ON THE HOMOTOPY TYPE OF INFINITE STUNTED PROJECTIVE SPACES

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1. INTRODUCTION

Consider the space $X_n = \mathbb{R}\mathbf{P}^\infty/\mathbb{R}\mathbf{P}^{n-1}$ together with the boundary map in the Barratt-Puppe sequence

$$X_n \longrightarrow \Sigma\mathbb{R}\mathbf{P}^{n-1}.$$

Francis Sergeraert and Vladimir Smirnov [6] have considered the low dimensional homotopy groups of X_n as well as their loop space homology. Their results are quite interesting, and their questions fit with several results that appeared previously in the literature [3, 8].

It is the purpose of this note to elaborate on a few of these remarks, as well as pointing out some natural associated questions and their connection to a recent result of Jie Wu [8], where he shows that the homotopy of the 3-sphere is a summand of the homotopy of $\Sigma\mathbb{R}\mathbf{P}^2$. That result is a consequence of the structure considered here. Moreover, it will be shown here that the homotopy type of the spaces X_n is closely related to that of certain finite complexes described below.

Here are some concrete results, the first of which is an observation that was also pointed out by Broto [2].

Theorem 1.1. *There is a fibration*

$$S^3 \longrightarrow X_2 \longrightarrow K(\mathbb{Z}, 2),$$

which is split after looping.

Notice that the calculation of loop space homology for X_2 follows trivially from this splitting result and the known loop space homology for S^3.

Theorem 1.2. *There is a 2-local fibration*

$$A_3 \longrightarrow X_3 \longrightarrow BS^3,$$

where A_3 is the 6-skeleton of the Lie group G_2. Furthermore, there is a splitting

$$\Omega_0^4 X_3 \simeq \Omega^3 S^3 \times \Omega_0^4 A_3,$$

where Ω_0^n denotes the component of the constant map in an iterated loop space.

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The calculation of loop space homology for X_3 is a bit more involved. A work of David Anick [1] gives the homology of ΩA_3. Let V denote the 5-skeleton of the Lie group G_2. Then A_3 is obtained from V by attaching a single 6-cell. It is easy to see that $H = H_*(\Omega V; \mathbb{F}_2)$ (in fact with any coefficients) is a tensor algebra on two generators a and b of dimensions 2 and 4 respectively. The obvious inclusion induces a H-module structure on $H_*(\Omega A_3; \mathbb{F}_2)$. Let $H\langle t\rangle$ denote the free associative H-algebra on one generator t in dimension 5. Thus $H\langle t\rangle$ is isomorphic to a tensor algebra on a, b and t. Define a differential d on $H\langle t\rangle$ by $d(t) = a^2$ and $d(a) = d(b) = 0$. This turns $H\langle t\rangle$ into a differential graded algebra and Anick's theorem now gives

Theorem 1.3. *The mod-2 loop space homology of A_3 is isomorphic as an H-module to the homology of the differential graded algebra $(H\langle t\rangle,\ d)$ defined above. Moreover the Poincaré series for $H_*(\Omega A_3, \mathbb{F}_2)$ is given by the formula*

$$P_{\Omega A_3}(t) = \frac{1-t^4}{1-t^2-t^4-t^7}.$$

The fibration

$$\Omega A_3 \longrightarrow \Omega X_3 \longrightarrow S^3$$

is not split because the connecting map $S^3 \longrightarrow A_3$ is essential. However, it is degree 2 on the bottom cell and hence trivial on mod-2 homology. Since the fibration is multiplicative, the mod-2 Serre spectral sequence collapses at the E^2-page and one obtains

Corollary 1.4. *There is an isomorphism of $H_*(\Omega A_3, \mathbb{F}_2)$-modules*

$$H_*(\Omega X_3; \mathbb{F}_2) \cong H_*(\Omega A_3, \mathbb{F}_2) \otimes E[x_3].$$

Remark 1.5. *The low dimensional homotopy of A_3 is quite easy to compute, as the natural map $A_3 \longrightarrow K(Z, 3)$ induces a homology isomorphism through dimension 6, and the homotopy fibre is easy to handle through low dimensions, as demonstrated below.*

Recall the following theorem of J. Wu.

