THE HOMOLOGY OF ITERATED LOOP SPACES

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APPENDIX of *F. Sergeraert*

Introduction

Last years to solve different problems in Agebraic Topology it needs to consider more and more complicate structures on the chain complex $C_*(X)$ of a topological space X and its homology $H_*(X)$.

One of the most difficult problem is the problem of calculating the homology groups of iterated loop spaces. The first steps to solve this problem were made by J.F.Adams. To calculate the homology $H_*(\Omega X)$ of the loop space ΩX over a topological space X he introduced the notion of cobar construction FK over a coalgebra K, [1].

Recall that a chain complex K is called a coalgebra if there is given a chain mapping $\nabla: K \to K \otimes K$ satisfying the associativity relation

$$(\nabla \otimes 1)\nabla = (1 \otimes \nabla)\nabla.$$

The cobar construction FK over a coalgebra K is a differential algebra, that coincides, as a graded algebra, with the tensor algebra $TS^{-1}K$ over the desuspension $S^{-1}K$ over K. The generators in FK are denoted by $[x_1, \dots, x_n]$ where $x_i \in K$, $1 \leq i \leq n$, and have dimensions

$$dim[x_1, \dots, x_n] = \sum_{i=1}^{n} dim(x_i) - n.$$

The product $\pi: FK \otimes FK \to FK$ is defined by the formula

$$\pi([x_1, \dots, x_n] \otimes [x_{n+1}, \dots, x_{n+m}]) = [x_1, \dots, x_{n+m}].$$

The differential on the generators $[x] \in FK$ is defined by the formula

$$d[x] = -[d(x)] + \sum (-1)^{\epsilon} [x', x''],$$

where $\nabla(x) = \sum x' \otimes x''$, $\epsilon = dim(x')$. On the other elements the differential is defined as on the products of such elements.

The chain complex $C_*(X)$ of a topological space X possesses a natural coalgebra structure. Therefore there is defined the cobar construction $FC_*(X)$.

J.F.Adams proved that for a simply connected topological space X there is a chain equivalence of differential algebras

$$C_*(\Omega X) \simeq FC_*(X).$$

In particular, if the topological space X is the suspension over a space Y, i.e. $X = SY$, then the coalgebra structure on $C_*(X)$ is trivial. Hence the chain complex

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$C_*(\Omega X)$ of the loop space ΩX will be chain equivalent to the tensor algebra $TC_*(Y)$ over the chain complex $C_*(Y)$.

Unfortunately, the cobar construction FK over a coalgebra K does not admit iteration, because passing to the cobar construction we lose a coalgebra structure. So there is no natural coalgebra structure on the Adams cobar construction FK over a coalgebra K.

However H.J.Baues [2] introduced a coalgebra structure on the Adams cobar construction $FC_*(X)$ over the chain complex $C_*(X)$ of a topological space X. This structure was determined using a family of operations

$$\nabla_{n,m}: C_*(X) \to C_*(X)^{\otimes n} \otimes C_*(X)^{\otimes m}$$

of dimensions $n+m-1$. So he defined the double cobar construction $F^2C_*(X)$ and proved that for a 2-connected topological space X there is a chain equivalence

$$C_*(\Omega^2 X) \simeq F^2 C_*(X).$$

But there is no appropriate coalgebra structure on the double cobar construction and therefore further iterations are not possible.

In [3] J.P.May introduced the notion of an operad and investigated the structure on the iterated loop spaces. This structure used in [4] to calculate the homology $H_*(\Omega^n S^n X)$. The homology $H_*(\Omega^n S^n X)$ investigated also in [5], [6] and others.

In [7], [8], [9] the operad methods were transfered from the category of topological spaces to the category of chain complexes. It was shown that on the chain complex $C_*(X)$ of a topological space X there is a natural E_∞-coalgebra structure which determines the weak homotopy type of X. Using this structure the chain complex $C_*(\Omega^n X)$ of the n-fold loop space $\Omega^n X$ over an n-connected topological space X was expressed through the chain complex $C_*(X)$ of a space X.

Our aim here is to construct the spectral sequence of the homology of the iterated loop spaces and produce some calculations.

To do it we use the E_∞-coalgebra structure on the chain complex $C_*(X)$ of a topological space X (Theorem 1). Then we express the chain complex $C_*(\Omega^n X)$ of the iterated loop space $\Omega^n X$ in terms of the cobar construction over $C_*(X)$ (Theorem 2).

After that we consider the spectral sequence of the cobar construction and calculate its first term over $\mathbb{Z}/2$-coefficients (Theorem 3), over $\mathbb{Z}/p$-coefficients (Theorem 3') and over a field of characteristic zero (Theorem 3"). Also we give the expression of the E^1-term as the free n-Poisson algebra generated by the homology of X (Theorems 4, 4').

In the cases of $\mathbb{Z}/2$, $\mathbb{Z}/p$ and characteristic zero coefficients, we calculate the differential d^1 on the first term and obtain the expresion of the E^2-term of the spectral sequence (Theorems 5, 5', 5").

Finally we apply these results to calculate the homology of the iterated loop spaces over the real and complex projective spaces (Theorem 6, 7, 8), which play important role in Algebraic Topology.

Note that general method to calculate the homology of the iterated loop space gives us Simplicial Theory. There is the simplicial construction GX of the loop space over a simplicial set X and its iteration G^nX. But this construction is very complicate and real calculations may be produced only using computer methods [10].

The computer calculations may be produced for any topological spaces but in low dimensions. Conversely the spectral sequence calculations may be produced for "nice" spaces in any dimensions. So the computer calculations complement the spectral sequence calculations.

Ones of the most important spaces in Algebraic Topology, besides spheres, are the real and complex projective spaces. The problem of comparing the computer and the spectral sequence calculations for these spaces was stated by F.Sergeraert. It seems to be very usefull for both sides of these calculations.

§1. Operads and algebras over operads

Consider the category of chain complexes over a ring R. By a symmetric family $\mathcal{E}$ in this category is meant a family $\mathcal{E} = \{\mathcal{E}(j)\}_{j\geq 1}$ of chain complexes $\mathcal{E}(j)$ operated on by the symmetric groups Σ_j.

Given two symmetric families $\mathcal{E}'$, $\mathcal{E}''$ we define the symmetric families $\mathcal{E}' \otimes \mathcal{E}''$, $\mathcal{E}' \times \mathcal{E}''$ putting

$$(\mathcal{E}' \otimes \mathcal{E}'')(j) = \mathcal{E}'(j) \otimes \mathcal{E}''(j),$$

and $(\mathcal{E}' \times \mathcal{E}'')(j)$ equal to the quotiont module of the Σ_j- free module generated by the module

$$\sum_{j_1+\cdots+j_k=j} \mathcal{E}'(k) \otimes \mathcal{E}''(j_1) \otimes \cdots \otimes \mathcal{E}''(j_k)$$

with respect to the equivalence generated by the relations

$$x'\sigma \otimes x''_1 \otimes \cdots \otimes x''_k \sim x' \otimes x''_{\sigma^{-1}(1)} \otimes \cdots \otimes x''_{\sigma^{-1}(k)} \cdot \sigma(j_1, \ldots, j_k),$$
$$x' \otimes x''_1\sigma_1 \otimes \cdots \otimes x''_k\sigma_k \sim x' \otimes x''_1 \otimes \cdots \otimes x''_k \cdot \sigma_1 \times \cdots \times \sigma_k,$$

where $\sigma(j_1, \ldots, j_k)$ is the permutation of a set of j elements obtained by partitioning the set into k blocks of $j_1, \ldots, j_k$ elements, respectively, and carrying out on these blocks the permutation σ, while $\sigma_1 \times \cdots \times \sigma_k$ means the image of the element $(\sigma_1, \ldots, \sigma_k)$ under the imbedding $\Sigma_{j_1} \times \cdots \times \Sigma_{j_k} \to \Sigma_j$.

It is easy to see that for the symmetric families $\mathcal{E}', \mathcal{E}'', \mathcal{F}', \mathcal{E}''$ there is the interchanging mapping

$$T\colon (\mathcal{E}' \otimes \mathcal{E}'') \times (\mathcal{F}' \otimes \mathcal{E}'') \to (\mathcal{E}' \times \mathcal{F}') \otimes (\mathcal{E}'' \times \mathcal{F}'').$$

A symmetric family $\mathcal{E}$ is called an operad if there is given a symmetric-family mapping $\gamma\colon \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ such that $\gamma(\gamma \times 1) = \gamma(1 \times \gamma)$, or that is the same, the following diagram is commutative

$$\begin{array}{ccc}
\mathcal{E} \times \mathcal{E} \times \mathcal{E} & \xrightarrow{\gamma \times 1} & \mathcal{E} \times \mathcal{E} \\
{\scriptstyle 1\times\gamma}\downarrow & & \downarrow{\scriptstyle \gamma} \\
\mathcal{E} \times \mathcal{E} & \xrightarrow{\gamma} & \mathcal{E}
\end{array}$$

If there exist an element $1 \in \mathcal{E}(1)$ such that $\gamma(1 \otimes x) = \gamma(x \otimes 1^{\otimes k}) = x$ for all $x \in \mathcal{E}(k)$ then we say that $\mathcal{E}$ is an operad with identity.

An operad $\mathcal{E}$ is called a Hopf operad if besides the operation $\gamma: \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ there is given an operation $\nabla: \mathcal{E} \to \mathcal{E} \otimes \mathcal{E}$ which is associative and satisfy the Hopf relation, i.e. the following diagram is commutative

$$\begin{array}{ccccc}
\mathcal{E} \times \mathcal{E} & \xrightarrow{\nabla \times \nabla} & (\mathcal{E} \otimes \mathcal{E}) \times (\mathcal{E} \otimes \mathcal{E}) & \xrightarrow{T} & (\mathcal{E} \times \mathcal{E}) \otimes (\mathcal{E} \times \mathcal{E}) \\
{\scriptstyle\gamma}\big\downarrow & & & & \big\downarrow{\scriptstyle\gamma \otimes \gamma} \\
\mathcal{E} & \xrightarrow{\nabla} & \mathcal{E} \otimes \mathcal{E} & \xrightarrow{=} & \mathcal{E} \otimes \mathcal{E}
\end{array}$$

A mapping of operads $f: \mathcal{E}' \to \mathcal{E}''$ is a mapping of symmetric families for which the folowing diagram is commutative

$$\begin{array}{ccc}
\mathcal{E}' \times \mathcal{E}' & \xrightarrow{\gamma'} & \mathcal{E}' \\
{\scriptstyle f \times f}\big\downarrow & & \big\downarrow{\scriptstyle f} \\
\mathcal{E}'' \times \mathcal{E}'' & \xrightarrow{\gamma''} & \mathcal{E}''
\end{array}$$

If $\mathcal{E}'$ and $\mathcal{E}''$ are operads with identities $1'$ and $1''$ respectively, then it is required that $f(1') = 1''$.

