Abelian varieties and theta functions as invariants for compact Riemannian manifolds; constructions inspired by superstring theory.

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1 Introduction

In some forms of superstring theory particular theta-functions come up as partition functions. The associated Abelian varieties come either from certain cohomology groups of the underlying universe or, in more recent theories (e.g. [Witten] [Mo-Wi]), are linked to their K-groups.

Expressed in mathematical terms, one canonically associates to the cohomology or the K-theory of an even dimensional compact spin manifold a principally polarized Abelian variety. Moreover, if the dimension is 2 mod 8 a particular line bundle is singled out whose first Chern class is the principal polarization. This bundle thus has a non-zero section represented by a theta function which, after suitable normalization, is indeed the partition function of the underlying theory.

Let me give some further detail on the physical motivation. There are several types of superstring theories, e.g. type I which is self-dual and types IIA and IIB wich are related via T-duality. The theories start from a space-time Y which in a first approximation can be taken to be $Y = X \times T$ where T is the time-"axis" [1] and X is some compact Riemannian manifold. In Type IIA theory the *Ramond-Ramond field* is a closed differential form $G = G_0 + G_2 + \cdots$ on X with components of all even degrees while in type IIB G is an odd degree closed differential form on X. Moreover, these forms are *integral* (that is they have integral periods over integral homology

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[1] T could be a circle in physical theories but it could even be a point ("absence of branes").

cycles). The reason is that they are Poincaré dual to certain submanifolds of X which are the "world"-part of a brane in Y. Such a field should be thought of as some configuration in the theory. The partition function assembles all possible configurations in some generating function which can in turn be used to derive further physical properties of the model. In type IIA theory this partition function is of the form $\Theta(0)/\Delta$ where Θ is some normalized theta-function. While Δ is canonically associated to the Riemannian manifold X, this is no longer the case for Θ. Instead, as suggested by Witten in [Witten] and later by Moore and Witten in [Mo-Wi] one should lift the discussion up to K-theory using the Chern character. But then, in order to make a canonical choice for Θ one has to assume that the manifold has a spin structure and has dimension 2 mod 8. See the discussion in § 2 for more details.

For algebraic geometers these constructions look a bit esoteric at first sight, the more since they are phrased in terms foreign to them. For instance, an algebraic geometer might ask: *is the construction related to the Weil jacobian?* This was the question posed to me by V. Srinivas and motivated the present note. Clearly, an answer entails a careful analysis of the construction proposed in [Mo-Wi]. This presupposes precise knowledge of topological K-theory related to the index theorems of Atiyah-Singer et. al., a subject not too well-known among algebraic geometers. To make this note readable for the average algebraic geometer I have placed these facts in 2 appendices.

Here is an outline of the paper. The background from physics is collected in § 2. It is not necessary for an understanding of the rest of the paper, but it purports to explain how physicist came to the particular Jacobians and the normalized theta-functions.

The basic construction implicitly used in [Mo-Wi] is really simple and given in §3. It is apparently well known among physicists but I could not find a reference for it in this precise form, although a variant is well known in symplectic geometry, cf. [McD-S, Prop. 2.48 (ii)].

Then in § 4.1 examples using cohomology are given and in § 4.2 examples using K-groups leading up to the example in [Mo-Wi]. It should be noted that the proof for the crucial unimodularity property is missing in loc. cit. I explain in this section how it can be viewed as a special case of an old result [AH3] on normalized multipliers.

Then follows a short digression on normalized theta functions and finally in § 5.2 this is applied to give a mathematical formulation of the pertaining results of [Mo-Wi, §3]. This uses in a crucial way some constructions from real K-theory.

The note ends with an appendix explaining K-theory and the real index theorems followed by an appendix where this is specialized to Dirac operators. Noteworthy here is a version of the Thom-isomorphism in real K-theory (=Theorem A.5) which extends the one found in the literature for spin-manifolds whose dimension is divisible by 8. This generalization can be extracted from [At2] but is not explicitly stated there. In the appendix it is explained how this can be deduced from results in loc. cit. This form of the Thom-isomorphism theorem is explicitly used in [Witten].