Theorem 1.6 (Wu). *Let $\gamma : P^3(2) \longrightarrow BSO(3)$ denote the map given by inclusion to the bottom skeleton. Let Y denote its homotopy fibre. Then*

1. *Y is homotopy equivalent to $\Sigma(\mathbb{R}\mathbf{P}^4/\mathbb{R}\mathbf{P}^1) \vee P^6(2)$ and*
2. *After looping 4 times and restricting to components of the constant map $\Omega^4\gamma$ has a right homotopy inverse.*

The boundary map in the Barratt-Puppe sequence

$$X_n \longrightarrow \Sigma\mathbb{R}\mathbf{P}^{n-1}$$

together with Theorem 1.2 and naturality gives a new proof part 2 the theorem. This theorem gives a way of computing the homotopy of $P^3(2)$ through a considerable range, as it is given by that of the 3-sphere and Y.

Theorem 1.7. *There is a fibration*

$$S^7 \vee P^6(2) \longrightarrow X_4 \longrightarrow BS^3,$$

which splits after looping.

Notice that the loop space homology of A_4 is the tensor algebra on the reduced homology of $S^7 \vee P^6(2)$. It also follows that there infinitely many elements of order 8 in the homotopy groups of X_4 [3]. It seems reasonable to conjecture that there does not exist an elements of order 16.

The maps described above fit in a more systematic context. Namely, consider the natural maps $g_n : \mathbb{R}\mathbf{P}^n \longrightarrow SO(n)$, which induce a surjection in cohomology. Using these maps one gets the following

Theorem 1.8. *There are maps*

$$\alpha_n : X_n \longrightarrow BSpin(n),$$

such that the fibre F_n of α_n is a finite complex.

One might be tempted to conjecture that the spaces X_n have homotopy exponents. In addition, one might wonder how the splittings of the loop space of the suspension of X_n impinge on features of the degree 2 map on spheres.

Indeed, the maps

$$X_n \longrightarrow \Sigma\mathbb{R}\mathbf{P}^{n-1}$$

yield factorisations of several useful maps related to the degree 2 map. In particular when n is even, the degree 2 map on S^n factors through X_n and $\Sigma\mathbb{R}\mathbf{P}^{n-1}$, and when n is odd the map

$$P^{n+1}(2) \longrightarrow S^n,$$

given by collapsing onto the top cell followed by η factors through X_n and $\Sigma\mathbb{R}\mathbf{P}^{n-1}$. A discussion on how to obtain these factorisations and a speculation on their possible utility is in the last section of the paper.

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2. The spaces X_n

Consider the natural map from $\mathbb{R}\mathbf{P}^\infty$ to $K(\mathbb{Z},2)$, given by the first non-vanishing integral cohomology class. Since $K(\mathbb{Z},2)$ is simply-connected this map factors through X_2. Thus there is a fibration

$$F \longrightarrow X_2 \longrightarrow K(\mathbb{Z},2).$$

Pulling this fibration back once, one obtains a principal fibration with fibre S^1 and base space X_2. Inspection of the integral cohomology Serre spectral sequence for this fibration gives that the cohomology of the fibre is isomorphic to the cohomology of the 3-sphere. Since F is obviously simply-connected, the first statement of Theorem 1.1 follows.

To see that the fibration in the theorem splits after looping, observe that the connecting map

$$\Omega K(\mathbb{Z},2)=S^1 \longrightarrow F=S^3$$

is null-homotopic for the obvious reason. Thus the projection from ΩX_2 to S^1 has a section and the fibration splits.

