The Eilenberg-Zilber Theorem via Discrete Vector Fields*

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Abstract

The Eilenberg-Zilber (EZ) Theorem is basic in Algebraic Topology. It connects the chain complex $C_*(A \times B)$ of the product of two simplicial sets to the tensor product $C_*(A) \otimes C_*(B)$, allowing one to compute $H_*(A \times B)$ from $H_*(A)$ and $H_*(B)$. Initially proved in 1953, it is stated and used in almost every elementary textbook of Algebraic Topology.

Sixty-five years after the initial EZ paper, we present this result via a totally new proof, based on the relatively recent tool called *Discrete Vector Field* (DVF, Robin Forman, 1998). The various available EZ proofs are not so trivial; the present proof on the contrary is intuitive and gives more information. The key point is the notion of *s-path*, a 2-dimensional description of every simplex of the standard simplicial decomposition of $\Delta^p \times \Delta^q$, p and q arbitrary, which allows us to use the *same* DVF for the *homotopical* and homological EZ theorem*s*.

The right notion of morphism between cellular chain complexes provided with discrete vector fields is given; its definition is really amazing and is here an essential tool.

Besides a new understanding of the EZ theorem, we obtain much better algorithms for the machine implementation of the EZ theorem, automatically avoiding the countless degenerate simplices produced by the Rubio-Morace formula for the EZ-homotopy. Other striking applications will be the subject of other papers.

1 Introduction.

The precise origin of the EZ theorem is in the following problem. Let Δ^p and Δ^q be the standard simplices of dimension p and q. What about the product $P := \Delta^p \times \Delta^q$? Two possible points of view. We may decide after all this product also is *elementary*, more generally we may decide every product of simplices is elementary. Obvious drawback: the various results of *simplicial* combinatorial topology cannot directly be applied.

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It is therefore natural to *triangulate* the product $P := \Delta^p \times \Delta^q$. The *canonical* triangulation of this product is in a sense the heart of our subject. This triangulation is elegant but has a handicap: it is difficult to see and use in it the underlying product structure.

The EZ-theorem settles the right connection between both viewpoints. At least for the *homological* problem. For every pair A and B of simplicial sets, the EZ-theorem defines chain complex morphisms between $C_*(A \times B)$ and $C_*(A) \otimes C_*(B)$, these chain complex morphisms being inverse homological equivalences. The first complex $C_*(A \times B)$ is the chain complex canonically associated to the *simplicial* set $A \times B$; in other words, every product $\sigma_A \times \sigma_B$, σ_A (resp. σ_B) being a (non-degenerate) p-simplex of A (resp. q-simplex of B) is triangulated as $\Delta^p \times \Delta^q$, in particular producing $\binom{p+q}{p}$ simplices of dimension $p + q$. The second chain complex has exactly *one* generator in dimension $p + q$ for every such pair (σ_A, σ_B): the chain complex $C_*(A \times B)$ is much bigger than $C_*(A) \otimes C_*(B)$. The second chain complex is in fact canonically associated to $A \times B$ considered as a union of "bisimplices" $\sigma_A \times \sigma_B$.

This article gives a new description of the Eilenberg-Zilber environment. In front of a product $A \times B$, the notion of *s-path* allows us to define a funny *discrete vector field* (Definition 24) on the cellular complex $C_*(A \times B)$. This discrete vector field automatically produces a *reduction*:

$$\rho = (f, g, h) = \boxed{h \circlearrowright C_*(A \times B) \underset{f}{\overset{g}{\leftrightarrows}} C_*(A) \otimes C_*(B)} \tag{1}$$

which reduction contains the EZ theorem.

We so find again the classical maps f = AW = Alexander-Whitney, g = EML = Eilenberg-MacLane. The composition fg = AW EML is the identity. We find also the less classical Rubio-Morace formula h = RM for the homotopy between gf and the identity, which map h cannot be avoided if a *constructive* result is required. Most often, only the lazy *existence* of this map is proved, preventing the EZ theorem from being constructively used.

The naturality of the Eilenberg-Zilber process is obtained here thanks to the notion of *morphism between cellular chain complexes provided with discrete vector fields*. This definition, to our knowledge so far unknown, is as surprising as efficient.

We will see that, according to the degeneracies in the faces of the simplices of A and B, many terms of the terrible Rubio-Morace formula (55) in fact do not contribute. It happens the vector field technology produces the optimal algorithm with respect to these degeneracies, explaining the strong time difference between both implementations, the first one using the Rubio-Morace formula, the second one using the canonical vector field defined in this text. It is tempting to see an analogy betweeen, on the one hand, the "efficicency" of the proof of the s-cobordism theorem by Forman [8], and on the other hand, the concrete efficiency of our computer program using the same method [14]. Also the vector field method is so simple that it is easily implemented, the corresponding computer program being simple, readable, easy to maintain.

Another point is to be mentioned. Theorem 23, the *theorem of the hollowed prism*, can seem insubstantial for topologists not used to the problems of *constructive* combinatorial topology. It happens it was a crucial point to make constructive the Bousfield-Kan spectral sequence [16]. The constructive Bousfield-Kan spectral sequence is nothing but the *complete* connection between homology groups and homotopy groups, and it is well known that "in general" homotopy is harder than homology. But understanding the deep nature of the theorem of the hollowed prism is simple: it is nothing but the *homotopical* version of the EZ theorem, whereas the usual EZ theorem is the homological version. It is remarkable the *same* discrete vector field produces both versions, homological and homotopical, of the EZ theorem. The perceptive reader will notice the same difference between both ways of using our vector field as the difference between the combinatorial definitions of π_1 and H_1.

Other papers are planned, using the same method, in fact really the *same* discrete vector field, to obtain constructive and efficient versions of the *twisted* EZ theorem [1, 2] and the Eilenberg-Moore spectral sequences [5].

The paper is organized as follows.

1. We recall the classical EZ theorem, the notion of (Whitehead) *collapse*, the quite elementary definition of discrete vector field in the comfortable framework of *cellular* chain complexes.

2. Then the key notion of s-path is described, it is a combinatorial 2-dimensional simple description of an arbitrary simplex of the canonical triangulation of $\Delta^p \times \Delta^q$, p and q being themselves arbitrary.

3. Then the *homotopical* EZ-theorem is stated and proved, thanks to a *filling sequence*, which sequence is nothing but a discrete vector field provided with an extra *order*. This vector field, forgetting this order, is our tool to produce *some* EZ theorem by a totally new way. There remains to connect "both" EZ-theorems.

4. The notion of morphism between cellular chain complexes provided with vector fields is also quite amazing and has a scope much wider than the subject of the present paper.

5. The long section 12 proves both EZ theorems produce the same (abstract) results. This section has its own introduction to explain why this work is justified.

6. Conclusion.

2 The Eilenberg-Zilber Theorem.

The standard notions of *simplicial set* is assumed known. The most common reference for the definition of and the basic results about the simplicial sets is the "little red[1] book" [9]. All the necessary formulas are available in this book, unfortunately without any illustrative examples. So that you can be helped by the text [20] which on the contrary contains many simple examples.

[1]Red because the unique copy of this book at the universitary library of Grenoble has been so used that a new cover has been necessary, and it happens this new cover is red.

Theorem 1 (Eilenberg-Zilber Theorem) — *Let X and Y be two simplicial sets. Then a canonical* reduction *ρ is defined:*

$$\rho = (f, g, h) = \boxed{h \circlearrowright C_*(X \times Y) \underset{f}{\overset{g}{\leftrightarrows}} C_*(X) \otimes C_*(Y)} \tag{2}$$

♣ The initial proof is the article [7]. ♣

All the chain complexes of simplicial sets $C_*(\cdots)$ are assumed *normalized*, that is, generated by the *non-degenerate simplices*.

In this text, a *reduction* is a homology equivalence between two chain complexes given as a triple $\rho = (f, g, h)$. The maps f and g are chain complex morphisms. The second one g is an injection of a "small" chain complex, here $C_*(X) \otimes C_*(Y)$, into a "big" one, here $C_*(X \times Y)$; the first one f is a projection of the big chain complex onto the small one. The composition fg is the identity of the small chain complex, and h is a homotopy operator between gf and the identity of the big chain complex. Also the compositions fh, hg and hh are null. These data induce a canonical decomposition of the big chain complex here $C_*(X \times Y) = \operatorname{im} g \oplus \ker f$, the first component is isomorphic to the small chain complex, and the second component is acyclic provided with a homological contraction h.

The general idea of a well defined reduction is the following: the big chain complex is "rich" but a little too complicated, and a "simplified" version is the small one. The small version is simplified with respect to some point of view, but must keep the homological nature of the big one.

The simplicial sets X and Y are (simplicially) triangulated, and then what about the product $X \times Y$? In fact this product naturally inherits a simplicial structure, but it happens this structure is in general relatively complex, generating many simplices. For example, if $X = \Delta^p$ and $Y = \Delta^q$, both factors are made only of *one* simplex, yet the canonical simplicial structure of $\Delta^p \times \Delta^q$ is made of $\binom{p+q}{p}$ simplices of dimension $p + q$. If product objects such that $\Delta^{p,q} := \Delta^p \times \Delta^q$ are allowed in the collection of "elementary" objects, then $\Delta^{p,q}$, a unique object, is sufficient to describe the product $\Delta^p \times \Delta^q$.

This is the philosophy of the EZ theorem. If you allow the objects $\Delta^{p,q}$, then the product $X \times Y$ is trivially a union of objects of this sort, and the standard definition of the differential for a tensor product of chain complexes produces the chain complex $C_*(X) \otimes C_*(Y)$ with the right homology for $X \times Y$. The Künneth theorem then gives easily the homology groups of the product. Well, but a priori you do not have a simplicial structure on the product, which can be an essential difficulty. On the contrary, the standard simplicial structure of $X \times Y$ could be used, but it can generate too many simplices to be convenient.

The EZ theorem gives a canonical connection between both chain complexes, $C_*(X \times Y)$ and $C_*(X) \otimes C_*(Y)$, allowing the topologist to use the advantages of both presentations, according to some or other concrete problem. And the EZ reduction allows one to transfer the results obtained on one side to the other side.

3 Discrete vector fields.

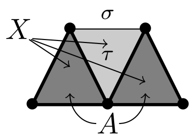
This section is devoted to the general notion of discrete vector field, and to its connection with collapses, under a form convenient for our interpretation of the EZ theorem.

3.1 Collapses.

Definition 2 — *An* elementary collapse *is a pair* (X, A) *of simplicial complexes, satisfying the following conditions:*

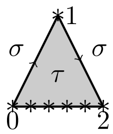
1. *The component* A *is a* simplicial subset *of the simplicial set* X*.*
2. *The difference* $X - A$ *is made of exactly* two *non-degenerate simplices* $\tau \in X_n$ *and* $\sigma \in X_{n-1}$*, the second one* σ *being a* face *of the first one* τ*.*
3. *The incidence relation* $\sigma = \partial_i \tau$ *holds*[2] *for a* unique *index* $i \in 0...n$*.*

For example, X could be made of three triangles and A of two only as in the next figure.



(3)

The condition 3 is required to assert the existence of a topological contraction of X onto A. Think for example of the minimal triangulation of the real projective plane $P^2\mathbb{R}$ as a simplicial set X, see the next figure: one vertex $*$, one edge σ and one triangle τ; no choice for the faces of σ; the faces of τ must be $\partial_0 \tau = \partial_2 \tau = \sigma$ and $\partial_1 \tau = \eta_0 *$ is the degeneracy of the base point. The realization of X is homeomorphic to $P^2\mathbb{R}$. If you omit the condition 3 in the definition of collapse, then $(X, *)$ would be a collapse, but $P^2\mathbb{R}$ is not contractible.



(4)

Definition 3 — A *collapse* is a pair (X, A) of simplicial sets satisfying the following conditions:

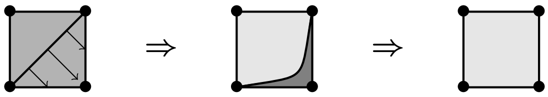
1. The component A is a *simplicial subset* of the simplicial set X.
2. There exists a sequence $(A_i)_{0 \leq i \leq m}$ with:
   (a) $A_0 = A$ and $A_m = X$.
   (b) For every $0 < i \leq m$, the pair (A_i, A_{i-1}) is an elementary collapse. ♣

In other words, a collapse is a finite sequence of elementary collapses. If (X, A) is a collapse, then a topological contraction $X \to A$ can be defined.

Another kind of modification when examining a topological object can be studied. Let us consider the usual triangulation of the square with two triangles, the square cut by a diagonal. Then it is tempting to modify this

[2]We denote $\partial_i \sigma$ (resp. $\eta_i \sigma$) the i-th face (resp. degeneracy) of the simplex σ.

triangulation by pushing the diagonal onto two sides as roughly described in this figure.



(5)

Why not, but this needs other kinds of cells, here a square with *four* edges, while in a simplicial framework, the only objects of dimension 2 that are provided are the triangles Δ^2. Trying to overcome this essential obstacle leads to two major subjects:

1. The Eilenberg-Zilber theorem, an algebraic translation of this idea, which consists in *algebraically* allowing the use of simplex products.
2. The CW-complex theory, where the added cells are attached to the previously constructed object through *arbitrary* attaching maps.

This paper systematically reconsiders these essential ideas through the notion of *discrete vector field.* The EZ theorem has an essentially *homological* nature. Our point of view will naturally lead also to a *homotopical* EZ theorem, with a unified approach of both contexts.

3.2 Algebraic discrete vector fields.

The notion of discrete vector field (DVF) is due to Robin Forman [8]; it is an essential component of the so-called *discrete Morse theory.*

This notion is usually described and used in combinatorial *topology*, but a purely algebraic version can also be given; we prefer this context.

Definition 4 — An *algebraic cellular complex* (ACC) is a family:

$$C = (C_p, d_p, \beta_p)_{p\in\mathbb{Z}}$$

of *free* $\mathbb{Z}$-modules and boundary maps. Every C_p is called a *chain group* and is provided with a *distinguished* $\mathbb{Z}$-basis β_p; every basis component $\sigma \in \beta_p$ is a *p-cell.* The boundary map $d_p : C_p \to C_{p-1}$ is a $\mathbb{Z}$-linear map connecting two consecutive chain groups. The usual boundary condition $d_{p-1}d_p = 0$ is satisfied for every $p \in \mathbb{Z}$. ♣

Most often we omit the index of the differential, so that the last condition can be denoted by $d^2 = 0$. The notation is redundant: necessarily, $C_p = \mathbb{Z}[\beta_p]$, but the standard notation C_p for the group of p-chains is convenient.

We consider only here the case of the ground ring $\mathbb{Z}$, the natural one for our EZ problem. Obvious extensions to more general cases, arbitray ground rings, in particular fields, may also be given.

The chain complex associated to any sort of topological cellular complex is an ACC. We are specially interested in the chain complexes associated to simplicial sets.

Important: we *do not* assume *finite* the distinguished bases β_p, the chain groups are not necessarily of *finite type.* This is not an artificial extension to the traditional Morse theory: this point will be often essential, but this extension is obvious.

Definition 5 — Let C be an ACC. A $(p-1)$-cell σ is said to be a *face* of a p-cell τ if the coefficient of σ in $d\tau$ is non-null. It is a *regular face* if this coefficient is +1 or -1. ♣

If Δ^p is the standard simplex, every face of every subsimplex is a regular face of this subsimplex. We gave after Definition 2 an example of triangulation of the real projective plane as a simplicial set; the unique non-degenerate 1-simplex σ is *not* a regular face of the triangle τ, for $d\tau = 2\sigma$.

Note also the *regular* property is *relative*: σ can be a regular face of τ but also a non-regular face of another simplex τ'.

Definition 6 — A *discrete vector field* V on an algebraic cellular complex $C = (C_p, d_p, \beta_p)_{p\in\mathbb{Z}}$ is a collection of pairs $V = \{(\sigma_i, \tau_i)\}_{i\in\beta}$ satisfying the conditions:

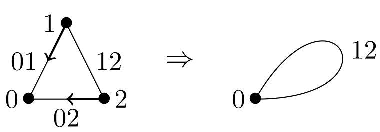
1. Every σ_i is some p-cell, in which case the other corresponding component τ_i is a $(p+1)$-cell. The degree p depends on i and in general is not constant.
2. Every component σ_i is a *regular* face of the corresponding component τ_i.
3. A cell of C appears *at most one time* in the vector field: if $i \neq j$, then $\{\sigma_i, \tau_i\} \cap \{\sigma_j, \tau_j\} = \varnothing$. ♣

It is not required all the cells of C appear in the vector field V. In particular the void vector field is allowed. In a sense the remaining cells are the most important.

Definition 7 — A cell χ which does not appear in a discrete vector field $V = \{(\sigma_i, \tau_i)\}_{i\in\beta}$ is called a *critical* cell. A component (σ_i, τ_i) of the vector field V is a *p-vector* if σ_i is a p-cell. Such a cell σ_i is a *source cell*, and the other cell τ_i is a *target cell*. ♣

We do not consider in this paper the traditional vector fields of differential geometry, which allows us to call simply a *vector field* which should be called a *discrete vector field*.

