exact and transforms direct sums into products.

That the conditions are necessary is obvious. We discuss the aspects of the other implication that are useful for us. Suppose that the problem is solved and that the functor is represented by an unstable module $B(R)$. Thus we have a natural equivalence

$$\gamma : R \to \mathrm{Hom}_{\mathcal{U}}(-, B(R))$$

In particular we have

$$R(F(n)) \cong \mathrm{Hom}_{\mathcal{U}}(F(n)), B(R)) = B(R)^n \quad .$$

which defines $B(R)$ as a graded vector space. An unstable module structure is defined on this graded vector space as follows: the operation $\theta \in \mathcal{A}_2$ acts as $R(u_\theta)$, where $u_\theta : F(n+|\theta|) \to F(n)$ is the $\mathcal{A}_2$-linear map associated to $\theta i_n \in F(n)^{n+|\theta|}$.

One defines the natural transformation as follows Let M be in $\mathcal{U}$ and x be in $R(M)$. An element $y \in M$ can be identified with a map $\hat{y} : F(|y|) \to M$. We set $\gamma_M(x)$ to be the ($\mathcal{A}_2$-linear) map which sends $y \in M$ to the element $R(\hat{y})(x) \in R(F(|y|) = B(R)^{|y|}$. By the very construction $\gamma_{F(n)}$ is an isomorphism. To show that γ_M is an isomorphism, M being given in $\mathcal{U}$, one considers the beginning of a projective resolution of M :

$$P_1 \to P_0 \to M \to 0 \quad ,$$

where P_0 and P_1 are direct sums of $F(n)$'s and one considers an appropriate the diagram using the properties of R.

Next we consider three examples of functors that satisfy the conditions of the lemma and are left exact.

Consider the following functor :

$$H_n : M \mapsto M^{n*} = \mathrm{Hom}_{\mathcal{E}}(M^n, \mathbf{F}_2) \quad .$$

It is obviously right exact and transforms direct sums into products. Therefore it is representable. As it is also left exact the representing module is injective.

Definition : The Brown-Gitler module $J(n)$ is the representing module for the functor H_n.

Next, consider the functor :

$$K_i : M \mapsto (\mathrm{dirlim}_q \{M^{2^q i}, \mathrm{Sq}^{2^q i}\})^* \quad .$$

It is obviously right exact and transforms direct sums into products. Therefore it is representable. As it is also left exact the representing module is injective.

Definition : The Carlsson module $K(i)$ is the representing module for the functor K_i.

Finally, consider :

$$L_{i,n} : M \mapsto (\mathrm{dirlim}_q \{M^{2^q i+n}, \mathrm{Sq}^{2^q i}\})^* \quad .$$

It is obviously right exact and transforms direct sums into products. Therefore it is representable. As it is also left exact the representing module is injective.

Definition : The Lannes-Zarati module $L(i,n)$ is the representing module for the functor $L_{i,n}$.

2.2 Brown-Gitler's modules

Here are basic facts about the unstable modules $J(n)$:

- $J(0)$ is isomorphic to the unstable module $\mathbf{F}_2$;

- $J(n)^m = F(m)^{n*}$, thus $J(n)$ is of finite type;

- $J(n)^m = 0$ if $m > n$;

- $J(n)^n = \mathbf{F}_2$,

  the generator of $H_n(J(n))$ corresponding to the identity map of $J(n)$, it will be denoted by b_n;

- the restriction to degree n :

$$\mathrm{Hom}_{\mathcal{U}}(M, J(n)) \to \mathrm{Hom}_{\mathcal{E}}(M^n, J(n)^n)$$

  identifies with the isomorphism $\mathrm{Hom}_{\mathcal{U}}(M, J(n)) \cong (M^n)^*$.

This remark implies that for any unstable module M the unstable modules there is a map

$$M \to \prod_n \prod_{u \in H_n(M)} J(n) \quad ,$$

where the map from M to the component labelled by u on the right hand side is the one determined by u. It is clearly injective. Therefore the category $\mathcal{U}$ has enough injective objects.

An operation $\theta \in \mathcal{A}_2$ determines an $\mathcal{A}_2$-linear map $\tilde{\theta} : J(n + |\theta|) \to J(n)$ as follows. It is the map corresponding to the linear form $i_n \circ \theta : J(n + |\theta|)^n \to \mathbf{F}_2$, where $\iota_n \in F(n)^n \cong (J(n)^n)^*$ is the generator. In the sequel we will consider in particular the map $\Lambda = \bar{\mathrm{Sq}}^{n/2} : J(n) \to J(n/2)$, which is understood to be trivial if $n/2$ is not an integer.

The class $\iota_n \otimes \iota_n$

in $H_{m+n}(J(m) \otimes J(n))$ determines an $\mathcal{A}_2$-linear

map

$$\mu_{m,n} : J(m) \bigotimes J(n) \to J(m+n) \quad .$$

In degree $m + n$ the map $\mu_{m,n}$ is just the identity of $\mathbf{F}_2$. If we set $m = 1$, since $J(1) = \Sigma\mathbf{F}_2$, we get an $\mathcal{A}_2$-linear map $\mu_{1,n} : \Sigma J(n) \to J(n+1)$.

Theorem : (Mahowald's exact sequences) The following is an exact sequence of unstable modules

$$0 \to \Sigma J(n-1) \xrightarrow{\mu_{1,n-1}} J(n) \xrightarrow{\Lambda} J(n/2) \to 0 \quad ,$$

where $J(n/2)$ is trivial if n is odd.

Proof : Apply the functor H_n to the exact sequence

$$0 \to \Phi F(m) \to F(m) \to \Sigma F(m-1) \to 0$$

for all $m \geq 0$.

The connectivity $c(M)$ of an unstable module M is the smallest n such that M^{n+1} is non zero. The preceding sequence allows computation of the connectivity of $J(n)$. For an integer n, denote by $\alpha(n)$ the sum of the coefficients of its 2-adic expansion.