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2 Physics background

The constructions are in fact inspired by quantum field theory. The fields in question come in three types: scalar fields, fermionic fields and gauge fields. These are defined on a space-world Y. The first type is just a function on Y, the second one is a smooth section in a spinor-bundle on Y, and the last are Lie-algebra valued forms on Y. The last two fields have been experimentally observed.

Given the fields, the physics is deduced from a Lagrangian density. The simplest case for a free scalar field ϕ would be (using Einstein summation)

$$L = \frac{1}{2}(\partial_\mu \phi(x))(\partial^\mu \phi(x) - \frac{1}{2}\phi(x)\phi(x).$$

Its integral over Y is the action:

$$S = \int_Y \mathrm{d}^n x \, L,$$

where $\mathrm{d}^n x$ is some suitable n-dimensional measure on Y, $n = \dim Y$. The action obviously depends on the field configuration.

In physics Y has a *time component* and one assumes that there is a Lorentzian metric on Y which is a genuine metric on word-sheets. To change it to a genuine metric, there is a trick, called *Wick-rotation*: one formally replaces the time parameter t by $\mathrm{i}t$. Note that this replaces S by $\mathrm{i}S$ as well.

Once this has been done, by definition, the partition function is the integral over all field configurations, where each field configuration is weighted by $\exp(\mathrm{i}S)$. For a free scalar field one gets $Z = \int \mathcal{D}\phi \cdot \exp(\mathrm{i}S)$. Such an integral, a path integral, although mathematically ill-defined has been interpreted in physics litterature in such a way that it yields meaningful answers which, moreover, in many cases are surprisingly close the experimentally observed values.

The theory that is important here is inspired by gauge fields on 4-manifolds Y. These have a nice geometric reformulation for the action. A gauge field can be viewed as a connection one-form on a given principal fibre bundle on Y. This is a Lie algebra valued one-form on Y. Its covariant derivative $F = DA$, i.e. the curvature of the fibre bundle is an ordinary 2-form on Y. The action for the gauge fields can be written in this geometric language as $S = \int F \wedge *F$, where $*F$ is the Hodge-star of F, another two-form on Y.

In string theory one replaces the ordinary 4-dimensional time-world Y by some other variety of dimension 10. There are several types of superstring theories: type I which is self-dual, and types IIA and IIB which are each others dual. In this note I will restrict myself to type IIA theories. For such a theory a gauge field F, an integral closed form, is replaced by an arbitrary even degree closed form $G = G_0 + G_2 + ...$ with integral periods, a *Ramond-Ramond field* and, in analogy to ordinary gauge theory, the action is

$$S = \int_Y G \wedge *G.$$

Next, one wants to define a partition function of the form

$$Z = \int \mathcal{D}C \cdot \exp(\mathrm{i}S), \tag{1}$$

where $\int \mathcal{D}C$ is the path integral over all C with $dC = G$ and where G runs over the even degree closed forms with integral periods. In order to make this more precise, one assumes that $Y = X \times T$ where T is the "time-axis"- which may or may not be compact and represents time (usually $T = \mathbb{R}$, but $T =$ a point is also a possibility) and $X = (X, g)$ is a Riemannian manifold of dimension d which is assumed to be compact or at least one on which the Hodge decomposition theorem holds which I recall now for convenience. The metric g induces one on the co-tangent bundle of X and their p-th exterior powers, the bundles of p-forms. The Hodge $*$-operator sends a p-form to a $(d-p)$-form. The operator d^* is the formal adjoint of d with respect to these metrics. It lowers the degree by one and a form G is co-closed, respectively co-exact if $d^*G = 0$, respectively $G = d^*H$ for some $(d+1)$-form H. A form is harmonic if it is closed and co-closed. The *Hodge decomposition theorem*