Corollary 2.1. *The torsion in the homotopy of X_2 has an exponent at any prime p.*

Next analyse X_3 and X_4. Consider the inclusion i of $\mathbb{Z}/2\mathbb{Z}$ as the centre of the Lie group S^3. This homomorphism induces a map

$$Bi : \mathbb{R}\mathbf{P}^\infty = B\mathbb{Z}/2\mathbb{Z} \longrightarrow BS^3.$$

In cohomology this map takes the generator $u_4 \in H^*(BS^3,\mathbb{F}_2)$ to z^4, where $z \in H^*(B\mathbb{Z}/2\mathbb{Z},\mathbb{F}_2)$ is the generator. Since BS^3 is 3-connected, Bi factors through both X_3 and X_4. Thus there are fibrations for $i=3,4$

$$A_i \longrightarrow X_i \longrightarrow BS^3. \tag{1}$$

The cohomology of A_i in both cases is easy to compute using the Serre spectral sequence for the pulled back fibrations

$$S^3 \longrightarrow A_i \longrightarrow X_i. \tag{2}$$

This can be done either with mod-2 coefficients or integrally. In either case the spaces A_i have only 2-torsion and their mod-2 cohomology is given by the proposition below. The explicit calculation is straight-forward and is left for the reader.

Proposition 2.2. *The mod-2 cohomology of A_3 is generated additively by classes a_3, b_5 and a_3^2, with $Sq^2a=b$ and $Sq^1b=a^2$.*

The mod-2 cohomology of A_4 is generated additively by classes x_5, y_6 and z_7, with $Sq^1x=y$.

The analysis of A_4 follows directly: It is clear by the cohomological calculation that its 6-skeleton is given by the Moore space $P^6(2)$. The fibre of the obvious map $P^6(2) \longrightarrow K(\mathbb{Z}/2\mathbb{Z},5)$ gives the 5-connected cover of $P^6(2)$ and makes it visible that $\pi_6(P^6(2))$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and is generated by η on the bottom cell. Attaching a 7-cell to $P^6(2)$ by this unique non-trivial element would have resulted in a Sq^2 connecting the 5 and 7 cells in the mod-2 cohomology of cofibre. The cohomological structure of A_4 thus implies the structure claimed in Theorem 1.7. That the fibration in the theorem splits after looping follows by pulling it back once and observing the the connecting map from S^3 to the fibre A_4 is null-homotopic for connectivity reasons.

Next analyse the homotopy type of A_3. The 5-skeleton of this space is seen to be given by $Y=S^3\cup_\eta e^5$ by inspection of cohomology. The natural map $Y \longrightarrow K(\mathbb{Z},3)$ induces an isomorphism on mod-2 cohomology up to dimension 5 and its fibre is the 4-connected cover of Y. By inspection of the associated Serre spectral sequence, $\pi_5(Y)=\mathbb{Z}$ and the cohomological structure implies that the attaching map is given by a multiple of the generator by an integer divisible by 2 exactly once. One can in fact verify by doing an integral Serre sequence calculation that the attaching map is exactly twice the generator and so the structure of A_3 is thus determined. Notice that it has been shown here that the homotopy type of a simply-connected space with the mod-2 and integral cohomology of A_3 is determined uniquely. Thus it follows that A_3 is homotopy equivalent to the 6-skeleton of the Lie group G_2 as claimed.

An old theorem of J. Harper [4] shows that the 2-local homotopy type of the Lie group G_2 is determined by its mod-2 cohomology. The result here by contrast is obtained by very elementary methods and does not overlap with Harper's calculation.

The splitting result claimed in Theorem 1.7 follows immediately from the fact that the connecting map $S^3 \longrightarrow A_4 = P^6(2) \vee S^7$ is null-homotopic for connectivity reasons.

For the splitting result claimed in Theorem 1.2 one needs the following

Lemma 2.3. *Let Z be a 2-connected space and let $g : S^3 \longrightarrow Z$ be any map. If $\pi_4(g)$ is trivial then $\Omega^2(2g)$ is null-homotopic when restricted to $\Omega^2 S^3\langle 3\rangle$ at the prime 2.*

Proof. Let f denote $2g$. Thus f is given by the composite

$$S^3 \xrightarrow{2} S^3 \xrightarrow{g} Z$$

Recall that the loops on the second Hilton-Hopf invariant $\Omega h_2 : \Omega^2 S^3 \longrightarrow \Omega^2 S^5$ has order 2 in the group $[\Omega^2 S^3, \Omega^2 S^5]$. Thus the composition

$$\Omega^2 S^3 \xrightarrow{2} \Omega^2 S^3 \xrightarrow{\Omega h_2} \Omega^2 S^5$$

is null-homotopic.