Theorem : The connectivity $c(J(n))$ of $J(n)$ is $\alpha(n) - 1$.

The proof is done by induction on n using Mahowald's exact sequences.

Finally we note the following ;

Proposition : A finite, or more generally a locally finite unstable module embeds into a direct sum (and not a product) of Brown-Gitler modules.

This is an exercice, it shows that the sub-category of locally finite unstable module is thick.

2.3 Miller's bigraded algebra J_*^*

To understand the structure of the Brown-Gitler modules it is convenient, following Miller, to consider the $\mathbf{N} \times \mathbf{N}$ bigraded object J_*^* defined by $J_k^\ell = J(k)^\ell$, where the bidegree of $x \in J_k^\ell$ is denoted by $||x|| = (|x|_1, |x|_2) = (\ell, k)$.

This object has a very rich structure :

- it is an unstable module, for it is just the direct sum of the unstable modules $J(k)$, the action of the Steenrod algebra leaves fixed the second degree and increases the first one;
- there is a map $\Lambda : J_*^* \to J_*^*$ which leaves fixed the first degree and divides the second one. This map is $\mathcal{A}_2$-linear;
- the maps $\mu_{m,n} : J(m) \bigotimes J(n) \to J(m+n)$ define the structure of a commutative, associative, unital, $\mathbf{F}_2$-bigraded algebra;

- the Cartan formula holds for this product, and the map Λ is multiplicative.

All the preceding properties are more or less straightforward to check and left to the reader.

The algebra J_*^* is not an unstable algebra because the second condition does not hold. In fact one has :

Lemma : For any $x \in J_*^*$ one has

$\mathrm{Sq}^0 x = (\Lambda x)^2$.

Proof :

We indicate here the main steps. Given an unstable module M let us denote by $S_2(M)$ its second symmetric power, *i.e.* the quotient of $M \otimes M$ by the sub-module of elements of the form $x \otimes y + y \otimes x$. If M is a commutative algebra (not necessary unstable) the product map $\mu : M \otimes M \to M$ factors as $\mu = \tilde{\mu} \circ \pi$, where $\pi : M \otimes M \to S_2(M)$ is the canonical projection and $\tilde{\mu}$ maps $S_2(M)$ into M. Consider now the following diagram :

$$\begin{tikzcd}
\Phi J(2k) \arrow[r, "i"] \arrow[d, "\Phi(\bar{\mathrm{Sq}}^k)"'] & S_2(J(2k)) \arrow[r, "\tilde{\mu}_{2k,2k}"] \arrow[d, "S_2(\bar{\mathrm{Sq}}^k)"'] & J(4k) \arrow[d] \\
\Phi J(k) \arrow[r] & S_2(J(k)) \arrow[r, "\tilde{\mu}_{k,k}"] & J(2k)
\end{tikzcd}$$

where we recall that Sq^k is nothing else that Λ, and i sends x to the image of $x \otimes x$ into $S_2(J(2k))$.

Lemma : This diagram is commutative.

Proof :

The fact that the left hand side square is commutative is just naturality. For the right hand side one it is enough to show commutativity of :

$$\begin{array}{ccc} J(2k)\bigotimes J(2k) & \xrightarrow{\mu_{2k,2k}} & J(4k) \\ {\scriptstyle \bar{\mathrm{Sq}}^k\otimes\bar{\mathrm{Sq}}^k}\downarrow & & {\scriptstyle \bar{S}q^k}\downarrow \\ J(k)\bigotimes J(k) & \xrightarrow{\mu_{k,k}} & J(2k) \end{array}$$

which follows from the Cartan formula, the instability, and the :

Lemma : Let θ be in $\mathcal{A}_2$, M an unstable module and $u\in H_n(M)$=$\mathrm{Hom}_{\mathcal{U}}(M,J(n))$. Then the map $\tilde{\theta}\circ u \in=\mathrm{Hom}_{\mathcal{U}}(M,J(n-|\theta|))$ as an element of $H_{n-|\theta|}(M)$ is the composite

$$M^{n-|\theta|}\xrightarrow{\theta}M^n\xrightarrow{u}\mathbf{F}_2 \quad .$$

The proof of the Lemma is left to the reader. It remains to observe that the composite from $\Phi J(2k)$ to $J(2k)$ in the first diagram is just $\lambda_{J(2k)}$.

We conclude by describing the structure of J_*^* :

Theorem : The algebra J_*^* is isomorphic to the bigraded polynomial

algebra

$$\mathbf{F}_2[x_0,\ldots,x_i,\ldots]$$

on generators

x_i of bidegree $(1,2^i)$. It has an $\mathcal{A}_2$-action

such that :

- it is an unstable module;
- the Cartan formula holds;
- $\mathrm{Sq}^1(x_i)=x_{i-1}^2$;
- the multiplicative map Λ determined by $\Lambda(x_0)=0$ and $\Lambda(x_i)=x_{i-1}$, $i>0$ is $\mathcal{A}_2$-linear.

Proof : The Mahowald exact sequences allow us to define non-zero elements $\alpha_i\in J(2^i)^1$. The polynomial algebra T_*^* abstractly defined by the theorem is mapped into J_*^*

in a unique way by declaring that x_i is sent to α_i. What we said before shows that all the structure is preserved. In particular the span of monomials whose second degree is n is mapped into $J(n)$. We have for the subspace T_*^n the exact sequences strictly analogous to the Mahowald exact sequences. The result follows by induction considering the commutative diagram :

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T^*_{n-1} & \xrightarrow{\times x_0} & T^*_n & \xrightarrow{\Lambda} & \Sigma T^*_{n/2} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J^*_{n-1} & \xrightarrow{\times \varepsilon_0} & J^*_n & \xrightarrow{\Lambda} & \Sigma J^*_{n/2} & \longrightarrow & 0
\end{array}$$

with the usual conventions.