[Warn, Chapter 6] states that there is an orthogonal decomposition

$$\begin{tikzcd}
A^p(X) \arrow[d, equal] & = & \mathsf{Har}^p(X) \arrow[d, equal] & \oplus & d\mathsf{A}^{p-1}(X) \arrow[d, equal] & \oplus & d^*\mathsf{A}^{p+1}(X) \arrow[d, equal] \\
\text{[p-forms]} & & \text{[harmonic p-forms]} & & \text{[exact p-forms]} & & \text{[co-exact p-forms].}
\end{tikzcd}$$

Moreover, the space of harmonic forms is *finite dimensional* and every De Rham cohomology class has a unique representing harmonic form. From harmonic theory it also follows (see loc. cit.) that $d \oplus d^*$ induces a self-adjoint operator on $\mathsf{Har}(X)^\perp \subset \mathsf{A}(X)$.

In the theory of Ramond-Ramond fields one is only interested in even degree forms that are already closed. So one writes down the decomposition $\mathsf{A}(X) = \mathsf{A}^+(X) \oplus \mathsf{A}^-(X)$ into even and odd degree forms and to understand the even exact forms, one looks at the action of d on $\mathsf{Har}^-(X)^\perp$ coupled with the action of d^* on $\mathsf{Har}^+(X)^\perp$:

$$\mathsf{D} := \begin{pmatrix} 0 & d\Big|_{\mathsf{Har}^-(X)^\perp} \\ d^*\Big|_{\mathsf{Har}^+(X)^\perp} & 0 \end{pmatrix}. \tag{2}$$

By [B-G-V, §9.6] using ζ-functions there is an exact way to define its regularized determinant $\det \mathsf{D}$ and its square root

$$\Delta := \sqrt{\det \mathsf{D}}, \tag{3}$$

is called the *determinant of the non-zero modes.* In view of the form (2) for the operator D, the determinant of the non-zero modes can be viewed as the determinant of the operator d on odd degree forms.

Let me now come back to the integral (1). To evaluate it, first one sums over the contributions for the *integral* harmonic forms which is called the *classical contribution.* Next, one does the path integral over the odd degree *real* forms C taking care of the exact forms dC. This gives the *quantum contribution.* The relevant calculations have been carried out in detail in [H-N-S] [2] and they can than be summarized as follows:

- the classical contribution is of the form

$$Z_{\text{classical}} = \text{Anomalous pre-factor} \cdot \Theta(0)$$

where $\Theta(z)$ is some classical theta-function.

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[2] Actually, in [H-N-S] the odd middle degree differential forms on a compact manifold of even dimension $d \equiv 2 \bmod 4$ are studied. However, exactly the same calculations apply in the setting of the article [Mo-Wi]

- in the total partition function the anomalous pre-factor must be discarded and the quantum contributions give the factor Δ^{-1} where Δ is the determinant of non-zero modes (3):

$$Z = \frac{1}{\Delta}\Theta(0).$$

Remark. In loc. cit. there is given no rule as to which Θ-function should be used. This is one of the issues which the two articles [Witten, Mo-Wi] address and which I want to discuss below in § 5.2.

3 The basic linear algebra construction

Proposition 3.1. *Let (V, g) be a finite dimensional euclidean $\mathbb{R}$-vector space and ω a non-degenerate $\mathbb{R}$-valued skew-symmetric form. There exists a unique complex structure J on V such that*

1. $g(Jx, Jy) = g(x, y)$ *for all* $x, y \in V$*;*
2. $\omega(Jx, Jy) = \omega(x, y)$ *for all* $x, y \in V$*;*
3. $\omega(x, Jx) > 0$ *for all* $x \in V - \{0\}$.

