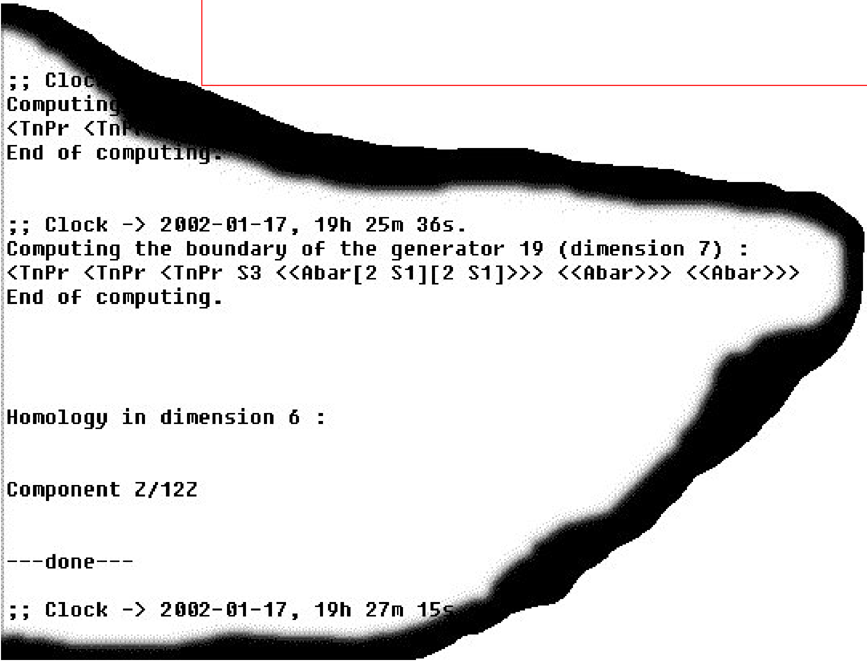
Constructive Homological Algebra III.

Koszul complexes



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Genova Summer School, 2006

k = commutative field. A = commutative k-algebra.

$x_1, \ldots, x_m \in A$. M = A-module.

<u>Definition</u>: The Koszul complex $K_A(M; x_1, \ldots, x_m)$ is a chain complex K_* of A-modules with:

$$K_n := M \otimes_k \wedge^n k^m$$

A generator of K_n is denoted by $m\ \delta x_{i_1} \cdots \delta x_{i_n}$.

Differential: $d : K_n \longrightarrow K_{n-1}$:

$$\begin{aligned} m\ \delta x_{i_1}\ \cdots\ \delta x_{i_n} \mapsto &+ m x_{i_1}\ \delta x_{i_2} \cdots \delta x_{i_n} \\ &- m x_{i_2}\ \delta x_{i_1}\ \delta x_{i_3} \cdots \delta x_{i_n} \\ &+ \cdots \\ &+ (-1)^{n-1}\ m x_{i_n}\ \delta x_{i_1}\ \delta x_{i_2} \cdots \delta x_{i_{n-1}} \end{aligned}$$

"Geometrical" interpretation of Koszul complexes.

Principal case:

$$K_A(A; x_1, \ldots) = A \otimes_t \wedge k^m \text{ (}\sim\text{ total space)}$$

$$\begin{aligned} A &= \text{structural algebra } (\sim \text{ structural group}); \\ \wedge k^m &= \text{base coalgebra } (\sim \text{ base space}); \\ t &= \text{twisting cochain } (\sim \text{ twisting function}); \end{aligned}$$

General case:

$$\begin{aligned} K_A(M; x_1, \ldots) &= M \otimes_A (A \otimes_t \wedge k^m) \\ &= \text{Fibration associated to } M \otimes_A A \to M. \end{aligned}$$

Particular case: $A = k[x_1, \ldots, x_m]$.

$$K(A; x_1, \ldots, x_m) =: K(A) := A \otimes_t \wedge k^m$$
$$= \text{canonical Koszul complex of } A \text{ is } \boxed{\text{acyclic}}.$$

$$K(A) \text{ acyclic} \Leftrightarrow K(A) = \text{universal fibration of } A$$
$$\Leftrightarrow K(A) = A\text{-resolution of } k:$$

$$0 \leftarrow k \leftarrow A \leftarrow A \otimes k^m \leftarrow A \otimes \wedge^2 k^m \leftarrow \ldots$$

$\Rightarrow K(A) =$ possible tool to compute $\mathrm{Tor}^A(M, k)$.