2.4 Carlsson's modules and the Carlsson algebra

We now consider the case of the functors K_i. The corresponding modules were introduced by Carlsson in his proof of the Segal conjecture for elementary abelian 2-groups. Let i be a given integer. First observe that by definition the functors K_i and K_{2i} are naturally equivalent. Thus the unstable module $K(i)$ is isomorphic to the unstable module $K(2i)$. The following lemma is a direct consequence of the definition, and of the fact that the Brown-Gitler modules are of finite type :

Lemma : The unstable module $K(i)$ is the inverse limit of the following system

$$\{J(2^q i),\ \tilde{\mathrm{Sq}}^{2^{q-1}i}; q \in \mathbf{N}\} \quad .$$

Proof : Indeed, one has for any unstable module :

$$\mathrm{Hom}_{\mathcal{U}}(M, K(i)) \cong \{\lim_q M^{2^q i}\}^* \cong \mathrm{Hom}_{\mathcal{U}}(M, \{\lim_q J(2^q i)\}) \quad ,$$

where the inverse limits are taken over the maps $\tilde{\mathrm{Sq}}^{2^q i}$. The first isomorphism is by definition, the second one follows from the interpretation of the maps $\tilde{\mathrm{Sq}}^{2^q i}$ given in 2.2. The result follows.

The following is a consequence of the preceding and of the Mahowald exact sequences that show that the connectivity of the maps $\tilde{\mathrm{Sq}}^{2^q i}$ increase with q.

Corollary : The unstable module $K(i)$ is of finite type.

Proposition : The unstable modules $K(i)$ are reduced.

Proof : It is enough that there are no non-trivial suspensions in $K(i)$. Let ΣM be a suspension contained in $K(i)$. By the definition of a nilpotent module $\mathrm{Hom}_{\mathcal{U}}(\Sigma M, K(i))$ is trivial, therefore and ΣM is trivial.

They are cogenerators for reduced modules in the sense that any reduced module embeds in a product of $K(i)$'s.

Because of the isomorphism $K(i) \cong K(2i)$ one can define of the $K(i)$ for all rationals in $\mathbf{N}[1/2]$ by the following formulas

$$K(i) = K(2^q i) \quad \text{for} \quad 2^q i \in \mathbf{N} \quad ,$$

it makes sense because the value does depend on q.

We can define a $\mathbf{N} \times \mathbf{N}[1/2]$-bigraded object K_*^* by $K_i^j = K(i)^j$.

Let us introduce for $q \geq 0$ the $\mathbf{N} \times \mathbf{N}[1/2]$ bigraded algebras $J(q)_*^*$ defined by :

- $J(q)_i^j = J(2^q i)^j$ if $2^q i \in \mathbf{N}$, 0 otherwise.

The bigraded object $J(q)_*^*$ should be thought of as a copy of J_*^* "shifted multiplicatively down" q-times with respect to the second grading. It has all properties of J_*^*, one has:

$$J(q)_*^* = \mathbf{F}_2[x_{q,i}, i \geq -q] \quad .$$

The action of the Steenrod algebra is determined by the Cartan formula and $\mathrm{Sq}^1 x_{q,i} = x_{q,i-1}^2$ is $i > -q$, and 0 if $i = -q$.

Then consider the inverse limit of these algebras taken over the algebra maps

$$\Gamma_q \ : \ J(q)_*^* \to J(q-1)_*^* \quad .$$

Where the map Γ_q is given by $\Gamma_q(x_{q,i}) = x_{q-1,i}$ if $i > -q$, and 0 if $i = -q$.

Note that the map Γ_q from $J(q)_{2i}^*$ to $J(q)_i^*$ identifies with the map $\tilde{\mathrm{Sq}}^{2^{q-1}} i \ : J(2^q i) \to J(2^{q-1} i)$. Thus, the above results implies that the inverse limit identifies with K_*^*.

The bigraded object K_*^* has an unstable module structure which preserves the second degree. It has a bigraded algebra structure. Moreover

The maps $\Lambda : J(q)_*^* \to J(q)_*^*$ determine a map $\Lambda : K_*^* \to K_*^*$ which preserves the first degree and divides the second by 2.

The following theorem due to J. Lannes and S. Zarati (it is implicit in Carlsson) describes the structure of K_*^* :

Theorem : The Carlsson algebra K_*^* is isomorphic to the polynomial algebra

$$\mathbf{F}_2[\hat{x}_i, i \in \mathbf{Z}]$$

where

- $||\hat{x}_i|| = (1, 2^i)$,
- $\Lambda\hat{x}_i = \hat{x}_{i-1}$, and
- the relation $\mathrm{Sq}^1\hat{x}_i = \hat{x}_{i-1}^2$ and the Cartan formula determine the unstable module structure.

2.5 Tensor products of injective unstable modules

In this section we give the following theorem due to Lannes and Zarati and describe the consequences. The proofs are technical and will not be given here and for them we refer to [Sc], nevertheless we note they are the logical sequel of the preceding ones.

Theorem : The representing unstable module for the functor $L_{i,n}$ is the unstable module $K(i) \bigotimes J(n)$. It is an injective unstable module.

In fact only the first statement deserves a proof.

Here is an essential Corollary :

Theorem : Let I and J be two $\mathcal{U}$-injectives such that J is a reduced unstable module, and I or J is of finite type;

Then $I \otimes J$ is injective.

2.6. Commutation of T_V with tensor products

After exactness the second main property of the functor T_V is its behaviour with respect to tensor product. The following theorem of Lannes shows that the result is the best possible :

Theorem : For any unstable modules M_1 and M_2, there is a natural isomorphism

$$\nu_{M_1,M_2} : T_V(M_1 \otimes M_2) \to T_V(M_1) \otimes T_V(M_2) \quad .$$

The proof of this result is to long to be given in these notes. Using right exactness of T_V it reduces easily to compute T_V on a tensor product of the form $F(m) \bigotimes F(n)$. In this context the fact that T commutes with Φ is very useful. The following lemma which has its own interest is needed to do the computation in degree zero :

Proposition : For q large enough the unstable module $F(m) \otimes \Phi^q F(n)$ is monogenic.