In case of an ACC coming from a topological cellular complex, a vector field is a recipe to cancel "useless" cells in the underlying space, useless with respect to the homotopy type. A component (σ_i, τ_i) of a vector field can vaguely be thought of as a "vector" starting from the center of σ_i, going to the center of τ_i. For example $\partial\Delta^2$ and the circle have the same homotopy type, which is described by the following scheme:

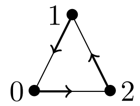


(6)

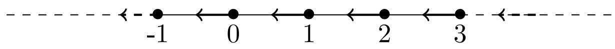
The initial simplicial complex is made of three 0-cells 0, 1 and 2, and three 1-cells 01, 02 and 12. The drawn vector field is $V = \{(1, 01), (2, 02)\}$, and this vector field defines a homotopy equivalence between $\partial\Delta^2$ and the minimal triangulation of the circle as a simplicial set. The last triangulation is made of the critical cells 0 and 12, attached according to a process which deserves to be seriously studied in the general case. This paper is devoted to a systematic use of this idea.

3.3 V-paths and admissible vector fields.

By the way, what about this vector field in $\partial\Delta^2$?



No critical cell and yet $\partial\Delta^2$ does not have the homotopy type of the void object. We must forbid possible *loops*. This is not enough. Do not forget the infinite case must be also covered; but look at this picture:



(7)

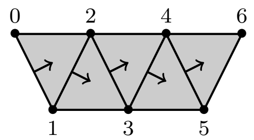
representing an infinite vector field on the real line triangulated as an infinite union of 1-cells connecting successive integers. No critical cell and yet the real line does not have the homotopy type of the void set. We must also forbid the possible *infinite* paths.

The notions of V-paths and admissible vector fields are the appropriate tools to define the necessary restrictions.

Definition 8 — If $V = \{(\sigma_i, \tau_i)\}_{i\in\beta}$ is a vector field on an algebraic cellular complex $C = (C_n, d_n, \beta_n)_n$, a V-path of degree p is a sequence $\pi = ((\sigma_{i_k}, \tau_{i_k}))_{0\le k<m}$ satisfying:

1. Every pair $((\sigma_{i_k}, \tau_{i_k}))$ is a component of the vector field V and the cell τ_{i_k} is a p-cell.
2. For every $0 < k < m$, the component σ_{i_k} is a face of $\tau_{i_{k-1}}$, non necessarily regular, but different from $\sigma_{i_{k-1}}$.

If $\pi = ((\sigma_{i_k}, \tau_{i_k}))_{0\le k<m}$ is a V-path, and if σ is a face of $\tau_{i_{m-1}}$ different from $\sigma_{i_{m-1}}$, then π *connects* σ_{i_0} and σ through the vector field V. ♣



(8)

A V-path connecting the edges 01 and 56.

In a V-path $\pi = ((\sigma_{i_k}, \tau_{i_k}))_{0\le k<m}$ of degree p, a $(p-1)$-cell σ_{i_k} is a regular face of τ_{i_k}, for the pair $(\sigma_{i_k}, \tau_{i_k})$ is a component of the vector field V, but the same σ_{i_k} is non-necessarily a regular face of $\tau_{i_{k-1}}$.

Definition 9 — The *length* of the path $\pi = ((\sigma_{i_k}, \tau_{i_k}))_{0\le k<m}$ is m. ♣

If (σ, τ) is a component of a vector field, in general the cell τ has *several* faces different from σ, so that the possible paths starting from a cell generate an oriented graph.

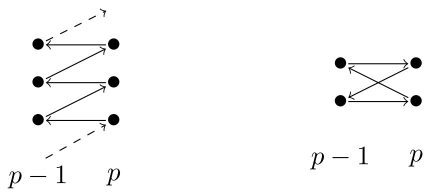
Definition 10 — A discrete vector field V on an algebraic cellular complex $C = (C_n, d_n, \beta_n)_{n\in\mathbb{Z}}$ is *admissible* if for every $n \in \mathbb{Z}$, a function $\lambda_n : \beta_n \to \mathbb{N}$ is provided satisfying the following property: every V-path starting from $\sigma \in \beta_n$ has a length bounded by $\lambda_n(\sigma)$. ♣

Excluding infinite paths is almost equivalent. The difference between both possibilities is measured by Markov's principle; we prefer our more constructive statement.

A circular path would generate an infinite path and is therefore excluded.

The next diagram, an *oriented bipartite graph*, can help to understand this notion of admissibility for some vector field V. This notion makes sense degree by degree. Between the degrees p and $p-1$, organize the *source* $(p-1)$-cells (resp. *target* p-cells) as a lefthand (resp. righthand) column of cells. Then every vector $(\sigma, \tau) \in V$ produces an oriented edge $\sigma \rightarrowtail \tau$. In the reverse direction, if τ is a target p-cell, the boundary $d\tau$ is a finite linear combination $d\tau = \sum \alpha_i \sigma_i$, and some of these σ_i's are source cells, in particular certainly the corresponding V-source cell σ. For every such source component σ_i, be careful *except* for the corresponding source σ, you install an oriented edge $\sigma_i \leftarrowtail \tau$.

Then the vector field is admissible between the degrees $p-1$ and p if and only if, starting from every source cell σ, all the (oriented) paths have a length bounded by some integer $\lambda_p(\sigma)$. In particular, the loops are excluded. We draw the two simplest examples of vector fields non-admissible. The lefthand one has an infinite path, the righthand one has a loop, a particular case of infinite path.



(9)

Definition 11 — Let $V = \{(\sigma_i, \tau_i)_{i \in \beta}\}$ be a vector field on an ACC. A *Lyapunov function* for V is a function $L : \beta \to \mathbb{N}$ satisfying the following condition: if σ_j is a face of τ_i different from σ_i, then $L(j) < L(i)$. ♣

It is the natural translation in our discrete framework of the traditional notion of Lyapunov function in differential geometry. It is clear such a Lyapunov function proves the admissibility of the studied vector field. Obvious generalizations to ordered sets more general than $\mathbb{N}$ are possible.

4 Elementary combinatorial topology.

In this section we present a combinatorial description of the simplices of a prism by means of the notion of *s-path*, introduced in [15]. The definitions and results of this section can also be found in [15], where these ideas are used to provide a constructive version of the Bousfield-Kan spectral sequence [16]. They are repeated here for the convenience of the reader. Furthermore, taking account of the symmetry of the definition of a simplicial set, you may reverse the order of indices of the face and degeneracy operators, several choices are possible for various formulas, for example two main choices for the Alexander-Whitney operator. It happens the choices made in [15] are not the right one to obtain the standard Eilenberg-Zilber formulas, essential with respect to the subject of this paper, still more important when we will study in a next paper the *twisted* Eilenberg-Zilber theorem through the same technology.

4.1 Triangulation of a prism.

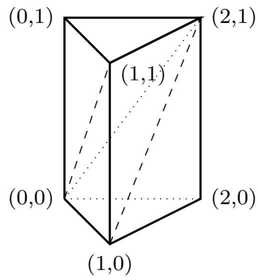
We have to work in the simplicial complex $\Delta^{p,q} = \Delta^p \times \Delta^q$. A vertex of Δ^p is an integer in $\underline{p} = [0 \dots p]$, a (non-degenerate) d-simplex of Δ^p is a strictly increasing sequence of integers $0 \le v_0 < \dots < v_d \le p$. The same for our second factor Δ^q.

The canonical triangulation of $\Delta^p \times \Delta^q$ is made of (non-degenerate) simplices $((v_0, v'_0), \dots, (v_d, v'_d))$ satisfying the relations:

- $0 \le v_0 \le v_1 \le \dots \le v_d \le p$.
- $0 \le v'_0 \le v'_1 \le \dots \le v'_d \le q$.
- $(v_i, v'_i) \ne (v_{i-1}, v'_{i-1})$ for $1 \le i \le d$.

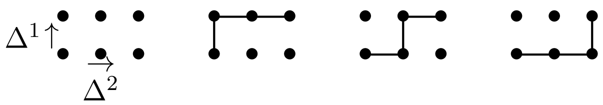
In other words, the canonical triangulation of $\Delta^{p,q} = \Delta^p \times \Delta^q$ is associated to the poset $\underline{p} \times \underline{q}$ endowed with the *product order* of the factors. For example the three maximal simplices of $\Delta^{2,1} = \Delta^2 \times \Delta^1$ are:

- $((0,0),(0,1),(1,1),(2,1))$.
- $((0,0),(1,0),(1,1),(2,1))$.
- $((0,0),(1,0),(2,0),(2,1))$.



4.2 Simplex = s-path.

We can see the poset $\underline{p} \times \underline{q}$ as a lattice where we arrange the first factor $\underline{p}$ in the horizontal direction and the second factor $\underline{q}$ in the vertical direction. The first figure below is the lattice $\underline{2} \times \underline{1}$ while the other figures are representations of the maximal simplices of $\Delta^{2,1} = \Delta^2 \times \Delta^1$ as *increasing* paths in the lattice.



(10)

Definition 12 — An *s-path* π of the lattice $\underline{p} \times \underline{q}$ is a finite sequence $\pi = ((a_i, b_i))_{0 \le i \le d}$ of elements of $\underline{p} \times \underline{q}$ satisfying $(a_{i-1}, b_{i-1}) < (a_i, b_i)$ for every $1 \le i \le d$ with respect to the product order. The d-simplex σ_π represented by the path π is the convex hull of the points (a_i, b_i) in the prism $\Delta^{p,q}$. ♣

The simplices Δ^p and Δ^q have affine structures which define a product affine structure on $\Delta^{p,q}$, and the notion of convex hull is well defined on $\Delta^{p,q}$.

"S-path" stands for "path representing a simplex", more precisely a non-degenerate simplex. Replacing the strict inequality between two successive vertices by a non-strict inequality would lead to analogous representations for degenerate simplices, but such simplices are not to be considered in this section.

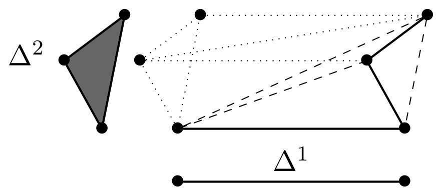
This representation of a simplex as an s-path running in a lattice is the key point to master the relatively complex structure of the canonical prism triangulations.

Definition 13 — The *last simplex* $\lambda_{p,q}$ of the prism $\Delta^{p,q}$ is the $(p+q)$-simplex defined by the path:

$$\lambda_{p,q} = ((0,0),(1,0),\ldots,(p,0),(p,1),\ldots,(p,q)). \tag{11}$$

♣

The path runs some edges of $\Delta^p \times 0$, visiting all the corresponding vertices in the right order; next it runs some edges of $p \times \Delta^q$, visiting all the corresponding vertices also in the right order. Geometrically, the last simplex is the convex hull of the visited vertices. The last simplex of the prism $P_{1,2} = \Delta^1 \times \Delta^2$ is shown in the figure below. The path generating the last simplex is drawn in full lines, the other edges of this last simplex are dashed lines, and the other edges of the prism are in dotted lines.



(12)

4.3 Subcomplexes.

Definition 14 — The *hollowed prism* $H\Delta^{p,q} \subset \Delta^{p,q}$ is the difference:

$$H\Delta^{p,q} := \Delta^{p,q} - \text{int(last simplex)}. \tag{13}$$

The faces of the last simplex are retained, but the interior of this simplex is removed.

Definition 15 — The *boundary* $\partial\Delta^{p,q}$ of the prism $\Delta^{p,q}$ is defined by:

$$\partial\Delta^{p,q} := (\partial\Delta^p \times \Delta^q) \cup (\Delta^p \times \partial\Delta^q) \tag{14}$$

It is the geometrical Leibniz formula.

We will give a detailed description of the pair $(H\Delta^{p,q}, \partial\Delta^{p,q})$ as a *collapse*, cf. Definition 3; it is a combinatorial version of the well-known topological contractibility of $\Delta^{p,q} - \{*\}$ on $\partial\Delta^{p,q}$ for every point $*$ of the interior of the prism. A very simple admissible vector field will be given to homologically annihilate the difference $H\Delta^{p,q} - \partial\Delta^{p,q}$. In fact, carefully ordering the components of this vector field will give the desired collapse.

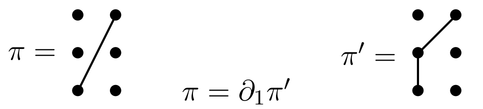
4.4 Interior and exterior simplices of a prism.

Definition 16 — A simplex σ of the prism $\Delta^{p,q}$ is said *exterior* if it is included in the boundary of the prism: $\sigma \subset \partial\Delta^{p,q}$. Otherwise the simplex is said *interior*. We use the same terminology for the s-paths, implicitly referring to the simplices coded by these paths. ♣

The barycenter of an exterior simplex of $\Delta^{p,q}$ is in $\partial\Delta^{p,q}$ whereas the barycenter of an interior simplex is not. The faces of an exterior simplex are also exterior, but an interior simplex can have faces of both sorts.

Proposition 17 — *An s-path π in $\underline{p} \times \underline{q}$ is interior if and only if the projection-paths π_1 on $\underline{p}$ and π_2 on $\underline{q}$ run all the respective vertices of $\underline{p}$ and $\underline{q}$.*

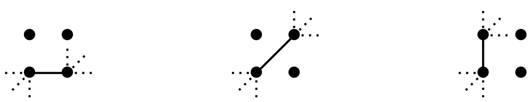
The first s-path π in the figure below represents a 1-simplex *in* $\partial\Delta^{1,2}$, it is *exterior*, for the point 1 is missing in the projection π_2 on the second factor $\underline{2}$. The second s-path π' represents an *interior* 2-simplex of $\Delta^{1,2}$, for both projections are surjective.



(15)

In particular, if $\pi = ((a_i, b_i))_{0\le i\le d}$ is an interior simplex of $\Delta^{p,q}$, then necessarily $(a_0, b_0) = (0, 0)$ and $(a_d, b_d) = (p, q)$: an s-path representing an interior simplex of $\Delta^{p,q}$ starts from $(0, 0)$ and arrives at (p, q).
♣ If for example the first projection of π is not surjective, this means the first projection of the generating path does not run all the vertices of Δ^p, and therefore is included in one of the faces $\partial_k\Delta^p$ of Δ^p. This implies the simplex σ_π is included in $\partial_k\Delta^p \times \Delta^q \subset \partial\Delta^{p,q}$. ♣

We so obtain a simple descrition of an interior simplex $((a_i, b_i))_{0\le i\le d}$: it starts from $(a_0, b_0) = (0, 0)$ and arrives at $(a_d, b_d) = (p, q)$; furthermore, for every $1 \le i \le d$, the difference $(a_i, b_i) - (a_{i-1}, b_{i-1})$ is $(0, 1)$ or $(1, 0)$ or $(1, 1)$: both components of this difference are non-negative, and if one of these components is ≥ 2, then the surjectivity property is not satisfied. In a geometrical way, the only possible *elementary steps* for an s-path π describing an *interior* simplex of $\Delta^{p,q}$ are:


(16)

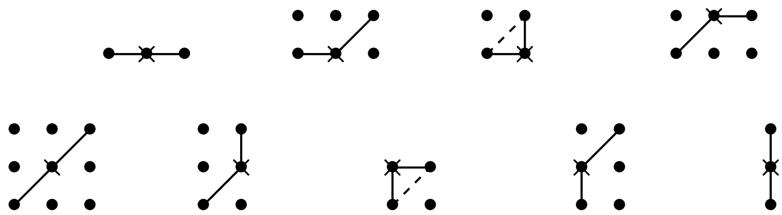
4.5 Faces of s-paths.

If $\pi = ((a_i, b_i))_{0\le i\le d}$ represents a d-simplex σ_π of $\Delta^{p,q}$, the face $\partial_k\sigma_\pi$ is represented by the same s-path except the k-th component (a_k, b_k) which is removed: we could say this point of $\underline{p} \times \underline{q}$ is *skipped*. For example in Figure (15) above, $\partial_1\pi' = \pi$. In particular a face of an interior simplex is not necessarily interior.

Proposition 18 — *Let $\pi = ((a_i, b_i))_{0 \leq i \leq d}$ be an s-path representing an interior d-simplex of $\Delta^{p,q}$. The faces $\partial_0 \pi$ and $\partial_d \pi$ are certainly* not *interior. For $1 \leq k \leq d-1$, the face $\partial_k \pi$ is interior if and only if the point (a_k, b_k) is a right-angle or of the s-path π in the lattice $\underline{p} \times \underline{q}$.*

♣ Removing the vertex $(a_0, b_0) = (0, 0)$ certainly makes non-surjective a projection π_1 or π_2 (or both if $(a_1, b_1) = (1, 1)$). The same if the last point (a_d, b_d) is removed.

If we examine now the case of $\partial_k \pi$ for $1 \leq k \leq d-1$, nine possible configurations for two consecutive elementary steps before and after the vertex (a_k, b_k) to be removed:



(17)

In these figures, the intermediate point of the displayed part of the considered s-path is assumed to be the point (a_k, b_k) of the lattice, to be removed to obtain the face $\partial_k \pi$. In the cases 1, 2, 4 and 5, skipping this point makes non-surjective the first projection π_1 on $\underline{p}$ '=' Δ^p. In the cases 5, 6, 8 and 9, the second projection π_2 on $\underline{q}$ becomes non-surjective. There remain the cases 3 and 7 where the announced right-angle bend is observed. ♣

5 Vector-Field Reduction Theorem.

A vector field defined on an algebraic cellular complex defines a reduction of this chain complex onto the so-called *critical* chain compex.