Proof: Define $A \in \mathrm{Gl}(V)$ by $\omega(x, y) = g(Ax, y)$. Then A is g-skew adjoint, i.e. $A^* = -A$, where the index means the g-adjoint. Hence $P = A^*A = AA^* = -A^2$ is self-adjoint and positive definite. In particular, V has an orthonormal basis of P-eigenvectors so that the matrix of P becomes diagonal with positive entries, say $\lambda_i > 0$, on the diagonal. Replacing these by the positive root $\sqrt{\lambda_i}$ defines the root $Q = P^{\frac{1}{2}}$ of P. Now write

$$A = QJ, \quad J := AQ^{-1}. \tag{4}$$

Since $Q^2 = P = -A^2$ and $Q^* = Q$ we have

$$g(Jx, Jy) = g(AQ^{-1}x, AQ^{-1}y) = g(Q^{-1}x, A^*AQ^{-1}y) = g(Q^{-1}x, Qy) = g(x, y)$$

so that J is g-orthogonal. Since Q is self-adjoint and positive definite this implies that (4) is the unique *polar decomposition* of A. Moreover, since $A^*A = AA^*$, one easily deduces that J and Q (and hence also A and Q) commute. It follows that

$$J^2 = (AQ^{-1})^2 = A^2Q^{-2} = -\,\mathrm{id}_V$$

and hence J is a complex structure. Next, $\omega(Jx, Jy) = g(AJx, Jy) = g(JAx, Jy) = g(Ax, y) = \omega(x, y)$ since J and A also commute. Finally, $\omega(x, Jx) = g(Ax, Jx) = g(QJx, Jx) = g(Qx, x) > 0$ for $x \neq 0$ since Q is positive definite.
To show uniqueness, note that A is uniquely defined by g and ω and hence so is its polar decomposition. We only have to see that J as characterized by 1)–3) gives a polar decomposition $A = RJ$. First of all $AJ = JA$ as can easily be seen from 1) and 2). But then $R := AJ^{-1} = J^{-1}A$ is seen to be self-adjoint. From this and 3) it follows that R is positive definite which finishes the proof of uniqueness. □

Remark 3.2. One can rephrase the Lemma in terms of symplectic geometry as follows: *A symplectic structure on a finite dimensional euclidean vector space is tamed by a unique almost complex structure compatible with the metric.*

Corollary 3.3. *ω is a Riemann-form for (V, J), i.e.*

1. *$\omega(Jx, Jy) = \omega(x, y)$ for all $x, y \in V$;*
2. *the form $b(x, y) := \omega(Jx, y)$ is a symmetric $\mathbb{R}$-valued positive definite bilinear form on V;*
3. *$b(Jx, Jy) = b(x, y)$.*

It follows that the $\mathbb{C}$-valued form

$$h(x, y) := b(x, y) + \mathrm{i}\omega(x, y) \tag{5}$$

is hermitian (with respect to J) and positive definite. Hence:

Corollary 3.4. *Suppose that $V = \Lambda \otimes \mathbb{R}$ for some lattice $\Lambda \subset V$ of maximal rank and that ω is* integer valued. *Then the torus V/Λ is an abelian variety with polarization h given by equation* (5). *This is a principal polarization if ω is unimodular.*

Definition 3.5. The abelian variety just constructed is denoted $J(\Lambda, g, \omega)$. This can equivalently be phrased in terms of Hodge theory: the triple (Λ, g, ω) defines a unique polarized weight one Hodge structure whose Jacobian is $J(\Lambda, g, \omega)$.

4 Examples

4.1 Examples from cohomology

1) Let (X, g) be a compact oriented Riemannian manifold of dimension $4q+2$ and consider the middle cohomology $H^{2q+1}(X)$ equipped with the cup-product pairing $\bullet$ and the Hodge metric. Recall that the latter is defined as follows.

Definition 4.1. The *Hodge metric* on $H^*(X)$ is the metric associated to g defined by

$$b^{(g)}([\alpha],[\beta]) = \int_X \alpha \cup *\beta,$$

where α, β are the unique harmonic forms in the classes $[\alpha], [\beta]$ (see also the discussion on harmonic theory in § 2.)

