1 Stable and unstable cohomology operations

In this section we explain what the cohomology operations are and what they are good for. As the emphasis of the summer school is on classification problem, we start with a simple example of the classification of homotopy type. However, we will need a definition first :

Definition 1 *Let E, F be cohomology theories. A natural transformation from $E^n(-)$ to $F^m(-)$ (considered as functors from the category of spaces to that of the sets) is called a cohomology operation from $E^n(-)$ to $F^m(-)$. A family of cohomology operations $o_k : E^k \to F^{l+k}$ that is compatible with the suspension isomorphisms is called a stable cohomology operation (of degree l) from $E^*(-)$ to $F^*(-)$.*

Here are some (rather trivial) examples.

Example 2 *(i) Let $h^*(-)$ be a multiplicative cohomology theory. The map $x \mapsto x^i$ defines a cohomology operations from h^j to h^{ij}.*

(ii) Let $\alpha \in F^i(pt)$. (In this example it is important to consider unreduced cohomology theory). The transformation which associates α (considered as an element of $F^i(X)$) to any element in $E^*(X)$ is a cohomology operation.

The above are examples of unstable cohomology operations. Here is an example of stable ones.

Example 3 *Let $h^*(-)$ be as above, and $\alpha \in h^*(pt)$. Then the multiplication by α(note that $h^*(X)$ is a module over $h^*(X)$) is a stable cohomology operation for $h^*(-)$.*

Here is something more interesting.

Example 4 *$KU^*(-)$ denotes the complex K-theory. $KU^*(X)$ is a λ-ring, thus it is provided with natural Adams' operations Ψ^i's for $i \in Z$.*

These operations satisfy following properties. We will give a (rather unusual) proof later.

Theorem 5 *(i) Ψ^i acts as a (unitary) ring automorphism on $KU^*(X)$.*

(ii) Ψ^i *acts as identity map on* $KU^*(pt)$.

(iii) $\Psi^i \circ \Psi^j = \Psi^{ij}$

(iv) if ξ *is a line bundle, then* $\Psi^i(\xi) = \xi^i$

(v) if p *is a prime number, then* $\psi^p(x) = x^p \bmod p$.

(vi) $\Psi^i \circ \sigma = i\sigma \circ \Psi^i$ *on* $KU^{odd}(X)$, $\Psi^i \circ \sigma = \sigma \circ \Psi^i$ *on* $KU^{even}(X)$.

Now, we use them to answer the following questions.

(i) Does $CP(2)$ and $S^2 \vee S^4$ have the same homotopy type?

(ii) Does there exist $i \in N$ such that $\Sigma^i CP(2)$ and $\Sigma^i S^2 \vee S^4$ have the same homotopy type?

(iii) Does there exist a space X with $\Sigma X = CP(2)$?

As we know that $KU^*(CP(2)) = KU^*[x]/(x^3)$, we see immediately that the question i) has the negative answer. Of course as we also know that the cup product vanishes on suspension we can conclude that the question iii) has the negative answer as well, however we use our Adams' operations to prove it. In $KU^*(CP(2))$, we have $\Psi^2(x) = x^2 \bmod 2$. Let y and z denote the elements in $KU^*(X)$ that correspond to x and x^2 by the suspension isomorphism. Thus one has

$$\Psi^2(y) = ay + bz$$

with $a, b \in KU^*(pt)$. Therefore,

$$\begin{aligned} \Psi^2(x) &= \Psi^2(\sigma(y)) \\ &= 2\sigma\Psi^2(y) \\ &= 2ax + 2bx^2, \end{aligned}$$

which contradicts what we just saw. For i), similarly if we still denote by y and z the elements that correspond to x and x^2 by the suspension isomorphism, then we see that $\Psi^k(y) = k^{[i/2]}z \bmod 2$. On the other hand, in $\tilde{KU}^*(S^{i+2} \vee S^{i+4})$, if we denote by y' and z' the generators that are image of generators of the generators of $\tilde{KU}^*(S^{i+2})$ and $\tilde{K}^*(S^{i+4})$ by appropriate pinching map then by naturality one sees that $\Psi^k(y')$ is contained in the image of $\tilde{KU}^*(S^{i+2})$ Therefore there can be isomorphism between

$KU^*(S^{i+2} \vee S^{i+4})$ and $KU^*(\Sigma^i CP(2))$ that commutes with Adams' operations.

Now what happens in ordinary cohomology? Well, to deal with this problem, the situation is not so different.

Theorem 6 (Steenrod) *There exists a family of stable cohomology operations $Sq^i : H^*(X; Z/2) \to H^{*+i}(X; Z/2)$ such that*

(i) $Sq^0 = id$,

(ii) $Sq^1 = \beta$ *(Bockstein)*

(iii) $Sq^i(xy) = \Sigma_{j+k=i} Sq^j(x) Sq^k(y)$

(iv) $Sq^i(x) = x^2$ *if* $deg(x) = i$

(v) $Sq^i(x) = 0$ *if* $deg(x) < i$

Remark 7 *These operations are subject to a family of relations called Adem relations, and they form an algebra called Steenrod algebra.*

Therefore, in $H^*(\Sigma^i CP(2); Z/2)$, the generators in degree $i+2$ and $i+4$ are connected by the operation Sq^2, whereas in $H^*(S^{i+2} \vee S^{i+4})$ they are not. Thus $\Sigma^i CP(2)$ and $S^{i+2} \vee S^{i+4}$ have different homotopy types. Similarly, as Sq^2 vanishes on elements in $H^1(X)$, there can be no space X with ΣX having same homotopy type as $CP(2)$.