<u>Definition</u>: M and N = A-modules $\Rightarrow$ $\mathrm{Tor}^A(M,N) =$???

Let $R_A(M)$ be an A-resolution of M,

$R_A(N)$ an A-resolution of N.

$$H_*(R_A(M) \otimes_A N) =: \mathrm{Tor}^A(M,N) := H_*(M \otimes_A R_A(N)).$$

Standard method computing $\mathrm{Tor}^A(M,k)$:

1. Compute an A-resolution $R_A(M)$ of M of A-finite type.

   (Syzygies)

2. $\Rightarrow R_A(M) \otimes_A k =$

   Chain complex of finite dimensional k-vector spaces.

3. $\Rightarrow H_*(R_A(M) \otimes_A k) = \mathrm{Tor}^A(M,k) =$

   elementary computation.

Drawbacks: 1) $R_A(M)$ = sygyzies ⇒ not so easy.

2) It happens $\mathrm{Tor}^A(M, k) := H_*(M \otimes_A R_A(k))$

can be much more interesting !!

Theorem (Serre): $\mathcal{S}$ = PDE local system in $0 \in k^m$.

$I_{\mathcal{S}}$ = canonical ideal associated to $\mathcal{S}$.

Then $\mathcal{S}$ involutive $\Leftrightarrow \mathrm{Tor}^A(I_{\mathcal{S}}, k)_+ = 0$.

But the theorem comes

from the explicit examination of $I \otimes_A R_A(k)$.

Using this theorem needs a complete solution

for the homological problem of $I \otimes_A R_A(k)$.

Previous results described about Effective Homology:

1. **Reductions;**
2. **Equivalences;**
3. **Basic perturbation Lemma;**
4. **Cones;**
5. **SES$_i$ theorems;**

$\Rightarrow$

A simple algorithm computes

the effective homology of $K(A/<g_1, \ldots, g_n>)$.

Typical simple example.

$I = \langle x - t^3, y - t^5 \rangle \subset A = \mathbb{Q}[x, y, t]$.

How to compute $H_*(K(A/I)) = H_*(K(A/I\,;\, x, y, t))$ **?**

Step 1: Compute a Groebner basis for I**.**

Choose a coherent monomial order,

for example DegRevLex = DRL.

$\Rightarrow \text{Groebner}(I, \text{DRL}) = \langle xt^2 - y, t^3 - x, x^2 - yt \rangle$.

Step 2: Consider $J = \langle xt^2, t^3, x^2 \rangle$

= the associated monomial ideal.

Then: 1. The $\mathbb{Q}$-vector spaces A/I and A/J

are canonically isomorphic.

2. $\Rightarrow$ $K(A/I)$ and $K(A/J)$ are

graded $\mathbb{Q}$-vector spaces canonically isomorphic,

but with non-compatible differentials:

$$d_{K(A/J)}(t^2\delta x) = 0 \quad ; \quad d_{K(A/I)}(t^2\delta x) = y.$$

Plan: 1. Compute $H_*(K(A/J))$.

2. Apply BPL to deduce $H_*(K(A/I))$.

How to compute $H_*(A/{<}xt^2, t^3, x^2{>})$?

Recursive process about the number of generators.

Relation between $H_*(A/{<}xt^2, t^3, x^2{>})$ and $H_*(A/{<}t^3, x^2{>})$?

Exact sequence of A-modules:

$$0 \to \frac{A}{<x, t>} \xrightarrow{\times xt^2} \frac{A}{<t^3, x^2>} \xrightarrow{\text{pr}} \frac{A}{<xt^2, t^3, x^2>} \to 0$$

Remark: $<x, t> = <t^3, x^2> : xt^2 = \{a \in A \ \underline{\text{st}} \ axt^2 \in <t^3, x^2>\}$.