Note that if we take $m_1 = \Sigma \mathbf{F}_2$ one gets the commutation with suspension.

We are only going to describe the above mentioned natural map.

Let M_1, M_2, L_1, L_2 be in $\mathcal{U}$. Consider the maps $\varphi_i : M_i \to L_i \otimes (M_i : L_i)_{\mathcal{U}}$ $(i = 0, 1)$ adjoint to the identity of $(M_i : L_i)_{\mathcal{U}}$. The adjoint of $(1 \otimes \tau \otimes 1) \circ (\varphi_1 \otimes \varphi_2)$ (τ being the obvious twisting map) is by definition

$$\nu^{L_1,L_2}_{M_1,M_2} : (M_1 \otimes M_2 \; : \; L_1 \otimes L_2)_{\mathcal{U}} \to (M_1 \; : \; L_1)_{\mathcal{U}} \otimes (M_2 \; : \; L_2)_{\mathcal{U}} \quad .$$

Let $L_1 = L_2 = L$ be an unstable algebra. Then the multiplication $\mu : L \otimes L \to L$ defines a map $\mu^*_M : (M : L)_{\mathcal{U}} \to (M : L \otimes L)_{\mathcal{U}}$.

If $L = H^* V$, one defines ν_{M_1,M_2} to be the composite $\nu^{H^* V, H^* V}_{M_1,M_2} \circ \mu^*_{M_1 \otimes M_2}$.

The fundamental application of what precedes shows that the behaviour of T_V on unstable algebras is, again, as best as possible, it is due to Lannes:

Theorem : Let K, L be unstable algebras. The unstable module $T_V(K)$ is in a natural way an unstable algebra and there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{K}}(K, H^* V \otimes L) \cong \mathrm{Hom}_{\mathcal{K}}(T_V(K), L) \quad .$$

We will not give the full proof of this statement, but comment certain important parts. We need to show :

- that the functor $\mathcal{K} \to \mathcal{K}$, $L \mapsto H^* V \otimes L$ has a left adjoint $K \mapsto (K : H^* V)_{\mathcal{K}}$; and

- that, $(K : H^*V)_{\mathcal{K}}$ identifies with $T_V(K)$.

Here is the first proposition:

Proposition : Let C be an unstable $\mathcal{A}$-algebra of finite type. The functor $L \mapsto L \otimes C$ from the category $\mathcal{K}$ to itself has a left adjoint $K \mapsto (K : C)_{\mathcal{K}}$. For any K, L in $\mathcal{K}$, there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{K}}(K, C \otimes L) \cong \mathrm{Hom}_{\mathcal{K}}((K : C)_{\mathcal{K}}, L) \quad .$$

As in the case of the category $\mathcal{U}$ this is implied by Freyd's adjoint functor theorem. Thus, the first part of the result is formal. What is not is the subsequent identification. The main point of the proof is :

Theorem : Let K be an unstable algebra. Then $T_V(K)$ is in a natural way an unstable algebra.

The product is defined to be the composite :

$$T_V(K) \otimes T_V(K) \cong T_V(K \otimes K) \to \xrightarrow{T_V\mu} T_V(K) \quad .$$

That this defines an unstable algebra algebra structure is not hard. Except for the axiom concerning Sq_0. We refer to [Sc]. We give now a very important application. Observe that if K is an unstable algebra K^0 is a Boolean algebra. Then :

Theorem : Let K be an unstable algebra which is noetherian as an algebra. Then the Boolean algebra $T_V(K)^0$ is isomorphic to the one of maps from $\mathrm{Hom}_{\mathcal{K}}(K, H^*V)$ to ${}_bfF_2$. In other words the canonical linear extension :

$$\mathbf{F}_2[\mathrm{Hom}_{\mathcal{K}}(K, H^*V)] \cong \mathrm{Hom}_{\mathcal{U}}(K, H^*V)$$

is an isomorphism.

This result extends if one removes the finiteness condition on K, but one needs to introduce profinite topology. We note also that it implies injectivity of H^*V in the categorical sense in $\mathcal{K}$.

Finally, we note that given an unstable algebra K and a primitive idempotent π in K^0, πK is a connected unstable algebra. In particular the above result shows that primitive idempotents in the Boolean algebra $T_V(K)^0$ are in one to one correspondence with elements of $\mathrm{Hom}_{\mathcal{K}}(K, H^* V)$. So for $\varphi \in \mathrm{Hom}_{\mathcal{K}}(K, H^* V)$, there is a primitive idempotent ε_φ and one can consider $\varepsilon_\varphi T_V(K)$.

2.7 Important examples

We collect here a few examples of important computations with T_V. The most basic example is the case of the mod p-cohomology of an elementary abelian 2-group W :

$$T_V H^* W \cong H^* W^{\mathrm{Hom}(V,W)} \quad .$$

The degree zero part of this result is known as the Adams-Gunawardena- Miller-Lannes-Zarati theorem. After dualization it writes as

$$\mathrm{Hom}_{\mathcal{U}}(H^* W, H^* V) \cong \mathbf{F}_2[\mathrm{Hom}(V, W)] \quad .$$

Let us consider the “case of very nice algebras”. Let M be an unstable module the algebra $U(M)$ is described as follows. The functor U is the Steenrod-Epstein functor which is left adjoint to the forgetful functor $\mathcal{K} \to \mathcal{U}$. Therefore, for any $K \in \mathcal{K}$ and $M \in \mathcal{U}$, one has

$$\mathrm{Hom}_{\mathcal{U}}(M, \mathcal{O}K) \cong \mathrm{Hom}_{\mathcal{K}}(U(M), K) \quad .$$

Consider the symmetric algebra $S(M)$, it is the quotient of the tensor algebra $T(M)$ by the ideal generated by the elements $x \otimes y \otimes x$ for all x and y in M. Therefore, $S(M)$ is an unstable module and has a product for which the Cartan formula holds. However it is not true that it is an unstable algebra. One defines the unstable algebra $U(M)$ to be the quotient of $S(M)$ by the ideal generated by the elements $\mathrm{Sq}_0 y + y^{\otimes 2}$ for any $y \in M$. It is nothing other than the universal enveloping algebra of M regarded as an abelian restricted Lie algebra. The functor $M \mapsto U(M)$ is clearly left adjoint to the forgetful functor $\mathcal{K} \to \mathcal{U}$.