This remains true if all spaces are replaced by their 1-connected covers and so the composition

$$\Omega^2(S^3\langle 3\rangle) \xrightarrow{2} \Omega^2(S^3\langle 3\rangle) \xrightarrow{\Omega h_2} \Omega^2 S^5$$

is null and 2 on $\Omega^2(S^3\langle 3\rangle)$ lifts to the fibre of Ωh_2, which is given by ΩS^3. Thus after passing to 1-connected covers $\Omega^2 f$ is homotopic to a composition

$$\Omega^2(S^3\langle 3\rangle) \xrightarrow{l} \Omega S^3 \xrightarrow{\Omega j} \Omega^2(S^3\langle 3\rangle) \xrightarrow{\Omega^2 g} \Omega^2(Z\langle 3\rangle).$$

But notice that j takes the fundamental class in $\pi_3(S^3)$ to $\eta \in \pi_3(\Omega(S^3\langle 3\rangle))$, which under Ωg is taken, by hypothesis, to 0. Hence $\Omega^2 g \Omega j$ is null-homotopic and so $\Omega^2 f$ is null-homotopic, restricted to 1-connected covers. $\square$

Notice that in the fibration

$$S^3 \xrightarrow{\delta} A_3 \longrightarrow X_3$$

the fibre inclusion δ is degree 2 on the bottom cell. Also the map $A_3 \longrightarrow K(\mathbb{Z}, 3)$ corresponding to the bottom cohomology class induces an isomorphism on cohomology up to dimension 5. Hence $\pi_4(A_3) = 0$ and the conditions of Lemma 2.3 are satisfied. Thus looping twice and passing to universal covers, the map

$$\Omega^2\delta : \Omega^2 S^3 \longrightarrow \Omega^2 A_3$$

is null-homotopic, implying the splitting result of Theorem 1.2.

Finally analyse X_n for higher values of n. Consider a map $g_n : \mathbb{R}\mathbf{P}^{n-1} \longrightarrow SO(n)$, which induces the inclusion of generators in homology. Then there is an induced diagram where the left column is a cofibration and the right column is a fibration.

$$\begin{tikzcd} \mathbb{R}\mathbf{P}^{n-1} \arrow[r] \arrow[d] & SO(n) \arrow[d] \\ \mathbb{R}\mathbf{P}^{\infty} \arrow[r, "="] \arrow[d] & \mathbb{R}\mathbf{P}^{\infty} \arrow[d] \\ X_n \arrow[r, "\alpha_n"] & BSpin(n) \end{tikzcd}$$

Consider the fibration

$$\mathbb{R}\mathbf{P}^{\infty} \longrightarrow BSpin(n) \longrightarrow BSO(n).$$

Using either the well known calculation of the cohomology of $BSpin(n)$ [5] or inspection of the Serre spectral sequence of this fibration, one observes that for some minimal positive integer r depending on n the class $z^{2^r} \in H^{2^r}(RP^{\infty})(\mathbb{F}_2)$ is an infinite cycle. Let $u_{2^r} \in H^{2^r}(BSpin(n), \mathbb{F}_2)$ be any class restricting to z^{2^r}.

Lemma 2.4. *Let F_n denote the fibre of α_n. Then there is an isomorphism*

$$H^*(F_n, \mathbb{F}_2) \cong H^*(X_n, \mathbb{F}_2)/(z^{2^r}) \otimes Tor_{H^*(BSpin(n),\mathbb{F}_2)/(u_{2^r})}(\mathbb{F}_2, \mathbb{F}_2)$$

of $H^*(X_n, \mathbb{F}_2)$-modules. In particular F_n has the homotopy type of a finite complex.

Proof. Observe that α_n^* takes any choice of u_{2^r} to the cohomology class in $H^*(X_n, \mathbb{F}_2)$ inflating to z^{2^r} and that any class in $H^{2^r}(BSpin(n), \mathbb{F}_2)$ which is in the image of inflation from $H^*(BSO(n), \mathbb{F}_2)$ is sent to 0 by α_n^*. The calculation now becomes direct by using the Eilenberg-Moore spectral sequence for the fibration

$$F_n \longrightarrow X_n \longrightarrow BSpin(n)$$

and Smith's "big collapse theorem" [7]. □

3. Some applications and sample calculations

Proposition 3.1. *For any positive integer n the degree 2 map on S^{2n} factors through X_{2n}. Also if n is odd then the map*

$$P^{2n}(2) \longrightarrow S^{2n-1},$$

given by collapsing onto the top cell followed by η factors through X_{2n+1}.