Before going on any further, we answer the most natural question to ask : how does one find cohomology operations? Well, in general, it is much easier to describe the set of all the cohomology operations than to construct a particular cohomology operation. As a matter of fact, using Brown's representability theorem and Yoneda's lemma, one sees :

Theorem 8 *Let E and F be spectra. Then the set of stable cohomology operations from E to F can be identified with $F^*(E)$.*

For unstable cohomology operations, we have,

Theorem 9 *Let E and F be cohomology theories. Then the set of cohomology operations from $E^n(-)$ to $F^m(-)$ can be identified with $F^m(\underline{E}_n)$ where $\underline{E}_n$ denotes the n-th space in the Ω spectrum for E. Furthermore, the subset of additive unstable operations corresponds to the subset of primitives $PF^m(\underline{E}_n)$.*

That is, by definition we have a natural isomorphism $E^n(X) \cong [X, \underline{E}_n]$, and thus the elements in $E^n(X)$ are pull-backs of the universal element in $E^n(\underline{E}_n)$. The elements on $E^m(\underline{E}_n) \cong [\underline{E}_n, \underline{E}_m]$ define a natural transformation as required by composition of maps. We also note that $\Omega \underline{E}_n \cong \underline{E}_{n-1}$. When we have a sequence of the form $X_0,\ X_1,\ X_2,\ \ldots$ with $\Omega X_i \cong X_{i+1}$, we call it an infinite loop space. Thus studying cohomology operations is equivalent to studying generalized cohomology of infinite loop spaces. When we are lucky, E- cohomology of infinite loop spaces $E^*(\underline{F}_*)$ becomes just the E^*-dual of E-homology $E_*(\underline{F}_*)$, that is why we are interested in generalized homology of infinite loop spaces, which we will deal with in the following section.

So why do we care about generalized cohomology theories (rather than just ordinary cohomology theory)? We will show several facts that illustrate the potential utility of unstable cohomology operations for generalized cohomology theories.

Fact 10 *(i) The map $HZ/p^*(HZ/p) \to PH^*(K(Z/p,n))$ is epi.*

(ii) The map $PH^*(K(Z/p,n)) \to QH^*(K(Z/p,n))$ is epi, in other words, the Hopf algebra $H^*(K(Z/p,n))$ is generated (as an algebra) by primitive elements.

Thus in ordinary mod p cohomology, all additive unstable operations come from stable operations, and they generate all the operations (under cup products). This is in sharp contrast with the case for Brown-Peterson cohomology :

Fact 11 *(i) The map $BP^*(BP) \to PBP^*(\underline{BP}_n)$ is mono, and the cokernel is torsion-free.*

(ii) The map $PBP^*(\underline{BP}_n) \to QBP^*(\underline{BP}_n)$ is not epi.

Thus for Brown-Peterson cohomology, there are far richer unstable cohomology operations than stable ones. On the other hand, cohomology operations in ordinary mod p cohomology lift to BP up to operations involving the Bockstein. Thus this facts doesn't mean the lack of stanble operations in BP, but it means abundance of unstable operations.

Next, we give an example to illustrate the utility of Adams' operations (that are, of course, unstable).

Let's go back to the question we have treated. We can use the result to conclude that the Hopf map $S^3 \to S^2$ whose cofiber is $CP(2)$ remains essential after arbitrary number of suspension. The induced map in cohomology

$\tilde{h}^*(S^2) \to \tilde{h}^*(S^3)$ is trivial, (for $h = KU$ or $h = HZ/2$) so that $\tilde{h}^*(CP(2))$ fits into a short exact sequence $\tilde{h}^*(S^4) \to \tilde{h}^*(CP(2)) \to \tilde{h}^*(S^2)$, i.e., an element of $Ext^1(\tilde{h}^*(S^2), \tilde{h}^*(S^4))$ in suitable category. This element was shown to be non-zero, to remain non-zero after suspension, and not to be in the image of suspension by considering the structure of this "suitable category". The reader may have noticed that Ext is the derived functor of Hom, and we used Ext^1 to detect something that doesn't get detected in Hom. As a matter of fact this pattern can be continued and one can set up a spectral sequence (Adams spectral sequence) that is treated in L. Schwartz's lectures. Now, we can ask the following question : for which n is there a map $S^{4n-1} \to S^{2n}$ such that the cohomology of the cofiber is isomorphic to $h^*(x)/x^3$? Adams solved this problem (by reducing it to the question concerning the spectral sequence we have just mentioned) using ordinary homology in [2]. The proof involves rather sophisticated homological algebra and we don't reproduce it here. On the other hand, the same result was reproved using KU-theory and Adams'operation [3]. We reproduce the argument here to convince the reader the utility of generalized cohomology theories.