We shall say that an operad $\mathcal{E}$ acts on a symmetric family $\mathcal{F}$ on the left (right) if there is given a mapping $\mu': \mathcal{E} \times \mathcal{F} \to \mathcal{F}$ ($\mu'': \mathcal{F} \times \mathcal{E} \to \mathcal{F}$) such that

$$\mu'(\gamma \times 1) = \mu'(1 \times \mu') \quad (\mu''(1 \times \gamma) = \mu''(\mu'' \times 1)),$$

or that is the same, the following diagrams are commutative

$$\begin{array}{ccc}
\mathcal{E} \times \mathcal{E} \times \mathcal{F} & \xrightarrow{\gamma \times 1} & \mathcal{E} \times \mathcal{F} \\
{\scriptstyle 1 \times \mu'}\big\downarrow & & \big\downarrow{\scriptstyle \mu'} \\
\mathcal{E} \times \mathcal{F} & \xrightarrow{\mu'} & \mathcal{F}
\end{array}
\qquad
\left(\begin{array}{ccc}
\mathcal{F} \times \mathcal{E} \times \mathcal{E} & \xrightarrow{1 \times \gamma} & \mathcal{F} \times \mathcal{E} \\
{\scriptstyle \mu'' \times 1}\big\downarrow & & \big\downarrow{\scriptstyle \mu''} \\
\mathcal{F} \times \mathcal{E} & \xrightarrow{\mu''} & \mathcal{F}
\end{array}\right)$$

For any chain complex X and symmetric family $\mathcal{E}$ we define chain complexes $\mathcal{E}(X)$, $\overline{\mathcal{E}}(X)$ putting

$$\mathcal{E}(X) = \sum_k \mathcal{E}(k) \otimes_{\Sigma_k} X^{\otimes k}, \quad \overline{\mathcal{E}}(X) = \prod_k Hom_{\Sigma_k}(\mathcal{E}(k); X^{\otimes k}).$$

If $\mathcal{E}$ is an operad, then the operad structure in $\mathcal{E}$ determines a mapping

$$\gamma: \mathcal{E}^2(X) = \mathcal{E}(\mathcal{E}(X)) \to \mathcal{E}(X)$$

such that the correspondence $X \longmapsto \mathcal{E}(X)$ is a monad in the category of chain complexes.

Dually, the operad structure in $\mathcal{E}$ determines a mapping

$$\overline{\gamma}: \overline{\mathcal{E}}(X) \to \overline{\mathcal{E}}^2(X) = \overline{\mathcal{E}}(\overline{\mathcal{E}}(X))$$

such that the correspondence $X \longmapsto \overline{\mathcal{E}}(X)$ is a comonad in the category of chain complexes.

A chain complex X is called an algebra over the operad $\mathcal{E}$, or simply $\mathcal{E}$-agebra, if there is given a mapping $\mu\colon \mathcal{E}(X) \to X$ satisfying the associativity relation:

$$\mu \circ \gamma(X) = \mu \circ \mathcal{E}(\mu),$$

or that is the same, the following diagram is commutative

$$\begin{array}{ccc} \mathcal{E}^2(X) & \xrightarrow{\gamma(X)} & \mathcal{E}(X) \\ {\scriptstyle \mathcal{E}(\mu)}\downarrow & & \downarrow{\scriptstyle \mu} \\ \mathcal{E}(X) & \xrightarrow{\mu} & X \end{array}$$

Dually, a chain complex X is called a coalgebra over the operad $\mathcal{E}$, or simply $\mathcal{E}$-coalgebra, if there is given a mapping $\tau\colon X \to \overline{\mathcal{E}}(X)$ satisfying the associativity relation:

$$\overline{\gamma}(X) \circ \tau = \overline{\mathcal{E}}(\tau) \circ \tau,$$

or that is the same, the following diagram is commutative

$$\begin{array}{ccc} X & \xrightarrow{\tau} & \overline{\mathcal{E}}(X) \\ {\scriptstyle \tau}\downarrow & & \downarrow{\scriptstyle \overline{\gamma}(X)} \\ \overline{\mathcal{E}}(X) & \xrightarrow{\overline{\mathcal{E}}(\tau)} & \overline{\mathcal{E}}^2(X) \end{array}$$

Consider some examples of operads and algebras over operads.

1. The Hopf operad $E_0 = \{E_0(j)\}$, where $E_0(j)$ – the free module with one zero dimensional generator $e(j)$ and trivial action of the symmetric group Σ_j. So $E_0(j) \cong R$. The operation $\gamma\colon E_0 \times E_0 \to E_0$ is given by the formula

$$\gamma(e(k) \otimes e(j_1) \otimes \cdots \otimes e(j_k)) = e(j_1 + \cdots + j_k).$$

The operation $\nabla\colon E_0 \to E_0 \otimes E_0$ is given by the formula

$$\nabla(e(j)) = e(j) \otimes e(j).$$

It is easy to see that the required relations are satisfied and algebras (coalgebras) over the operad E_0 are simply commutative and associative algebras.

2. The Hopf operad $A = \{A(j)\}$, where $A(j)$ – the Σ_j-free module with one zero dimensional generator $a(j)$. So $A(j) \cong R(\Sigma_j)$. The operation $\gamma\colon A \times A \to A$ is given by the formula

$$\gamma(a(k) \otimes a(j_1) \otimes \cdots \otimes a(j_k)) = a(j_1 + \cdots + j_k).$$

The operation $\nabla\colon A \to A \otimes A$ is given by the formula

$$\nabla(a(j)) = a(j) \otimes a(j).$$

It is easy to see that the required relations are satisfied and that algebras (coalgebras) over the operad A are simply associative algebras.

3. For a symmetric family $\mathcal{E}$ define the suspension $S\mathcal{E}$ putting $(S\mathcal{E})(j) = S^{j-1}\mathcal{E}(j)$ – $(j-1)$-fold suspension over $\mathcal{E}(j)$. It is clear that if $\mathcal{E}$ is an operad then the suspension $S\mathcal{E}$ will be also an operad, and if X is an algebra (coalgebra) over an operad $\mathcal{E}$ then the suspension SX will be an algebra (coalgebra) over the operad $S\mathcal{E}$.

4. For operads $\mathcal{E}'$, $\mathcal{E}''$ its tensor product $\mathcal{E}' \otimes \mathcal{E}''$ evidently is an operad.

5. Let L be the suboperad of the operad A generated by the element

$$b(2) = a(2) - a(2)T, \quad T \in \Sigma_2.$$

Then algebras (coalgebras) over the operad L will be simply Lie algebras (coalgebras).

Similary, let L_n be the suboperad of the operad S^nA generated by the element

$$b_n(2) = s^n a(2) + (-1)^n s^n a(2)T.$$

Then algebras (coalgebras) over the operad L_n will be simply n-Lie algebras (coalgebras).

6. Let $P_n = E_0 \times L_n$ and the operad structure γ is determined by the corresponding structures in E_0, L_n and by the formulas

$$\gamma(b_n(2) \otimes 1 \otimes e(2)) = e(2) \otimes b_n(2) \otimes 1 + e(2) \otimes 1 \otimes b_n(2) \cdot (213).$$

The operad P_n is called the n-Poisson operad. Algebras (coalgebras) over this operad are called n-Poisson algebras (coalgebras).

7. For any chain complex X define the operads $\mathcal{E}_X$, $\mathcal{E}^X$, putting

$$\mathcal{E}_X(j) = Hom(X^{\otimes j}; X); \quad \mathcal{E}^X(j) = Hom(X; X^{\otimes j}).$$

The actions of the symmetric groups are determined by the permutations of factors of $X^{\otimes j}$ and the operad structures are defined by the formulas

$$\gamma_X(f \otimes g_1 \otimes \cdots \otimes g_k) = f \circ (g_1 \otimes \cdots \otimes g_k), \quad f \in \mathcal{E}_X(k),\ g_i \in \mathcal{E}_X(j_i);$$
$$\gamma^X(f \otimes g_1 \otimes \cdots \otimes g_k) = (g_1 \otimes \cdots \otimes g_k) \circ f, \quad f \in \mathcal{E}^X(k),\ g_i \in \mathcal{E}^X(j_i).$$

Directly from the definition it follows that a chain complex X is an algebra (coalgebra) over the operad $\mathcal{E}$ if and only if there is given an operad mapping $\xi: \mathcal{E} \to \mathcal{E}_X$ ($\xi: \mathcal{E} \to \mathcal{E}^X$).

Analogously, for chain complexes X and Y there are defined the symmetric families $\mathcal{F}_{X,Y}$, $\mathcal{F}^{X,Y}$

$$\mathcal{F}_{X,Y}(j) = Hom(X^{\otimes j}; Y), \quad \mathcal{F}^{X,Y}(j) = Hom(X; Y^{\otimes j})$$

and actions

$$\mu': \mathcal{E}_Y \times \mathcal{F}_{X,Y} \to \mathcal{F}_{X,Y},\ \mu'': \mathcal{F}_{X,Y} \times \mathcal{E}_X \to \mathcal{F}_{X,Y};$$
$$\mu': \mathcal{E}^X \times \mathcal{F}^{X,Y} \to \mathcal{F}^{X,Y},\ \mu'': \mathcal{F}^{X,Y} \times \mathcal{E}^Y \to \mathcal{F}^{X,Y}.$$

8. One of the most important topological operad is the Bordman and Vogt's "little n-cubes" operad E_n, [2]. Let J denote the open interval $(0,1)$ and J^n the open n-dimensional cube. By an n-dimensional little cube it calles a liner embedding $f\colon J^n \to J^n$ with parallel axes. Then $E_n(j)$ is the set of ordered j-tuples $(f^1,\dots,f^j)$ of n-dimensional little cubes $f^i\colon J^n \to J^n$ that images don't intersect. This operad acts on the n-fold loop space $\Omega^n X$ over a space X.

The direct limite of the operads E_n over the inclusions $E_n \subset E_{n+1}$ denoted by E_∞. It is acyclic operad with the free actions of the symmetric groups.

9. It is easy to see that if $\mathcal{E}$ is a topological operad then it's singular chain complex $C_*(\mathcal{E})$ is an operad in the category of chain complexes, and if $\mathcal{E}$ acts on a space X then $C_*(\mathcal{E})$ acts on $C_*(X)$.

Similary, the homology $H_*(\mathcal{E})$ of a topological operad $\mathcal{E}$ is an operad in the category of graded modules, and if $\mathcal{E}$ acts on a space X then $H_*(\mathcal{E})$ acts on the homology $H_*(X)$. In particular, the homology $H_*(E_n)$ of the topological n-cubes operad E_n is isomorphic to the n-Poisson operad P_n. So the homology $H_*(\Omega^n X)$ of the n-fold loop space $\Omega^n X$ is an algebra over the Poisson operad P_n.

The operad $C_*(E_\infty)$ gives us an example of acyclic operad in the category of chain complexes with the free actions of the symmetric groups.

Note that all acyclic operads with the free actions of the symmetric groups consist of the Σ_j-homotopy equivalent chain complexes. We will call such operads as E_∞-operads.

10. Another example of the E_∞-operad give us simplicial resolutions of the symmetric groups. Denote by $E\Sigma_*(j)$ the free simplicial resolution of the symmetric group Σ_j, i.e.

$$E\Sigma_*(j) : \Sigma_j \leftarrow \Sigma_j \times \Sigma_j \leftarrow \dots$$

The mappings $\gamma\colon \Sigma_k \times \Sigma_{j_1} \times \dots \times \Sigma_{j_k} \to \Sigma_{j_1+\dots+j_k}$ induce the operad structure

$$\gamma_*\colon E\Sigma_*(k) \times E\Sigma_*(j_1) \times \dots \times E\Sigma_*(j_k) \to E\Sigma_*(j_1 + \dots + j_k).$$

So $E\Sigma_*$ will be acyclic operad with the free actions of the symmetric groups in the category of simplicial sets.