Proof. Let $i : S^{2n} \longrightarrow X_{2n}$ denote the inclusion to the bottom cell. Let $\delta : X_{2n} \longrightarrow \Sigma\mathbb{R}\mathbf{P}^{2n-1}$ denote the connecting map in the Barratt-Puppe sequence. Then by projecting to the top cell, one gets a map $q : X_{2n} \longrightarrow S^{2n}$. It is immediate that δ is degree 2 in dimension $2n$. Thus the first part of the proposition follows.

For any n the $2n$-skeleton of X_{2n-1} is given by $P^{2n}(2)$. By composing with δ and collapsing to the top cell, one gets a map $P^{2n}(2) \longrightarrow S^{2n-1}$. Restricted to the bottom cell, this map is immediately seen to be degree 0 and is hence null-homotopic. Thus it factors through the top cell via a map $\alpha : S^{2n} \longrightarrow S^{2n-1}$. The map α can be either η or the null map.

Consider the commutative diagram of cofibrations

$$\begin{tikzcd}
P^{2n}(2) \arrow[r] \arrow[d, "="'] & X_{2n-1} \arrow[r] \arrow[d] & X_{2n+1} \arrow[d, "f"'] \\
P^{2n}(2) \arrow[r] \arrow[d] & \Sigma\mathbb{R}\mathbf{P}^{2n-2} \arrow[r] \arrow[d] & C \arrow[d, "h"'] \\
* \arrow[r] & \Sigma\mathbb{R}\mathbf{P}^{\infty} \arrow[r] & \Sigma\mathbb{R}\mathbf{P}^{\infty}
\end{tikzcd}$$

Notice that C is $(2n+1)$-dimensional whereas X_{2n+1} is $2n$-connected. Since $\Sigma\mathbb{R}\mathbf{P}^{2n-2}$ is $(2n-1)$-dimensional, f induces the zero map on mod-2 cohomology. Thus h^* surjects in all dimensions. Notice that additively, $H^*(C, \mathbb{F}_2)$ is isomorphic to that of $H^*(\Sigma\mathbb{R}\mathbf{P}^{2n}, \mathbb{F}_2)$. Thus h^* is in fact an isomorphism through dimension $2n+1$. Now, let $\sigma x^{2n-2} \in H^{2n-1}(\Sigma\mathbb{R}\mathbf{P}^{\infty}, \mathbb{F}_2)$ denote the generator. Then

$$Sq^2(\sigma x^{2n-2}) = \sigma Sq^2(x^{2n-2}) = \sigma(Sq^1(x^{n-1})^2).$$

Thus $Sq^2(\sigma x^{2n-2}) = 0$ if n is even and is equal to σx^{2n} if n is odd. The claim now follows by commutativity of the diagram

$$\begin{tikzcd}
P^{2n}(2) \arrow[r] \arrow[d] & \Sigma\mathbb{R}\mathbf{P}^{2n-2} \arrow[r] \arrow[d] & C \arrow[d] \\
S^{2n} \arrow[r, "\alpha"] & S^{2n-1} \arrow[r] & K
\end{tikzcd}$$

using the fact that η is detected by Sq^2.

□

A remark related to the last proposition is the following. By suspending the factorisation of the degree 2 map on an even sphere one obtains a factorisation for the same map on an odd sphere.

$$S^{2n+1} \longrightarrow \Sigma X_{2n} \longrightarrow \Sigma^2 RP^{2n-1} \longrightarrow S^{2n+1}.$$

Barratt's distributivity formula for the degree 2 map on an odd sphere states that

$$\Omega[2] = 2 + (\Omega[\iota_{2n+1}, \iota_{2n+1}] \circ H_2),$$

where the second summand is the second Hilton-Hopf invariant composed with the loops on the Whitehead square

$$\Omega S^{2n+1} \xrightarrow{H_2} \Omega S^{4n+1} \xrightarrow{\Omega[\iota_{2n+1}, \iota_{2n+1}]} \Omega S^{2n+1}.$$

This identity holds in the non-abelian group $[\Omega S^{2n+1}, \Omega S^{2n+1}]$. One may wonder if this map is null-homotopic after looping $2n$ times more.