Theorem 12 *There exists a map $S^{4n-1} \to S^{2n}$ such that the cohomology ring of the cofiber is isomorphic to $h^*(x)/x^3$ if and only if $n = 1, 2, 4$.*

The "if" part is a consequence of existence of the associative algebra structure on R^{2n}. Here is the proof of the "only if" part. By the properties of The Adams' operations, one sees immediately that on $\tilde{KU}^*(S^{2n})$ Ψ^k acts as multiplication by k^n. We note by a and b the generators of $\tilde{KU}^*(C)$ (where C denotes the cofiber) such that $b = \pi^*(g_{4n})$, $i^*(a) = g_{2n}$ with $i : S^{2n} \to C$ the obvious inclusion, $\pi : C \to \Sigma S^{4n-1}$ the pinch map, g_k generator of $\tilde{KU}^*(S^k)$. Thus by the naturality one sees that

$$\exists q(k) \in KU^*(pt) \text{ such that } \Psi^k(a) = k^n a + q(k) b$$

Now by computing $\Psi^2 \circ \Psi^k(a) = \Psi^k \circ \Psi^2$ one sees that

$$k^n(k^n - 1)q(2) = 2^n(2^n - 1)q(k),$$

however, by the property v) of Ψ's, q(2) is odd. Thus one sees that for any odd number k, $k^n = 1 \bmod 2^n$. Let $k = -1$ and one sees that n has to be equal to 1 or even. Set $n = 2m$. Then by letting $k = 2^m + 1$ one sees that m is divisible by 2^{m-1}. This occurs only if $m = 1, 2$.

These two examples illustrate general philosophy. Let $F : C \to D$ be a functor. (In our case, C is the homotopy category, i.e., the category whose objects are topological spaces and whose morphisms are homotopy classes of maps among them. F is a generalized cohomology theory $h^*(-)$, D is the category of modules over $h^*(pt)$.) We want to use F and D to study the category C. Suppose we can factorize $F = G \circ H$, $H : C \to D'$, $G : D' \to D$. (In our case. D' is the category of $h^*(-)$-algebra on which cohomology operations act.) Then H and D' reflect better the structure of C and thus are more suitable to study C (provided that we can get hold of them! Of course, taking $H = id$ and $D' = C$ doesn't do anything...) G may not be injective on objects so H can distinguish the objects better (the first example), and G may not be surjective on objects so we can find certain objects in D are not in the image of F by studying the category D' (the second example). In the section 3, we will give more precise description of what we use as the category D'. However, we shall first study objects that are dual to the set of the unstable operations.

2 Hopf rings and homology of infinite loop spaces

In this section we describe the algebraic setting in which homology of infinite loop spaces can be best dealt with. We first recall some general notions.

2.1 Group objects, ring objects, and so on.

Definition 13 *A group object in a category C is an object G of C such that $\exists \mu : G \times G \to G$, $u : . \to G$, and $t : G \to G$ that induce the structure of the group on the functor $Hom_C(-, G)$ where . is the terminal object. (The map induced by μ and t correspond to the multiplication and the inverse, and for any object X of C, the unit element of the group $Hom_C(X, G)$ is the composition $X \to . \to G$.)*

Obviously for this definition to make sense, we have to assume that G is closed under finite products and possesses a terminal object. This definition comes down to the following : the identities that define a group can be expressed in terms of commutative diagram. For example, the associativity of the multiplication in a group A is equivalent to the commutativity of the following diagram :

$$\begin{tikzcd} A \times A \times A \arrow[r, "{(-\times -), -}"] \arrow[d, "{-, (-\times -)}"'] & A \times A \arrow[d] \\ A \times A \arrow[r] & A \end{tikzcd}$$

requiring G to be a group object in the category C is equivalent to requiring that the maps that correspond to the structure of the group exist in the category C and that the commutative diagram that characterises groups commute in C.

We can similarly define the notion of monoid object, ring object, semi-ring object, etc.

Now we recall the notion of coalgebras.

Definition 14 *A coalgebra over a ring R (called ground ring) is a R-module C together with the comultiplication map $\Delta : C \to C \otimes C$ that is coassociative, i.e. that makes the diagram dual to the one that defines the associativity of algebras commutative. It is counitary if there is a counit $\epsilon : C \to R$ that makes appropriate diagram commutative, and cocommutative if Δ is invariant by the permutation of two factors.*

Example 15 *(i) Let S be a set. Then the free R-module generated by elements of S (which we denote by $R[S]$) has a natural coalgebra structure with comultiplication coming from the diagonal $S \to S \times S$ and counit coming from the map $R[S] \to R[*] \cong R$.*

(ii) Let X be a space. Then $H_*(X; Z/p)$ is a coalgebra with coproduct induced by the diagonal map.

These examples are special cases of the following.