Taking the chain complex $C_*(E\Sigma_*)$ we obtain the E_∞-operad in the category of chain complexes. Denote it simply by $E\Sigma$.

Note that for any chain operad $\mathcal{E}$ the operad $\mathcal{E} \otimes E\Sigma$ has the same homology and the free actions of the symmetric groups. The projection $E\Sigma \to E_0$ induces the projection $\mathcal{E} \otimes E\Sigma \to \mathcal{E}$. So $\mathcal{E} \otimes E\Sigma$ may be considered as the Σ-free resolution of the operad $\mathcal{E}$. If $\mathcal{E}$ is an acyclic operad then $\mathcal{E} \otimes E\Sigma$ will be the E_∞-operad.

§2. On the chain complex of a topological space

Here we consider the structure on the singular chain complex $C_*(X)$ of a topological space X, and dually on the singular cochain complex $C^*(X)$.

Besides the coalgebra structure

$$\nabla\colon C_*(X) \to C_*(X) \otimes C_*(X)$$

on the chain complex of a topological space there are coproducts

$$\nabla_i\colon C_*(X) \to C_*(X) \otimes C_*(X)$$

increasing dimensions by i and such that

$$d(\nabla_i) = \nabla_{i-1} + (-1)^i T \nabla_{i-1},$$

where $T\colon C_*(X) \otimes C_*(X) \to C_*(X) \otimes C_*(X)$ permutes factors.

Dually, on the cochain complex $C^*(X)$ besides the algebra structure

$$\cup\colon C^*(X) \otimes C^*(X) \to C^*(X)$$

there are products

$$\cup_i\colon C^*(X) \otimes C^*(X) \to C^*(X)$$

such that $d(\cup_i) = \cup_{i-1} + (-1)^i \cup_{i-1} T$.

To describe all operations on the singular chain complex $C_*(X)$ of a toplogical space X and on its dual cochain complex $C^*(X)$ we consider the corresponding operad.

For $n \geq 0$ denote by Δ^n the normalized chain complex of the standard n-dimensional simplex. Then $\Delta^* = \{\Delta^n\}$ is the cosimplicial object in the category of chain complexes. Consider also the cosimplisial object $(\Delta^*)^{\otimes j} = \Delta^* \otimes \cdots \otimes \Delta^*$ and its realization

$$E^\Delta(j) = Hom(\Delta^*; (\Delta^*)^{\otimes j}),$$

where Hom is considered in the category of cosimplicial objects.

The family $E^\Delta = \{E^\Delta(j)\}$ will be the operad for which the actions of the symmetric groups and the operad structure are defined similary to the corresponding structures for the above defined operad $\mathcal{E}^X$, where instead of X we take Δ^*.

Note that the complexes Δ^n are acyclic and hence the operad E^Δ is also acyclic.

Theorem 1. *On the chain complex $C_*(X)$ of a topological space X there exists a natural coalgebra structure $\tau\colon C_*(X) \to \overline{E}^\Delta(C_*(X))$ with the following universal property: if for any operad $\mathcal{E}$ there is a natural $\mathcal{E}$ - coalgebra structure $\widetilde{\tau}\colon C_*(X) \to \overline{\mathcal{E}}(C_*(X))$, then there exist a unique operad mapping $\xi\colon \mathcal{E} \to E^\Delta$ such that $\widetilde{\tau} = \overline{\xi}(C_*(X)) \circ \tau$.*

Proof. Our aim is to define the natural operations

$$\tau\colon E^\Delta(j) \otimes C_*(X) \to C_*(X)^{\otimes j}.$$

Let $x_n \in C_n(X)$, $y \in E^\Delta(j) = Hom(\Delta^*; (\Delta^*)^{\otimes j})$. The element $x_n \in C_n(X)$ determines a chain mapping $\overline{x}_n\colon \Delta^n \to C_*(X)$ such that the generator $u_n \in \Delta^n$ maps to x_n.

Note that there is the operation $\tau^n\colon E^\Delta(j) \otimes \Delta^n \to (\Delta^n)^{\otimes j}$. Define the required operation τ putting

$$\tau(y \otimes x_n) = (\overline{x}_n)^{\otimes j} \circ \tau^n(y \otimes u_n).$$

Then we will have the following commutative diagram

$$\begin{array}{ccc}
E^\Delta(j) \otimes C_*(X) & \xrightarrow{\tau} & C_*(X)^{\otimes j} \\
{\scriptstyle 1\otimes\overline{x}_n}\uparrow & & \uparrow{\scriptstyle (\overline{x}_n)^{\otimes j}} \\
E^\Delta(j) \otimes \Delta^n & \xrightarrow{\tau^n} & (\Delta^n)^{\otimes j}
\end{array}$$

It is easy to see that the required relations are satisfied.

The E^Δ-coalgebra structure on the chain complex $C_*(X)$ of a topological space X induces the E^Δ-algebra structure on the cochain complex

$$C^*(X) = Hom(C_*(X); R).$$

The corresponding operations $\mu: E^\Delta(j) \otimes C^*(X)^{\otimes j} \to C^*(X)$ are defined by the formulas

$$\mu(y \otimes f_1 \otimes \cdots \otimes f_j)(x) = (f_1 \otimes \cdots \otimes f_j) \circ \tau(y \otimes x),$$

where $y \in E^\Delta(j)$, $f_i: C_*(X) \to R$, $x \in C_*(X)$. So we have

Theorem 1'. *On the cochain complex $C^*(X)$ of a topological space X there exists a natural algebra structure $\mu: E^\Delta(C^*(X)) \to C^*(X)$ with the following universal property: if for any operad $\mathcal{E}$ there is a natural $\mathcal{E}$ - algebra structure $\widetilde{\mu}: \mathcal{E}(C^*(X)) \to C^*(X)$, then there exists a unique operad mapping $\xi: \mathcal{E} \to E^\Delta$ such that $\widetilde{\mu} = \mu \circ \xi(C^*(X))$.*

Let $R(\Sigma_2)$ – the Σ_2-free resolution with generators e_i of dimensions i and the differential defined by the formula

$$d(e_i) = e_{i-1} + (-1)^i e_{i-1}T, \quad T \in \Sigma_2.$$

Since $E^\Delta(2)$ is acyclic, there is a Σ_2-chain mapping $R(\Sigma_2) \to E^\Delta(2)$ and hence a mapping

$$R(\Sigma_2) \otimes_{\Sigma_2} C^*(X)^{\otimes 2} \to C^*(X).$$

Its restriction on the elements e_i usually denoted by

$$\cup_i: C^*(X) \otimes C^*(X) \to C^*(X)$$

and called cup-i product.

Let $p^n: \Delta^n \to S\Delta^{n-1}$ be the projection obtained by contracting the $(n-1)$-dimensional face spanned by the vertixes with numbers $0, 1, \ldots, n-1$. These projections induce the projection of operads $E^\Delta \to SE^\Delta$. For a topological space X the suspension $SC_*(X)$ will be the coalgebra over the operad SE^Δ and the following diagram commutes

$$\begin{array}{ccc} SC_*(X) & \longrightarrow & \overline{SE}^\Delta(SC_*(X)) \\ \downarrow & & \downarrow \\ C_*(SX) & \longrightarrow & \overline{E}^\Delta(C_*(SX)) \end{array}$$

Iterating this construction we obtain the projections $E^\Delta \to S^n E^\Delta$ and the commutative diagrams

$$\begin{array}{ccc} S^n C_*(X) & \longrightarrow & \overline{S^n E}^\Delta(S^n C_*(X)) \\ \downarrow & & \downarrow \\ C_*(S^n X) & \longrightarrow & \overline{E}^\Delta(C_*(S^n X)) \end{array}$$

Note that the operad E^{Δ} may be not Σ-free and so it is not the E_{∞}-operad. To obtain the E_{∞}-operad we consider the operad $E^{\Delta} \otimes E\Sigma$ and denote it simply by E.

The projection $E \to E^{\Delta}$ induces the E-coalgebra structure on the chain complex $C_*(X)$ of a topological space X. The mapping $E^{\Delta} \to SE^{\Delta}$ induces the operad mapping $E \to SE$ and for the chain complexe $C_*(S^n X)$ there is the corresponding commutative diagram similar to the diagram for the operad E^{Δ}.

§3. Bar and cobar constructions over operads

Let $\mathcal{E}$ be an operad with right action on a symmetric family $\mathcal{F}$, $\nu\colon \mathcal{F} \times \mathcal{E} \to \mathcal{F}$, and let X be an algebra over the operad $\mathcal{E}$, $\mu\colon \mathcal{E}(X) \to X$. Consider the simplicial object $B_*(\mathcal{F}, \mathcal{E}, X) = \{B_n(\mathcal{F}, \mathcal{E}, X)\}$ for which $B_n(\mathcal{F}, \mathcal{E}, X) = \mathcal{F}\mathcal{E}^n(X)$ with face and degeneracy operators given by the formulas

$$d_0 = \nu 1^n,\ d_i = 1^i \gamma 1^{n-i},\ 0 < i < n;$$
$$d_n = 1^n \mu,\ s_j = 1^{j+1} i 1^{n-j+1}, 0 \le j \le n.$$

Its realization

$$B(\mathcal{F}, \mathcal{E}, X) = |B_*(\mathcal{F}, \mathcal{E}, X)| = \sum_n \Delta^n \otimes B_n(\mathcal{F}, \mathcal{E}, X)/\sim$$

is called the bar construction.

In the case of trivial $\mathcal{F}$, i.e. $\mathcal{F}(1) = R$ and $\mathcal{F}(j) = 0$ if $j \ge 2$, the corresponding bar construction denoted by $B(\mathcal{E}, X)$.

If X is an A-algebra, i.e. simply an algebra then the bar construction $B(A, X)$ will be chain equivalent to the desuspension over the usual Adams bar construction $B(X)$, i.e.

$$B(A, X) \simeq S^{-1} BX,\ [8].$$

Dually, if X is a coalgebra over the operad $\mathcal{E}$, $\tau\colon X \to \overline{\mathcal{E}}(X)$, then we can consider the cosimplicial object $F^*(\mathcal{F}, \mathcal{E}, X) = \{F^n(\mathcal{F}, \mathcal{E}, X)\}$ for which $F^n(\mathcal{F}, \mathcal{E}, X) = \overline{\mathcal{F}\mathcal{E}}^n(X)$ with coface and codegeneracy operators given by the formulas

$$\delta^0 = \overline{\nu} 1^n,\ \delta^i = 1^i \overline{\gamma} 1^{n-i},\ 0 < i < n;$$
$$\delta^n = 1^n \tau,\ \sigma^j = 1^{j+1} p 1^{n-j+1}, 0 \le j \le n.$$

Its realization

$$F(\mathcal{F}, \mathcal{E}, X) = |F^*(\mathcal{F}, \mathcal{E}, X)| = Hom(\Delta^*; F^*(\mathcal{F}, \mathcal{E}, X)),$$

where Hom is considered in the category of cosimplicial objects, is called the cobar construction.

In the case of trivial $\mathcal{F}$ the corresponding bar construction denoted by $F(\mathcal{E}, X)$.