The cohomology of Eilenberg-MacLane spaces has this form, recall that Serre has shown

Theorem :

$$H^*(K(\mathbf{Z}/2)) \cong U(F(n)) \quad .$$

If $K \cong U(M)$ for some unstable module M, then

$$T_V U(M) \cong U(T_V M) \quad .$$

Let K be an unstable $\mathcal{A}$ -algebra provided with the action of a finite group G. Then, by naturality, G acts on $T_V K$ and

$$T_V(K^G) \cong (T_V K)^G \quad .$$

Consider now a subgroup G of the automorphism group $Aut V$ of an elementary abelian p-group V, $\gamma \in G$ acts on $H^* V$ by $x \to (\gamma^{-1})^* x$. Let K be the subalgebra of invariants $(H^* V)^G$, i the inclusion of K in $H^* V$ and φ an element of $\mathrm{Hom}(W, V)$.

Theorem : The unstable $\mathcal{A}$ -algebra $T_W K$ is isomorphic to the (graded) set of G-equivariant maps from $\mathrm{Hom}(W, V)$ to $H^* V$. The component $(T_W K)_{\varphi^* \circ i}$ is isomorphic to the subalgebra of invariants $H^* V^{G_\varphi}$, where G_φ is the subgroup of G of elements γ such that $\gamma \circ \varphi = \varphi$

Let G be a finite group, recall a theorem of Quillen. Let $\mathcal{C}(G)$ be the category whose objects are the abelian p-subgroups E of G; the morphisms being subconjugacy. There is a natural map

$$q_G : H^* BG \to lim_{\mathcal{C}(G)} H^* BE \quad .$$

Quillen's theorem says that q_G is an F-isomorphism. It means that any element in the kernel of q_G is nilpotent, and that for any element y in the target, there exist n such that y^{2^n} is in the image. In our terminology this means that

Ker q_G and Coker q_G are nilpotent unstable modules.

Proposition : Let M be a nilpotent unstable module. Then $T_V^0 M$ is trivial.

Indeed the operation Sq_0 is injective on $H^* \mathbf{Z}/2$ and nilpotent on any element of M. Let $\mathrm{Rep}(V, G)$ be the set of homomorphisms $\mathrm{Hom}(V, G)$ modulo conjugacy at the target. There is a canonical bijection

$$\mathrm{colim}_{\mathcal{C}(G)} \mathrm{Hom}(V, E) \to Rep(V, G) \quad .$$

The following consequence of Quillen's theorem has been proved by various people, among them J.-F. Adams, Lannes, Miller and C. Wilkerson.

Theorem : The Boolean algebra $T_V^0 H^* BG$ is isomorphic to the p-Boolean algebra of functions from $\mathrm{Rep}(V, G)$ to $\mathbf{F}_2$.

Let φ be in $\mathrm{Rep}(V,G)$ and let

$$Z_\varphi = (g \in G/g\varphi(v)g^{-1} = \varphi(v)\forall v \in V) \quad .$$

be the centralizer of φ. The homomorphism $V \times Z_\varphi \to G$ determines a map a_φ : $T_V H{*}BG \to H^*BZ_\varphi$, Lannes proved :

Theorem : The sum map

$$\oplus_\varphi\, a_\varphi : T_V H^*BG \to \oplus\, H^*BZ_\varphi$$

is an isomorphism.

Our last example is of a different nature. Let K be an unstable augmented algebra, let M be an unstable module, which is moreover a K-module. Suppose also that the structure map $K \otimes M \to M$ is $\mathcal{A}_2$-linear. Then the bar resolution determines on the functors $\mathrm{Tor}^*_K(M, \mathbf{F}_2)$ an unstable module structure. Moreover $T_V(M)$ comes with a $T_V(K)$-module structure, $T_V(K)$ being augmented.

Theorem : One has an isomorphism :

$$T_V(\mathrm{Tor}^*_K(M, \mathbf{F}_2) \cong \mathrm{Tor}^*_{T_V(K)}(T_V(M), \mathbf{F}_2) \quad .$$

This follows from exactness of T and commutation with tensor products.

2.8 The functor T, locally finite and nilpotent modules

We will only state two results. The first one is :

Theorem : An unstable module is nilpotent if and only if $T_V(M)^0$ is trivial for any V.

The second one concerns locally finite modules :

Theorem : An unstable module is locally finite if and only if $\bar{T}(M) = 0$.

Both are consequences of the classification of $\mathcal{U}$-injectives which essentially says that an indecomposable $\mathcal{U}$-injective embeds in $H^*V \otimes J(n)$ for some V and some n.

Applications of the functor T

In this chapter we give homotopical and cohomological applications of what precedes. Some of them use also the Eilenberg-Moore spectral sequence that we briefly recall. They are :

- a conjecture of Serre proved by C. McGibbon and J. Neisendorfer;
- a result on the cohomology of fibrations;
- the computation of the cohomology of mapping spaces whose source is the classifying space of an elementary abelian 2-groups;

We mention also the possibility to prove Quillen's theorem on the cohomology of finite groups mentioned above using this machinery, and in fact to extend it.