Let $C_* = (C_n, d_n, \beta_n)$ be an ACC, and $V = \{(\sigma_i, \tau_i)\}$ be a vector field defined on C_*. The *source* (resp. *target*) cells are the σ_i's (resp. the τ_i's). A cell which is neither a source cell nor a target cell is called *critical.* The critical chain complex $K_* = (K_n, ?, \beta_n^K)_n$ is generated by the critical cells, is provided in dimension n with the distinguished basis β_n^K made of the ciritical cells of dimension n, but what about the desired differential?

Theorem 19 (Forman's Theorems). *Given the context of the paragraph above, a canonical reduction is defined*

$$\rho = (f, g, h) = \boxed{h \circlearrowleft (C_*, d, \beta_*) \underset{f}{\overset{g}{\leftrightarrows}} (K_*, d_K, \beta_*^K)} \tag{18}$$

where d_K is a canonical differential for the critical chain complex.

The proof of Forman's theorems, presented in a relatively different way, is in the sections 6 to 8 of [8]. The article [22] gives a proof essentially equivalent, but presented in the framework of *homological reductions* deduced from the fundamental Homological Perturbation Theorem. When we later apply Forman's theorems to the EZ theorem, we will have opportunities to detail the construction of the components f, g and h of the reduction and also of the critical differential d_K.

6 From vector fields to collapses.

Let (X, A) be an elementary collapse, cf Definition 2. The difference $X - A$ is made of two non-degenerate simplices σ and τ, the first one being a face of the second one with a *unique* face index. The pair (σ, τ) is nothing but the unique vector of a vector field V, a vector field which, via Forman's Theorems 19, defines also the reduction $C_*(X) \Rrightarrow C_*(A)$.

Definition 20 — A simplicial pair (X,A) is an elementary *filling* if the difference $X - A$ is made of a *unique* non-degenerate simplex σ, all the faces of which are therefore simplices of A. ♣

You might think $\partial\sigma$ is the initial state of a decayed tooth in the body A, to be restored by adding $\text{int}(\sigma)$, obtaining X.

Definition 21 — Let (X, A) be a simplicial pair. A *description by a filling sequence* of this pair, more precisely of the difference $X - A$, is an ordering $(\sigma_i)_{0<i\leq r}$ of the non-degenerate simplices of $X - A$ satisfying the following condition: if $A_i = A \cup (\cup_{j=1}^{i} \sigma_j)$, then every pair (A_i, A_{i-1}) is an elementary filling. ♣

Every pair (X, A) with a finite number of non-degenerate simplices in $X - A$ can be described by a filling sequence: order the missing simplices according to their dimension. In particular, adding an extra vertex is a particular filling.

It is convenient to describe the general collapses, see Definition 3, by *special* filling sequences.

Proposition 22 — *Let (X,A) be a simplicial pair. This pair is a* collapse *if and only if it admits a description by a filling sequence $F = (\sigma_i)_{0<i\leq 2r}$ satisfying the extra condition: for every* even *index $2i$, the simplex σ_{2i-1} is a face of σ_{2i} with a unique face index.*

♣ Such a description is nothing but the vector field $V = \{(\sigma_{2i-1}, \sigma_{2i})_{0<i\leq r}\}$ with an *extra information*: the vectors are *ordered* in such a way they justify also the collapse property. Such a vector field is necessarily *admissible*: all the V-paths go to A and cannot loop. ♣

This extra information given by the order on the elements of the vector field is an avatar of the traditional difference between homotopy and homology.

7 The theorem of the hollowed prism.

Theorem 23 — *The pair: $(H\Delta^{p,q}, \partial\Delta^{p,q})$ is a collapse.*

The hollowed prism can be collapsed onto the boundary of the same prism.

♣ The proof is recursive with respect to the pair (p, q).

We start from $p = 0$ and $q = 0$. If $p = 0$, the boundary of $\Delta^0 = *$ is void, so that the boundary of $\Delta^{0,q} \cong \Delta^q$ is simply $\partial\Delta^q$; the last simplex in this case is the unique q-simplex, the hollowed prism $H\Delta^{0,q}$ is also $\partial\Delta^q$: the desired collapse is trivial, the corresponding vector field is empty. The same if $q = 0$ for the pair $(H\Delta^{p,0}, \partial\Delta^{p,0})$.

We recall the difference $H\Delta^{p,q} - \partial\Delta^{p,q}$ is made of all the interior simplices of $\Delta^{p,q}$, except the last simplex. More precisely we speak of the difference between the respective collection of simplices.

We prove the general case (p, q) with $p, q > 0$, assuming the proofs of the cases $(p-1, q-1)$, $(p, q-1)$ and $(p-1, q)$ are available, cases respectively concerning $\Delta^{p-1,q-1}$, $\Delta^{p,q-1}$ and $\Delta^{p-1,q}$. We think of these prisms as simplicial subsets of $\Delta^{p,q}$ as follows:

$$\begin{aligned} \Delta^{p-1,q-1} &\cong \partial_0\Delta^p \times \partial_0\Delta^q \\ \Delta^{p,q-1} &\cong \Delta^p \times \partial_0\Delta^q \\ \Delta^{p-1,q} &\cong \partial_0\Delta^p \times \Delta^q \end{aligned} \tag{19}$$

The corresponding lattices $\underline{p-1} \times \underline{q-1}$, $\underline{p} \times \underline{q-1}$ and $\underline{p-1} \times \underline{q}$ are accordingly to be considered respectively as top righthand, top or righthand sublattices of $\underline{p} \times \underline{q}$.

Three justifying filling sequences are available; it is more convenient to see the sequences of simplices as sequences of *s-paths*:

- $F^1 = (\pi_i^1)_{0<i\leq 2r_1}$ for $\Delta^{p-1,q-1} = \partial_0\Delta^p \times \partial_0\Delta^q$.
- $F^2 = (\pi_i^2)_{0<i\leq 2r_2}$ for $\Delta^{p,q-1} = \Delta^p \times \partial_0\Delta^q$.
- $F^3 = (\pi_i^3)_{0<i\leq 2r_3}$ for $\Delta^{p-1,q} = \partial_0\Delta^p \times \Delta^q$.

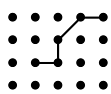
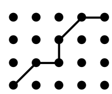
A simplex of each filling sequence must be interior in the corresponding $\Delta^{*,*}$. Therefore, all the components of these filling sequences are s-paths starting from $(1, 1)$ (resp. $(0, 1)$, $(1, 0)$), going to (p, q).

These filling sequences are made of all the interior non-degenerate s-paths (simplices) of the difference $H\Delta^{*,*} - \partial\Delta^{*,*}$, except the last simplex, ordered in such a way every face of an s-path is either interior *and* present *beforehand* in the list, or exterior; furthermore, for the s-paths of even index, the previous one is one of its faces. Using these sequences, we must construct an analogous sequence for the bidimension (p, q).

The process goes as follows. Every s-path of F^1, F^2 or F^3 will be completed to become an element of the filling sequence F to be constructed. This completion process is simple. We will then see two interior simplices of $\Delta^{p,q}$ are missing and must be added. Ordering all the s-paths so obtained gives the solution.

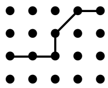
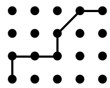
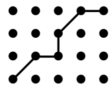
Every s-path π_i^j of dimension d can be completed into an interior s-path $\overline{\pi}_i^j$ of dimension $d+1$ in $\underline{p} \times \underline{q}$ in a unique way, adding a first diagonal step $((0, 0), (1, 1))$ if $j = 1$, or a first vertical step $((0, 0), (0, 1))$ if $j = 2$, or a first horizontal step $((0, 0), (1, 0))$ if $j = 3$. Conversely, every interior s-path of $\Delta^{p,q}$ can be obtained from an interior s-path of $\Delta^{p-1,q-1}$, $\Delta^{p,q-1}$ or $\Delta^{p-1,q}$ in a unique way by this completion process.

For example, in the next figure, we illustrate how an s-path π_i^1 of $\underline{3} \times \underline{2}$ can be completed into an s-path $\overline{\pi}_i^1$ of $\underline{4} \times \underline{3}$:

$\pi_i^1 =$  $\overline{\pi}_i^1 =$  (20)

Adding such a first diagonal step to an element of F^1 *does not add* any right-angle bend in the s-path, so that the assumed incidence properties of the initial sequence $F^1 = (\pi^1_1, \ldots, \pi^1_{2r_1})$ are preserved in the completed sequence $\overline{F}^1 = (\overline{\pi}^1_1, \ldots, \overline{\pi}^1_{2r_1})$: the faces of each s-path are already present in the sequence or are exterior; in the even case $\pi^1_{2i} \in F^1$, the previous s-path π^1_{2i-1} is a face of π^1_{2i} and hence $\overline{\pi}^1_{2i-1}$ is a face of $\overline{\pi}^1_{2i}$. For example in the illustration above, if i is even, certainly $\partial_1 \pi^1_i = \pi^1_{i-1}$ (for this face is the only interior face) and this implies also $\partial_2 \overline{\pi}^1_i = \overline{\pi}^1_{i-1}$.

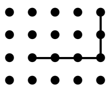
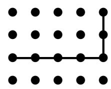
On the contrary, in the case $j = 2$, the completion process *can* add one right-angle bend, nomore. For example, in this illustration:

$\pi^2_i =$  $\overline{\pi}^2_i =$  $\partial_1 \overline{\pi}^2_i =$  (21)

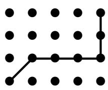
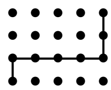
If the index i is even, then $\partial_2 \pi^2_i = \pi^2_{i-1}$ and the relation $\partial_3 \overline{\pi}^2_i = \overline{\pi}^2_{i-1}$ is satisfied as well. But another face of $\overline{\pi}^2_i$ is interior, namely $\partial_1 \overline{\pi}^2_i$, generated by the new right-angle bend; because of the diagonal nature of the first step of this face, this face is present in the list $\overline{F}^1$, see the previous illustration.

Which is explained about $\overline{F}^2$ with respect to $\overline{F}^1$ is valid as well for $\overline{F}^3$ with respect to $\overline{F}^1$.

The so-called last simplices, see Definition 13, must not be forgotten! The last simplex $\lambda_{p-1,q-1}$ (resp. $\lambda_{p,q-1}$) *is not* in the list F^1 (resp. F^2): these lists describe the contractions of the *hollowed* prisms over the corresponding boundaries: all the interior simplices are in these lists except the last ones. The figure below gives these simplices in the case $(p, q) = (4, 3)$:

$\lambda_{3,2} =$  $\lambda_{4,2} =$  (22)

Examining now the respective completed paths:

$\overline{\lambda}_{3,2} =$  $\overline{\lambda}_{4,2} =$  (23)

shows that $\partial_1 \overline{\lambda}_{p,q-1} = \overline{\lambda}_{p-1,q-1}$; also the faces $\partial_p \overline{\lambda}_{p-1,q-1}$ and $\partial_{p+1} \overline{\lambda}_{p,q-1}$ are respectively in $\overline{F}^1$ and $\overline{F}^2$.

Finally the completion of $\lambda_{p-1,q}$ is $\lambda_{p,q}$, respectively absent in F^3 and in the list to be constructed.

Putting together all these facts leads to the conclusion: If F^1, F^2 and F^3 are respective filling sequences for $(H\Delta^{(*,*)} - \partial\Delta^{(*,*)})$, with $(*, *) = (p-1, q-1)$, $(p, q-1)$ and $(p-1, q)$ then the following list *is a filling sequence proving the desired collapse property* for the indices (p, q):

$$\overline{F}^1 \;||\; \overline{F}^2 \;||\; (\overline{\lambda}_{p-1,q-1}, \overline{\lambda}_{p-1,q}) \;||\; \overline{F}^3 \tag{24}$$

where '||' is the list concatenation. Fortunately, the last simplex $\lambda_{p,q} = \overline{\lambda}_{p-1,q}$ is the only interior simplex of $\Delta^{p,q}$ missing in this list. ♣

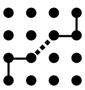
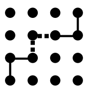
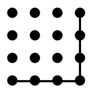
In [15] one can find some examples of the results of the algorithm for the small dimensions $(p, q) \leq (2, 2)$, with indices reversed.

8 If homology is enough.

The proposed proof of Theorem 23 is elementary but a little technical. If you are only interested by the *homological* version of the same result, precisely if you intend to prove the pair $(H\Delta^{p,q}, \partial\Delta^{p,q})$ does not have any homology, it is sufficient to prove the *difference* chain complex $C_*(H\Delta^{p,q})/C_*(\partial\Delta^{p,q})$ is acyclic. An opportunity to understand how a vector field can be sometimes the right tool, the one used here being close to the vector field which will be used for the EZ theorem.

Definition 24 — Let σ be an *interior* simplex of $\Delta^{p,q}$, represented by an s-path denoted by π. Then the *status* of σ (or π), *source* or *target* or *critical*, is defined as follows. You run the examined path π *from* (p, q) *backward* to $(0, 0)$ in the lattice $\underline{p} \times \underline{q}$ and you are interested by the *first* "event":

1. Either you run a diagonal elementary step $\swarrow$, in which case the path π is a *source s-path*;
2. Or you pass a bend  (not a bend ) in which case the path π is a *target s-path*.
3. Otherwise it is a *critical s-path* and only one *interior* s-path has this status, it is the s-path corresponding to the *last simplex* of $\Delta^{p,q}$. ♣

$\pi_1 =$  $\in S$ $\qquad \pi_2 =$  $\in T$ $\qquad \pi_3 =$  $\in C$ (25)

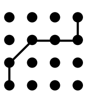
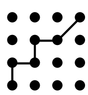
The figure above displays one example in either case when $p = q = 3$; the set S (resp. T, C) is the set of the *source* (resp. *target*, *critical*) cells. The deciding "event" is signalled by dotted lines. Observe $\partial_3\pi_2 = \pi_1$. More generally it is clear the operator assigning to every path of T the face in S corresponding to the bend  is a bijection organizing all these paths by pairs defining a discrete vector field, a good candidate to construct an interesting reduction: the unique path without any event fortunately is the last simplex!

Proposition 25 — *The relative chain complex* $C_*(H\Delta^{p,q}, \partial\Delta^{p,q})$ *admits a reduction to the null complex.*

♣ This relative chain complex is generated by all the interior simplices σ of $\Delta^{p,q}$ except the last one. Representing such a simplex σ by the corresponding s-path π allows us to divide all these simplices into two disjoint sets S (source) and T (target).

There remains to prove this vector field is admissible. It is a consequence of the organization of this vector field as a filling sequence given in Section 7, but it is possible to prove it directly and simply.

The example of the vector (π_1, π_2) above is enough to understand. We have to consider the faces of π_2 which are *sources*, in other words which are in S, therefore in particular *interior*, and *different* from π_1. In general at most two faces satisfy these requirements, here these faces $\pi_4 = \partial_2\pi_2$ and $\pi_5 = \partial_5\pi_2$:

$\pi_4 = \partial_2\pi_2 =$  $\in S$ $\qquad \pi_5 = \partial_5\pi_2 =$  $\in S$ (26)

This gives a Lyapunov function, see Definition 11. If $\pi \in S$, decide $L(\pi)$ is the number of points of the (p, q)-lattice strictly *above* the path. Observe $L(\pi_1) = 5$ while $L(\pi_3) = L(\pi_4) = 4$. A reader having reached this point of the text will probably prefer to play in ending the proof by himself. ♣

9 The Eilenberg-Zilber reduction.

9.1 The (p,q) Eilenberg-Zilber reduction.

Proposition 25 can be arranged to produce a sort of top-dimensional Eilenberg-Zilber reduction for the prism $\Delta^{p,q}$.

Proposition 26 — *The discrete vector field used in Proposition 25 induces a reduction $C_*(\Delta^{p,q}) \Rrightarrow C^c_*(\Delta^{p,q})$ where in particular the chain group $C_{p+q}(\Delta^{p,q})$ of rank $\binom{p+q}{p\,,\,q}$ is replaced by a critical chain group with a* unique *generator, the so-called last simplex $\lambda_{p,q}$.*

♣ This vector field makes sense as well in this context as in Proposition 25 and the admissibility property remains valid. A reduction is therefore generated, where in dimension $(p+q)$ the only critical simplex is the last simplex $\lambda_{p,q}$. ♣

9.2 Products of simplicial sets.

Let us recall the product $X \times Y$ of two simplicial sets X and Y is very simply defined. These simplicial sets X and Y are nothing but contravariant functors $\underline{\Delta} \to \underline{\text{Sets}}$, and the simplicial set $X \times Y$ is the *product functor.* In particular $(X \times Y)_p = X_p \times Y_p$ and if $\alpha : \underline{p} \to \underline{q}$ is a $\underline{\Delta}$-morphism, then $\alpha^*_{X\times Y} = \alpha^*_X \times \alpha^*_Y : (X \times Y)_q \to (X \times Y)_p$. It is not obvious when seeing this definition for the first time this actually corresponds to the standard notion of topological product but, except in esoteric cases when simplex sets are not countable, the topological realization of the product is homeomorphic to the product of realizations. In these exceptional cases, the result remains true under the condition of working in the category of compactly generated spaces [24].

The Eilenberg-Zilber lemma gives for every simplex a canonical expression from a unique non-degenerate simplex.