The abelian variety

$$J^{2q+1}(X,g) := J(H^{2q+1}(X), b^{(g)}, \bullet)$$

is an intrinsic invariant of the pair (X, g). The polarization is a principal polarization. When X is a compact Riemann surface ($q = 0$) the datum of g (up to conformal equivalence) is equivalent to the datum of a complex structure on X (up to isomorphism) and we find back the classical Jacobian of the Riemann surface.

2) Suppose more generally that (X, g) is an even dimensional compact oriented Riemannian manifold and consider the odd cohomology

$$H^{\text{odd}}(X) = H^1(X) \oplus H^3(X) \oplus \cdots H^{2d-1}(X), \quad 2d = \dim X.$$

Again, we have the Hodge metric $b^{(g)}$ on each of the $H^j(X)$ and hence on $H^{\text{odd}}(X)$. There is also a unimodular skew-form, the top degree cup-product form $\bullet$ defined by

$$a \bullet b = \int_X a \cup b.$$

It pairs $H^{2j+1}(X)$ perfectly to $H^{2d-2j-1}(X)$. So depending on whether d is odd or even two cases arise:

1. $\dim(X) = 4q = 2d$:

   $$J^{\text{odd}}(X,g) := J(H^{\text{odd}}(X), b, \bullet) = J^1 \times J^2 \cdots \times J^{2q-1},$$

   where $J^k = J^k(X,g) = J(H^k(X) \oplus H^{4q-k}(X), b, \bullet)$.

2. $\dim(X) = 4q+2$:

$$J^{\mathrm{odd}}(X,g) = J(H^{\mathrm{odd}}(X), b^{(g)}, \bullet) = J^1 \times J^2 \cdots \times J^{2q+1},$$

   where all but the last factors are as before and the last factor is the invariant $J^{2q+1}(X,g)$ from Example 1)

3) Next, consider the even cohomology. Define an involution $\iota : H^{\mathrm{even}}(X) \to H^{\mathrm{even}}(X)$ by letting ι act as id on $H^{2j}(X)$ is j is even and as $-\,\mathrm{id}$ if j is odd. The problem is to find the symplectic form. If $\dim X = 4q+2$ there is an easy solution. One then checks that

$$(x,y) \mapsto x \bullet^{\iota} y := x \bullet \iota(y) = \int_X x \cup \iota(y)$$

is a perfect skew-symmetric pairing. The resulting principally polarized variety is denoted $J^{\mathrm{even}}(X,g)$. It is a product whose factors are principally polarized varieties associated to two summands of the form $H^{2k}(X) \oplus H^{4q-2k+2}(X)$.

4) We now consider a twisted version of the above. Fix $\mathbf{a} \in H^{\mathrm{even}}(X;\mathbb{Q})$ which is invertible in the ring $H^*(X;\mathbb{Q})$ and define two $\mathbb{Q}$-valued pairings

$$\begin{aligned} \bullet_{\mathbf{a}}, \bullet_{\mathbf{a}}^{\iota} \quad : \quad & H^{\mathrm{even}}(X;\mathbb{Q}) \times H^{\mathrm{even}}(X;\mathbb{Q}) \to \mathbb{Q} \\ & (x,y) \mapsto x \bullet_{\mathbf{a}} y := \int_X [x \cup y \cup \mathbf{a}]. \qquad (6) \\ & (x,y) \mapsto x \bullet_{\mathbf{a}}^{\iota} y := \int_X [x \cup \iota(y) \cup \mathbf{a}]. \qquad (7) \end{aligned}$$

Now, since $\mathbf{a}$ is invertible, if y varies over all elements of H^{even} so do $y \cup \mathbf{a}$ and $\iota(y) \cup \mathbf{a}$. Hence, since the cup-product pairing is non-degenerate, the same holds for the two preceding twisted versions. If $\dim(X) = 4q+2$ the second pairing is skew-symmetric but need no longer be integral. Choose $N = N(\mathbf{a})$ to be a minimal integer such that $N \bullet_{\mathbf{a}}^{\iota}$ becomes integral. We can likewise use an invertible $\mathbf{b} \in H^*(X;\mathbb{R})$ to modify the Hodge metric: we define

$$b_{\mathbf{b}}^{(g)}(x,y) := \int_X [\mathbf{b} \cup x] \cup *[\mathbf{b} \cup y].$$