If X is an A-coalgebra, i.e. simply a coalgebra then the cobar construction $F(A, X)$ will be chain equivalent to the suspension over the usual Adams cobar construction $F(X)$, i.e.

$$F(A, X) \simeq SFX,\ [8].$$

Let now X be an n-connected topological space. As it was shown above, on the chain complex $S^nC_*(\Omega^n X)$ there is the S^nE-coalgebra structure

$$\tau: S^nC_*(\Omega^n X) \to \overline{S^nE}(S^nC_*(\Omega^n X)).$$

This structure and the mapping $j: S^n\Omega^n X \to X$ induce the mapping

$$S^nC_*(\Omega^n X) \to \overline{S^nE}(S^nC_*(\Omega^n X)) \to \overline{S^nE}(C_*(X)).$$

This mapping is a coaugmentation of the cosimplicial object $F^*(S^nE, E, C_*(X))$ and hence it induces the mapping of S^nE-coalgebras

$$S^nC_*(\Omega^n X) \to F(S^nE, E, C_*(X)).$$

Theorem 2. *For any n-connected topological space X the mapping*

$$S^nC_*(\Omega^n X) \to F(S^nE, E, C_*(X))$$

is a chain equivalence of S^nE-coalgebras.

Proof. For $n = 1$ the chain equivalence $SC_*(\Omega X) \to F(SE, E, C_*(X))$ follows from the Adams chain equivalence

$$SC_*(\Omega X) \simeq SFC_*(X)$$

and from the chain equivalence

$$SFC_*(X) \simeq F(SE, E, C_*(X)),\ [8].$$

Suppose that for any $(n-1)$-connected topological space X we have a chain equivalence

$$S^{n-1}C_*(\Omega^{n-1}X) \to F(S^{n-1}E, E, C_*(X)).$$

Then for an n-connected topological space X we will have the following sequence of chain equivalences

$$S^nC_*(\Omega^n X) \simeq SF(S^{n-1}E, E, C_*(\Omega X)) \simeq F(S^nE, SE, SC_*(\Omega X)) \simeq$$
$$F(S^nE, SE, F(SE, E, C_*(X))) \simeq F(S^nE, E, C_*(X)).$$

Its composition will be the desirable chain equivalence

$$S^nC_*(\Omega^n X) \simeq F(S^nE, E, C_*(X)).$$

§4. The spectral sequence for the homology of iterated loop spaces

Let X be an E-coalgebra. Consider the spectral sequence of the cobar construction $F(S^nE, E, X)$ over the filtration determined by the operad grading. Namely, for any operad $\mathcal{E}$ we define the grading of mappings $f: \mathcal{E}(j) \to X^{\otimes j}$ equal to j. So we will have the grading of the elements of $\overline{S^nE}(X)$, $\overline{S^nE} \circ \overline{E}(X)$ and so on. Thus we will have the grading of the elements of the cobar construction $F(S^nE, E, X)$ which determines in it the decreasing filtration.

Note that the mappings $\overline{\mu}: \overline{S^nE} \to \overline{S^nE} \circ \overline{E}$, $\overline{\gamma}: \overline{E} \to \overline{E} \circ \overline{E}$ preserve gradings and the mapping $\tau: X \to \overline{E}(X)$ increases it. Therefore the first term of the corresponding spectral sequence will be isomorphic to the homology of the cobar construction $F(S^nE, E, X)$, where X is considered as a trivial E-coalgebra. To calculate its homology recall the notion of the iterated Adams cobar construction.

Let K be a cocommutative coalgebra. Then the Adams cobar construction will be a cocommutative Hopf algebra with a coproduct $\nabla: FK \to FK \otimes FK$ defined on the generators $[x] \in FK$ by the formula

$$\nabla[x] = [x] \otimes 1 + 1 \otimes [x].$$

Note that a commutative coalgebra K may be considered as an E-coalgebra. The required E-coalgebra structure is induced by the projection $E \to E_0$. Moreover, the chain equivalence $SFK \simeq F(SE, E, K)$ will be a chain equivalence of SE-coalgebras.

Thus, in this case the Adams cobar construction may be iterated and there are chain equivalences

$$S^nF^nK \simeq F(S^nE, E, K).$$

In particular, if a chain complex X has the trivial E-coalgebra structure then there are the chain equivalences

$$S^nF^nX \simeq F(S^nE, E, X).$$

So to calculate the homology of the cobar construction $F(S^nE, E, X)$ over a trivial E-coalgebra X it is sufficient to calculate the homology of the iterated Adams cobar construction F^nX.

Consider the case of $Z/2$-coefficients. Denote $X_* = H_*(X)$. The homology $H_*(FX)$ is isomorphic to the tensor algebra $TS^{-1}X_*$ over the desuspension $S^{-1}X_*$ over X_*, i.e.

$$H_*(FX) \cong TS^{-1}X_*.$$

The double cobar construction F^2X will be chain equivalent to the cobar construction $FTS^{-1}X_*$, where $TS^{-1}X_*$ is considered as a Hopf algebra with a coproduct

$$\nabla: TS^{-1}X_* \to TS^{-1}X_* \otimes TS^{-1}X_*$$

determined on the generators $[x] \in S^{-1}X_*$ by the formula

$$\nabla[x] = [x] \otimes 1 + 1 \otimes [x].$$

Denote by $LS^{-1}X_*$ the free Lie algebra generated by the module $S^{-1}X_*$. Then by the Poincare-Birkhoff-Witt theorem there is an isomorphism of coalgebras

$$TS^{-1}X_* \cong T_sLS^{-1}X_*,$$

where $T_sLS^{-1}X_*$ denotes the free commutative algebra generated by the module $LS^{-1}X_*$.

The homology of $FT_sLS^{-1}X_*$ is isomorphic to the free commutative algebra generated by the module $S^{-1}PT_sLS^{-}X_*$, i.e.

$$H_*(F^2X) \cong T_sS^{-1}PT_sLS^{-1}X_*,$$

where $PT_sLS^{-1}X_*$ denotes the module of primitive elements of $TS^{-1}X_*$, i.e.

$$PT_sLS^{-1}X_* = \{y \in T_sLS^{-1}X_* | \ \nabla(y) = y \otimes 1 + 1 \otimes y\}.$$

Note that the module PT_sM of primitive elements of the Hopf algebra T_sM is generated by the elements of the form x^{2^k}, $x \in M$, $k \geq 0$.

So this construction may be iterated and by induction we obtain isomorphisms

$$H_*(F^nX) \cong T_s(S^{-1}PT_s)^{n-1}LS^{-1}X_*,$$

and hence isomorphisms

$$S^{-n}H_*(F(S^nE, E, X)) \cong T_s(S^{-1}PT_s)^{n-1}LS^{-1}X_*.$$

Recall that a graded module L (over $Z/2$) is called an n-Lie algebra if there is given an operation $[\ ,\ ]: L \otimes L \to L$ called an n-Lie bracket of dimension n and satisfying the relations

$$\begin{gathered} [x, x] = 0; \\ [x, y] + [y, x] = 0; \\ [x, [y, z]] = [[x, y], z] + [y, [x, z]]. \end{gathered}$$

For a graded module M denote by $\mathcal{E}_nM$ the module generated by the elements $e_{i_1} \dots e_{i_k} x$, where $x \in M$, $0 \leq i_1 \leq \dots \leq i_k \leq n$, and the dimensions of these elements are defined equal to $i_1 + 2i_2 + \dots + 2^{k-1}i_k + 2^k dim(x)$.

These elements $e_{i_1} \dots e_{i_k} x$ of $\mathcal{E}_nM$ may be rewritten using the Dyer-Lashof algebra $\mathcal{R}$ in the form

$$Q^{j_1} \dots Q^{j_k} x; \quad j_1 \leq 2j_2, \dots, j_{k-1} \leq 2j_k, dim(x) \leq j_k \leq dim(x) + n,$$

where

$$\begin{gathered} j_k = i_k + dim(x), \\ j_{k-1} = i_{k-1} + i_k + 2dim(x), \\ \dots \\ j_1 = i_1 + i_2 + 2i_3 + \dots + 2^{k-2}i_k + 2^{k-1}dim(x). \end{gathered}$$

It is clear that the sequences $Q^{j_1} \dots Q^{j_k}$ are admissible and represent elements of the Dyer-Lashof algebra $\mathcal{R}$.

Denote by $\mathcal{R}_nM$ the submodule of $\mathcal{R} \otimes M$ generated by the elements

$$Q^{j_1} \dots Q^{j_k} \otimes x$$

where $Q^{j_1} \dots Q^{j_k}$ – admissible sequences with $dim(x) \le j_k \le dim(x) + n$. Then there is an isomorphism $\mathcal{E}_n M \cong \mathcal{R}_n M$.

The correspondence $M \longmapsto \mathcal{R}_n M$ determines the monad $\mathcal{R}_n$ in the category of graded modules. Algebras over the monad $\mathcal{R}_n$ we will call $\mathcal{R}_n$-modules.

For an n-Lie algebra L_n we will have the module $\mathcal{R}_n L_n$. Denote also by $T_s \mathcal{R}_n L_n$ the quotient algebra of the free commutative algebra generated by the module $\mathcal{R}_n L_n$ over the relations $x \cdot x = e_0 x$.

From the above considerations it follows that if X is a chain complex (over $Z/2$) considered as the trivial E-coalgebra, then there are the isomorphisms

$$S^{-n} H_*(F(S^n E, E, X)) \cong T_s \mathcal{R}_{n-1} L_{n-1} S^{-n} X_*.$$

Hence we have

Theorem 3. *If X is an n-connected topological space then there is the spectral sequence which converges to $H_*(\Omega^n X)$ and for the first term of this spectral sequence (over $Z/2$) there is the isomorphism*

$$E^1 \cong T_s \mathcal{R}_{n-1} L_{n-1} S^{-n} H_*(X).$$

Consider now Z/p-coefficients, $p > 2$. Note that the module $PT_s M$ of primitive elements of the Hopf algebra $T_s M$ in this case is generated by the elements $x \in M$ and the elements x^{p^k} for which $k > 0$ and $dim(x)$ – even. However, the homology $H_*(FT_s M)$ is generated not only by the primitive elements of $T_s M$ but also by the elements βx^{p^k} where β – the Bockstein homomorphism. Denote the module generated by the primitive elements and the elements βx^{p^k} by $P_\beta T_s M$. Then the homology of $FT_s M$ will be isomorphic to $T_s S^{-1} P_\beta T_s M$. Thus for the homology $H_*(F^n X)$ of the iterated cobar constructions $F^n X$ over Z/p-coefficients there are the isomorphisms

$$H_*(F^n X) \cong T_s (S^{-1} P_\beta T_s)^{n-1} L S^{-1} X_*,$$

and hence the isomorphisms

$$S^{-n} H_*(F(S^n E, E, X)) \cong T_s (S^{-1} P_\beta T_s)^{n-1} L S^{-1} X_*.$$

Recall that a graded module L (over Z/p) is called an n-Lie algebra if there is given an operation $[\ ,\ ] \colon L \otimes L \to L$ of dimension n, called an n-Lie bracket and satisfying the relations

$$[x, y] + (-1)^\epsilon [y, x] = 0;$$
$$[x, [y, z]] = [[x, y], z] + (-1)^\epsilon [y, [x, z]];$$

where $\epsilon = (dim(x) + n) \cdot (dim(y) + n)$.

For a graded module M denote by $\mathcal{E}_n(M)$ the module generated by the sequences $\beta^{\epsilon_1} e_{i_1} \dots \beta^{\epsilon_k} e_{i_k} x$, where $x \in M$, $\epsilon = 0, 1$, $0 \le i_1 \le \dots \le i_k \le n$. For the elements

$e_i y$ it is demanded that i and $dim(y)$ have the same parity and the dimensions of the elements $e_i y$ are defined equal to $p \cdot dim(y) + (p-1)i$.