Proposition 16 *Let C be a category with product. If F is a functor from C to the category of R-module that sends the product to the tensor product. Then F takes its values in the category of R-coalgebras.*

Let's go back to the first example. It is well known that when the set S has the structure of the group, then $R[S]$ is naturally a Hopf algebra, i.e., a group object in the category of coalgebras. Similarly if X is an H-space, i.e., a space with multiplication up to homotopy, then $H_*(X, Z/p)$ is a Hopf algebra. These are special cases of the following :

Proposition 17 *Let C and F be as above. If X is a group object in C then $F(X)$ is a Hopf algebra over R.*

Now, what if S has a ring structure? Of course, then $R[S]$ becomes a ring object in the category of coalgebras. Similarly if X is a space that is a ring up to homotopy then $H_*(X; Z/p)$ is a a ring object in the category of coalgebras. As in the case of Hopf algebras, we have the following.

Definition 18 *A (commutative) Hopf ring is a (commutative) ring object in the category of coalgebras.*

Proposition 19 *Let F and C be as above. If X is a ring object in C, then $F(C)$ is a Hopf ring.*

Remark 20 *The example $R[S]$ may look rather trivial, however this is quite important example. The reason is that if $E^*(-)$ and $F^*(-)$ are multiplicative cohomology theories, then $E_*(\underline{F}_*)$ always contain $E_*([F^*])$: Let $\alpha \in F^*(pt) \cong [pt, \underline{F}_*]$, then we note by $[\alpha]$ the element $E_*(\alpha)(1) \in E_*(\underline{F}_*)$ where 1 is the unit in $E_*(pt)$. Then the E_*-submodule generated by $[F^*]$ is contained in $E_*(\underline{F}_*)$, and it is a sub Hopf ring if $E_*(\underline{F}_*)$ is a Hopf ring.*

We can reformulate the definition of Hopf ring in more concrete terms:

Definition 21 *A Hopf ring over the ground ring R is a Hopf algebra C_* (whose product will be denoted by $\star$) together with the pairing $\circ : C_* \otimes C_* \to C_*$ satisfying usual properties and the distributivity law :*

$$x \circ (y * z) = \Sigma x' \circ y \circ x'' \circ z$$

where

$$\Delta(x) = \Sigma x' \otimes x''$$

(with appropriate sign when C is graded).

Note that when $\Delta(a) = a \otimes a$ this is the usual distributivity law.

So what is the point of considering such objects? There are three main reasons. First of all, having two products instead of one, these objects tend to have simpler description as Hopf rings than as Hopf algebras. At least they always have fewer generators, but quite often the relations tend to be simple too. Furthermore when $E_*(\underline{F}_*)$ is a Hopf ring and $E^*(\underline{F}_*)$ its dual, the circle product determines the behavior of the unstable cohomology operations on cup products. Finally it is a powerful computational tool, especially in the computation of the bar spectral sequence : however, we don't deal with this point in this lecture. We will shortly illustrate the first point.

2.2 Remark on bigraded Hopf rings

Quite often we deal with the graded ring object in the homotopy category. Thus its homology is equipped with two gradings. Such an object is often called (abusively) bigraded object. However, they should rather be called "graded graded object", i.e., a graded object in the category of graded objects... Thus the nature of two gradings do differ a little bit. We will explain this in detail.

First of all, for fixed j, $h_*(\underline{X}_j)$ is a graded abelian group. By the symbol $h_*(-)$, strictly speaking, we mean the direct sum (which is also categorical sum) $\oplus_i h_i(-)$. Of course, as we consider only maps that preserve the grading, it comes down to considering all of $h_i(-)$'s at once. Now, when this sum happens to have Hopf algebra structure, the product goes from $h_{i_1}(\underline{X}_j) \otimes h_{i_2}(\underline{X}_j)$ to $h_{i_1+i_2}(\underline{X}_j)$. Now, the expression $h_*(\underline{X}_*)$ means the sum in the categorical sense of these Hopf algebras, i.e., their tensor product. Of course, as above it comes down to considering all of these objects in the same time. Thus in graded Hopf ring, addition map goes from $h_i(\underline{X}_j) \oplus h_i(\underline{X}_j)$ to $h_i(\underline{X}_j)$, $\star$ product from $h_i(\underline{X}_j) \otimes h_k(\underline{X}_j)$ to $h_{i+k}(\underline{X}_j)$, and the $\circ$ product from $h_i(\underline{X}_j) \otimes h_k(\underline{X}_l)$ to $h_{i+k}(\underline{X}_{j+l})$.

2.3 Hopf rings in algebraic topology

Now, before discussing examples of real-life Hopf rings with few generators and relations, we need to discuss the notion of generators and relations. Let's first make a brief digression to the corresponding notions for abelian groups. Let's say, an abelian group G is generated by elements $\{g_\lambda\}$ $(\lambda \in \Lambda)$, subject to relations $\{r_\mu\}$ $(\mu \in M)$. This means that there is a right exact sequence $F_1 \to F_0 \to G$ where F_i is the free abelian group generated by elements $\{f_{i,\lambda}\}$ for $(\lambda \in \Lambda)$. The map $F_0 \to G$ is given by $f_{0,\lambda} \mapsto g_\lambda$, $F_1 \to F_0$ is given by $f_{1,\mu} \mapsto r_\mu(f)$. Furthermore, a free abelian group is characterized by the following property : if S is a set, M is any abelian group, then

$$Hom_{Ab}(F(S), M) \cong Hom_{ens}(S, M)$$

where Ab is the category of abelian groups and ens is that of sets. In other words, the free group functor is the left adjoint to the forgetful functor. So, to get a correct notion of generators and relations, one needs the notion of free objects and that of quotients. Furthermore, the notion of free objects can be expressed in terms of left adjoint to a suitable forgetful functor. Now, as to free object, we know the following :