The splittings here do not seem to inform on the problem, but factoring the degree 2 map through the double suspension of the projective space might be useful with regards to it.

Next consider the space A_3 obtained as the fibre of the map $X_3 \longrightarrow BS^3$, constructed above. The calculation of loop space homology becomes easy using Anick's technique described in [1]. Specifically, A_3 is obtained from $V = S^3 \cup_\eta e^5$ by attaching a 6-cell. The cohomology of A_3 determines the attaching map $f : S^5 \longrightarrow V$. One observes easily that on mod-2 homology the adjoint $ad(f) : S^4 \longrightarrow \Omega V$ takes the generator to the element $a^2 \in H_4(\Omega V, \mathbb{F}_2)$. In this situation Anick's theorem [1, 3.7] applies and the first claim of Theorem 1.3 follows. One also obtains the formula

$$\frac{1}{P_{\Omega A_3}(t)} = \frac{1+t}{P_N(t)} - \frac{t}{P_{\Omega V}(t)} - t^5,$$

where N is defined to be the quotient of the algebra $H_*(\Omega V, \mathbb{F}_2)$ by the two sided ideal generated by a^2. The calculation of the Poincaré series for ΩA_3 thus follows from knowing the series for the algebra N.

Notice that there is a shost exact sequence of graded vector spaces

$$0 \longrightarrow \Sigma^4 T[a,b] \longrightarrow T[a,b] \longrightarrow N \longrightarrow 0.$$

From this exact sequence one obtains that

$$P_N(t) = P_{\Omega V}(t) - t^4 P_{\Omega V}(t) = \frac{1-t^4}{1-t^2-t^4}.$$

Pluging this in the formula above and simplifying, the proof of Theorem 1.3 is complete.

Wu's splitting theorem is also implied by the observations made here. We recall the relevant part of the theorem.

Theorem 3.2 (Wu). *Let $\gamma : P^3(2) \longrightarrow BSO(3)$ denote the map given by inclusion to the bottom skeleton. Let Y denote its homotopy fibre. Then After looping four times and restricting to components of the constant map $\Omega_0^4\gamma$ has a right homotopy inverse.*

Proof. Consider the following commutative diagram of fibrations:

$$\begin{tikzcd}
A_3 \arrow[r] \arrow[d] & X_3 \arrow[r] \arrow[d] & BS^3 \arrow[d] \\
Y \arrow[r] & P^3(2) \arrow[r, "\gamma"] & BSO(3)
\end{tikzcd}$$

where the vertical map in the centre is the connecting map in the Barratt-Puppe sequence in the cofibration defining X_3 and the right vertical map is the obvious one. Looping this diagram twice and taking connected components the right vertical map becomes an equivalence.

Thus looping four times and taking connected components there is a diagram

$$\begin{tikzcd}
\Omega_0^4 A_3 \arrow[r] \arrow[d] & \Omega_0^4 X_3 \arrow[r] \arrow[d] & \Omega_0^3 S^3 \arrow[r] \arrow[d, "="'] & \Omega_0^3 A_3 \arrow[d] \\
\Omega_0^4 Y \arrow[r] & \Omega_0^4 P^3(2) \arrow[r, "\Omega_0^4\gamma"] & \Omega_0^3 S^3 \arrow[r] & \Omega_0^3 Y
\end{tikzcd}$$

By Theorem 1.2 the top right horizontal map is null-homotopic. Hence the same applies to the bottom right horizontal map and the result follows.

□

Finally, the following low dimensional homotopy calculations are obvious from the splitting results presented here and well known facts on the homotopy groups of the spaces involved. These calculations are motivated by some computer oriented questions of Sergeraert. The program developed by Sergeraert and his coauthors however can certainly handle spaces much more general than the specific idiosyncratic cases here. The remarks here give a corroboration of those calculations.