Theorem 27 (Eilenberg-Zilber lemma) — *Let σ be a p-simplex of a simplicial set X. Then there exists a unique triple (q, η, τ), the* Eilenberg triple *of σ, satisfying:*

1. $0 \leq q \leq p$.
2. $\eta : \underline{p} \to \underline{q}$ *is a surjective $\underline{\Delta}$-morphism.*
3. $\tau \in X_q$ *is non-degenerate and* $\eta^*\tau = \sigma$.

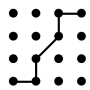
♣ [6, (8.3)] ♣

A p-simplex of the product $X \times Y$ is therefore a *pair* $\rho = (\sigma, \tau)$ of p-simplices of X and Y, and it is important to understand when this simplex is degenerate or not. Taking account of the Eilenberg-Zilber

Lemma 27, we prefer to express both components of this pair as the degeneracy of some non-degenerate simplex, which produces the expression $(\eta_{i_{s-1}} \cdots \eta_{i_0}\sigma', \eta_{j_{t-1}} \cdots \eta_{j_0}\tau')$ for our p-simplex ρ, where σ' (resp. τ') is a non-degenerate $(p-s)$-simplex of X (resp. $(p-t)$-simplex of Y); also the sequences i_* and j_* must be *strictly increasing*[3] with respect to their indices. Then the algebra of the elementary degeneracies η_i shows the simplex $\rho = (\sigma, \tau)$ is non-degenerate if and only if the intersection $\{i_{s-1}, \ldots, i_0\} \cap \{j_{t-1}, \ldots, j_0\}$ is empty.

The next definition is a division of all the non-degenerate simplices of the product $Z = X \times Y$ into three parts: the target simplices Z^t_*, the source simplices Z^s_* and the critical simplices Z^c_*. This division corresponds to a discrete vector field V, the natural extension to the whole product $Z = X \times Y$ of the vector field constructed in Sections 7 and 8 for the top bidimension (p, q) of the prism $\Delta^{p,q}$.

We must translate the definition of the vector field V in Section 8 into the language of non-degenerate product simplices expressed as pairs of possible degeneracies. To prepare the reader at this translation, let us explain the recipe which translates an s-path into such a pair. Let us consider this s-path:



(27)

This s-path represents an interior 5-simplex of $\Delta^{3,3}$ to be expressed in terms of the maximal simplices $\sigma, \tau \in \Delta^3_3$. Run this s-path from $(0,0)$ to $(3,3)$; every vertical elementary step, for example from $(1,0)$ to $(1,1)$ produces a degeneracy in the first factor, the index being the time when this vertical step is started, here 1. Another vertical step starts from $(2,2)$ at time 3, so that the first factor will be $\eta_3\eta_1\sigma$. In the same way, examining the horizontal steps produces the second factor $\eta_4\eta_0\tau$. Finally our s-path codes the simplex $(\eta_3\eta_1\sigma, \eta_4\eta_0\tau)$. The index 2 is missing in the degeneracies, meaning that at time 2 the corresponding step is diagonal: the lists of degeneracy indices are directly connected to the structure of the corresponding s-path. We will say the *degeneracy configuration* of this simplex is $((3,1),(4,0))$; a degeneracy configuration is a pair of disjoint decreasing integer lists.

Conversely, reading the indices of the degeneracy operators in the canonical writing $\rho = (\eta_{i_{s-1}} \cdots \eta_{i_0}\sigma, \eta_{j_{t-1}} \cdots \eta_{j_0}\tau)$ of a simplex ρ of $X \times Y$ unambiguously describes the corresponding s-path.

This process settles a canonical bijection between $S_{p,q}$ and $D_{p,q}$ if:

1. The set $S_{p,q}$ is the collection of all the *interior* s-paths running from $(0,0)$ to (p,q) in the (p,q)-lattice.
2. The set $D_{p,q}$ is the collection of all the configurations of degeneracy operators which can be used when writing a non-degenerate simplex in its canonical form $\rho = (\eta_{i_{s-1}} \cdots \eta_{i_0}\sigma, \eta_{j_{t-1}} \cdots \eta_{j_0}\tau)$, when $\sigma \in X_p$ and $\tau \in Y_q$. A configuration is a pair of integer lists $((i_{s-1}, \ldots, i_0), (j_{t-1}, \ldots, j_0))$ satisfying the various coherence conditions explained before: $p + s = q + t$, every component i_- and j_-

[3]Taking account of the reverse numbering of the indice!

is in $[0 \ldots (p+q-1)]$, both lists are disjoint, and their elements are increasing with respect to their respective indices.

9.3 The Eilenberg-Zilber vector field.

Definition 28 — Let $\rho = (\eta_{i_{s-1}} \cdots \eta_{i_0}\sigma, \eta_{j_{t-1}} \cdots \eta_{j_0}\tau)$ be a non-degenerate p-simplex of $X \times Y$ written in the canonical form. The degeneracy configuration $((i_{s-1}, \ldots, i_0), (j_{t-1}, \ldots, j_0))$ is a well-defined element of $D_{p-s,p-t}$ which in turn defines an s-path $\pi(\rho) \in S_{p-s,p-t}$. Then ρ is a *target* (resp. *source*, *critical*) simplex if and only if the s-path $\pi(\rho)$ has the corresponding property. ♣

Definition 29 — If $\rho = (\eta_{i_{s-1}} \cdots \eta_{i_0}\sigma, \eta_{j_{t-1}} \cdots \eta_{j_0}\tau)$ is a p-simplex of $X \times Y$ as in the previous definition, the pair $(p-s, p-t)$ is called the *bidimension of* ρ. It is the bidimension of the smallest prism of $X \times Y$ containing this simplex. ♣

For example the diagonal of the square $\Delta^{1,1}$ has the bidimension $(1, 1)$. The sum of the components of the bidimension can be bigger than the dimension.

Definition 30 — Let $X \times Y$ be the product of two simplicial sets. The division of the non-degenerate simplices of $X \times Y$ according to Definition 28 into *target* simplices, *source* simplices and *critical* simplices, combined with the pairing described in the proof of Proposition 25, defines the *Eilenberg-Zilber vector field* $V_{X\times Y}$ of $X \times Y$. ♣

Theorem 31 (Eilenberg-Zilber Theorem) — *Let $X \times Y$ be the product of two simplicial sets. The Eilenberg-Zilber vector field $V_{X\times Y}$ induces the Eilenberg-Zilber reduction:*

$$EZ : C_*(X \times Y) \Rrightarrow C_*(X) \otimes C_*(Y). \tag{28}$$

The reader may wonder why all these technicalities to reprove a well-known sixty-five years old theorem. Two totally different reasons.

On the one hand, the Eilenberg-Zilber reduction is time consuming when concretely programmed. In particular, profiler examinations of the effective homology programs show the terrible homotopy component $h : C_*(X\times Y) \to C_*(X \times Y)$ of the Eilenberg-Zilber reduction, rarely seriously considered[4], is the kernel program unit the most used in concrete computations. Our description of the Eilenberg-Zilber reduction makes the corresponding program unit simpler and more efficient.

On the other hand, maybe more important, the *same* (!) vector field can be used to process in the same way the *twisted* products, leading to totally

[4]With two notable exceptions. In the landmark papers by Eilenberg and... MacLane [3, 4], more useful than the standard reference [7], a nice recursive description of this homotopy operator is given. Forty years later (!), when a computer program was at last available to make experiments, Julio Rubio found a closed formula for this operator, proved by Frédéric Morace a little later [12]. We reprove this formula and others in the next section, by a totally elementary process depending only on our vector field, independent of Eilenberg-MacLane's and Shih's recursive formulas, reproved as well.

elementary *effective* versions of the Serre and Eilenberg-Moore spectral sequences. Tempting to hope a similar result for the Bousfield-Kan spectral sequence.

♣ The vector field $V_{X\times Y}$ has a layer for every bidimension (p,q). The admissibility proof given in Proposision 25 shows that every V-path starting from a source simplex of bidimension (p,q) goes after a finite number of steps to sub-layers. The Eilenberg-Zilber vector field is admissible.

If σ (resp. τ) is a non-degenerate p-simplex of X (resp. q-simplex of Y), we can denote by $\sigma\times\tau$ the corresponding (generalized) prism in $X\times Y$, made of all the simplices of bidimension (p,q) with respect to σ and τ. The collection of the *interior* simplices of this prism $\sigma\times\tau$ is nothing but an *exact* copy of the collection of the *interior* simplices of the standard prism $\Delta^{p,q}$. In particular only one *interior* critical cell in every prism. You are attending the birth of the tensor product $C_*(X)\otimes C_*(Y)$: exactly one generator $\sigma\otimes\tau$ for every prism $\sigma\times\tau$, namely the last simplex of this prism: $\sigma\otimes\tau$ "=" $\lambda_{p,q}(\sigma\times\tau)$.

There remains to prove the small chain complex so obtained is not only the right graded module $C_*(X)\otimes C_*(Y)$, but is endowed by the reduction process with the right differential. This is a corollary of the next section, devoted to a detailed study of the Eilenberg-Zilber vector field. ♣

10 Vector Fields and the Homomological Perturbation Theorem.

This section is devoted to necessary explanations about the proof of Forman's theorems based upon the Homological Perturbation Theorem (HPT). First we recall the statement of the last one and a few details about the resulting formulae.

The data of the HPT are as follows. An *initial* (homological) reduction is given:

$$\rho = (f,g,h) = \boxed{h \circlearrowright (\widehat{C}_*,\widehat{d}) \underset{f}{\overset{g}{\leftrightarrows}} (C_*,d)} \tag{29}$$

A "big" (resp. "small") chain complex $\widehat{C}_*$ (resp. C_*) is given. Both chain complexes are connected by a *reduction* $\rho=(f,g,h)$. This means f and g are chain complex morphisms, the composition fg is the identity and the composition gf is homotopic to the identity via the homotopy operator h: $\mathrm{id}_{\widehat{C}_*} - gf = \widehat{d}h + h\widehat{d}$ if $\widehat{d}$ is the differential of $\widehat{C}_*$.

Next, we introduce a *perturbation* $\widehat{\delta}$ of $\widehat{d}$, that is, we replace the differential $\widehat{d}$ of $\widehat{C}_*$ by the new differential $\widehat{d}+\widehat{\delta}$, needing of course $(\widehat{d}+\widehat{\delta})^2=0$. In general, the reduction ρ is no longer valid, in particular f and g do not remain chain complex morphisms. Notice the composition $\widehat{\delta}h$ has degree 0.

Theorem 32 (HPT). — *If the composition $\widehat{\delta}h$ is pointwise nilpotent, that is, for every $x\in\widehat{C}_*$ there exists an integer ν_x satisfying $(h\widehat{\delta})^{\nu_x}(x)=0$, then* *a new reduction ρ' is canonically defined:*

$$\rho' = (f', g', h') = \boxed{h' \circlearrowright (\widehat{C}_*, \widehat{d} + \widehat{\delta}) \underset{f'}{\overset{g'}{\rightleftarrows}} (C_*, d + \delta)} \tag{30}$$

In particular, it is necessary in general to perturb also the differential d of the small complex, becoming $d + \delta$.

The history of this result is detailed in [22], which also proposes which seems to be the right proof. In particular the following explicit formulae are valid for h', $g'$and f':

$$h' = \sum_{i=0}^{\infty} (-1)^i h(\widehat{\delta} h)^i = h - h\widehat{\delta} h + h\widehat{\delta} h\widehat{\delta} h - \cdots \tag{31}$$

$$g' = \big(\sum_{i=0}^{\infty} (-1)^i (h\widehat{\delta})^i\big) g = g - h\widehat{\delta} g + h\widehat{\delta} h\widehat{\delta} g - \cdots \tag{32}$$

$$f' = f\big(\sum_{i=0}^{\infty} (-1)^i (\widehat{\delta} h)^i\big) = f - f\widehat{\delta} h + f\widehat{\delta} h\widehat{\delta} h - \cdots \tag{33}$$

$$\delta = f\widehat{\delta} g - f\widehat{\delta} h\widehat{\delta} g + f\widehat{\delta} h\widehat{\delta} h\widehat{\delta} g - \cdots \tag{34}$$

A lovely application of the HPT is a new orgnization of the proof of Forman's theorems.

Let (C_*, d, β_*) be an algebraic cellular complex, provided with a discrete vector field $V = \{(\sigma_i, \tau_i)\}_i$. The σ_i's (resp. τ_i)'s are the *source* (resp. *target*) cells. The remaining cells, neither source nor target, are the *critical* cells. Let K_* be the graded $\mathbb{Z}$-module generated by the critical cells, and β_*^K the corresponding distinguished basis, made of the critical cells. Forman's theorems claim, *if the vector field V is admissible*, the existence of a canonical reduction:

$$\rho = (f, g, h) = \boxed{h \circlearrowright (C_*, d, \beta_*) \underset{f}{\overset{g}{\rightleftarrows}} (K_*, d_K, \beta_*^K)} \tag{35}$$

for an appropriate differential d_k on K_*. Let us sketch the simple proof of this reduction based on the HPT.

Definition 33 — Let $V = \{(\sigma_i, \tau_i)\}_i$ be a vector field on the cellular chain complex C_*. Then the *differential* $d_V : C_* \to C_{*-1}$ (resp. the *codifferential* $d'_V : C_* \to C_{*+1}$) is defined as follows: if τ_i is the target cell of the vector (σ_i, τ_i), then $d_V \tau_i = \varepsilon(\sigma_i, \tau_i)\sigma_i$ (resp. $d'_V \tau_i = 0$); if σ_i is the source cell of the vector (σ_i, τ_i), then $d'_V \sigma_i = \varepsilon(\sigma_i, \tau_i)\tau_i$ (resp. $d_V \tau_i = 0$); finally, if χ is a critical cell, then $d_V \chi = d'_V \chi = 0$. ♣

The number $\varepsilon(\sigma_i, \tau_i) = \pm 1$ is the *incidence number* between σ_i and τ_i, it is the coefficient of σ_i in the boundary $d\tau_i$. The properties of the vector field V ensure d_V (resp. d'_V) really is a differential (resp. codifferential).

The *graded module* (not the chain complex) C_* is decomposed as a direct sum $C_* = T_* \oplus S_* \oplus K_*$ of three graded modules respectively generated by the target cells, the source cells and the critical cells.

Our ACC (C_*, d, β_*) is provided with an initial differential d. The vector field V allows us to temporarily replace the differential d by d_V and to start with (C_*, d_V, β_*). For the last one, a reduction is easily defined:

$$\rho = (f, g, h) = \boxed{h \circlearrowright (C_*, d_V) \underset{f}{\overset{g}{\rightleftarrows}} (K_*, 0)} \tag{36}$$

with $f\sigma = \sigma$ if σ is critical, 0 otherwise; $g\sigma = \sigma$, necessarily critical, and finally $h = d'_V$. The map f is the canonical projection pr_3 of the direct sum over the third component K_*, while g is the canonical injection of K_* into C_*. Checking the various properties of a reduction is direct. But it is not the right differential of C_*. This leads to consider the *perturbation* $\widehat{\delta} = d - d_V$. We could apply the HPT if the nilpotency condition about $\widehat{\delta}h$ is satisfied. It happens this is satisfied if and only if the vector field V is *admissible*, it is in fact the reason of the definition of admissibility. If so, the HPT can be applied, producing at once Forman's theorems, with explicit convenient formulas for the Forman reduction. Details in [22].

The formula 31 implies a convenient recursive formula for the new homotopy h':

$$\begin{aligned} h' &= h - h\widehat{\delta}h + h\widehat{\delta}h\widehat{\delta}h - \cdots \\ &= h - (h - h\widehat{\delta}h + h\widehat{\delta}h\widehat{\delta}h - \cdots)\widehat{\delta}h \\ &= h - h'\widehat{\delta}h \end{aligned} \tag{37}$$

If $x \in \widehat{C}_*$, then $(\widehat{\delta}h)^\nu(x) = 0$ for some integer ν, and $\widehat{\delta}h(x)$ has a nilpotency degree one less than x. The recursiveness starts when $\widehat{\delta}h(x) = 0$. In particular $h'(x) = 0$ when $h(x) = 0$.

We translate these results in the framework of our vector field V applied to the ACC C_*. Then h, the initial homotopy, becomes d'_V, the perturbation $\widehat{\delta}$ becomes $\widehat{\delta} = d - d_V$, and the final homotopy h' becomes the Forman homotopy h. We obtain:

$$\begin{aligned} h(x) &= d'_V(x) - h(d - d_V)d'_V(x) \\ &= d'_V(x) - h(dd'_V(x) - d_V d'_V(x)) \end{aligned} \tag{38}$$

If x is a critical cell χ or a target cell τ, then $h(x) = 0$. If x is a source cell σ, then $d_V d'_V(\sigma) = \sigma$, giving in this case the key formula:

$$h(\sigma) = d'_V(\sigma) - h(dd'_V(\sigma) - \sigma) \tag{39}$$

easy to understand as a simple recipe: the homotopy of the source cell σ is the corresponding target cell τ, corrected by the homotopy applied to all the source cells of $d\tau$ except σ itself. The game consists simply in considering all the paths starting from σ, possible only if there are a finite number of such paths, all of finite length.

Analogous calculations using the formulae 32, 33 and 34 give the following expressions for the f, g and d_K components of the Forman reduction:

$$g = \mathrm{id} - hd \tag{40}$$

$$f = \mathrm{pr}_3(\mathrm{id} - dh) \tag{41}$$

$$d_K = \mathrm{pr}_3(d - dhd) \tag{42}$$

You must in particular use that $\widehat{\delta} = d - d_V$ and that d_V goes from T_* to S_*

We have now the good tool to study the naturality of Forman's theorems in general, the naturality of "our" EZ theorem in particular.