Theorem 22 *The forgetful functor from the category of Hopf rings to that of coalgebras admits a left adjoint, which we will note by F. For a coalgebra C we call $F(C)$ the free Hopf ring generated by C.*

As a matter of fact, this is just what one would naively expect what it should be : it suffices to take all possible star products and circle products, and quotient it out (as a Hopf algebra) by the relation generated by the distributivity law. On the other hand, the notion of the quotient is slightly more complicated. We content ourselves to say that given a subset I of C, the quotient of C by I is the maximal Hopf algebra quotient of C in which I is killed and that inherits the structure of Hopf ring. Now we can list examples

Theorem 23 *$H_*(K(Z/p,*) : Z/p)$ is the quotient of the free Hopf ring generated by $Z/p[Z/p]$ (in bidegree (0,0)), $H_*(K(Z/p,1) : Z/p)$ (in bidegree (*,1)) amd $H_*(CP(\infty) : Z/p)$ (in bidegree (*,2)) by the relation $e_1 \circ e_1 = \beta_1$ where $e_1 \in H_1(K(Z/p,1) : Z/p)$, $\beta_1 \in H_2(CP(\infty) : Z/p)$ are generators. When $p = 2$ the description simplifies : it is just the free Hopf ring generated by $Z/2[Z/2]$ and $H_*(K(Z/2,1) : Z/2)$.*

This is to be compared with the following :

Theorem 24 *For an odd prime p,*

$$H_*(K(Z/p,n); Z/p) \cong \otimes_{I,J} E(e1 \circ \alpha_I \circ \beta_J) \otimes_{I,J} P(\alpha_I \circ \beta_J)/(-)^p$$

where $I = (i_1, i_2, \ldots)$, $0 \leq i_1 < i_2 < \cdots$, $\alpha_I = \alpha_{(i_1)} \circ \alpha_{(i_2)} \circ \cdots$, $\alpha_{(\phi)} = [1] - [0]$, $\alpha_{(i)}$'s are dual elements of y^{p^i} where $y \in H^2(K(Z/p,1) : Z/p)$ is any non-trivial element, $\beta^{(0)} = [1] - [0]$, $\beta^J = \beta_{(0)}^{\circ j_0} \circ \beta_{(1)}^{\circ j_1} \circ \cdots$, $\beta_{(i)}$ is the element dual to x^{p^i} where $x \in H^2(CP(\infty) : Z/p)$ is a generator. e_1 is a non-trivial element in $H_1(K(Z/p,1) : Z/p)$, and we choose x in such a way that we have $\beta_{(0)} = e_1 \circ e_1$.

We show that the two expressions are in fact equivalent. First of all, $H_*(K(Z/p,1); Z/p)$ is isomorphic to $E(e_1) \otimes P(\alpha_{(0)}, \alpha_{(1)}, \ldots)/(-)^p$. The commutativity of $\circ$ product implies $a \circ b = [-1] \circ b \circ a$ if a and b are in $H_{even}(K(Z/p, odd) : Z/p)$. However, $[-1]\circ$ is nothing but the conjugation of the Hopf algebra, and on $H_*(K(Z/p,1) : Z/p)$, it is just multiplication by (-1). Thus we have $\alpha_{(i)} \circ \alpha_{(j)} = -\alpha_{(j)} \circ \alpha_{(i)}$ and $\alpha_{(i)} \circ \alpha_{(i)} = 0$. Furthermore, $H_*(CP(\infty) : Z/p)$ is isomorphic to $P(\beta_{(0)}, \beta_{(1)}, \ldots)/(-)^p$. Now, the commutativity of $\star$ product implies the elements $(e1 \circ \alpha_I \circ \beta_J)^{\star 2}$ are trivial. On the other hand, in $Z/p[Z/p]$, we have $[p] = [0]$, thus

$$0 = \epsilon(a) = [0] \circ a = [p] \circ a = V(a)^{\star p}$$

if a is in $Ker\epsilon$, where ϵ denotes the augmentation and V the Verscheibung. As $V(\beta_{(i)}) = \beta_{(i-1)}$ and $V(\alpha_{(i)}) = \alpha_{(i-1)}$, we see that $\alpha_I \circ \beta_J$ is truncated of height p. There is no other relation, so that the two descriptions are equivalent. In the argument above, we have used the following fact, whose proof is left to the reader as exercise :

Proposition 25 *(i) In any Hopf ring, the following equality holds :*

$$[0] \circ a = \epsilon(a).$$

(ii) In a Hopf ring over a ground field of characteristic p, the following equality holds :

$$[p] \circ a = V(a)^{\star p}.$$

We give another example where formal group laws do not get involved.