These elements $\beta^{\epsilon_1} e_{i_1} \dots \beta^{\epsilon_k} e_{i_k} x$, may be rewritten using the mod-p Dyer-Lashof algebra $\mathcal{R}$ in the form $\beta^{\epsilon_1} Q^{j_1} \dots \beta^{\epsilon_k} Q^{j_k} x$, where

$$\begin{gathered}2j_k = i_k + dim(x),\\ 2j_{k-1} = i_{k-1} + (p-1)i_k + pdim(x) - \epsilon_k,\\ \dots\\ 2j_1 = i_1 + (p-1)i_2 - \epsilon_2 + \dots + p^{k-1} dim(x).\end{gathered}$$

It is clear that the sequences $\beta^{\epsilon_1} Q^{j_1} \dots \beta^{\epsilon_k} Q^{j_k}$ are admissible and represent elements of the Dyer-Lashof algebra $\mathcal{R}$.

Denote by $\mathcal{R}_n M$ the submodule of $\mathcal{R} \otimes M$ generated by the elements

$$\beta^{\epsilon_1} Q^{j_1} \dots \beta^{\epsilon_k} Q^{j_k} \otimes x$$

where $\beta^{\epsilon_1} Q^{j_1} \dots \beta^{\epsilon_k} Q^{j_k}$ are admissible sequences with $dim(x) \leq 2j_k \leq dim(x) + n$. Then there is the isomorphism $\mathcal{E}_n M \cong \mathcal{R}_n M$.

The correspondence $M \longmapsto \mathcal{R}_n M$ determines the monad $\mathcal{R}_n$ in the category of graded Z/p-modules. Algebras over the monad $\mathcal{R}_n$ we will call $\mathcal{R}_n$-modules.

For an n-Lie algebra L_n we will have the module $\mathcal{R}_n L_n$. Denote also by $T_s \mathcal{R}_n L_n$ the quotient algebra of the free commutative algebra generated by the module $\mathcal{R}_n L_n$ over the relations $e_0 x = x^p$.

From the above considerations it follows that if X is a chain complex (over Z/p) considered as the trivial E-coalgebra, then there are the isomorphisms

$$S^{-n} H_*(F(S^n E, E, X)) \cong T_s \mathcal{R}_{n-1} L_{n-1} S^{-n} X_*.$$

Hence we have

Theorem 3'. *For the first term of the considered spectral sequence of $H_*(\Omega^n X)$ (over Z/p) there is the isomorphism*

$$E^1 \cong T_s \mathcal{R}_{n-1} L_{n-1} S^{-n} H_*(X).$$

Note that in the case of characteristic zero coefficients the module $PT_s M$ of primitive elements of $T_s M$ is isomorphic to M. Hence we have

Theorem 3". *The first term of the considered spectral sequence of $H_*(\Omega^n X)$ (over a field of characteristic zero) is isomorphic to the free $(n-1)$-Poisson algebra generated by $S^{-n} H_*(X)$, i.e.*

$$E^1 \cong T_s L_{n-1} S^{-n} H_*(X) = P_{n-1} S^{-n} H_*(X).$$

§5. The second term of the spectral sequence of the homology

of iterated loop spaces

To determine the second term of the spectral sequence of the homology of iterated loop spaces firstly we define the notion of a $\mathcal{P}_n$-algebra over $Z/2$-coefficients generalizing the notion of an n-Poisson algebra.

A graded module M (over $Z/2$) will be called a $\mathcal{P}_n$-algebra if

1. There is given a structure of a commutative algebra

$$x \otimes y \longmapsto x \cdot y, \quad x, y \in M.$$

2. There is given a structure of an n-Lie algebra

$$x \otimes y \longmapsto [x, y], \quad x, y \in M,$$

and the n-Lie algebra structure with the commutative algebra structure form an n-Poisson algebra structure.

3. There is given a structure of an $\mathcal{R}_n$-module

$$\mathcal{R}_n M \to M,$$

compatible with the n-Poisson algebra structure [4].

Denote by $\mathcal{P}_n$ the monad which corresponds to a graded module M the free $\mathcal{P}_n$-algebra generated by M. Then there is the isomorphism

$$\mathcal{P}_n(M) \cong T_s \mathcal{R}_n L_n M$$

and so we have

Theorem 4. *The first term of the considered spectral sequence of $H_*(\Omega^n X)$ (over $Z/2$) is isomorphic to the free $\mathcal{P}_{n-1}$-algebra generated by $S^{-n}H_*(X)$, i.e.*

$$E^1 \cong \mathcal{P}_{n-1} S^{-n} H_*(X).$$

Consider the relation between the Steenrod algebra and the Dyer-Lashof algebra. Let $\mathcal{A}$ be the mod-2 Steenrod algebra with $Sq^0 \neq 1$ and $\mathcal{K}$ – the Milnor coalgebra (dual to the Steenrod algebra). We will consider $\mathcal{K}$ as a family $\mathcal{K}(m)$ of polinomial algebras $\mathcal{K}(m)$, $m \geq 1$ generated by the elements $\xi_i(m)$, $1 \leq i \leq m$ of dimensions $2^i - 1$.

Define products $\mathcal{K} \otimes \mathcal{K}$, $\mathcal{K} \times \mathcal{K}$ to be the families

$$(\mathcal{K} \otimes \mathcal{K})(m) = \mathcal{K}(m) \otimes \mathcal{K}(m),$$
$$(\mathcal{K} \times \mathcal{K})(m) = \sum_k \mathcal{K}(k) \otimes \mathcal{K}(m-k).$$

We denote the multiplication by $\pi: \mathcal{K} \otimes \mathcal{K} \to \mathcal{K}$. The comultiplication $\nabla: \mathcal{K} \to \mathcal{K} \times \mathcal{K}$ is defined on the generators by the formula

$$\nabla(\xi_i(m)) = \sum_{j,k} \xi_{i-j}^{2^j}(k) \otimes \xi_j(m-k).$$

On the other elements the comultiplication is determined by the Hopf relation. So we can consider $\mathcal{K}$ as a Hopf algebra.

Let now $\mathcal{R}$ be the Dyer-Lashof algebra. Define a homomorphism $\varphi\colon\mathcal{K}\to\mathcal{R}$ of dimension -1 putting

$$\varphi(x)=\begin{cases} Q^{i-1}, & x=\xi_1^i(1),\\ 0, & \textit{otherwise}.\end{cases}$$

It is easy to see that $\varphi\cup\varphi=0$ and hence φ is a twisted cochain.

Denote by $\overline{E}_*$ the comonad in the category of graded modules which corresponds to a graded module M the graded module $H_*(\overline{E}(M))$. The E-coalgebra structure on the chain complex $C_*(X)$ of a topological space X induces on its homology $H_*(X)$ the $\overline{E}_*$-coalgebra structure. This structure consists of the commutative coalgebra structure

$$\nabla_*\colon H_*(X)\to H_*(X)\otimes H_*(X)$$

and of the coaction

$$H_*(X)\to\mathcal{K}\otimes H_*(X).$$

For an n-connected space X these structures induce on $\mathcal{P}_{n-1}S^{-n}H_*(X)$ a differential d_φ defined on the generators $s^{-n}x_{i+n}\in H_{i+n}(X)$ by the formula

$$d_\varphi(s^{-n}x_{i+n})=\sum_{x'<x''}[s^{-n}x',s^{-n}x'']+\varphi\cap s^{-n}x_{i+n}.$$

Denote the corresponding differential $\mathcal{P}_{n-1}$-algebra by $\mathcal{P}_{n-1\varphi}S^{-n}H_*(X)$. Then we will have

Theorem 5. *For the second term of the spectral sequence of $H_*(\Omega^nX)$ (over $Z/2$) there is the isomorphism*

$$E^2\cong H_*(\mathcal{P}_{n-1\varphi}S^{-n}H_*(X)).$$

Consider now Z/p-coefficients. A graded module M (over Z/p) will be called a $\mathcal{P}_n$-algebra if

1. There is given a structure of a commutative algebra

$$x\otimes y\longmapsto x\cdot y,\quad x,y\in M.$$

2. There is given a structure of an n-Lie algebra

$$x\otimes y\longmapsto[x,y],\quad x,y\in M,$$

and the n-Lie algebra structure with the commutative algebra structure form an n-Poisson algebra structure.

3. There is given a structure of an $\mathcal{R}_n$-module

$$\mathcal{R}_nM\to M,$$

compatible with the n-Poisson algebra structure [4].

Denote by $\mathcal{P}_n$ the monad which corresponds to a graded module M the free $\mathcal{P}_n$-algebra generated by M. Then there is the isomorphism

$$\mathcal{P}_n(M) \cong T_s \mathcal{R}_n L_n M$$

and hence we have

Theorem 4'. *The first term of the considered spectral sequence of $H_*(\Omega^n X)$ (over Z/p) is isomorphic to the free $\mathcal{P}_{n-1}$-algebra generated by $S^{-n}H_*(X)$, i.e.*

$$E^1 \cong \mathcal{P}_{n-1} S^{-n} H_*(X).$$

Consider the relation between the mod-p Steenrod algebra and the mod-p Dyer-Lashof algebra. Let $\mathcal{A}$ be the mod-p Steenrod algebra with $Sq^0 \neq 1$ and $\mathcal{K}$ – the Milnor coalgebra (dual to the Steenrod algebra). We will consider $\mathcal{K}$ as a family $\mathcal{K}(m)$ of polinomial algebras $\mathcal{K}(m)$, $m \geq 1$ generated by the elements $\xi_i(m)$, $\tau_i(m)$, $1 \leq i \leq m$ of dimensions $2(p^i - 1)$ and $2p^i - 1$ correspondingly.

There are a multiplication

$$\pi \colon \mathcal{K} \otimes \mathcal{K} \to \mathcal{K}$$

and a comultiplication

$$\nabla \colon \mathcal{K} \to \mathcal{K} \times \mathcal{K}$$

defined on the generators by the formulas

$$\nabla(\xi_i(m)) = \sum_{j,k} \xi_{i-j}^{p^j}(k) \otimes \xi_j(m-k).$$

$$\nabla(\tau_i(m)) = \tau_i(m) \otimes 1 + \sum_{j,k} \xi_{i-j}^{p^j}(k) \otimes \tau_j(m-k).$$

On the other elements the comultiplication is determined by the Hopf relation. So we can consider $\mathcal{K}$ as a Hopf algebra.

Let now $\mathcal{R}$ be the mod-p Dyer-Lashof algebra. Define the homomorphism $\varphi \colon \mathcal{K} \to \mathcal{R}$ of dimension -1 putting

$$\varphi(x) = \begin{cases} \beta P^i, & x = \xi_1^i(1), \\ P^i, & x = \xi_1^{i-1}(1)\tau_1(1), \\ 0, & otherwise. \end{cases}$$

It is easy to see that $\varphi \cup \varphi = 0$ and hence φ is the twisted cochain.