11 Naturality of the Eilenberg-Zilber reduction.

We consider here four simplicial sets X, X', Y and Y' and two simplicial morphisms $\varphi : X \to Y$ and $\varphi' : X' \to Y'$. These morphisms induce a simplicial morphism $\varphi \times \varphi' : X \times X' \to Y \times Y'$. Also the products $X \times X'$ and $Y \times Y'$ carry their respective Eilenberg-Zilber vector fields V and W.

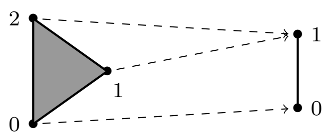
Theorem 34 — *With these data, the morphisms φ and φ' induce a natural morphism between both Eilenberg-Zilber reductions:*

$$\varphi \times \varphi' : [\rho = (f, g, h) : C_*(X \times X') \Rrightarrow C_*(X) \otimes C_*(X')] \longrightarrow \\ [\rho' = (f', g', h') : C_*(Y \times Y') \Rrightarrow C_*(Y) \otimes C_*(Y')] \qquad (43)$$

Let us recall all the chain complexes are normalized. At this time of the process, we do not have much information for the small chain complexes: we know the underlying graded modules are (isomorphic to) those of $C_*(X) \otimes C_*(Y)$ and $C_*(X') \otimes C_*(Y')$, but we do not yet know their differentials.

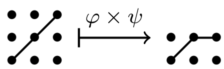
It is... natural to ask for such a result, but another goal is looked for. As it is common in a simplicial environment, once such a naturality result is available, it is often enough to prove some desired result in the particular case of an appropriate *model*, maybe a prism $\Delta^{p,q}$, and then to use an obvious simplicial morphism to transfer this result to an arbitrary product and obtain the general result. In fact, this method will also be an essential ingredient of the proof.

The statement of Theorem 34 is... natural, but the proof is more difficult than we could expect. The morphism $\varphi \times \psi$ can be *not at all compatible* with the respective Eilenberg-Zilber vector fields of $X \times Y$ and $X' \times Y'$, which generates essential obstacles. Consider for example the morphisms $\varphi = \mathrm{id} : \Delta^2 \to \Delta^2$ and $\psi : \Delta^2 \to \Delta^1$ defined by $\psi(012) = (011)$; the map ψ is nothing but the map $\psi = \eta_1 : \Delta^2 \to \Delta^1$ canonically associated to the $\underline{\Delta}$-map $\eta_1 : [0 \dots 2] \to [0 \dots 1]$, see Figure (44).



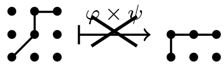
(44)

Then the simplicial morphism $\varphi \times \psi$ sends the "diagonal" 2-simplex σ of $\Delta^{2,2} = \Delta^2 \times \Delta^2$ to some 2-simplex of $\Delta^{2,1}$ as displayed in the next figure, where a simplex is represented by an s-path.



(45)

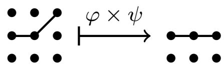
Both simplices are *source* cells, but for “reasons” which do no match. The lefthand cell is source because of the diagonal component $(1,1)-(2,2)$ of the s-path, which component is sent by $\varphi \times \psi$ to the horizontal component $(1,1)-(2,1)$. While the righthand cell is source because of the diagonal component $(0,0)-(1,1)$. The corresponding target cells are below:



(46)

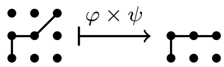
and they do not match by $\varphi \times \psi$: the image by $\varphi \times \psi$ of the lefthand target cell is in fact degenerate.

It can also happen the image of a source cell is critical:



(47)

or target:



(48)

In other words the tempting relation $(\varphi \times \psi)V(\sigma) = V(\varphi \times \psi)(\sigma)$ in general *is false*, or even does not make sense. It is then clear the “natural” Theorem 34 requires a “real” proof. It depends on a very general result, the statement of which is rather amazing.

Definition 35 — A *cellular morphism* $\varphi : A_* \to B_*$ between two cellular (chain) complexes A_* and B_* is a *chain complex morphism* which maps every cell of A_* to *zero* or to a *cell* of B_*. ♣

Definition 36 — A *vectorious* cellular complex is a pair (A_*, V) where A_* is a cellular complex A_* provided with an *admissible* discrete vector field V. ♣

Definition 37 — Let $\varphi : (A_*, V) \to (B_*, W)$ be a cellular morphism between the vectorious cellular complexes (A_*, V) and (B_*, W). The morphism φ is an *admissible morphism* if the following conditions are satisfied:

- The morphism φ maps every critical cell of V to 0 or to a critical cell of W.
- The morphism φ maps every target cell of V to 0 or to a target cell of W. ♣

In particular, no condition is required for a source cell of V!

Theorem 38 — *Let* $\varphi : (A_*, V) \to (B_*, W)$ *be an* admissible *morphism between the vectorious cellular complexes* $(A_*, V) \to (B_*, W)$. *Then the morphism* φ *induces a natural morphism between the reductions* $\rho : A_* \Rrightarrow A^c_*$ *and* $\rho' : B_* \Rrightarrow B^c_*$ *where* A^c_* *and* B^c_* *are the corresponding critical complexes.*

♣[38] Let $h : A_* \to A_*$ and $h' : B_* \to B_*$ be the respective homotopies of ρ and ρ'. We must in particular prove $\varphi h = h'\varphi$. We use the recursive construction of h given by the formula (39).

The relation $\varphi h(x) = h'\varphi(x)$ is obvious if x is a critical or target cell: the morphism φ is admissible and maps such a cell to a cell of the same sort, or to 0; in any case, both members are null. There remains to prove $\varphi h = h'\varphi$ for a source cell σ.

The formula (39) for a source cell:

$$h(\sigma) = d'_V(\sigma) - h(dd'_V(\sigma) - \sigma) \tag{49}$$

allows us to recursively assume $\varphi h(dd'_V(\sigma) - \sigma) = h'\varphi(dd'_V(\sigma) - \sigma)$: every source cell has a "level" $\lambda(\sigma)$, namely the maximal length of a path starting from σ, but every term of $(dd'_V(\sigma) - \sigma)$ is made of critical cells, or target cells, or source cells of level $< \lambda(\sigma)$, at least one of them being $\lambda(\sigma) - 1$.

Now we may compute:

$$\begin{aligned} \varphi h(\sigma) &= \varphi d'_V(\sigma) - \varphi h(dd'_V(\sigma) - \sigma) \\ &= \varphi d'_V(\sigma) - h'\varphi(dd'_V(\sigma) - \sigma) \\ &= \varphi d'_V(\sigma) - h'd'\varphi d'_V(\sigma) + h'\varphi(\sigma) \\ &= (1 - h'd')\varphi d'_V(\sigma) + h'\varphi(\sigma) \\ &= d'h'\varphi d'_V(\sigma) + h'\varphi(\sigma) \\ &= h'\varphi(\sigma) \end{aligned} \tag{50}$$

for $\varphi d'_V(\sigma)$ is null or is a target cell.

We have the formula $g = \mathrm{id} - hd$ for the injection of the critical complex A^c_* in the whole complex A_* and the same for B^c_* and B_*. Therefore: $\varphi g = \varphi(\mathrm{id} - hd) = (\mathrm{id} - h'd')\varphi = g'\varphi$ and the morphism φ is also compatible with the injections g and g' of ρ and ρ'.

There remains to obtain $\varphi f = f'\varphi$ for the respective projections of A_* and B_* over the critical complexes A^c_* and B^c_*. The map g' is injective and the desired relation is equivalent to $g'\varphi f (= \varphi g f) \stackrel{?}{=} g'f'\varphi$. But $gf = 1 - dh - hd$, the same for $g'f'$ and the relation is a consequence of $\varphi d = d'\varphi$ and $\varphi h = h'\varphi$. ♣[38]

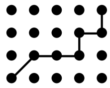
♣[34] Let us prove now Theorem 34. We may work simplex by simplex, more precisely prism by prism; if $\sigma \in X$ (resp. $\sigma' \in X'$) is a non-degenerate simplex, then the prism $\sigma \times \sigma' \in X \times X'$ is mapped inside a prism $\tau \times \tau' \in Y \times Y'$ depending on the image simplices $\varphi(\sigma)$ and $\varphi'(\sigma')$.

A map $\varphi : \sigma \to \tau$ between simplices is a composition of face and degeneracy operators. So that it is enough to consider the case with only a face operator, or only a degeneracy operator. A face operator is injective and the result is then obvious.

There remains to consider for example the case $X = Y = \Delta^p$, $X' = \Delta^q$, $Y' = \Delta^{q-1}$, $\varphi = \mathrm{id}_{\Delta^p}$ and φ' is the degeneracy $\eta_i : \Delta^q \to \Delta^{q-1}$ for some $0 \leq i < q$ which maps the vertex $\#j$ to itself if $j \leq i$ and to the vertex $\#(j-1)$ if $j > i$. The hoped-for result is then *not at all obvious*.

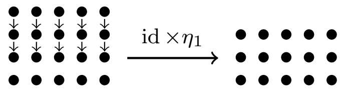
The technology of s-paths explained at Section 4.2 leads to understand a simplex of $\Delta^p \times \Delta^q$ as an oriented path in the lattice $[0 \ldots p] \times [0 \ldots q]$.

For example, the s-path:



(51)

represents the 6-simplex spanned by the vertices $(0,0)$, $(1,1)$, $(2,1)$, $(3,1)$, $(3,2)$, $(4,2)$ and $(4,3)$ in the prism $\Delta^4 \times \Delta^3$. We refer the reader to Definition 24 for the process dividing these simplices in source, target and critical simplices. A map $\mathrm{id} \times \eta_i$ is the identity for the vertices (j,k) satisfying $k \leq i$ and maps (j,k) to $(j,k-1)$ for $k > i$. For example the map $\mathrm{id} \times \eta_1 : \Delta^4 \times \Delta^3 \to \Delta^4 \times \Delta^2$ could be represented as follows:



(52)

We let the reader check himself the image of a critical cell is critical or degenerate, therefore in this case null in the normalized chain complex. The same for a target cell: essentially a morphism $\mathrm{id} \times \eta_i$ cannot destroy the bend characterizing a target cell without mapping it to a degenerate cell, cancelled in the normalized cell complex. And fortunately, thanks to the strange definition of admissible morphism, Definition 37, we do not have to study the case of the source cells, totally anarchic, see the comments after the statement of Theorem 34. ♣[34]

12 Two Eilenberg-Zilber reductions.

12.1 Introduction.

Our EZ environment has now two methodologies: the classical one and the present one. The classical one is most often based on *acyclic models*, the AW and EML formulas being rarely given with proofs; a noticeable exception is the landmark paper [4] with a level of details that can be compared with the next sections of the present paper.

Our own methodology is based on the discrete vector fields, designed by Robin Forman about 45 years after [4]. We will see that proving both methods produce the same results, in particular the same formulas, is quite technical. A tired reader may ask why so much work to produce the same result.

On the one hand, once our EZ DVF is defined, this DVF is too natural, too amusing also, to allow us not to study the exact connection with [7, 4]. There is another argument. The "distance" between both methods is a striking measure of the difference between the RM formula, really complex, and the EZ DVF: our vector field is nothing but a process which "magically" avoids the countless degenerate terms of the RM formula, as soon as a factor of the studied product has degenerate faces. The reader will see the RM formula has rich cancellation properties, so far never considered, cancellation properties that are automatically and efficiently processed by our vector field. This explains the enormous difference of efficiency between the previous version of our program of constructive algebraic topology and the last one. See [14] for a report in a computer science meeting.

Let X and Y be two simplicial sets. We have now in our toolbox two reductions $EZ_1 = (f_1, g_1, h_1) : C_*(X \times Y) \Rrightarrow C_*(X) \otimes C_*(Y)$ and $EZ_2 = (f_2, g_2, h_2) : C_*(X \times Y) \Rrightarrow C_*(X) \otimes C_*(Y)$. The first one was determined by Eilenberg and MacLane in [3, 4], the second one is obtained from our "Eilenberg-Zilber" vector field, see Definition 30 and Theorem 31. In fact both reductions are the same, it is the goal of this section.

Both reductions have quite different definitions and a real task is in front of us. The structure of the proof is as follows:

1. We recall the standard formulas for the Eilenberg-MacLane reduction EZ_1, that is, the AW-formula for f_1 (Alexander-Whitney), the EML-formula for g_1 (Eilenberg-MacLane) and the RM-formula for h_1 (Rubio-Morace).
2. The homotopy operator h_2 defined by our vector field satisfies also the RM-formula, it is the key point. And the most difficult.
3. We *deduce* from this fact that $g_2 = g_1$ and $f_2 = f_1$: our vector field reduction also satisfies the traditional Alexander-Whitney and Eilenberg-MacLane formulas.
4. We give the appropriate formula for the composition $g_2 f_2 = g_1 f_1$.
5. We prove the RM-formula satisfies the recursive Eilenberg-MacLane definition of h_1.

The last point is redundant with respect to which is available elsewhere, but the vector field understanding of the Eilenberg-MacLane recursive formula becomes too simple to be omitted.

12.2 Alexander-Whitney, Eilenberg-MacLane and Rubio-Morace.

The three components of the reduction EZ_1 are:

$$f_1 = AW : C_*(X \times Y) \to C_*(X) \otimes C_*(Y) \quad (AW = \text{Alexander-Whitney})$$
$$g_1 = EML : C_*(X) \otimes C_*(Y) \to C_*(X \times Y) \quad (EML = \text{Eilenberg-MacLane})$$
$$h_1 = RM : C_*(X \times Y) \to C_*(X \times Y) \quad (RM = \text{Rubio-Morace})$$

The explicit formulas for these components are:

$$AW(x_p \times y_p) = \sum_{i=0}^{p} \partial_{p-i+1} \cdots \partial_p x_p \otimes \partial_0 \cdots \partial_{p-i-1} y_p \tag{53}$$

$$EML(x_p \otimes y_q) = \sum_{(\eta,\eta') \in Sh(p,q)} \varepsilon(\eta,\eta')\,(\eta' x_p \times \eta y_q) \tag{54}$$

$$RM(x_p \times y_p) = \sum_{0 \le r \le p-1, 0 \le s \le p-r-1, (\eta,\eta') \in Sh(s+1,r)} (-1)^{p-r-s}\,\varepsilon(\eta,\eta') \tag{55}$$
$$(\uparrow^{p-r-s}(\eta')\eta_{p-r-s-1}\partial_{p-r+1} \cdots \partial_p x_p \times \uparrow^{p-r-s}(\eta)\partial_{p-r-s} \cdots \partial_{p-r-1} y_p)$$

where x_p and y_p are respective p-simplices of X and Y, which can be degenerate, but their cartesian product $(x_p \times y_p)$ is not. We denote $(x_p \times y_p)$ the p-*simplex* of $X \times Y$ defined by its *projections* x_p and y_p, with the separator $\times$, more readable in our relatively complex formulas than the traditional comma in (x_p, y_p). So that $(x_p \times y_p)$ denotes here a *simplex*, not a prism.

The diagonal map $C_*(X) \to C_*(X \times X)$ composed with the AW formula defines in particular the standard coproduct $\Delta : C_*(X) \to C_*(X) \otimes C_*(X)$ of a simplicial chain complex.

Two symmetric formulas are possible for AW, and any affine combination of both is also possible. The paper [11] gives a further condition with respect to some order compatibility to obtain a unique choice for AW. The EML formula is unique [10].

It is clear four symmetric possible choices are possible for our Eilenberg-Zilber vector field. Examine the process described in Definition 24 to decide whether an s-path is source, target or critical. Instead of running the s-path backward from (p, q) to $(0, 0)$ you could run this path forward from $(0, 0)$ to (p, q). Instead of replacing the diagonal $\swarrow$ by a bend to define the target associated to some source s-path, you could once for all prefer to replace the diagonal by a bend, modifying accordingly the criterion for a target s-path. Finally four possible natural vector fields. Giving four different symmetric RM-formulas, two different symmetric AW-formulas and only one EML-formula. Our choice gives the most standard AW-, EML- and RM-formulas given above.

In the EML formula above, the set $Sh(p, q)$ is made of all the (p, q)-*shuffles* of $(0 \cdots (p+q-1))$, that is, all the partitions of these $p+q$ integers in two increasing sequences of length p and q. Every shuffle produces in turn a pair of multi-degeneracy operators denoted in the same way; for example the shuffle $((0\,3), (1\,2\,4))$ produces the pair $(\eta, \eta') = (\eta_3\eta_0, \eta_4\eta_2\eta_1)$: the factors are to be in the right degeneracy order. Such a pair (η, η') is so associated to a permutation, producing a signature $\varepsilon(\eta, \eta')$. For example, the shuffle $((0\,1\,5), (2\,3\,4))$ in the case $p = q = 3$ produces the term $-(\eta_4\eta_3\eta_2 x_3 \times \eta_5\eta_1\eta_0 y_3)$, for the permutation $(0\,1\,5\,2\;3\,4)$ is negative.