Theorem 26 *$H_*(QS^0; Z/2)$ is the quotient of the free Hopf ring generated by $Z/2[Z] \otimes H_*(RP(\infty))$ by the following relation :*

$$e(s) \circ e(s+t) = e(s) \circ e(t)$$

where $e(X) = \Sigma_i e_i X^i$, e_i is the generator of $H_i(RP(\infty); Z/2)$.

The equality $e(s) \circ e(s+t) = e(s) \circ e(t)$ takes place in $H_*(QS^0; Z/2)[[s,t]]$, which is a Hopf ring over the ground ring $Z/2[[s,t]]$, that is, the products are defined in such a way as $f \star g = f \circ g = fg$ if $f, g \in Z/2[[s,t]]$. We obtain the relations among e_i's by extracting the coefficients of $s^i t^j$ on both side of the equality.

This equality can be proved in the following way. Let's consider the map

$$BZ/2 \times BZ/2 \to B\Sigma_4 \to B\Sigma_\infty \subset Z \times B\Sigma_\infty \cong QS^0$$

(where the first map is induced by the special case of the map $\Sigma_i \times \Sigma_j \to \Sigma_{ij}$). The image of $a_i \otimes a_j$ is just $e_i \circ e_j$ by definition, where a_i is the generator of $H_i(RP(\infty); Z/2)$. Multiply this with $s^i t^j$ and add them up to see that the image of $\Sigma a_i \otimes a_j s^i t^j$ is $e(s) \circ e(t)$. On the other hand the map $Z/2 \times Z/2 \to \Sigma_4$ is invariant under the action of $GL(Z/2, 2)$, and in particular the action by the element that maps u to $u+v$ and v to v (where u, v are appropriate generators of $Z/2 \times Z/2$). However this element sends $\Sigma a_i \otimes a_j s^i t^j$ to $\Sigma a_i \otimes a_j s^i (s+t)^j$, thus we have the equality in question.

To conclude the section, we take a look at examples where the formal group laws get involved.

A generalized cohomology theory E is said to be complex orientable if $E^*(CP(\infty)) \cong E^*[[x]]$, in which case x is called the orientation class. The addition map : $CP(\infty) \times CP(\infty) \to CP(\infty)$ induces a map in E-cohomology $E^*[[x]] \to E^*[[x_1, x_2]]$. We call the image of x under this map the formal sum of x_1 and x_2, and we note it by $F_E(x_1, x_2)$ or $x_1 +_E x_2$. Then we have the following :

Theorem 27 *Let F, G be a multiplicative cohomology theory with complex orientations. Then in $F_*(\underline{G}_*)[[s,t]]$ we have the following equality :*

$$b(s +_F t) = b(s) +_{[G]} b(t)$$

where $b(X) = \Sigma_i b_i X^i$, b_i being the image of $\beta_i \in F_{2i}(CP(\infty))$ by the map induced by $F_*(x^G)$, where x^G is the orientation map for G $CP(\infty) \to \underline{G}_*$, and the symbol $X +_{[G]} Y$ means $\star_{i,j}[a^G_{i,j}]X^{\circ i} \circ Y^{\circ j}$, where $a^G_{i,j}$ are coefficients in the formal group laws for G.

This leads to the following definition.

Definition 28 *Let $E^R_*(\underline{F}_{2*})$ denote the quotient of the free Hopf ring generated by $E_*[F^*] \otimes E_*(CP(\infty))$ (where $E_*(CP(\infty))$ lies in bidegree $(*, 2)$ by the relation $b(s +_E t) = b(s) +_{[F]} b(t)$. Let $E^R_*(\underline{F}_*)$ denote the Hopf ring obtained by adding a generator e_1 and the relation $e_1 \circ e_1 = b_1$ to it.*

Thus there is a natural map $E^R_*(\underline{F}_*)$ to $E_*(\underline{F}_*)$. The main theorem of [13] says :

Then we have :

Theorem 29 (Ravenel-Wilson) *Let F be MU or BP (or KU, which is classic). Then for reasonable E, the map $E^R_*(\underline{F}_*) \to E_*(\underline{F}_*)$ is isomorphism.*

We mention that $E^R_*(\underline{BP}_*)$ is a free commutative algebra, and that the set of generators can be described explicitly. This result has been generalized for other cohomology theories ([7] for example).

3 Algebras over unstable operations

We start this section by describing the set of unstable operations in several cases of interest, namely the case of mod p ordinary cohomology and Brown-Peterson cohomology. For this sake, we introduce yet another way to obtain a Hopf ring.

3.1 Colimit model Hopf rings and unstable operations

Definition 30 Let C be a subcategory of the homotopy category and X any space. Then by C/X we mean the category whose objects are *homotopy classes of maps from objects of C to X* and whose morphisms are *morphisms of C which makes the triangular diagram commutative.* If ϕ is any functor from the homotopy category ($Htop$) to some other category D, then define $\phi^C(X)$ to be the colimit of the functor $C/X \stackrel{Domain}{\rightarrow} Htop \stackrel{\phi}{\rightarrow} D$. Furthermore, denote by $\eta_C(X)$ the natural transformation $\phi^C(X) \to \phi(X)$ induced by $\phi(f) : \phi(Domain(f)) \to \phi(X)$.