Denote by $\overline{E}_*$ the comonad in the category of graded Z/p-modules which corresponds to a graded module M the graded module $H_*(\overline{E}(M))$. The E-coalgebra structure on the chain complex $C_*(X)$ of a topological space X induces on its homology $H_*(X)$ the $\overline{E}_*$-coalgebra structure. This structure consists of the commutative coalgebra structure

$$\nabla_* \colon H_*(X) \to H_*(X) \otimes H_*(X)$$

and of the coaction of the Milnor coalgebra

$$H_*(X) \to \mathcal{K} \otimes H_*(X).$$

For an n-connected space X these structures induce on $\mathcal{P}_{n-1}S^{-n}H_*(X)$ a differential d_φ defined on the generators $s^{-n}x_{i+n} \in H_{i+n}(X)$ by the formula

$$d_\varphi(s^{-n}x_{i+n}) = \sum_{x'<x''} (-1)^\epsilon [s^{-n}x', s^{-n}x''] + \varphi \cap s^{-n}x_{i+n},$$

where $\sum x' \otimes x'' = \nabla_*(x_{i+n})$, $\epsilon = n \cdot dim(x')$.

Denote the corresponding differential $\mathcal{P}_{n-1}$-algebra by $\mathcal{P}_{n-1\varphi}S^{-n}H_*(X)$. Then we will have

Theorem 5'. *For the second term of the considered spectral sequence of $H_*(\Omega^n X)$ (over Z/p) there is the isomorphism*

$$E^2 \cong H_*(\mathcal{P}_{n-1\varphi}S^{-n}H_*(X)).$$

Note that in the case of characteristic zero coefficients the $\overline{E}_*$-coalgebra structure on the homology $H_*(X)$ of a topological space X consists only of the commutative coalgebra structure

$$\nabla_*: H_*(X) \to H_*(X) \otimes H_*(X).$$

Hence we have

Theorem 5". *For the second term of the considered spectral sequence of $H_*(\Omega^n X)$ (over a field of characteristic zero) there is the isomorphism*

$$E^2 \cong H_*(P_{n-1\varphi}S^{-n}H_*(X)),$$

where the differential d_φ defined on the generators $s^{-n}x_{i+n} \in H_{i+n}(X)$ by the formula

$$d_\varphi(s^{-n}x_{i+n}) = \sum_{x'<x''} (-1)^\epsilon [s^{-n}x', s^{-n}x''],$$

where $\sum x' \otimes x'' = \nabla_*(x_{i+n})$, $\epsilon = n \cdot dim(x')$.

§6. The homology of iterated loop spaces over the real projective spaces

Ones of the most important spaces in Algebraic Topology, besides spheres, are the real projective spaces RP^n and RP^∞. In some sense these spaces are opposite to spheres. The homology $H_*(RP^\infty)$ (over $Z/2$) is the free $\overline{E}_*$-coalgebra with one 1-dimensional generator, inspite of spheres that homology are trivial $\overline{E}_*$-coalgebras.

From the other side the Adams spectral sequence of stable homotopy groups of RP^∞ is very similar to the corresponding spectral sequence of spheres, and it is the problem to find the relations between them.

Here we consider the problem of calculation the homology

$$H_*(\Omega^m(RP^\infty/RP^n)), \quad m \leq n,$$

of the iterated loop spaces over the space RP^∞/RP^n with $Z/2$ coefficients.

Denote the i-dimensional generator of $H_*(RP^\infty)$ as e_i. A coalgebra structure on this generators is determined by the formula

$$\nabla(e_i) = \sum_j e_j \otimes e_{i-j}.$$

An action of the Steenrod algebra is determined by the formula

$$Sq^j(e_i) = \binom{i-j}{j} e_{i-j}.$$

Since $H_*(RP^\infty)$ is the free $\overline{E}_*$-coalgebra, there are no higher $\overline{E}_*$-operations. The homology $H_*(RP^n)$ is the $\overline{E}_*$-subcoalgebra of $H_*(RP^\infty)$ and hence the $\overline{E}_*$-coalgebra structure on $H_*(RP^\infty)$ induces the $\overline{E}_*$-coalgebra structure on

$$H_*(RP^\infty)/H_*(RP^n) \cong H_*(RP^\infty/RP^n).$$

Therefore on the homology $H_*(RP^\infty/RP^n)$ there are no higher $\overline{E}_*$-operations. So the spectral sequence of the homology of the iterated loop spaces over RP^∞/RP^n has no higher differentials and we have

Theorem 6. *The homology $H_*(\Omega^m(RP^\infty/RP^n))$, $n \geq m$ (over $Z/2$) is isomorphic to the homology of the differential $\mathcal{P}_{m-1}$-algebra $\mathcal{P}_{m-1\varphi}S^{-m}H_*(RP^\infty/RP^n)$, where the differential d_φ on the generators $s^{-m}e_i$ is defined by the formula*

$$d_\varphi(s^{-m}e_i) = \sum_{\substack{j \\ 2j<i}} [s^{-m}e_j, s^{-m}e_{i-j}] + \sum_{\substack{j \\ 2j\leq i}} \binom{i-j}{j} e_{2j-i+m-1}(s^{-m}e_{i-j}).$$

Note that in the case $m = 1$ we have the isomorphisms

$$\mathcal{P}_0 S^{-1}H_*(RP^\infty/RP^n) \cong T_sLS^{-1}H_*(RP^\infty/RP^n) \cong TS^{-1}H_*(RP^\infty/RP^n),$$

and the differential in the tensor algebra $TS^{-1}H_*(RP^\infty/RP^n)$ on the generators $u_i = s^{-1}e_{i+1}$ has very simple form

$$d_\varphi(u_i) = \sum_j u_j \otimes u_{i-j-1}.$$

From here it follows that the homology $H_*(\Omega(RP^\infty/RP^n))$ (over $Z/2$) is isomorphic to the algebra generated by the elements u_i, $n \leq i \leq 2n$, of dimensions i and relations

$$u_n \cdot u_n = 0;$$
$$u_n \cdot u_{n+1} + u_{n+1} \cdot u_n = 0;$$
$$\dots$$
$$u_n \cdot u_{2n} + \dots + u_{2n} \cdot u_n = 0.$$

So as a graded module the homology $H_*(\Omega(RP^\infty/RP^n))$ is generated by the noncommutative products $u_{n_1}\cdot\ldots\cdot u_{n_k}$ with $n\leq n_1\leq 2n$, $n<n_2,\ldots,n_k\leq 2n$.

Consider the homology $H_*(\Omega^2(RP^\infty/RP^2))$. It is isomorphic to the homology of the differential $\mathcal{P}_1$- algebra $\mathcal{P}_{1\varphi}\{u_i|\ i\geq 1\}$, where $u_i=s^{-2}e_{i+2}$ and the differential is determined by the formulas

$$d_\varphi(u_{2i+1})=\sum_{j<i}[u_j,u_{2i-j-1}]+\binom{i+2}{i+1}e_0(u_i);$$
$$d_\varphi(u_{2i+2})=\sum_{j<i}[u_j,u_{2i-j}]+e_1(u_i).$$

In small dimensions we have
$d(u_1)=0;$
$d(u_2)=0;$
$d(u_3)=u_1u_1;$
$d(u_4)=e_1(u_1);$
$d(u_5)=[u_1,u_2];$
$d(u_6)=[u_1,u_3]+e_1(u_2);$
$d(u_7)=[u_1,u_4]+[u_2,u_3]+u_3u_3;$
$d(u_8)=[u_1,u_5]+[u_2,u_4]+e_1(u_3).$

From these formulas it follows that the homology $H_i=H_i(\Omega^2(RP^\infty/RP^2))$ in small dimensions i has the following generators
$H_1:\quad u_1;$
$H_2:\quad u_2;$
$H_3:\quad u_1u_2;$
$H_4:\quad u_2^2;$
$H_5:\quad u_1u_2^2,\ e_1(u_2);$
$H_6:\quad u_1e_1(u_2),\ u_2^3,\ [u_2,u_3],\ u_3^2;$
$H_7:\quad u_1u_2^3,\ u_1u_3^2,\ u_1[u_2,u_3],\ u_2e_1(u_2),\ e_1(u_3).$

Consider the homology $H_*(\Omega^3(RP^\infty/RP^3))$. It is isomorphic to the homology of the differential $\mathcal{P}_2$- algebra $\mathcal{P}_{2\varphi}\{u_i|\ i\geq 1\}$, where $u_i=s^{-3}e_{i+3}$ and the differential is determined by the formulas

$$d_\varphi(u_{2i+1})=\sum_{j<i-1}[u_k,u_{2i-j-2}]+\binom{i+3}{i+1}e_0(u_i)+e_2(u_{i-1});$$
$$d_\varphi(u_{2i+2})=\sum_{j\leq i-1}[u_j,u_{2i-j-1}]+\binom{i+3}{i+2}e_1(u_i).$$

In small dimensions we have
$d(u_1)=0;$
$d(u_2)=0;$
$d(u_3)=0;$
$d(u_4)=0;$
$d(u_5)=e_2(u_1);$

$d(u_6) = [u_1, u_2] + e_1(u_2);$
$d(u_7) = [u_1, u_3] + e_2(u_2) + u_3u_3;$
$d(u_8) = [u_1, u_4] + [u_2, u_3].$

From these formulas it follows that the homology $H_i = H_i(\Omega^3(RP^\infty/RP^3))$ in small dimensions i has the following generators
$H_1: \quad u_1;$
$H_2: \quad u_1^2,\ u_2;$
$H_3: \quad u_1^3,\ u_1u_2,\ e_1(u_1),\ u_3;$
$H_4: \quad u_1^4,\ u_1^2u_2,\ u_1e_1(u_1),\ u_1u_3,\ u_2^2,\ u_4;$
$H_5: \quad u_1^5,\ u_1^3u_2,\ u_1^2u_3,\ u_1^2e_1(u_1),\ u_1u_2^2,\ u_1u_4,\ u_2u_3,\ u_2e_1(u_1),\ e_1(u_2);$
$H_6: \quad u_1^6,\ u_1^4u_2,\ u_1^3u_3,\ u_1^3e_1(u_1),\ u_1^2u_2^2,\ u_1^2u_4,\ u_1u_2u_3,\ u_1u_2e_1(u_1),\ u_1e_1(u_2),$ $u_2u_4,\ u_2^3,\ u_3e_1(u_1),\ u_3^2,\ e_2(u_2),\ e_1(u_1)^2;$
$H_7: \quad u_1^7,\ u_1^5u_2,\ u_1^4u_3,\ u_1^4e_1(u_1),\ u_1^3u_2^2,\ u_1^3u_4,\ u_1^2u_2u_3,\ u_1^2u_2e_1(u_1),\ u_1^2e_1(u_2),$ $u_1u_2u_4,\ u_1u_2^3,\ u_1u_3e_1(u_1),\ u_1u_3^2,\ u_1e_2(u_2),\ u_1e_1(u_1)^2,\ u_2^2u_3,\ u_2^2e_1(u_1),$ $u_2e_1(u_2),\ u_3u_4,\ [u_2, u_3],\ e_1(u_3),\ e_1e_1(u_1).$

Consider also the homology $H_*(\Omega^4(RP^\infty/RP^4))$. It is isomorphic to the homology of the differential $\mathcal{P}_3$- algebra $\mathcal{P}_{3\varphi}\{u_i|\ i \geq 1\}$, where $u_i = s^{-4}e_{i+4}$ and the differential is determined by the formulas

$$d_\varphi(u_{2i+1}) = \sum_{j<i-1} [u_k, u_{2i-j-3}] + \binom{i+4}{i+1} e_0(u_i) + \binom{i+3}{i+2} e_2(u_{i-1});$$
$$d_\varphi(u_{2i+2}) = \sum_{j<i-1} [u_j, u_{2i-j-2}] + \binom{i+4}{i+2} e_1(u_i) + e_3(u_{i-1}).$$