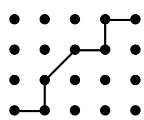
Julio Rubio, using numerical results computed by the EAT program [19], produced without proof in his thesis [17] the lovely formula RM (there called SHI) for the Eilenberg-Zilber homotopy operator. This formula was also called SHI in [12], this time with a proof (due to Frédéric Morace) based on the recursive formula given for Φ_n at [23, Page 25]. In fact this recursive formula is already at [4, Formula (2.13)].

The $\uparrow$-operator in the RM-formula shifts the indices of the multi-degeneracy operator, for example $\uparrow(\eta_3\eta_1) = \eta_4\eta_2$, $\uparrow^3(\eta_4\eta_2) = \eta_7\eta_5$.

This section is devoted to a careful analysis of the Eilenberg-Zilber vector field, leading to new proofs of all these formulas. Nothing more than a combinatorial game with the s-paths of prisms, that is, with the degeneracy operators. The key point is to obtain first the RM-formula, the others being in fact consequences.

12.3 Collisions between degeneracies.

A simplex in a prism is represented by an s-path such as this one:



(56)

This is the s-path representation of the 6-simplex spanning the vertices $(0 \times 0) - (1 \times 0) - (1 \times 1) - (2 \times 2) - (3 \times 2) - (3 \times 3) - (4 \times 3)$ of the (4×3)-prism $\Delta^4 \times \Delta^3$.

The last vertical or horizontal segments of an s-path will play an essential role. For example, for the above s-path, the fact the last two segments are a vertical one followed by a horizontal one implies the RM-value for this simplex is null. A sequence of lemmas are to be devoted to various situations.

Let us carefully examine the generic term of the RM-formula, where we forget the sign:

$$(\uparrow^{p-r-s}(\eta')\eta_{p-r-s-1}\partial_{p-r+1}\cdots\partial_p x_p \times \uparrow^{p-r-s}(\eta)\partial_{p-r-s}\cdots\partial_{p-r-1}y_p) \qquad (57)$$

Its dimension is $p+1$ and the configuration of the degeneracy operators will be essential. The pair (η, η') is a $(s+1, r)$-shuffle, invoking the indices $0\cdots(r+s)$, so that $\uparrow^{p-r-s}(\eta')$ *and* $\uparrow^{p-r-s}(\eta)$ *collectively* invoke the indices $(p-r-s)\cdots p$. Taking account also of the isolated operator $\eta_{p-r-s-1}$, finally all the indices of $(p-r-s-1)\cdots p$ are *explicitly* invoked in the formula.

But in general the initial factors x_p and y_p can also contain degeneracy operators, often generating "collisions" with the just considered explicit operators, then cancelling the corresponding term. For example if you consider a term $(\eta_6\eta_4 x', \eta_5\eta_3 y')$ of dimension 7, then if ever x' or y' contains an η_4 in his canonical expression, then this term is degenerate. If $x' = \eta_4 x''$, then $(\eta_6\eta_4 x', \eta_5\eta_3 y') = (\eta_6\eta_4\eta_4 x'', \eta_5\eta_3 y') = (\eta_6\eta_5\eta_4 x'', \eta_5\eta_3 y') = \eta_5(\eta_5\eta_4 x'', \eta_3 y')$ is degenerate; something analogous if $y' = \eta_4 y''$. This is due to the permutation rule $\eta_i\eta_j = \eta_{j+1}\eta_i$ if $j \geq i$ which tends to *increase* the indices when you sort the degeneracy operators. These examples allow us to state without any proof the next lemma.

Lemma 39 (Collision lemma) — *In an expression* $(\eta' x' \times \eta y')$ *of dimension* $p+1$ *where the multidegeneracies* η' *and* η *contain all the indices of* $(p-r-s-1)\cdots p$*, then, if* x' *or* y' *contains a degeneracy with an index in the same range, the term* $(\eta' x', \eta y')$ *is in fact degenerate.* ♣

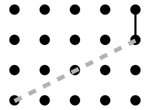
The minimal size of the range $(p-r-s-1)\cdots p$ is $(p-1)\cdots p$ for $r = s = 0$.

12.4 Null terms in the *RM*-formula.

A sequence of elementary lemmas playing with degeneracies will give *shorter* RM-formulas according to the nature of the simplex $(x_p \times y_p)$ considered. The general nature of these lemmas is roughly the following: if x_p and/or y_p contains degeneracy operators with high indices, then the corresponding terms of the RM-formula, because of the collision lemma, will be null *unless* the face operators previously annihilate these degeneracies.

Lemma 40 — *If* $x_p = \eta_{p-1}x'$*, then the terms* $r = 0$ *of the RM-formula are null.*

♣ This corresponds to this situation:

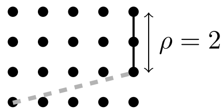


(58)

where the pale dashed line between $(0,0)$ and $(4,2)$ means we do not have any specific information about the s-path between the points $(0,0)$ and $(4,2)$.

Anyway, the relation $p-r-s-1 \leq p-1$ is satisfied in the generic term of the RM-formula. Because of the collision lemma, if the η_{p-1} of $x_p = \eta_{p-1}x'$ is not swallowed by a face operator, the term of the RM-formula will be null, which required $r \geq 1$. ♣

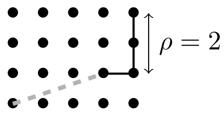
Lemma 41 — *If $x_p = \eta_{p-1}\eta_{p-2}\cdots\eta_{p-\rho}x'$, then the terms $r < \rho$ of the RM-formula are null.*



(59)

♣ Induction with respect to ρ, the previous lemma being the case $\rho = 1$. If the lemma is known for $\rho-1$, this implies a non-null term satisfies $r \geq \rho-1$; also the degeneracies $\eta_{p-1}\cdots\eta_{p-\rho+1}$ are annihilated by the face operators $\partial_{p-\rho+2}\cdots\partial_p$. Then the relation $p-r-s-1 \leq p-\rho$ is satisfied. To avoid a collision, it is necessary to annihilate also the $\eta_{p-\rho}$, which requires $r \geq \rho$. ♣

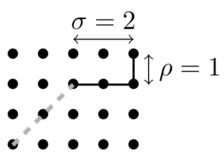
Lemma 42 — *If $x_p = \eta_{p-1}\eta_{p-2}\cdots\eta_{p-\rho}x'$ and $y_p = \eta_{p-\rho-1}y'$ then the terms $r \neq \rho$ <u>or</u> $s = 0$ of the RM-formula are null. This is valid even if $\rho = 0$.*



(60)

♣ We already know a non-null term satisfies $r \geq \rho$. So that $p-r-s-1 \leq p-\rho-1$ and the $\eta_{p-\rho-1}$ in the expression of y_p is to be annihilated to avoid a collision. This needs at least a face operator in the second component, that is, $s > 0$, <u>and</u> the last face index $(p-r-1)$ is $\geq p-\rho-1$, that is, $r \leq \rho$, so that finally $r = \rho$. ♣

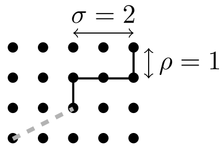
Lemma 43 — *If $x_p = \eta_{p-1}\eta_{p-2}\cdots\eta_{p-\rho}x'$ and $y_p = \eta_{p-\rho-1}\cdots\eta_{p-\rho-\sigma}y'$ <u>with $\sigma > 0$</u> then the terms $r \neq \rho$ <u>or</u> $s < \sigma$ of the RM-formula are null. This is valid even if $\rho = 0$.*



(61)

♣ Induction with respect to σ. ♣

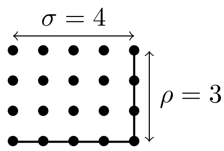
Lemma 44 — *If $x_p = \eta_{p-1}\eta_{p-2}\cdots\eta_{p-\rho}\eta_{p-\rho-\sigma-1}x'$ and $y_p = \eta_{p-\rho-1}\cdots\eta_{p-\rho-\sigma}y'$ <u>with $\sigma > 0$</u> then $RM(x_p, y_p) = 0$. This is valid even if $\rho = 0$.*



(62)

♣ A non-null term would satisfy $r = \rho$ and $s \geq \sigma$, so that the relation $p - r - s - 1 \leq p - \rho - \sigma - 1$ is satisfied. The $\eta_{p-\rho-\sigma-1}$ must therefore be annihilated to produce a non-null term, but nomore face operator is available in the corresponding component. Note $\sigma > 0$ is crucial while on the contrary $\rho = 0$ is possible. ♣

Lemma 45 — *If* $x_p = \eta_{p-1}\eta_{p-2}\cdots\eta_{p-\rho}x'$ *and* $y_p = \eta_{p-\rho-1}\cdots\eta_0 y'$*, then* $RM(x_p \times y_p) = 0$*. This is valid even if* $\rho = 0$ *or* $\sigma = 0$*.*



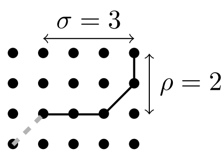
(63)

♣ A non-null term would satisfy $r \geq \rho$ and $s \geq \sigma$ but the required inequality $r + s \leq p - 1$ cannot be satisfied. ♣

Corollary 46 — *If* $(x_p \times y_p)$ *is a target or critical cell in the product* $X \times Y$*, then* $RM(x_p \times y_p) = 0$*.*

♣ Restatement of the last two lemmas. ♣

Lemma 47 — *If* $x_p = \eta_{p-1}\cdots\eta_{p-\rho+1}x'$ *and* $y_p = \eta_{p-\rho-1}\cdots\eta_{p-\rho-\sigma+1}y'$ *for* $\sigma \geq 2$*, then a non-null term in the RM-formula must satisfy* $r = \rho$ *and* $\sigma - 1 \leq s \leq p - \rho - 1$*, or* $r = \rho - 1$ *and* $0 \leq s \leq p - \rho$*.*



(64)

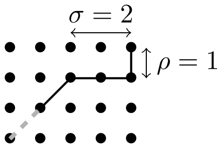
♣ If $r > \rho$, the first degeneracy $\eta_{p-\rho-1}$ of the second factor remains alive and becomes $\eta_{p-\rho-s-1}$; the condition $p - \rho - s - 1 < p - r - s - 1$ is required to avoid a collision, that is, $r < \rho$, contradiction. Now if $r = \rho$ and $s < \sigma - 1$, then a degeneracy remains in the second factor, the first one being again $\eta_{p-\rho-s-1}$, requiring also $r < \rho$ to obtain a non-null term. ♣

12.5 Examining a source cell.

The important part of a source cell is the last diagonal, certainly followed *first* by horizontal segments, *then* by vertical segments; but these vertical and/or horizontal parts can be missing. A generic expression for such a source cell is therefore:

$$(\eta_{p-1}\ldots\eta_{p-\rho}x' \times \eta_{p-\rho-1}\cdots\eta_{p-\rho-\sigma}y') \tag{65}$$

with the degeneracy operator $\eta_{p-\rho-\sigma-1}$ absent in x' and y', so that the segment of the s-path between times $p-\rho-\sigma-1$ and $p-\rho-\sigma$ is *diagonal*:



(66)

We noticed above the non-totally symmetric role of ρ and σ. This is the reason why two different cases are to be considered in our final expression for the RM-formula when evaluated on a source cell.

Proposition 48 — *Let* $(x_p \times y_p)$ *be a* source *p-simplex of a simplicial product* $X \times Y$*, detailed as above. Then there are two different situations. If* $\sigma = 0$*, then an RM-term is non null only if* $r \geq \rho$*. The RM-formula therefore is then:*

$$RM(x_p \times y_p) = \sum_{\rho \leq r \leq p-1, 0 \leq s \leq p-r-1, (\eta,\eta') \in Sh(s+1,r)} (-1)^{p-r-s}\, \varepsilon(\eta,\eta') \tag{67}$$
$$(\uparrow^{p-r-s}(\eta')\eta_{p-r-s-1}\partial_{p-r+1}\cdots\partial_p x_p \times \uparrow^{p-r-s}(\eta)\partial_{p-r-s}\cdots\partial_{p-r-1}y_p)$$

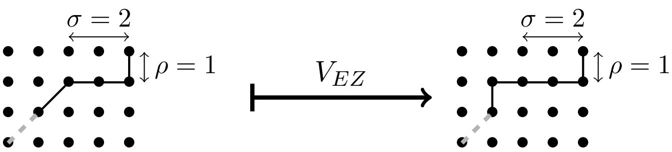
If $\sigma > 0$*, then to obtain a non-null term, the conditions* $r = \rho$ *and* $s \geq \sigma$ *are required The RM-formula therefore is then:*

$$RM(x_p \times y_p) = \sum_{\sigma \leq s \leq p-\rho-1, (\eta,\eta') \in Sh(s+1,\rho)} (-1)^{p-\rho-s}\, \varepsilon(\eta,\eta') \tag{68}$$
$$(\uparrow^{p-\rho-s}(\eta')\eta_{p-\rho-s-1}\partial_{p-\rho+1}\cdots\partial_p x_p, \uparrow^{p-\rho-s}(\eta)\partial_{p-\rho-s}\cdots\partial_{p-\rho-1}y_p)$$

Some of the preserved terms can be also null because we do not have any information for the faces of x_p and y_p, and also because of other collisions as it will be observed later. But at least one term is certainly non-null, we call it the *principal term*, the one corresponding to the parameters $r = \rho$, $s = \sigma$ and (η, η') the trivial shuffle $((0\cdots\sigma),((\sigma+1)\cdots(\rho+\sigma)))$; it is:

$$(-1)^{p-\rho-\sigma}(\eta_p\cdots\eta_{p-\rho+1}\eta_{p-\rho-\sigma-1}x_{p-\rho} \times \eta_{p-\rho}\cdots\eta_{p-\rho-\sigma}y_{p-\sigma}) \tag{69}$$

which is nothing but the *target cell* associated to our source cell in our Eilenberg-Zilber vector field V_{EZ}.



(70)

In other words, the "first non-null term" of the RM-formula is the corresponding target cell. Someone who knows only the RM-formula can *guess* our Eilenberg-Zilber vector field, in fact unique to be compatible with a homotopy operator, for it can be easily proved that the homotopy h operator of the reduction $\rho = (f, g, h)$ defined by a vector field V is enough to recover this vector field.

12.6 EZ-Homotopy = RM-formula

Theorem 49 — *Let $X \times Y$ be the product of two simplicial sets X and Y. The Eilenberg-Zilber vector field V_{EZ} defines a reduction (f_2, g_2, h_2) and in particular a homotopy operator h_2. Then $h_2 = RM = h_1$.*

Cf. Definition 28 for the Eilenberg-Zilber vector field V_{EZ}.

♣ The naturality property obtained in Section 11 allows us to consider only $X = Y = \Delta^p$ and to prove the RM-formula for the operator h for the diagonal simplex $(\delta_p \times \delta_p)$. But the necessary recursive process leads to consider more generally any non-degenerate simplex $(x_p \times y_p) \in \Delta^p \times \Delta^p$ where x_p and/or y_p can be degenerate.

If $(x_p \times y_p)$ is a target or critical cell, we say its *level* is 0. If it is a source cell, its *level* is the length of the longest V_{EZ}-path starting from this simplex. The proof is recursive with respect to the level of $(x_p \times y_p)$. Corollary 46 proves the result if the level is null.

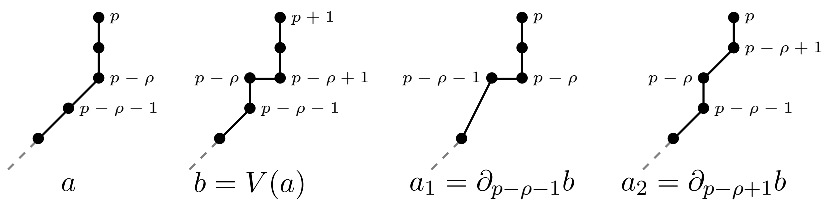
We assume now the result is known for a source cell of level $\ell - 1$ and we assume $(x_p \times y_p)$ is a source cell of level ℓ. The homotopy operator is defined by the recursive formula (39): $h(\sigma) = d'_V(\sigma) - h(dd'_V(\sigma) - \sigma)$ if σ is a source cell, if d'_V is the associated codifferential. The term $dd'_V(\sigma) - \sigma$ is made of cells of levels $<$ level(a), which justifies the recursive process.

In our Eilenberg-Zilber situation, in fact a maximum of two components of $dd'_V(\sigma) - \sigma$ are source cells, certainly with a smaller level. The game now is the following: we (recursively) assume $h(dd'_V(\sigma) - \sigma)$ can be computed by the RM-formula and we will prove $h(\sigma)$ can be computed by the same formula. Which can seem a priori a little strange, but it will be a consequence of the results detailed in Section 12.4.

12.6.1 The case $\sigma = 0$.

A source cell admits a canonical presentation as explained before Proposition 48, producing in particular two important indices ρ and σ. The situation is very different in the cases $\sigma = 0$ or $\sigma \neq 0$. We begin with the first one $\sigma = 0$.

The essential part of the corresponding s-paths are drawn below.



(71)

The s-path a represents a simplex denoted also by a finishing by a vertical segment of length ρ, preceded by a diagonal segment between times $p - \rho - 1$ and $p - \rho$. Its associated target cell b is obtained by replacing this diagonal segment by a vertical one followed by a horizontal one. All the faces of b are also target cells except the faces of index $p - \rho - 1$ and $p - \rho + 1$. So that the recursive formula for the homotopy operator is simply in this case: $h(a) = b + h(a_1) + h(a_2)$ if we forget the signs. The values of $h(a_1)$ and $h(a_2)$ are recursively given by the RM-formula: $h(a_1) = RM(a_1)$, $h(a_2) = RM(a_2)$ and we must prove $h(a) = RM(a)$.