$\Rightarrow$ Exact sequence of chain complexes:

$$0 \to K\left(\frac{A}{<x, t>}\right) \xrightarrow{\times xt^2} K\left(\frac{A}{<t^3, x^2>}\right) \xrightarrow{\text{pr}} K\left(\frac{A}{<xt^2, t^3, x^2>}\right) \to 0$$

$\Rightarrow$

Effective homologies of $K(A/<x,t>)$ and $K(A/<t^3,x^2>)$
give effective homology of $K(A/<xt^2,t^3,x^2>)$

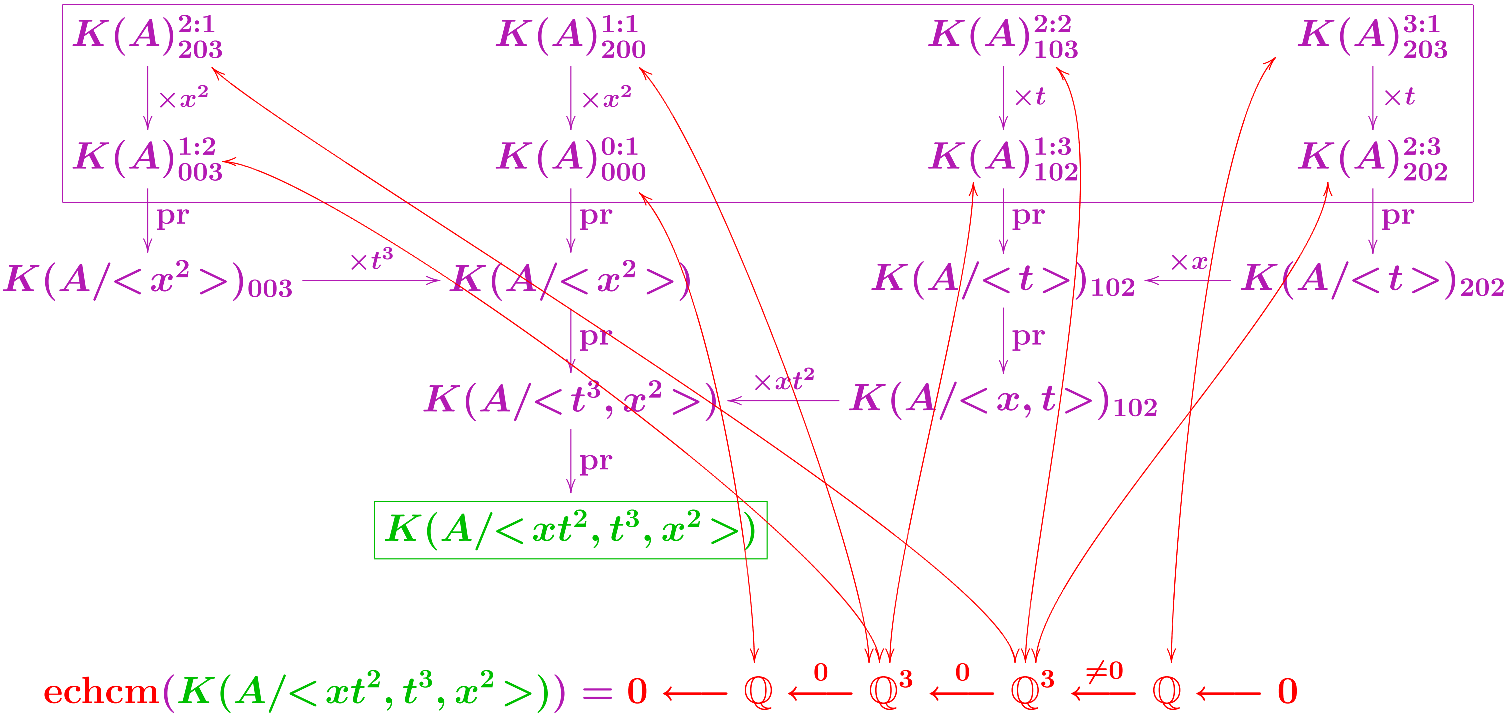
What about the first step of the recursive process?

Continuing in the same way $\Rightarrow$ short exact sequence:

$$0 \rightarrow K\left(\frac{A}{<>}\right) \xrightarrow{\times x^2} K\left(\frac{A}{<>}\right) \xrightarrow{\text{pr}} K\left(\frac{A}{<x^2>}\right) \rightarrow 0$$

$\Rightarrow$

It is enough to know the effective homology of $K(A)$.