Lemma 31 *Let D be a category that is closed under finite products and that contains the terminal object. If a functor $\phi : Htop \to D$ respects the product and the terminal object, i.e., if there exists a natural equivalence of bifunctors $\phi(-\times, -) \cong \phi(-) \times \phi(-)$, and $\phi(pt)$ is the terminal object, then ϕ^C respects the product and the terminal object, too.*

Definition 32 *If A, B, are categories and ϕ is a functor from A to B, we denote by (f, b) the element of the $colim_A(\phi)$ that corresponds to $b \in \phi(f)$.*

Proposition 33 *Let C be a subcategory of the homotopy category that is closed under finite products, (including the 0-fold power of a space. Thus its objects are pointed.) F_* a ring space (i.e., a ring object in the homotopy category) and let E_* be a homology theory that satisfies the Künneth isomorphism for objects in C. Then $E_*^C(F_*)$ becomes an Hopf ring.*

This "colimit model" turns out to be quite useful. We will see some examples later. We spell out the two products in the Hopf ring $E_*^C(F_*)$. $\star$ and $\circ$ respectively represent addition and multiplication in the ring-valued functor $E_*^C(F_*)$ represents.

$$\begin{aligned} (f,b) \star (g,c) &= (p_1^* f + p_2^* g, i_{1*}(b) \otimes i_{2*}(c)) \\ (f,b) \circ (g,c) &= (p_1^* f \times p_2^* g, i_{1*}(b) \otimes i_{2*}(c)) \end{aligned}$$

Here p_j's and i_j's denote appropriate projections and inclusions.

Remark 34 Let C be the category of "all" spaces. Then C/F_* has the terminal object $id : F_* \to F_*$. Thus the colimit is nothing but $E_*(F_*)$. However, unless the coefficient ring E^* is a field, our hypothesis is not satisfied.

Definition 35 $\mathcal{CP}$ is the category whose objects are *finite products* of CP^∞'s, and whose morphisms are *homotopy classes of maps* among them.

Note that the category $\mathcal{CP}/\underline{E}_{2n}$ can be considered as the category of *pairs* $(CP^\infty \times \ldots \times CP^\infty, s)$, where s is an element of $E^{2n}(CP^\infty \times \ldots \times CP^\infty)$ and maps which "respect" a *marked cohomology class.* We will recall some definitions from [13] (or [15]). Now we can state the main result of this section.

Proposition 36 *We have an isomorphism of Hopf rings $E_*^R(\underline{F}_{2*}) \cong E_*^{CP}(\underline{F}_{2*})$*

Remark 37 To get the version that works for odd spaces as well, (i.e., to add the suspension element $e_1 \in E_1(\underline{F}_1)$ with the relation $e_1 \circ e_1 = b_1$), one only needs to replace the category $\mathcal{CP}$ with the full subcategory of the homotopy category whose objects are finite products of S^1's and CP^∞'s.

$\mathcal{CP}$ is not the only category that is useful. For example, one knows

Fact 38 *Let $\mathcal{RP}$ denote the category whose objects are finite products of RP^∞'s. Then we have*

- $HZ/2_*^{\mathcal{RP}}(K(Z/2,*)) \cong H_*(K(Z/2,*);Z/2)$.
- $HZ/2_*^{\mathcal{RP}}(QS^0) \cong H_*(QS^0;Z/2)$.

One of the reasons for which the colimit model is useful is as follows :

Theorem 39 *Let E, F be cohomology theories such that $E_*^{\mathcal{C}}(\underline{F}_*) \cong E_*(\underline{F}_*)$. Then the set of unstable cohomology operations from F to E can be identified with the set of the natural transformations from $F \circ \iota$ to $E \circ \iota$ where ι is the inclusion functor from $\mathcal{C}$ to the homotopy category of spaces.*

As an application, we show the existence and properties of Adams' operations as promised earlier. First we only deal with even degree part. Thus by the Theorem above, it suffices to construct such operations for $\mathcal{CP}$. However, by noting that the "multiplication by p" map on product of CP^∞'s commute with any other maps, so that the induced map in E-cohomology is natural transformation. The properties i),ii),iii) are clear. To prove iv), by naturality it suffices to prove the statement on the universal line bundle on $BU(1)$. However, as the "multiplication by k" map of $BU(1)$ is nothing but the classifying map for k-fold tensor product, the property follows. To prove v), note that $1 + ux \mapsto (1+ux)^p = 1 + (ux)^p \bmod p$. As Ψ^p is a ring homomorphism, the property follows. Instead of vi), we first prove that the equality $\Psi^i \circ \sigma^2 = i\sigma^2\Psi^i$. By the multiplicativity of Ψ^i, it suffices to prove that Ψ^i acts as a multiplication by i on $\tilde{KU}(S^2)$. However, the "multiplication by i" map on $CP(\infty)$ restricts to the degree i self map on S^2, which induces multiplication by i on any cohomology theory. Thus we get the desired result. To extend the result to the odd degree part, it suffices to notice that the $\circ$ multiplication with e_1 (which is the same thing as the homology suspension) induces an isomorphism $QKU_*(\underline{KU}_{2n}) \cong QKU_{*+1}(\underline{E}_{2n+1})$, thus the set of additive cohomology operations on $KU^{2n+1}(-)$ is isomorphic to that on $KU^{2n}(-)$.