Proposition 3.3. *The 2-primary low dimensional homotopy of X_3 is given as follows:*

$$\pi_i(X_3) = \begin{cases} 0 & i \leq 2 \\ \mathbb{Z}/2\mathbb{Z} & i = 3 \\ 0 & i = 4 \\ \mathbb{Z}/2\mathbb{Z} & i = 5, 6 \\ \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & i = 7 \end{cases}$$

The 2-primary low dimensional homotopy of x_4 is given as follows:

$$\pi_i(X_4) = \begin{cases} 0 & i \leq 3 \\ \mathbb{Z} & i = 4 \\ \pi_{i-1}(S^3) \oplus \mathbb{Z}/2\mathbb{Z} & i = 5, 6 \\ \pi_6(S^3) \oplus \mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} & i = 7 \\ \pi_{i-1}(S^3) \oplus (\mathbb{Z}/2\mathbb{Z})^3 & i = 8, 9 \\ \pi_9(S^3) \oplus (\mathbb{Z}/8\mathbb{Z})^2 & i = 10 \end{cases}$$

Proof. That the space X_3 is 2-connected and $\pi_3(X_3) \cong \mathbb{Z}/2\mathbb{Z}$ follows at once by looking at the cohomology of X_3. By Theorem 1.2, there is a splitting

$$\Omega_0^4 X_3 \simeq \Omega^3 S^3 \times \Omega_0^4 A_3,$$

where Ω_0^n denotes the zero component and A_3 is the 6-skeleton of the Lie group G_2. Thus it suffices to compute the homotopy of A_3 through the specified range. Notice that A_3 is also the 6-skeleton of $K(\mathbb{Z}, 3)$. Hence the inclusion induces an isomorphism on integral homology though dimension 5 and so $\pi_i(A_3) = 0$ for $i = 4, 5$. The fibre of this inclusion is the 3-connected cover of A_3, which is seen by inspection of the Serre spectral sequence to be 6-connected with π_7 given by $\mathbb{Z}/2\mathbb{Z}$. It follows that $\pi_4(X_3) = 0$ and $\pi_i(X_3)$ are given by $\pi_{i-1}(S^3)$ for $i = 5, 6$. Finally for $i = 7$ there is an exact sequence

$$\pi_7(A_3) \longrightarrow \pi_7(X_3) \longrightarrow \pi_6(S^3),$$

which is split because of Theorem 1.2. The result for X_3 follows.

The calculation for X_4 uses the splitting of Theorem 1.7

$$\Omega X_4 \simeq S^3 \times \Omega(S^7 \vee P^6(2)).$$

Thus

$$\pi_i(X_4) \cong \pi_{i-1}(S^3) \oplus \pi_i(S^7 \vee P^6(2)) \cong \pi_{i+1}(S^3) \oplus \pi_i(S^7) \oplus \pi_i(P^6(2))$$

through dimension 10. The right hand side equality follows from looking at the fibre of the inclusion $S^7 \vee P^6(2) \longrightarrow S^7 \times P^6(2)$, given by $\Sigma(\Omega S^7 \wedge \Omega P^6(2))$ which is 10-connected. The result for X_4 follows from the known homotopy of S^7 and $P^6(2)$ in this range. $\square$

The homotopy of the 3-sphere is of course known through a larger range than is dealt with here. With some extra effort one can work out a few more homotopy groups for A_3 as well as the cross terms that occur in bigger dimensions arising from the homotopy groups of the wedge $S^7 \vee P^6(2)$.

Corollary 3.4. *There are infinitely many summands of $\mathbb{Z}/8\mathbb{Z}$ in the homotopy of X_4.*

This follows from a calculation by the J. Wu and the first author [3] for the homotopy of mod-2 Moore spaces. Explicit dimensions of an infinite family of $\mathbb{Z}/8\mathbb{Z}$ summands are listed in this article.

Conjecture 3.5. *There do not exist any elements of order 16 in the homotopy of X_4.*

The conjecture fits with the Barratt conjecture for the Moore space. The 2-primary component of the homotopy of S^3 has exponent 4. The best known exponent for the 2-primary homotopy of S^7 is 32, but a conjecture of Barratt and Mahowald gives 8 as an upper bound.

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