In small dimensions we have
$d(u_1) = 0;$
$d(u_2) = 0;$
$d(u_3) = 0;$
$d(u_4) = 0;$
$d(u_5) = e_2(u_1);$
$d(u_6) = e_1(u_2) + e_3(u_1);$
$d(u_7) = [u_1, u_2] + u_3u_3;$
$d(u_8) = [u_1, u_3] + e_1(u_3) + e_3(u_2).$

From these formulas it follows that the homology $H_i = H_i(\Omega^4(RP^\infty/RP^4))$ in small dimensions i has the following generators
$H_1: \quad u_1;$
$H_2: \quad u_1^2,\ u_2;$
$H_3: \quad u_1^3,\ u_1u_2,\ e_1(u_1),\ u_3;$
$H_4: \quad u_1^4,\ u_1^2u_2,\ u_1e_1(u_1),\ u_1u_3,\ u_2^2,\ u_4;$
$H_5: \quad u_1^5,\ u_1^3u_2,\ u_1^2u_3,\ u_1^2e_1(u_1),\ u_1u_2^2,\ u_1u_4,\ u_2u_3,\ u_2e_1(u_1),\ e_1(u_2);$
$H_6: \quad u_1^6,\ u_1^4u_2,\ u_1^3u_3,\ u_1^3e_1(u_1),\ u_1^2u_2^2,\ u_1^2u_4,\ u_1u_2u_3,\ u_1u_2e_1(u_1),\ u_1e_1(u_2),$ $u_2^3,\ u_2u_4,\ u_3e_1(u_1),\ u_3^2,\ e_2(u_2),\ e_1(u_1)^2;$
$H_7: \quad u_1^7,\ u_1^5u_2,\ u_1^4u_3,\ u_1^4e_1(u_1),\ u_1^3u_2^2,\ u_1^3u_4,\ u_1^2u_2u_3,\ u_1^2u_2e_1(u_1),\ u_1^2e_1(u_2),$ $u_1u_2^3,\ u_1u_2u_4,\ u_1u_3e_1(u_1),\ u_1u_3^2,\ u_1e_2(u_2),\ u_1e_1(u_1)^2,\ u_2^2u_3,\ u_2^2e_1(u_1),$ $u_2e_1(u_2),\ u_3u_4,\ e_1(u_3),\ e_3(u_2),\ e_1e_1(u_1).$

§7. The homology of iterated loop spaces over the complex projective spaces

Here we consider the problem of calculation the homology

$$H_*(\Omega^m(CP^\infty/CP^n)),\quad m \le 2n+1,$$

of iterated loop spaces over the space CP^∞/CP^n. We begin with $Z/2$ coefficients.

Denote the $2i$-dimensional generator of $H_*(CP^\infty)$ as c_i. A coalgebra structure on this generators is determined by the formula

$$\nabla(c_i) = \sum_j c_j \otimes c_{i-j}.$$

An action of the Steenrod algebra is determined by the formula

$$Sq^{2j}(c_i) = \binom{i-j}{j} c_{i-j}.$$

Since $H_*(CP^\infty)$ is the free $\overline{E}_*$-coalgebra there are no higher $\overline{E}_*$-operations. The homology $H_*(CP^n)$ is the $\overline{E}_*$-subcoalgebra of $H_*(CP^\infty)$ and hence the $\overline{E}_*$-coalgebra structure on $H_*(CP^\infty)$ induces the $\overline{E}_*$-coalgebra structure on

$$H_*(CP^\infty)/H_*(CP^n) \cong H_*(CP^\infty/CP^n)$$

Therefore on the homology $H_*(CP^\infty/CP^n)$ there are no higher $\overline{E}_*$-operations. So the spectral sequence of the homology of the iterated loop spaces over CP^∞/CP^n has no higher differentials and we have

Theorem 7. *The homology $H_*(\Omega^m(CP^\infty/CP^n))$, $m \le 2n+1$ (over $Z/2$) is isomorphic to the homology of the $\mathcal{P}_{m-1}$-algebra $\mathcal{P}_{m-1\varphi}S^{-m}H_*(CP^\infty/CP^n)$, where the differential d_φ on the generators $s^{-m}c_i$ is defined by the formula*

$$d_\varphi(s^{-m}c_i) = \sum_{\substack{j\\2j<i}} [s^{-m}c_j, s^{-m}c_{i-j}] + \sum_{\substack{j\\2j\le i}} \binom{i-j}{j} e_{2(2j-i)+m-1}(s^{-m}c_{i-j}).$$

Note that in the case $m=1$ we have the isomorphisms

$$\mathcal{P}_0 S^{-1}H_*(CP^\infty/CP^n) \cong T_s L S^{-1}H_*(CP^\infty/CP^n) \cong TS^{-1}H_*(CP^\infty/CP^n),$$

and the differential in the tensor algebra $TS^{-1}H_*(CP^\infty/CP^n)$ on the generators $v_i = s^{-1}c_{i+1}$ of dimensions $2i+1$ has very simple form

$$d_\varphi(v_i) = \sum_j v_j \otimes v_{i-j-1}.$$

From here it follows that the homology $H_*(\Omega(CP^\infty/CP^n))$ (over $Z/2$) is isomorphic to the algebra generated by the elements v_i, $n \le i \le 2n$, and relations

$$v_n \cdot v_n = 0;$$
$$v_n \cdot v_{n+1} + v_{n+1} \cdot v_n = 0;$$
$$\dots$$
$$v_n \cdot v_{2n} + \dots + v_{2n} \cdot v_n = 0.$$

So as a graded module the homology $H_*(\Omega(CP^\infty/CP^n))$ is generated by the noncommutative products $v_{n_1}\cdot\ldots\cdot v_{n_k}$ with $n\le n_1\le 2n$, $n<n_2,\ldots,n_k\le 2n$.

Consider the homology $H_*(\Omega^2(CP^\infty/CP^n))$. It is isomorphic to the homology of the differential $\mathcal{P}_1$- algebra $\mathcal{P}_{1\varphi}\{v_i|\ i\ge n\}$, where $v_i = s^{-2}c_{i+1}$, $dim(v_i)=2i$ and the differential is determined by the formulas

$$d_\varphi(v_{2i}) = \sum_{j<i}[v_j, v_{2i-j-1}];$$
$$d_\varphi(v_{2i+1}) = \sum_{j<i}[v_j, v_{2i-j}] + e_1(v_i).$$

From these formulas it follows that the homology $H_*(\Omega^2(CP^\infty/CP^n))$ is the $\mathcal{P}_1$-algebra generated by the elements $v_n,\ldots,v_{2n}$ and relations

$$e_1(v_n)=0;$$
$$[v_n,v_{n+1}]=0;$$
$$[v_n,v_{n+2}]+e_1(v_{n+1})=0;$$
$$\ldots$$
$$[v_n,v_{2n}]+[v_{n+1},v_{2n-1}]+\cdots=0.$$

Consider the homology $H_*(\Omega^4(CP^\infty/CP^2))$. It is isomorphic to the homology of the differential $\mathcal{P}_3$- algebra $\mathcal{P}_{3\varphi}\{v_i|\ i\ge 1\}$, where $v_i = s^{-4}c_{i+2}$, $dim(v_i)=2i$ and the differential is determined by the formulas

$$d_\varphi(v_{2i+1}) = \sum_{j<i}[v_j, v_{2i-j-1}] + \binom{i+2}{i+1}e_1(v_i);$$
$$d_\varphi(v_{2i+2}) = \sum_{j<i}[v_j, v_{2i-j}] + e_3(v_i).$$

In small dimensions we have
$d(v_1)=0;$
$d(v_2)=0;$
$d(v_3)=e_1(v_1);$
$d(v_4)=e_3(v_1);$
$d(v_5)=[v_1,v_2];$
$d(v_6)=[v_1,v_3]+e_3(v_2);$
$d(v_7)=[v_1,v_4]+[v_2,v_3]+e_1(v_3);$

From these formulas it follows that the homology $H_i = H_i(\Omega^4(CP^\infty/CP^2))$ in small dimensions i has the following generators
$H_1=0;$
$H_2:\quad v_1;$
$H_3=0;$
$H_4:\quad v_1^2,\ v_2;$
$H_5=0;$

$H_6:\quad v_1^3,\ v_1v_2,\ e_2(v_1);$
$H_7 = 0;$
$H_8:\quad v_1^4,\ v_1^2v_2,\ v_1e_2(v_1),\ v_2^2;$
$H_9:\quad e_1(v_2);$
$H_{10}:\quad v_1^5,\ v_1^3v_2,\ v_1^2e_2(v_1),\ v_1v_2^2,\ v_2e_2(v_1),\ e_2(v_2);$
$H_{11}:\quad v_1e_1(v_2),\ e_3(v_2);$
$H_{12}:\quad v_1^6,\ v_1^4v_2,\ v_1^3e_2(v_1),\ v_1^2v_2^2,\ v_1v_2e_2(v_1),\ v_1e_2(v_2);$
$H_{13}:\quad v_1^2e_1(v_2),\ v_1e_3(v_2),\ e_1(v_3),\ [v_2,v_3].$

Consider now Z/p coefficients, $p>2$. The coalgebra structure on the generators is determined by the formula

$$\nabla(c_i) = \sum_j c_j \otimes c_{i-j}.$$

The action of the mod-p Steenrod algebra is determined by the formula

$$P^j(c_i) = \binom{i-(p-1)j}{j} c_{i-(p-1)j}.$$

Since $H_*(CP^\infty)$ is the free $\overline{E}_*$-coalgebra there are no higher $\overline{E}_*$-operations. The homology $H_*(CP^n)$ is the $\overline{E}_*$-subcoalgebra of $H_*(CP^\infty)$ and hence the $\overline{E}_*$-coalgebra structure on $H_*(CP^\infty)$ induces the $\overline{E}_*$-coalgebra structure on

$$H_*(CP^\infty)/H_*(CP^n) \cong H_*(CP^\infty/CP^n).$$

Therefore on the homology $H_*(CP^\infty/CP^n)$ there are no higher $\overline{E}_*$-operations. So the spectral sequence of the homology of the iterated loop spaces over CP^∞/CP^n has no higher differentials and we have

Theorem 8. *The homology $H_*(\Omega^m(CP^\infty/CP^n))$, $m \le 2n+1$ (over Z/p) is isomorphic to the homology of the $\mathcal{P}_{m-1}$-algebra $\mathcal{P}_{m-1\varphi}S^{-m}H_*(CP^\infty/CP^n)$, where the differential d_φ on the generators $s^{-m}e_i$ is defined by the formula*

$$d_\varphi(s^{-m}c_i) = \sum_{\substack{j\\ 2j\le i}} [s^{-m}c_j, s^{-m}c_{i-j}] + \\ + \sum_{\substack{j\\ pj<i}} \binom{i-(p-1)j}{j} \beta e_{2(pj-i)+m}(s^{-m}c_{i-(p-1)j}).$$

Note that in the case $m=1$ we have the isomorphisms

$$\mathcal{P}_0S^{-1}H_*(CP^\infty/CP^n) \cong T_sLS^{-1}H_*(CP^\infty/CP^n) \cong TS^{-1}H_*(CP^\infty/CP^n),$$

and the differential in the tensor algebra $TS^{-1}H_*(CP^\infty/CP^n)$ has very simple form

$$d_\varphi(s^{-1}c_i) = \sum_j s^{-1}c_j \otimes s^{-1}c_{i-j}.$$

From here it follows that the homology $H_*(\Omega(CP^\infty/CP^n))$ (over Z/p) is isomorphic to the algebra generated by the elements $v_i = s^{-1}c_{i+1}$, $n \le i \le 2n$, of dimensions $2i+1$ and relations

$$\begin{gathered} v_n \cdot v_n = 0; \\ v_n \cdot v_{n+1} + v_{n+1} \cdot v_n = 0; \\ \dots \\ v_n \cdot v_{2n} + \dots + v_{2n} \cdot v_n = 0. \end{gathered}$$

So as a graded module the homology $H_*(\Omega(CP^\infty/CP^n))$ is generated by the noncommutative products $v_{n_1} \cdot \ldots \cdot v_{n_k}$ with $n \le n_1 \le 2n$, $n < n_2, \ldots, n_k \le 2n$.