3.2 Unstable algebras

Sometimes it is nice to be able to consider the category of all algebraic objects that "looks like" E-cohomology of topological spaces. That is we require that all the unstable cohomology operations act on it as if they acted on E-cohomology of spaces. What do we mean by " as if they acted on E-cohomology of spaces" ? Well let's trace proofs of the properties of Adams' operation backwards : we proved that certain relations among the operations hold on products of CP^∞'s, which implies that the relations in question hold on the universal class in $E^{2i}(\underline{E}_{2i})$ as it injects to $lim_{\mathcal{CP}/\underline{E}_{2i}} E^{2i}(-)$. Now every elements in $E^{2i}(X)$ is pull-back of this element, so by naturality the relations hold on any element. So, the main point is that all the relations that hold on the universal class in $E^*(\underline{E}_*)$ should hold on any other element. There are two ways to describe this : the one is simply "to write down" all the relations and actually do require these relations. Although theoretically this definition works, it is quite far from elegant. Here is another definition of such objects :

Definition 40 *Assume that $E_*(\underline{E}_k)$ is free E-module for each k (which is true for $E = HZ/p, BP, MU, KU$, or $K(n)$). Then An unstable (E-cohomology) algebra is a compete Hasudorff filtered E^* -algebra M equipped with a continuous morphism $\rho_M : M \to UM$ of E^*-algebras that makes the following diagrams commutative :*

$$\rho_M M \quad \begin{tikzcd} M \arrow[r, "\rho_M"] \arrow[d] & U \arrow[d, "\rho_{UM}"] \\ UM \arrow[r, "U_{\rho_M}"] & {} \end{tikzcd}$$

$$\epsilon M \quad \begin{tikzcd} M \arrow[r, "\rho_M"] \arrow[dr] & HM \arrow[d] \\ M & {} \end{tikzcd}$$

here U is the functor defined as follows : for any filtered algebra M, $U(M)^k$ is the set of continuous algebra homomorphisms from $E^*(\underline{E}_k)$ to M, We define the natural transformation $\epsilon : U \to id$ by the evaluation on the universal class.

We refer the readers to [5] for details. The idea is that the elements in $E^*(X)$ defines a map from X to $\underline{E}_*$, which induces an algebra map $E^*(\underline{E}_*)$ with desired property. Or, in more algebraic terms, given a class in $E^*(X)$, the evaluation on this class gives an algebra homomorphism from $E^*(\underline{E}_*)$ to $E^*(X)$ with the desired properties.

In the case of HZ/p, this agrees with the classical definition. We only consider the case when $p = 2$.

Definition 41 *An unstable algebra over the Steenrod algebra is an algebra on which the Steenrod algebra acts in such a way as the conditions i),iii),iv),v) of Theorem 6 are satisfied.*

This category has an inconvenience of being non-linear. To avoid the difficulty arising from it, one often works with the the category of unstable modules over the Steenrod algebra. Namely:

Definition 42 *An unstable module over the Steenrod algebra is an algebra on which the Steenrod algebra acts in such a way that the conditions i) and v) are satisfied.*

These two categories are studied in detail in the lectures by L. Schwartz, so we just mention the following result, which first appeared in litterature in this form in the paper [9].

Definition 43 Let *vect* be the *category of the vector spaces* over $Z/2$, and *f.vect* be its full subcategory whose objects are *finite dimensional vector spaces.* For any additive category A, $Rep(A)$ will denote the category whose objects are *functors* from A to *vect*, and whose morphisms are *additive natural transformations.* Let S^i be the i-th symmetric power functor, i.e., $S^i(V) = \overbrace{V \otimes \cdots \otimes V}^{i-factors}/\Sigma_i$. Denote by S^* the *full subcategory* of $Rep(f.vect)$ whose objects consist of the functor S^i. *Denote by U the category of unstable modules over the Steenrod algebra.*

Theorem 44 $U = RepS^*$.

This is just a reformulation of Theorem 37 in view of the Fact 36, as $S^i = H^i(K(-,1) : Z/2)$, and $(K(-,1)$ is an equivalence of the categories *f.vect* and $\mathcal{RP}$.

4 For further study

We recommend the reading of [15] (that we followed in part) for introduction to the Hopf ring method and quick introduction to BP-cohomology and related cohomology theories, [4] and [5] for thorough treatment of theory of cohomology operations. To learn more about Steenrod operation, we recommend [10] and [14]. [1] is a classical introduction to stable homotopy theory, The Adams'spectral sequence that we mentioned in section 1 is one of the most active area of research in stable homotopy theory, [11, 12] should give you an idea of what is going on in this area. As to formal group laws, the most comprehensive reference is [6], which also has a very extensive bibliography. In section 3, we followed the approach taken in [8].

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