$K(A)^{2:1}_{203}$
$K(A)^{1:1}_{200}$
$K(A)^{2:2}_{103}$
$K(A)^{3:1}_{203}$
$\times x^2$
$\times x^2$
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$\times t$
$K(A)^{1:2}_{003}$
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$K(A)^{1:3}_{102}$
$K(A)^{2:3}_{202}$
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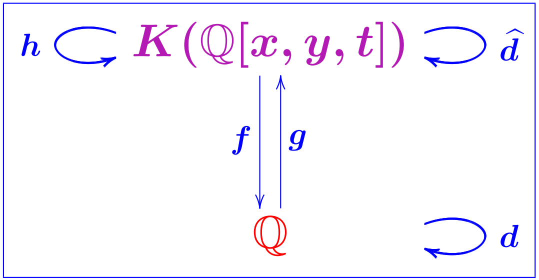
$K(A/<x^2>)_{003}$
$\times t^3$
$K(A/<x^2>)$
$K(A/<t>)_{102}$
$\times x$
$K(A/<t>)_{202}$
$K(A/<t^3, x^2>)$
$\times xt^2$
$K(A/<x, t>)_{102}$
$K(A/<xt^2, t^3, x^2>)$
$\mathbf{echcm}(K(A/<xt^2, t^3, x^2>)) = 0 \longleftarrow \mathbb{Q} \xleftarrow{0} \mathbb{Q}^3 \xleftarrow{0} \mathbb{Q}^3 \xleftarrow{\neq 0} \mathbb{Q} \longleftarrow 0$

Theorem: **A multi-homogeneous reduction can be produced:**



with all the maps $\widehat{d}$, d, f, g and h homogeneous with respect to a $[x, y, t]$-multi-grading.

Proof.

Multi-grading of $x^\alpha y^\beta t^\gamma\ \delta x\ \delta t = [\alpha+1, \beta, \gamma+1]$

$\Rightarrow$ Koszul differential $\widehat{d}$ is multi-homogeneous.

$$h(x^\alpha y^\beta t^\gamma\ \delta x\ \delta t) = 0$$
$$h(x^\alpha y^\beta t^3\ \delta x) = -x^\alpha y^\beta t^2\ \delta x\ \delta t$$
$$h(x^\alpha y^4\ \delta x) = -x^\alpha y^3\ \delta x\ \delta y$$
$$h(x^3\ \delta x) = 0$$
$$h(x^3) = x^2\ \delta x$$

$\Rightarrow$ Contraction h is multi-homogeneous.

The trivial morphisms f and g

are trivially multi-homogeneous.

Easy complements of Effective Homology Theorems:

If every input is multi-homogeneous,

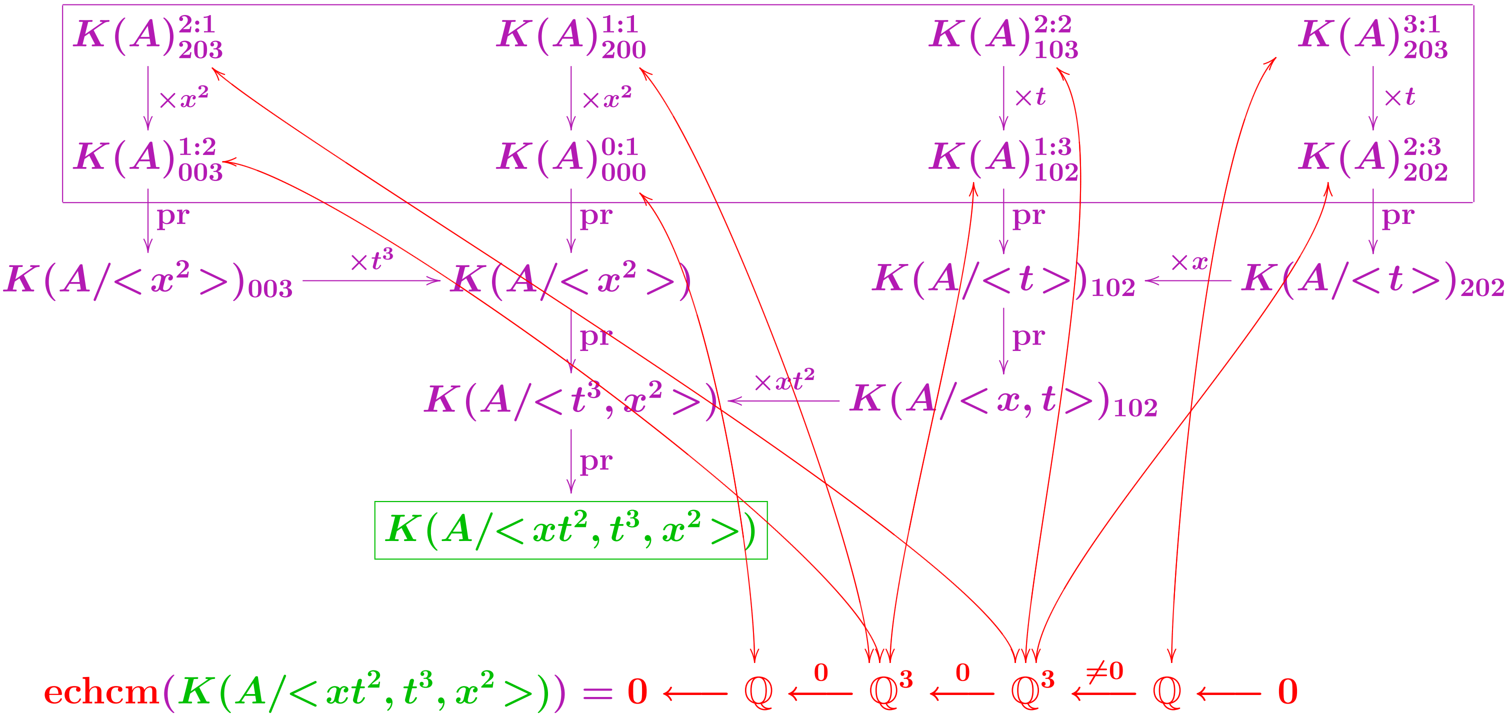
then every output is multi-homogeneous.