3.1 The Eilenberg-Moore spectral sequence

First, we recall some facts about the Eilenberg-Moore spectral sequence. The basic references are [Sm] and [R]. Consider the fibre square

$$\begin{array}{ccc} F & \xrightarrow{i} & E \\ \downarrow & & \downarrow p \\ pt & \longrightarrow & B \end{array}$$

The Eilenberg-Moore spectral sequence is a second quadrant cohomological spectral sequence. It converges strongly towards $H^*\Omega F$ as soon as $\pi_1 B = 0$. A more general result has been given by W. Dwyer, in fact the spectral sequence converge as soon as $\pi_1(B)$ is a finite 2-group. The exact condition is that the fundamental group of the base acts nilpotently on the cohomology (supposed of finite type) of the fiber.

Its $E_2^{s,*}$-term identifies, as an $\mathcal{A}_2$-module, with $\mathrm{Tor}^s_{H^*X}(\mathbf{F}_2, \mathbf{F}_2)$ provided with the $\mathcal{A}_2$-module structure defined by the Cartan formula. The columns $E_r^{s,*}$, $s \leq 0$, $2 \leq r \leq$

∞ are unstable modules and it is convenient to interpret the differentials as $\mathcal{A}_2$-linear of degree zero from $E_r^{s,*}$ to $\Sigma^{r-1}E_r^{s,*}$:

$$d_r : E_r^{s,*} \to \Sigma^{r-1}E_r^{s+r,*} \quad .$$

There is a convergent filtration by unstable modules on H^*F :

$$H^*F \supset \ldots F_s \supset F_{s+1} \supset \ldots \supset F_{-1} \supset F_0 \quad ,$$

such that $\Sigma^{-s}F_s/F_{s+1} \cong E_\infty^{s,*}$. The unstable module F_0 is the image of H^*E.

The kernel K of the edge homomorphism

$$E_2^{0,*} \cong H^*E \bigotimes_{H^*B} \mathbf{F}_2 \to \Sigma H^*F$$

has a convergent filtration : $K \supset \ldots K_s \supset K_{s+1} \supset \ldots \supset K_{-1} = 0$ such that

$$\Sigma^{-s-1}K_s/K_{s+1} \cong d_{-s}E_{-s}^{s,*} \quad .$$

3.2 A conjecture of Serre

The following theorem of McGibbon and Neisendorfer was conjectured by J.-P. Serre :

Theorem : Let X be a 1-connected finite space whose reduced mod-2 cohomology is non-trivial. Then, an infinite number of homotopy groups of X contain 2-torsion.

J.-P Serre proved that infinitely many homotopy groups have to contain 2-torsion or $\mathbf{Z}$. The proof of McGibbon and Neisendorfer depends on a fundamental theorem of Miller that we will state later. The proof of J. Lannes and the author depends on :

Theorem : Let X be a 1-connected space such that H^*X is of finite type. Then, if H^*X is locally finite as an $\mathcal{A}_2$-module, so is $H^*\Omega^nX$ for any n.

Proof : Let us do the case $n = 1$. The result follows from what we said above on the Eilenberg-Moore spectral sequence. Indeed assume that the cohomology of X is locally finite. Then each column of the E_2-term of the spectral sequence is also locally finite, and also each column of the E_∞-term. Next as the category of locally finite unstable module is thick this is true for the term F_t in the filtration> Hence this is true for $H^*\Omega X$ which is the direct limit.

When one iterates to get $H^*\Omega^n X$ this argument has to be modified a bit to deal with the problem of a non-trivial fundamental group of the base. However this is easily done by observing that :

- if we have a factor $\mathbf{Z}$ in the fundamental group and as we are dealing now with loop spaces an S^1 will split off the space, so we can neglect it;
- we can localize at 2, therefore ignore odd primary torsion;
- finally we are left with a π_1 that is a finite 2-group, thus we can apply Dwyer's criterion.

In fact an elementary proof can also be given using Gysin exact sequences for 2-sheeted covering spaces :

Proposition : Let $\widetilde{X} \to X$ a 2-sheeted covering of spaces. Assume that the mod-2 cohomology of X is locally finite and of finite type. Then so is the cohomology of $\widetilde{X}$.

It follows by induction that if we have a space whose cohomology is locally finite, of finite type and whose fundamental group is a finite 2-group, the cohomology of its universal cover is also locally finite.

Then the proof of the theorem is the same as in McGibbon and Neisendorfer. One supposes that the theorem is false. Using Serre's theorem by looping down enough times we get a space $\Omega^n Y_{(2)}$ so that the localisation at 2 of its universal cover satisfies :

- $\pi_2 \widetilde{Y}_{(2)}$ contains $\mathbf{Z}_{(2)}$;
- the homotopy groups do not contain 2-torsion.

Then standard obstruction theory shows that $\widetilde{Y}_{(2)}$ writes as a product $K(\mathbf{Z}_{(2)} \times Z$. Indeed the obstructions to lift the map $\widetilde{\Omega}^n Y_{(2)} \to K(\mathbf{Z}/2,2)$ are in torsion free groups : $H^n(K(\mathbf{Z}/2,2), \pi_{n+1}(F)$ where F is the homotopy fiber. Moreover they are zero rationally, indeed $\widetilde{\Omega}^n Y_{(2)}$ is an H-space and an H-space rationally is a product of Eilenberg-MacLane spaces.

But the cohomology of this space is not locally finite, this is a contradiction.

3.3 A result on the cohomology of fibrations

The result we have in mind is the following :

Theorem : Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration. Assume that all spaces have cohomology of finite type and that $\pi_1(B)$ is trivial. Then the kernel of the edge homomorphism of the Eilenberg-Moore spectral sequence

$$E_2^{0,*} \cong \mathbf{F}_2 \bigotimes_{H^*B} H^*E \to H^*F$$

is a nilpotent unstable module.

Proof : We need to introduce the definition of nilpotent module of class ℓ.

Definition : An unstable module is said to be nilpotent of class ℓ if it is the direct limit of unstable modules which have a finite filtration whose quotients are ℓ-suspensions.

One has the following characterization :

Proposition : An unstable module M is nilpotent of class ℓ if and only if $T_V(M)$ is $(\ell - 1)$-connected for any V.