Consider the homology $H_*(\Omega^2(CP^\infty/CP^n))$. It is isomorphic to the homology of the differential $\mathcal{P}_1$- algebra $\mathcal{P}_{1\varphi}\{v_i |\ i \ge 1\}$, where $v_i = s^{-2}c_{i+1}$, $i \ge n$ and the differential is determined by the formulas

$$d_\varphi(v_i) = \sum_{2j<i} [v_j, v_{i-j-1}].$$

From these formulas it follows that the homology $H_i = H_i(\Omega^2(CP^\infty/CP^n))$ is the $\mathcal{P}_1$-algebra generated by the elements $v_n, \ldots, v_{2n}$ and relations

$$\begin{gathered} [v_n, v_n] = 0; \\ [v_n, v_{n+1}] = 0; \\ [v_n, v_{n+2}] + [v_{n+1}, v_{n+1}] = 0; \\ \dots \\ [v_n, v_{2n}] + [v_{n+1}, v_{2n-1}] + \dots = 0. \end{gathered}$$

Consider the homology $H_*(\Omega^4(CP^\infty/CP^2))$. It is isomorphic to the homology of the differential $\mathcal{P}_3$- algebra $\mathcal{P}_{3\varphi}\{v_i |\ i \ge 1\}$, where $v_i = s^{-4}c_{i+2}$, $dim(v_i) = 2i$ and the differential is determined by the formulas

$$\begin{aligned} d_\varphi(v_{pi-1}) &= \sum_{2j<pi-2} [v_j, v_{pi-j-3}] + \binom{i+1}{i} \beta e_2(v_{i-1}); \\ d_\varphi(v_k) &= \sum_{2j<k-1} [v_j, v_{k-j-2}],\ k \ne pi-1. \end{aligned}$$

In small dimensions we have
$d(v_1) = 0;$
$d(v_2) = 0;$
$d(v_3) = 0;$
$d(v_4) = [v_1, v_1];$
$d(v_5) = [v_1, v_2];$
$d(v_6) = [v_1, v_3] + [v_2, v_2];$
$\dots$

$d(v_{2p-1}) = [v_1, v_{2p-4}] + \cdots + [v_{p-2}, v_{p-1}] + 3\beta e_2(v_1);$

$\dots$

From these formulas it follows that the homology $H_i = H_i(\Omega^4(CP^\infty/CP^2))$ with, for example, $Z/3$-coefficients in small dimensions i has the following generators

$H_1 = 0;$

$H_2 : \quad v_1;$

$H_3 = 0;$

$H_4 : \quad v_1^2,\ v_2;$

$H_5 = 0;$

$H_6 : \quad v_1^3,\ v_1v_2,\ v_3;$

$H_7 = 0;$

$H_8 : \quad v_1^4,\ v_1^2v_2,\ v_1v_3,\ v_2^2;$

$H_9 : \quad \beta e_2(v_1);$

$H_{10} : \quad v_1^5,\ v_1^3v_2,\ v_1^2v_3,\ v_1v_2^2,\ v_2v_3,\ e_2(v_1);$

$H_{11} : \quad v_1\beta e_2(v_1),\ [v_2, v_2].$

APPENDIX

F. Sergeraert

The general ideas of the paper [10] led Julio Rubio and myself to a simple method solving the computability problem for the homology groups of iterated loop spaces, when the initial space is sufficiently reduced. The main ingredient is *functional programming*, allowing us to constructively apply three particular cases of the basic homological perturbation lemma in situations involving highly infinite simplicial sets; see [17].

This is not only a theoretical result. The algorithm, the existence of which has been so proved, has been concretely written in Common Lisp with significant results. The first version of our program, named *EAT* (= Effective Algebraic Topology), is Web-reachable [18] with a rich documentation (250pp.). The most recent version of this program named *Kenzo* has just been finished when this appendix is written (January 1999). See [17] for a few explanations and a small demonstration file. A public version of this program, with a reasonably complete user guide, will be soon distributed at the same Web-address.

Let us give a typical example of use of Kenzo, related to the main subject of this paper. What about the first $\mathbb{Z}$-homology groups of the iterated loop spaces $\Omega_n^p = \Omega^p(RP^\infty/RP^n)$, if $p \le n$? The situation is difficult: the projective space is not a suspension and the classical results about the homology groups of spaces of the form $\Omega^n S^n X$ cannot be applied; see [12] for a survey of this subject. The only general solution known at this time to determine the groups $H_n(\Omega_n^p; \mathbb{Z}_2)$, due to Vladimir Smirnov, is the subject of the main part of this paper. We explain here how the results of Smirnov can be verified with the Kenzo program for the first homology groups.

Kenzo is the name of my *cat*, and CAT = *Constructive Algebraic Topology*.

Let us consider for example the computation of $H_5(\Omega^2(RP^\infty/RP^2))$. In an interactive Lisp session where the Kenzo program has been loaded, you can execute:

```
USER(1):  (setf trunc3-proj-space (r-proj-space 3)) ==>
[K1 Simplicial-Set]
```

This Lisp dialog must be understood as follows. The string `USER(1):` is the *Lisp prompt*; Lisp is waiting for the next expression you want to evaluate. In this case the instruction is :

```
(setf trunc3-proj-space (r-proj-space 3))
```

which means you want to assign to the symbol $\boxed{trunc3-proj-space}$ the result of the evaluation of $\boxed{(r-proj-space3)}$. The last evaluation *constructs* in the Lisp environment our version of the truncated projective space RP^∞/RP^2, and the resulting object is assigned to the symbol; furthermore a simple external form of the assigned object is displayed; this form can be read: this is the Kenzo object $\#1$ $\boxed{K1}$, which is a simplicial set. Of course the internal form is much more complicated: the internal object *is* the looked-for space, or more precisely *codes* this space.

But we want to consider the second loop-space of the truncated projective space. The Kenzo program constructs it in this way:

```
USER(2):  (setf omega2-trunc3-proj-space
                 (loop-space trunc3-proj-space 2)) ==>
[K22 Simplicial-Group]
```

The result, the Kenzo object $\#$ 22, is a simplicial *group*, namely the Kan model of the second loop space [15], a highly infinite object.

The homology groups of an object such as a chain complex, a simplicial set, are computed by the Kenzo function $\boxed{homology}$. For example if you want to verify the truncated projective space has the right homology in dimension 5:

```
USER(3):  (homology trunc3-proj-space 5) ==>
Computing boundary-matrix in dimension 5.
Rank of the source-module :  1.
Computing boundary-matrix in dimension 6.
Rank of the source-module :  1.
Homology in dimension 5 :
Component Z/2Z
---done---
```

The homology group is Z_2. The homology group $H_5(\Omega^2(RP^\infty/RP^2))$ is obtained in the same way:

```
USER(4):  (homology omega2-trunc3-proj-space 5) ==>
Computing boundary-matrix in dimension 5.
Rank of the source-module :  29.
Computing boundary-matrix in dimension 6.
Rank of the source-module :  70.
Homology in dimension 5 :
Component Z/2Z
Component Z/2Z
---done---
```

So that $H_5(\Omega^2(RP^\infty/RP^2)) = Z_2^2$. As you see, the program determines a *Hirsch Z-complex* for our loop space, with 29 generators in dimension 5 and 70 generators in dimension 6. Note the program gives you the Z-homology, not the Z_2-homology, like in Smirnov's text. The complexity is very high: the highest group computed by Kenzo for this second loop space is $H_7 = Z_2^2 \oplus Z_8$; one day of CPU time has been needed on a powerful Linux PC to obtain it.

The topologists frequently ask for some homology groups determined by the Kenzo program that are not reachable by human computers. The first example of this sort was given by the previous version of the Kenzo program, the EAT program [18]. We repeat it here, because it is very simple and ten years after its first computation, we have not yet found a topologist knowing how to compute this group by hand. Please try to do it!

The construction is the following. The homotopy group $\pi_2\Omega S^3$ is Z, so that attaching a 3-cell to ΩS^3 by a map of degree 2 makes sense. Let $D\Omega S^3$ the space. Question: what about the homology groups of $\Omega D\Omega S^3$? Let us show how the first homology groups of this strange loop space are computed by the Kenzo program. Firstly the loop space ΩS^3 is constructed as before:

```
USER(5):  (setf s3 (sphere 3)) ==>
[K262 Simplicial-Set]
USER(6):  (setf os3 (loop-space s3)) ==>
[K267 Simplicial-Group]
```

How to attach a 3-cell? This 3-cell will be a 3-simplex and, in order to attach it, we must describe what 2-simplices of ΩS^3 are its faces. The list of faces is defined and used as follows:

```
USER(7):  (setf faces (list (loop3 0 's3 1)
                    (absm 3 +null-loop+)
                    (loop3 0 's3 1)
                    (absm 3 +null-loop+))) ==>
(<<Loop[S3]>> <AbSm 1-XL:Loop>>> <<Loop[S3]>> <AbSm 1-0 <<Loop>>>)
USER(8):  (setf dos3 (disk-pasting os3 3 'new faces))
[K380 Simplicial-Set]
```

The $\boxed{disk - pasting}$ function atttaches a 3-cell of "name" $\boxed{new}$ to $\boxed{os3}$, using the face list $\boxed{faces}$; this list is:

[0] The "fundamental" simplex of ΩS^3;
[1] The second face is collapsed on the base point;
[2] The same as 0;
[3] The same as 2;

so that, taking account of the usual sign rules, the attaching map has degree 2. The space so constructed is assigned to the symbol $\boxed{dos3}$. The second homology group of this space is verified:

```
USER(9):  (homology dos3 2) ==>
Computing boundary-matrix in dimension 2.
Rank of the source-module :  1.
Computing boundary-matrix in dimension 3.
Rank of the source-module :  1.
Homology in dimension 2 :
```

```
Component Z/2Z
---done---
```

Finally we construct the loop space $\Omega D\Omega S^3$ and compute the sixth homology group:

```
USER(10):  (setf odos3 (loop-space dos3)) ==>
[K398 Simplicial-Group]
USER(11):  (homology odos3 6) ==>
Computing boundary-matrix in dimension 6.
Rank of the source-module :  26.
Computing boundary-matrix in dimension 7.
Rank of the source-module :  50.
Homology in dimension 6 :
Component Z/6Z
Component Z/2Z
Component Z/2Z
Component Z/2Z
Component Z/2Z
Component Z/2Z
Component Z/2Z
Component Z/2Z
Component Z/2Z
Component Z/2Z
Component Z/2Z
Component Z/2Z
Component Z/2Z
---done---
```

$H_6(\Omega D\Omega S^3) = Z_2^{12} \oplus Z_6$. The Kenzo program has determined these groups up to dimension 9 in a few days.

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