Applying to the SES_3 theorem for:

$$0 \rightarrow K(\mathbb{Q}[x,y,t]) \xrightarrow{\times x^{\boxed{2}}} K(\mathbb{Q}[x,y,t]) \xrightarrow{\mathrm{pr}} K\left(\frac{\mathbb{Q}[x,y,t]}{<x^2>}\right) \rightarrow 0$$

Multiplication by $x^{\boxed{2}}$ $\Rightarrow$ you must shift the multi-grading of the lefthand $K(\mathbb{Q}[x,y,t])$ to get $\times x^{\boxed{2}}$ multi-homogeneous:

$$\mathrm{Multigrading}(x^\alpha y^\beta t^\gamma \; \delta x \; \delta t) = [\alpha + 1 + \boxed{2}, \beta, \gamma + 1]$$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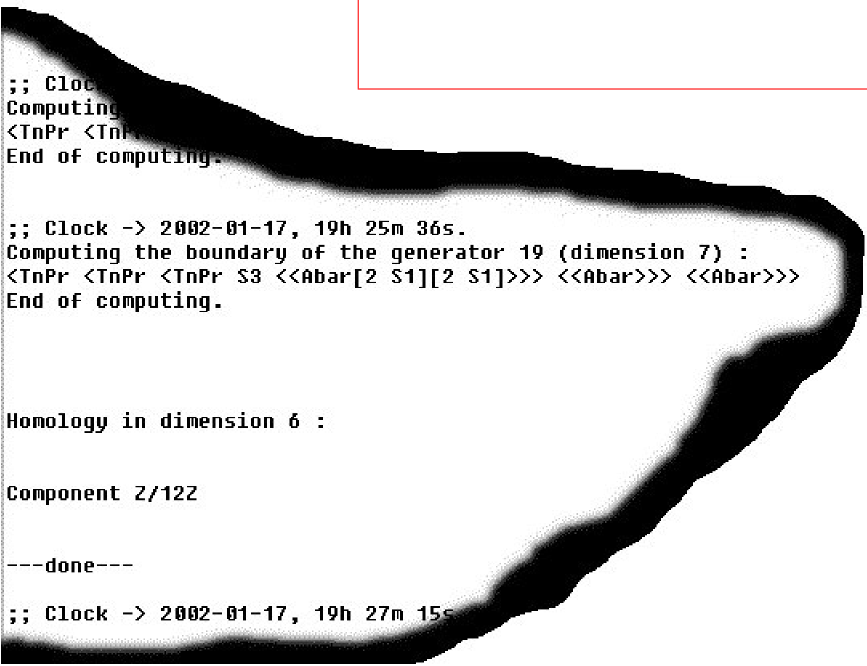

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