The proof of the theorem starts by observing that :

Theorem : Let K be a connected ($K^0 \cong \mathbf{F}_2$) unstable algebra, and let M be an unstable module which is also a K-module so that the map $K \otimes M \to M$ is $\mathcal{A}_2$-linear. Then, for $\ell \geq 0$, the unstable module :

$$\mathrm{Tor}_K^{-\ell,*}(M, \mathbf{F}_2)$$

is nilpotent of class ℓ.

The bar resolution shows that the preceding group is $(\ell - 1)$-connected.

One applies the preceding criteria, recall that :

$$T_V(\mathrm{Tor}_K^{-\ell,*}(M, \mathbf{F}_2)) \cong \mathrm{Tor}_{T_V(K)}^{-\ell,*}(T_V(M), \mathbf{F}_2) \quad .$$

However, it is not clear that this group $(\ell - 1)$-connected because the unstable algebra $T_V(K)$ is no longer connected. We now use the fact that $T_V(K)^0$ is a Boolean algebra. Let $\varepsilon : T_V(K) \to \mathbf{F}_2$ be the augmentation of $T_V(K)$ induced by the one of K, consider also the restriction of ε to $T_V(K)^0$. Let $T_V(K)_\varepsilon$ be $T_V(K) \bigotimes_{T_V(K)^0} \mathbf{F}_2$ and $T_V(M)_\varepsilon$ be $T_V(M) \bigotimes_{T_V(K)^0} \mathbf{F}_2$. A change of ring spectral sequence, using the fact that on the Boolean algebra $T_V(K)^0$ all modules are projective show that :

$$\mathrm{Tor}^{-\ell,*}_{T_V(K)}(T_V(M), \mathbf{F}_2) \cong \mathrm{Tor}^{-\ell,*}_{T_V(K)_\varepsilon}(T_V(M)_\varepsilon, \mathbf{F}_2) \quad .$$

But $T_V(K)_\varepsilon$ is connected, so it finishes the proof. We come back now to the proof of the initial theorem. To complete it we have to come back to the description of the differentials and of the kernel of the edge homomorphism. One observes that this kernel is nilpotent because it is the direct limit of unstable modules who have finite filtrations whose quotients are suspensions. Indeed the $-\ell$-column is the direct limit of unstable modules who have finite filtrations whose quotients are ℓ-suspensions. Its image under the differential d_ℓ is the direct limit of unstable modules who have finite filtrations whose quotients are suspensions.

We conclude this section by observing that a proof of Serre's theorem has been given by Lannes and author using that kind of techniques. It depends on evaluating the growth rate, as a function of d, of $\mathrm{Hom}_{\mathcal{K}}(H^*X, H^*\mathbf{Z}/2^{\oplus d})$. If X is finite connected this function is constant to 1. But one can show that for a finite Postnikov *tower*, non-trivial in the sense that some homotopy group either contains 2-torsion either has a non-trivial reduction mod-2, this function is never constant. This is proved by induction on the tower. Note that the argument does not work for Postnikov system.

3.4 The cohomology of mapping spaces

In this section we will consider spaces that have a finite number of connected components and whose homotopy groups are finite 2-groups. We will call them 2-finite. We will moreover suppose that they are complete at the prime 2. We refer to [BK] for all the basics about completion.

This is not a serious restriction because an Eilenberg-MacLane space whose homotopy group is a finite 2-group has these properties. Moreover these properties are preserved, *i.e.* they hold for F, if one consider a fibration : $F \to E \to B$, such that the conditions hold for E and B and B is 1-connected. And it is exactly what we will need.

Next, we consider the construction by D. Rector of the Eilenberg-Moore spectral sequence. Let again

$$F \xrightarrow{i} E \xrightarrow{p} B \quad ,$$

be a fibration. We suppose that our spaces are pointed. One constructs a cosimplicial space **F**.

Recall that a cosimplicial **X** is a family of spaces X^n with cofaces $d^j : X^{n-1} \to X^n$, $0 \leq j \leq n$ and codegeneracies $s^i : X^n \to X^{n-1}$, $0 \leq i \leq n-1$ such that the identities dual of those satisfied by the simplicial maps are satisfied. We will only recall that $d^j d^i = d^i d^{j-1}$ if $i < j$.

In our case one has :

- $F^n = B^{\times n} \times E$;
- $d^0\,(x_1, \ldots, x_{n-1}, e) = (*, x_1, \ldots, x_n, e)$,
- $d^i\,(x_1, \ldots, x_{n-1}, e) = (x_1, \ldots, x_i, x_i, \ldots, x_n, e)$

  if $0 < i < n$,
- $d^n\,(x_1, \ldots, x_{n-1}, e) = (x_1, \ldots, x_n, p(e), e)$.

Here $*$ means the base point.

Then one consider the spaces $\bar{F}^n = F^n/(\mathrm{Im} d^1 \cup \ldots \mathrm{Im} d^n)$. Because of the cosimplicial relations the maps d^0 induce maps $\delta^n : \bar{F}^{n-1} \to \bar{F}^n$. The composite $\delta^n \circ \delta^{n-1}$ is homotopically trivial. Thus, from the complex of spaces :

$$\bar{F}^0 \xrightarrow{\delta^1} \bar{F}^1 \xrightarrow{\delta^2} \bar{F}^2 \ldots$$

we can derive an exact couple, and it turns out that this leads to the “ordinary” Eilenberg-Moore spectral sequence.

The following theorem is due to E. Dror and J. Smith.

Theorem : Let us assume that the natural natural maps

$$T_V H^* B \to H^* \mathrm{map}(BV, B) \text{ and } T_V H^* E \to H^* \mathrm{map}(BV, E)$$

are isomorphisms so is the natural map

$$T_V H^* F \to H^* \mathrm{map}(BV, F) \quad .$$

In particular, let X be a connected 2-finite space which has only finitely many non-trivial homotopy groups. Then

$$T_V H^* X \cong H^* \mathrm{map}(BV, X) \quad .$$

From this point using a lemma of Bousfield to do a “limit process” one gets the mapping space theorem of Lannes :

Theorem : Let X be a 2-finite space. Assume the moreover that the $T_V H^* X$ is of finite type and trivial in degree zero. Then the natural map $T_V H^* X \to H^*(\mathrm{map}(BV, X)$ is an isomorphism.