We must appropriately express b, a_1, a_2, $h(a_1)$ and $h(a_2)$ and observe which is finally obtained is exactly which is expected for $h(a)$. We collect first the simplices:

$$
\begin{aligned}
a &= (\eta_{p-1} \cdots \eta_{p-\rho} x' \times y_p) && (72)\\
b &= (\eta_p \cdots \eta_{p-\rho+1} \eta_{p-\rho-1} x' \times \eta_{p-\rho} y) && (73)\\
a_1 &= (\eta_{p-1} \cdots \eta_{p-\rho} x' \times \eta_{p-\rho-1} \partial_{p-\rho-1} y_p) && (74)\\
a_2 &= (\eta_{p-1} \cdots \eta_{p-\rho+1} \eta_{p-\rho-1} x' \times y_p) && (75)
\end{aligned}
$$

The components x' and y_p can contain other degeneracies, but they do not concern the rest of the computation.

We systematically forget the signs, always easily checked correct. The hoped-for formula for $h(a)$ is:

$$h(a) = \sum (\uparrow^{p-r-s}(\eta')\eta_{p-r-s-1}\partial_{p-r+1} \cdots \partial_{p-\rho} x' \times \uparrow^{p-r-s}(\eta)\partial_{p-r-s} \cdots \partial_{p-r-1} y_p)$$

the parameters satisfying $\rho \leq r \leq p-1$, $0 \leq s \leq p-r-1$, $(\eta, \eta') \in Sh(s+1, r)$.

First the simplex b is the term in this formula corresponding to $r = \rho$, $s = 0$ and $(\eta, \eta') = ((0), (1 \cdots \rho))$.

The simplex a_1 has $\sigma > 0$, which implies the *known* formula for $h(a_1)$ is:

$$h(a_1) = \sum (\uparrow^{p-\rho-s}(\eta')\eta_{p-\rho-s-1} x' \times \uparrow^{p-\rho-s}(\eta)\partial_{p-\rho-s} \cdots \partial_{p-\rho-1} y_p)$$

because of a happy relation $\partial_{p-\rho-1}\eta_{p-\rho-1} = \text{id}$ in the second factor. The valid parameters are $1 \leq s \leq p-\rho-1$ and $(\eta, \eta') \in Sh(s+1, \rho)$. This implies $h(a_1)$ produces all the desired terms for the $h(a)$-formula satisfying $r = \rho$ and $s \geq 1$.

Processing $h(a_2)$ is more complex. The initial formula for $h(a_2)$ is:

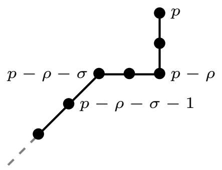
$$h(a_2) = \sum (\uparrow^{p-r-s}(\eta')\eta_{p-r-s-1}\partial_{p-r+1} \cdots \partial_{p-\rho+1}\eta_{p-\rho-1} x' \times \\ \times \uparrow^{p-r-s}(\eta)\partial_{p-r-s} \cdots \partial_{p-r-1} y_p)$$

for $\rho - 1 \leq r \leq p-1$, $0 \leq s \leq p-r-1$ and $(\eta, \eta') \in Sh(s+1, r)$. For $r = \rho - 1$, the degeneracy $\eta_{p-\rho-1}$ survives in the first factor, so that the condition $p - \rho - 1 < p - r - s - 1$ is required to avoid a collision, that is, $s = 0$. This produces all the terms of the desired formula for $h(a)$ satisfying $\rho = r$, $s = 0$, except the term b. Observe in particular how the shuffles in $Sh(1, \rho - 1)$ in $h(a_2)$ produce the shuffles in $Sh(1, \rho)$ in $h(a)$.

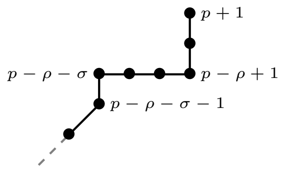
If $r = \rho$, the degeneracy $\eta_{p-\rho-1}$ in the first factor again remains alive, but this time, the collision cannot be avoided and all the corresponding terms in fact are null. Finally, if $r > \rho$, the degeneracy $\eta_{p-\rho+1}$ is annihilated by a face operator and an elementary computation shows all the obtained terms of this sort are exactly the same as those with the same indices in $h(a)$.

12.6.2 The case $\sigma > 0$.

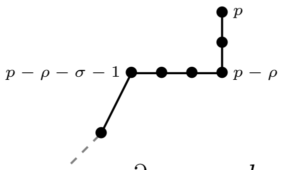
Consider the figures below to understand the involved degeneracies. Again the simplex a produces a target cell b, and only two faces of b are source cells, so that we again have to deduce $h(a) = RM(a)$ from $h(a_1) = RM(a_1)$ and $h(a_2) = RM(a_2)$ and from the recursive formula $h(a) = b + h(a_1) + h(a_2)$, signs omitted.



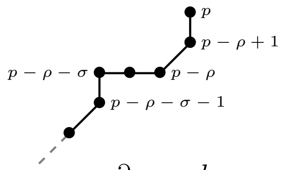
a



b (76)



$a_1 = \partial_{p-\rho-\sigma-1} b$



$a_2 = \partial_{p-\rho+1} b$ (77)

The simplices are:

$$\begin{aligned} a &= (\eta_{p-1} \cdots \eta_{p-\rho} x' \times \eta_{p-\rho-1} \cdots \eta_{p-\rho-\sigma} y') && (78) \\ b &= (\eta_p \cdots \eta_{p-\rho+1} \eta_{p-\rho-\sigma-1} x' \times \eta_{p-\rho} \cdots \eta_{p-\rho-\sigma} y') && (79) \\ a_1 &= (\eta_{p-1} \cdots \eta_{p-\rho} x' \times \eta_{p-\rho-1} \cdots \eta_{p-\rho-\sigma-1} \partial_{p-\rho-\sigma-1} y') && (80) \\ a_2 &= (\eta_{p-1} \cdots \eta_{p-\rho+1} \eta_{p-\rho-\sigma-1} x' \times \eta_{p-\rho-1} \cdots \eta_{p-\rho-\sigma} y') && (81) \end{aligned}$$

Again the subcomponents x' and y' can have other degeneracies. The simplex a has $\sigma > 0$, so that the formula to be proved is:

$$h(a) = \sum (\uparrow^{p-\rho-s}(\eta') \eta_{p-\rho-s-1} x' \times \uparrow^{p-\rho-s}(\eta) \partial_{p-\rho-s} \cdots \partial_{p-\rho-\sigma-1} y')$$

for $\sigma \leq s \leq p-\rho-1$ and $(\eta, \eta') \in Sh(s+1, \rho)$. The corresponding target cell b is the term of this sum with $s = \sigma$ and $(\eta, \eta') = ((0 \cdots \sigma), ((\sigma+1) \cdots (\rho+\sigma))$.

The simplex a_1 has also $\sigma > 0$, which gives us the expression:

$$h(a_1) = \sum (\uparrow^{p-\rho-s}(\eta') \eta_{p-\rho-s-1} x' \times \uparrow^{p-\rho-s}(\eta) \partial_{p-\rho-s} \cdots \partial_{p-\rho-\sigma-1} y')$$

for $\sigma + 1 \leq s \leq p - \rho - 1$ and $(\eta, \eta') \in Sh(s+1, \rho)$; always at least one face operator $\partial_{p-\rho-\sigma-1}$ in the second factor. We see this $h(a_1)$ gives all the terms of the desired expression for $h(a)$ satisfying $s \geq \sigma + 1$.

Again processing $h(a_2)$ is more complex, for a_2 has $\sigma = 0$. The initial expression for $h(a_2)$ is:

$$h(a_2) = \sum (\uparrow^{p-r-s}(\eta') \eta_{p-r-s-1} \partial_{p-r+1} \cdots \partial_{p-\rho+1} \eta_{p-\rho-\sigma-1} x' \times \uparrow^{p-r-s}(\eta) \partial_{p-r-s} \cdots \partial_{p-r-1} \eta_{p-\rho-1} \cdots \eta_{p-\rho-\sigma} y')$$

for $r = \rho - 1$ and $0 \leq s \leq p - \rho$, or $r = \rho$ and $\sigma \leq s \leq p - \rho - 1$, see Lemma 47.

For $r = \rho$, the tail of the first factor is $\partial_{p-\rho+1} \eta_{p-\rho-\sigma-1} x' = \eta_{p-\rho-\sigma-1} \partial_{p-\rho} x'$, and to avoid a collision, the relation $p-\rho-\sigma-1 < p-\rho-s-1$ is necessary, that is, $s < \sigma$, contradiction.

For $r = \rho - 1$, all the face operators of the first factor are absent and there remains for the first component: $\uparrow^{p-\rho-s+1}(\eta') \eta_{p-\rho-s} \eta_{p-\rho-\sigma-1} x'$. Again, to avoid a collision, the relation $p - \rho - \sigma - 1 < p - \rho - s$ is necessary, that is, $s \leq \sigma$. So that all the face operators are also absent of the second factor which becomes: $\uparrow^{p-\rho-s+1}(\eta)\eta_{p-\rho-s-1} \cdots \eta_{p-\rho-\sigma} y'$. A careful examination then shows we have so found all the remaining terms of the desired formula for $h(a)$ satisfying $s = \sigma$, except the term corresponding to b. In particular a term $s = \sigma$, $(\eta, \eta') \in Sh(\sigma + 1, \rho)$ of $h(a)$ corresponds to a term of $h(a_2)$ with $s = \sigma - \ell$ for ℓ the length of the longest "trivial" initial sequence $(0 \cdots (\ell - 1)) \subset \eta$, the maximal length $\sigma + 1$ being excluded, coming on the contrary of the target cell b. Also the position of this removed trivial initial sequence does not modify the signature of the underlying permutation. ♣

12.7 Eilenberg-MacLane formula.

Theorem 50 — *Let (f_2, g_2, h_2) be the reduction associated to the Eilenberg-Zilber vector field on the product $X \times Y$ of two simplicial sets. Then the chain complex morphism g_2 is given by the Eilenberg-MacLane formula:*

$$EML(x_\sigma \otimes y_\rho) = \sum_{(\eta,\eta') \in Sh(\sigma,\rho)} \varepsilon(\eta, \eta')\, (\eta' x_\sigma \times \eta y_\rho) \tag{82}$$

We prefer our favorite indices σ and ρ instead of the traditional p and q: they had essentially the same interpretations as in the previous sections.

Taking account of [10], Theorem 50 was already proved thirty years ago. But it is good training for further tasks to obtain the result directly using this game of vector fields and s-paths.

♣ The g-component of our vector field reduction (f, g, h) is a chain complex morphism $g : C_*(X) \otimes C_*(Y) \to C_*(X \times Y)$. More precisely, a generator $x_\sigma \otimes y_\rho \in C_\sigma(X) \otimes C_\rho(Y)$ of the source chain complex must first be translated into the so-called "last simplex" $\lambda(x_\sigma, y_\rho) = (\eta_{p-1} \cdots \eta_\sigma x_\sigma \times \eta_{\sigma-1} \cdots \eta_0 y_\rho) \in C_*(X \times Y)$; see the explanations after Theorem 31.

(83)

The formula $g = \mathrm{id} - hd$ for the injection of the critical complex into the whole complex was given at (11). In particular $\mathrm{id}(\lambda(x_\sigma, y_\rho))$ is the term with $(\eta, \eta') = ((0 \cdots (\sigma - 1)), (\sigma \cdots (\rho + \sigma - 1)))$ of the EML-formula.

The $hd(\lambda(x_\sigma, y_\rho))$ must produce the other terms with the right signs. The homotopy operator is null except for the source cells, and it happens we have only one source cell in $d(\lambda(x_\sigma, y_\rho))$, drawn above, with the expression:

$$d(\lambda(x_\sigma, y_\rho)) = (\eta_{\rho+\sigma-2} \cdots \eta_\sigma x_\sigma, \eta_{\sigma-2} \cdots \eta_0 y_\rho) \tag{84}$$

We apply the RM-formula to this term to obtain:

$$\sum (\uparrow^{\rho+\sigma-r-s-1}(\eta') \eta_{\rho+\sigma-r-s-2} \partial_{\rho+\sigma-r} \cdots \partial_{\rho+\sigma-1} \eta_{\rho+\sigma-2} \cdots \eta_\sigma x_\sigma \times \\ \uparrow^{\rho+\sigma-r-s-1}(\eta) \partial_{\rho+\sigma-r-s-1} \cdots \partial_{\rho+\sigma-r-2} \eta_{\sigma-2} \cdots \eta_0 y_\rho) \tag{85}$$

for $r = \rho$ and $\sigma - 1 \leq s \leq \rho + \sigma - r - 2 = \sigma - 2$, impossible, and there remain only the parameters $r = \rho - 1$, $0 \leq s \leq \sigma - 1$. The face operators of the first factor are exactly cancelled by the following degeneracies, and the face operators of the second factor are totally cancelled by the following degeneracies, but some of these degeneracies in general remain alive. We obtain finally:

$$\sum(\uparrow^{\sigma-s}(\eta')\eta_{\sigma-s-1}x_\sigma \times \uparrow^{\sigma-s}(\eta)\eta_{\sigma-s-2}\cdots\eta_0 y_\rho) \tag{86}$$

for $0 \leq s \leq \sigma - 1$ and $(\eta, \eta') \in Sh(s+1, \rho-1)$. The generic term of this sum corresponds to the term $(\overline{\eta}' x_\sigma \times \overline{\eta} y_\rho)$ of the standard Eilenberg-MacLane formula for $\sigma - s - 1$ the length of the maximal "trivial" initial segment $(0\cdots(\sigma-s-2))$ in $\overline{\eta}$. The last simplex $(\eta_{\rho+\sigma-1}\cdots\eta_\sigma x_\sigma \times \eta_{\sigma-1}\cdots\eta_0 y_\rho)$ would correspond to $s = -1$, not possible, but this last simplex had been previously produced by the id term of the formula $g = \mathrm{id} - hd$. For further reference, we give the formula obtained:

$$hd(\lambda(x_\sigma, y_\rho)) = \sum(\eta' x_\sigma \times \eta x_\rho) \tag{87}$$

where $(\eta, \eta') \in Sh(\rho, \sigma) - \{\mathrm{id}\}$, that is, all the shuffles are to be used except the trivial one.

We let the reader check the signs are also correct. ♣

12.8 Alexander-Whitney formula.

Theorem 51 — *Let (f_2, g_2, h_2) be the reduction associated to the Eilenberg-Zilber vector field on the product $X \times Y$ of two simplicial sets. Then the chain complex morphism f_2 is given by the Alexander-Whitney formula:*

$$AW(x_p \times y_p) = \sum_{i=0}^{p} \partial_{p-i+1}\cdots\partial_p x_p \otimes \partial_0\cdots\partial_{p-i-1} y_p \tag{88}$$

Same remark with respect to [11] as with respect to [10] in the previous section for the EML-formula.

♣ This morphism f_2 is *natural* and it is sufficient to consider the particular case $X = Y = \Delta^p$ and $(x_p \times y_p) = (\delta_p \times \delta_p)$ if δ_p is the maximal simplex of Δ_p and $(\delta_p \times \delta_p)$ the diagonal p-simplex of $\Delta_p \times \Delta_p$.

The formula 41 is $f = \mathrm{pr}_3(\mathrm{id} - dh)$ with pr_3 being the canonical projection on the critical subcomplex. For example $\mathrm{pr}_3\,\mathrm{id}(\delta_p \times \delta_p) = 0$, for this diagonal simplex is a source cell, giving a null projection on the critical complex. Except if $p = 0$ where the AW-formula is then obvious, for $h(\delta_0 \times \delta_0) = 0$.

Now we have to compute $dh(\delta_p \times \delta_p)$ and to extract from a lot of terms those that are critical. Figure 1 should help to understand a generic term of $h(\delta_p \times \delta_p)$, that is:

$$(\uparrow^{p-r-s}(\eta')\eta_{p-r-s-1}\partial_{p-r+1}\cdots\partial_p\delta_p \times \uparrow^{p-r-s}(\eta)\partial_{p-r-s}\cdots\partial_{p-r-1}\delta_p)$$

The figure represents the particular case $p = 7$, $r = 2$ and $s = 2$. In general the face operators of this generic term cancel the vertices $(p-r+1)\cdots(p)$ of the first factor and $(p-r-s)\cdots(p-r-1)$ for the second factor, so that the s-path is allowed to run among a subset of $(0\cdots p)\times(0\cdots p)$ represented by $\bullet$'s in the figure, whereas the forbidden points are represented by $\times$'s.