We just describe the natural map. One starts with the evaluation map

$$\mathrm{map}(BV, X) \times BV \to X \quad .$$

It induces a map in cohomology $H^* X \to H^* BV \otimes H^* \mathrm{map}(BV, X)$ and by adjunction a map $T_V(H^* X) \to H^* \mathrm{map}(BV, X)$. This is the natural map.

Before giving application of Lannes' theorem we describe shortly the proof of the Dror-Smith Theorem. On first observes that applying the functor $\mathrm{map}(BV, -)$ to the cosimplicial space $F^{\cdot}$ we get the cosimplicial space associated to the fibration :

$$\mathrm{map}(BV, E) \to \mathrm{map}(BV, B) \quad .$$

This is purely formal. Next if we choose as base point in $\mathrm{map}(BV, B)$ the constant map at the base point of B the fiber, at this base point, is just $\mathrm{map}(BV, Y)$. Note that the mapping spaces are not connected in general.

The second observation is that one can apply the functor T_V to the Eilenberg-Moore spectral sequence of the fibration $F \to E \to B$. Indeed, all the objects in sight are unstable modules. We get a new spectral sequence that converges to $T_V(H^* F)$.

Finally there is a natural transformation of this last spectral sequence to the spectral sequence of the fibration :

$$\mathrm{map}(BV, E) \to \mathrm{map}(BV, B) \quad .$$

This transformation is given as above by the evaluation map :

$$\mathrm{map}(BV, \mathbf{F}) \times BV \to \mathbf{F} \quad ,$$

that generalizes the above map.

The proof of the second part if the Dror-Smith theorem is done by induction on the number of non-trivial homotopy groups. The induction starts with the case of Eilenberg-MacLane spaces that has been known for a long time.

The following, now immediate, corollary is essentially Miller's celebrated theorem :

Theorem : Let X be a 1-connected space such that $H^* X$ is of finite type and locally finite. Then the mapping space $\mathrm{Map}(BV, X)$ is homotopically equivalent to X.

In fact the converse is true (Lannes and the author) :

Theorem : Let X be a 1-connected space such that H^*X is of finite type. Then if the mapping space $\mathrm{Map}(BV, X)$ is homotopically equivalent to X the cohomology of X is locally finite.

Indeed, we have seen that an unstable module M is locally finite if and only if $\bar{T}M{=}0$.

Here is a corollary (conjectured by Miller and proved by Lannes).

Theorem : Let X be a nilpotent space with finite fundamental group. Then the map :

$$[BV, X] \to \mathrm{Hom}_{\mathcal{K}}(H^*X, H^*BV),\ \ f \mapsto f^*$$

is a bijection.

3.5 Realizibility of polynomial algebras on one generator

The question of determining which polynomial algebras are the cohomology of a space was raised by Steenrod. Let X be a 2-complete space and assume that $H^*X \cong \mathbf{F}_2[x]$. One has :

Theorem : The degree of x is either 1, 2 or 4.

It is easy to show that the degree of x is a power of 2. Indeed as $\mathrm{Sq}^{|x|}x = x^2$, now :

- the cohomology is trivial in degrees strictly between $|x|$ and $2|x|$,
- the operation Sq^i is indecomposable if and only if i is a power of 2, thus necessarily $|x|$ is a power of 2.

It was proved by J.-F. Adams that the degree can only be 1, 2 or 4. We will give here a proof of Aguade, Broto and Notbohm [ABN].

We start by considering the space $\mathrm{map}(B\mathbf{Z}/2, X)$. The preceding paragraph shows that it has 2-connected component : the one of the trivial map, the maps contained in the other one induce the obvious inclusion $\mathbf{F}_2[x] \to \mathbf{F}_2[u]$ in cohomology. Let E =

$\mathrm{map}(B\mathbf{Z}/2, X)_i$ be the connected component of these last maps. The theorem on the cohomology of mapping spaces allows us to compute the cohomology of this space and it is isomorphic to H^*X, and in fact E is homotopically equivalent to X.

Recall now that there is a model for $B\mathbf{Z}/2$ which is a topological abelian group. The space E comes with an action of the group $B\mathbf{Z}/2$, it acts at the source by translation. We can do the Borel construction and get a fibration :

$$E \cong X \xrightarrow{i} E \times_{B\mathbf{Z}/2} E(B\mathbf{Z}/2) \xrightarrow{p} K(\mathbf{Z}/2, 2) \quad ,$$

where $E(B\mathbf{Z}/2)$ is a contractible space on which $B\mathbf{Z}/2$ acts freely.

The induced map $\omega : \Omega K(\mathbf{Z}/2, 2) \cong B\mathbf{Z}/2 \to E$ induces the inclusion in cohomology. Next recall that the cohomology $H^*K(\mathbf{Z}/2, 2)$ is polynomial on the elements $x_n = \mathrm{Sq}^{\Delta_n}\iota_2$, where ι_2 is the generator in degree 2. Consider now the Serre spectral sequence for the above fibration. What we said implies that the image of the transgression τ on x has to be of the form $x_n +$ decomposable. Let us assume that $|x| > 1$, then $\mathrm{Sq}^1 x = 0$. The element $\mathrm{Sq}^1 x_n + \mathrm{Sq}^1(\text{decomposable})$ cannot be the image of a differential for degree reasons , thus it is zero. Assume that $n > 1$, we get that : $\mathrm{Sq}^1 x_n = x_{n-1}^2$ is in the image of Sq^1. This is only possible if $n = 2$.

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The given references do not claim to be complete in any way. In particular most recent papers concerning further works are not given here.

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