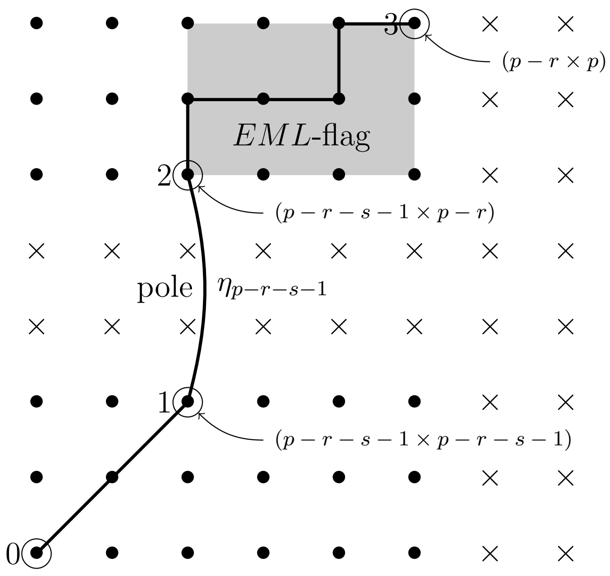


Figure 1: Understanding $RM(\delta_p \times \delta_p)$

The "generic" simplex is represented by an s-path with essentially three parts:

1. A diagonal part starting at (0×0) up to the point $(p-r-s-1 \times p-r-s-1)$; this diagonal part can be a point, if $r+s = p-1$.
2. A *vertical* segment, always present, devoted to this very particular degeneracy of the first factor $\eta_{p-r-s-1}$, starting at $(p-r-s-1 \times p-r-s-1)$ up to $(p-r-s-1 \times p-r)$ in a *unique step*, for the expected intermediary vertices are in fact cancelled by the face operators of the second factor. We call this segment the *pole* of the s-path, think of the pole of a flag, a flag to be described soon.
3. Finally an "Eilenberg-MacLane" step starting from $(p-r-s-1 \times p-r)$ up to $(p-r \times p)$ by an arbitrary combination of vertical (toward north)) and horizontal segments (toward east), combination depending only on the shuffle (η, η'). For example, in our figure, it is the shuffle $((1 \cdot 2 \cdot 4)(0 \cdot 3))$. We call "$EML$-flag" the rectangle between $(p-r-s-1 \times p-r)$ and $(p-r \times p)$, for this part of the path, parametrized by a shuffle, mimics an s-path produced by the EML-formula. And this flag is "carried" by the "pole" $\eta_{p-r-s-1}$.

Remember we have to select all the critical components of the expression $dh(\delta_p \times \delta_p)$. The figure represents one term of $h(\delta_p \times \delta_p)$. A component of $dh(\delta_p \times \delta_p)$ is obtained by cancelling one of the vertices of the s-path, and which remains must be a *critical* cell, that is, an s-path made first only of horizontal segments and then only of vertical segments.

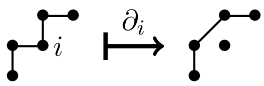
For example, for the simplex represented by Figure 1, no face of this simplex is critical, for a simple reason: there certaily will remain at least one diagonal component if we apply a face operator. In other words we just have to consider the cases $p-r-s-1 = 0$ or 1.

Another obstacle for a face to be critical is the vertical segment $\eta_{p-r-s-1}$

followed by an Eilenberg-MacLane path made of horizontals and verticals joining the extreme points of the rectangle $(((p-r-s-1)\cdots(p-r))\times((p-r)\cdots p))$: it is difficult for such a path not to have a bend qualifying this path as a target cell, and the same for its faces.

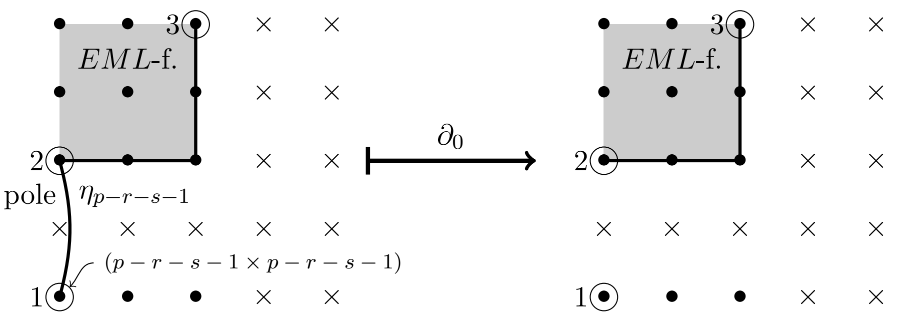
The width of the EML-area is $(p-r)-(p-r-s-1) = s+1 \geq 1$, which implies the certain presence of at least one horizontal segment for our path in this area. More precisely, after the pole, there certainly will be a bend. This bend, or another possible one, later in the flag, qualifies our s-path as a target cell, which is not amazing: a value of the RM-formula is a combination of target cells. This already implies the terms for which $p-r-s-1 = 1$ can be given up, because for these terms we will have to destroy by *one* face operator a diagonal *and* a bend, impossible. For $p-r-s-1 = 0$, that is, for $r+s = p-1$, no initial diagonal part for our s-path.

But we have to apply now to this *target* cell a face operator to get a *critical* cell, not easy! First we must have in our target cell a *unique* bend, for it is impossible to destroy *two* such bends with *one* face operator without creating a diagonal:



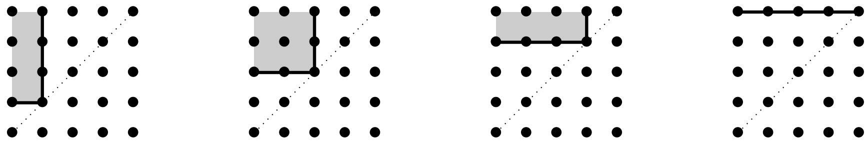
(89)

Remember also a critical cell has a unique bend or no bend at all. If you remove the precise point of the unique bend of our target cell, you create a diagonal and the result is a source cell, forbidden. If you remove another point, then you first think this cannot suppress the bend ? Correct, except a special case, when the bend has before (resp. after) the bend the *initial* vertical (resp. *final* horizontal) segment of the path. This is the key point of our subject which, once understood, immediately products the Alexander-Whitney formula. An example of this sort is below $(p=4, r=2, s=1, (\eta,\eta') = (0\cdot 1, 2\cdot 3))$:



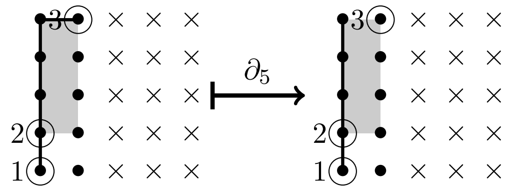
(90)

For an arbitrary $p \geq 1$, we so obtain p terms of the Alexander-Whitney formula, those corresponding in the RM-formula to the parameters $0 \leq r \leq p-1$, $s = p-r-1$, $(\eta,\eta') = ((0\cdots s),((s+1)\cdots(p-1)))$, to which we apply the ∂_0-operator. For example we obtain for $p = 4$:



(91)

One term of the Alexander-Whitney formula is still missing. It is obtained by another *unique* particular case, when $r = p - 1$, $s = 0$, $(\eta, \eta') = ((p-1), (0 \cdots (p-2)))$. The figure when $p = 4$:



(92)

There remains to apply the canonical correspondance between critical cells of the product $\Delta^p \times \Delta^p$ and the generators of $C_*\Delta^p \otimes C_*\Delta^p$. That is:
$(\eta_{p-1} \cdots \eta_{p-r}\partial_{p-r+1} \cdots \partial_p\delta_p \times \eta_{p-r-1} \cdots \eta_0\partial_0 \cdots \partial_{p-r-1}\delta_p) \longmapsto$
$\partial_{p-r+1} \cdots \partial_p\delta_p \otimes \partial_0 \cdots \partial_p\delta_p$.
and also to check the final sign is always positive... ♣

12.9 The critical differential.

Several times in the previous sections we used the expression "critical subcomplex" to designate the graded *submodule* generated by the critical cells. Which submodule in general *is not* a subcomplex.

In the Eilenberg-Zilber particular case, we must now study the differential to be installed on $C^c_*(X \times Y)$ and check, taking account of the isomorphism $C^c(X \times Y) \cong C_*(X) \otimes C_*(Y)$, that we find the usual differential of the tensor product. It is as funny (!?) as in the previous sections. Because of the naturality of our process, it is sufficient to consider the case of $C_*(\Delta^p \times \Delta^p)$.

A generic critical cell to be considered is:

$$s^c = (\eta_{p-1} \cdots \eta_\sigma\delta_\sigma \times \eta_{\sigma-1} \cdots \eta_0\delta_\rho) \cong \delta_\sigma \otimes \delta_\rho \qquad (93)$$

where as usual δ_σ and δ_ρ are simplices of respective dimensions σ and ρ with $\sigma + \rho = p$.

The formula (42) $d' = \mathrm{pr}_3(d - dhd)$ is to be applied to our critical generic simplex s^c. The game is similar to the one of the previous section for Alexander-Whitney. Anyway we must compute the initial differential ds^c (signs omitted):

$$ds^c = \sum_{i=0}^{p} \partial_i(\eta_{p-1} \cdots \eta_\sigma\delta_\sigma \times \eta_{\sigma-1} \cdots \eta_0\delta_\rho) \qquad (94)$$

and study which happens according to the situation of i with respect to σ. If $i < \sigma$, the face operator on the one hand is to be applied to the δ_σ of the first factor, and on the other hand annihilates a degeneracy of the second factor, to produce:

$$\partial_i s^c = (\eta_{p-2} \cdots \eta_{\sigma-1} \partial_i \delta_\sigma \times \eta_{\sigma-2} \cdots \eta_0 \delta_\rho) \cong \partial_i \delta_\sigma \otimes \delta_\rho \tag{95}$$

where we recognize the expected $\partial_i \delta_\sigma \otimes \delta_\rho$; this is valid for $i < \sigma$, just one face of this sort $\partial_\sigma \delta_\sigma \otimes \delta_\rho$ is missing.

Symmetrically, if $i > \sigma$, in the same way, we obtain:

$$\partial_i s^c = (\eta_{p-2} \cdots \eta_\sigma \delta_\sigma \times \eta_{\sigma-1} \cdots \eta_0 \partial_{i-\sigma} \delta_\rho) \cong \delta_\sigma \otimes \partial_{i-\sigma} \delta_\rho \tag{96}$$

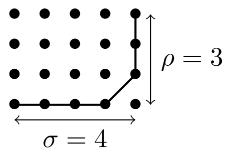
We recognize the expected $\delta_\sigma \otimes \partial_j \delta_\rho$ for $j = i - \sigma > 0$; one face of this sort is missing: $\delta_\sigma \otimes \partial_0 \delta_\rho$.

All these simplices are critical cells and they contribute to the studied differential d' only via the component $\mathrm{pr}_3 d$ in the expression $d' = \mathrm{pr}_3(d - dhd)$, for the homotopy operator h is null on the critical cells.

Processing the face of index σ is very particular. First the face operator cancels the last degeneracy of the first factor and the first one of the second factor, without forgetting how the degeneracies of the first factor are renumbered:

$$\partial_\sigma s^c =: s' := (\eta_{p-2} \cdots \eta_\sigma \delta_\sigma \times \eta_{\sigma-2} \cdots \eta_0 \delta_\rho) \tag{97}$$

We are again in front of this "almost critical" simplex which generates the EML-formula, in fact a source simplex. Example for $\sigma = 4$ and $\rho = 3$:

$s' =$ 

(98)

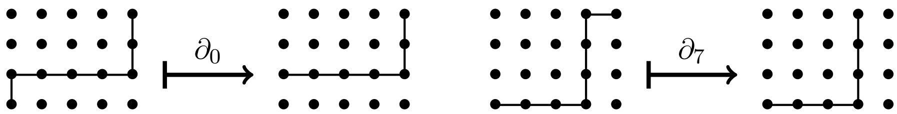
It is a source cell, so that the contribution of this face in $d's^c = \mathrm{pr}_3(d - dhd)(s^c)$ is only $-\mathrm{pr}_3 dh\partial_\sigma s^c = dhs'$, sign omitted. We rewrite as follows the formula (87):

$$hs' = \sum (\eta' x_\sigma \times \eta x_\rho) \tag{99}$$

for$(\eta, \eta') \in Sh(\rho, \sigma) - \{\mathrm{id}\}$. The final step consists in computing the differential of this sum and keeping in the obtained terms the critical cells.

Every term of this sum is a target cell; again it is a little funny, for in the Eilenberg-MacLane formula, all the terms are target cells except the trivial one which is critical. In the case 4×3 drawn above at (98), an Eilenberg-MacLane term must go from (0×0) to (4×3) following only horizontals toward east and verticals toward north. A unique path of this sort is without a bend, the critical cell $(0 \times 0) \to (4 \times 0) \to (4 \times 3)$. All the other paths necessarily have somewhere a bend and this is why they are target cells.

Now we have to differentiate such a cell and keep the critical cells. Remember the discussion page 40 for the Alexander-Whitney formula where we already meet such a situation. A target cell containing a critical cell in his differential must have a unique bend and this bend must be either preceded by the initial vertical segment of the path or followed by its final horizontal segment. Only two possibilities drawn below in the case 4×3:



(100)

which fortunately exactly produces the two terms missing in the differential of our critical complex. ♣

12.10 Eilenberg-MacLane's recursive formula.

The recursive formula [4, (2.13)]:

$$\Phi c_q = -\Phi' c_q + h' D_0 c_q \tag{101}$$

is essential when Eilenberg and MacLane process the Eilenberg-Zilber equivalence. The pseudo-derivative operator to be applied to Φ and h in the second member, these operators Φ and h being mainly made of face and degeneracy operators, consists in fact in taking the same operator for one dimension less and to shift all the face and degeneracy indices by +1. For example, if $k(s) := \eta_2\partial_3 s + \eta_1\partial_4 s$ for a 5-simplex s, then $k'(s) = \eta_3\partial_4 s + \eta_2\partial_5 s$ for a 6-simplex s. The Φ of Eilenberg-MacLane is our homotopy operator $h = RM$ and *their* h is our composition $gf = EML \circ AW$. Finally D_0 is our degeneracy η_0.

So that the translation of Eilenberg-MacLane's recursive formula into our notations is:

$$RM(x_p \times y_p) = -RM'(x_p \times y_p) + EML'AW'\eta_0(x_p \times y_p) \tag{102}$$

Also, possibly degenerate terms in the final result are to be cancelled.

We deduce from the AW formula (53) and the EML-formula (54) the following expression for the composition $EML \circ AW$ (signs omitted):

$$\sum(\eta'\partial_{p-r+1}\cdots\partial_p x_p \times \eta\partial_0\cdots\partial_{p-r-1}y_p) \tag{103}$$

for $0 \leq r \leq p$ and $(\eta, \eta') \in Sh(p-r, r)$. Replacing x_p and y_p by $\eta_0 x_p$ and $\eta_0 y_p$, and shifting the indices to take account of the pseudo-derivations gives:

$$\sum(\uparrow(\eta')\partial_{p-r+2}\cdots\partial_{p+1}\eta_0 x_p \times \uparrow(\eta)\partial_1\cdots\partial_{p-r}\eta_0 y_p) \tag{104}$$

We must now install the η_0's at the right place. The right one is always killed by a face operator, except for $r = p$; the left one always remains alive, so that the unique term corresponding to $r = p$ disappears, it is 0-degenerate, and there remains:

$$\sum(\uparrow(\eta')\eta_0\partial_{p-r+1}\cdots\partial_p x_p \times \uparrow(\eta)\partial_1\cdots\partial_{p-r-1}y_p) \tag{105}$$

for $0 \leq r \leq p-1$ and $(\eta, \eta') \in Sh(p-r, r)$. We recognize here all the terms of the RM-formula (55) for which $s = p-r-1$ or, simpler, all the terms for which $p-r-s-1=0$, that is, exactly these terms where this very specific degeneracy operator $\eta_{p-r-s-1}$ is in fact η_0.

Which orients the study of the Eilenberg-MacLane recursive formula to Figure 1. All the terms of the RM-formula can be divided in two simple classes. The first one contains all the terms starting by a vertical, that is, those terms where $p-r-s-1=0$. An example of this sort is the lefthand figure of (90). These terms are produced by $EML'AW'\eta_0(x_p \times y_p)$ in (102). The second class of terms are those satisfying $p-r-s-1>0$, that is, those starting by a diagonal segment in Figure 1; they are produced by $-RM'(x_p \times y_p)$ in (102), for a possible pattern for the s-path after the first diagonal segment is a RM-pattern for one dimention less. ♣

13 Conclusion.

As it was briefly explained in the introduction, the previous text settles a complete correspondence between, on the one hand the reduction deduced from what was called the Eilenberg-Zilber vector field, and on the other hand the "old" Eilenberg-Zilber reduction defined by the AW-EML-RM formulae. It was the role of the long Section 12, consisting mainly in carefully examining the numerous degeneracy operators of the RM-formula and taking account of the so-called "collisions" between these operators, cancelling many terms of the RM-formula.

The terms of the RM-formula corresponding to these collisions are automatically ignored in the vector field environment, explaining the efficiency of the implementation of the EZ theorem based upon the EZ vector field.

In another paper, we will explain how the same strategy can be applied to the *twisted* EZ-theorem. This time it is the terrible Szczarba formula [25] which is involved, still more complicated than the formulae of the present text. In fact Szczarba's formula is a sort of twisted EML formula. No close formula is yet known for a hypothetical twisted RM-formula. But the twisted EZ vector field is *exactly the same* (!!) as the non-twisted one, allowing to easily apply the vector field method to obtain the corresponding reduction. This really amazing property of the EZ vector field definitely implies this vector field is in fact the real *kernel* of this subject.

Producing our simple solution for *constructive* versions of Serre and Eilenberg-Moore spectral sequences. To be explained in forthcoming papers.

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