So

Proposition 4.2. *Let (X,g) be a compact Riemannian manifold of dimension $2 \bmod 4$ and $\mathbf{a} \in H^{\mathrm{even}}(X;\mathbb{Q})$, invertible in the ring $H^*(X;\mathbb{Q})$ and*

$\mathbf{b}$ *invertible in* $H^*(X;\mathbb{R})$*. Using the notation of Definition 3.5, the above construction yields an abelian variety*

$$J(H^{\text{even}}(X), b_{\mathbf{b}}^{(g)}, N(\mathbf{a})\bullet_{\mathbf{a}}^{\iota}),$$

canonically associated to $(X, g, \mathbf{a}, \mathbf{b})$*.*

5) Let me show how Weil's Jacobian fits in this picture as well. Let (X,ω) be a compact Kähler manifold of dimension d. Introducing

$$\mathbf{a}_\omega := \sum_{k=0}^{d} (-1)^{\frac{1}{2}k(k-1)} \underbrace{\omega \wedge \cdots \wedge \omega}_{k \text{ times}} \in H^{\text{even}}(X;\mathbb{R}),$$

the Riemann form can be written as $\bullet_{\mathbf{a}_\omega}$ which is a real-valued skew-symmetric form on $H^{\text{odd}}(X)$. If X is projective we can choose ω to be an integral class: take it to be the first Chern form of an ample line bundle on X. Now the Hodge decomposition $H(X;\mathbb{C}) = \oplus H^{p,q}(X)$ endows $H(X)$ with a pure Hodge structure and the *Weil-operator* C is the unique real operator which equals multiplication by i^{p-q} on the $H^{p,q}(X)$. Since $C^2 = -\,\mathrm{id}$ on $H^{\text{odd}}(X)$ this gives a complex structure. However, we also need a compatible metric. The Riemann forms give such a metric, but only on the *primitive* real cohomology: there one sets

$$g_\omega(x,y) = (Cx) \bullet_{\mathbf{a}_\omega} y.$$

It is easy to verify that the complex structure on $H^{\text{odd}}(X) \cap H^*_{\text{prim}}(X)$ given by C satisfies the two properties of Prop.3.1 relative $(\bullet_{\mathbf{a}_\omega}, g)$ and so we get a polarized abelian variety which is a product whose factors are the *Weil jacobians*

$$J^k(X,\omega) := J(H^{2k-1}_{\text{prim}}(X), g_\omega, \bullet_{\mathbf{a}_\omega}).$$

To extend this to all odd degree cohomology, following [Weil, Ch. IV,§ 7, Corr. of Thm. 7] one first has to define a modified product

$$* : H^k(X;\mathbb{Q}) \times H^k(X;\mathbb{Q}) \to H^{2k}(X;\mathbb{Q})$$

using the primitive decomposition on $H^*(X;\mathbb{Q})$ by writing $x = \sum x_r \omega^{\wedge r}$, $y = \sum y_r \omega^{\wedge r}$ with x_r, y_r primitive and then defining

$$x * y := \sum (-1)^r x_r y_r \omega^{\wedge(2r)}.$$

The form

$$x *_{\mathbf{a}_\omega} y := \int_X x * y \cup \mathbf{a}_\omega$$

is also skew symmetric on $H^{2k+1}(X)$ but in general will no longer be integral. However, there is a universal integer $N_{d,k}$ depending on d and k making the form $N_{d,k} *_{\mathbf{a}_\omega}$ integral (see [Weil, Ch. 2.4 Th. 1]).

Then one has to redefine

$$g_\omega(x, y) := (Cx) x *_{\mathbf{a}_\omega} y$$

which now is a metric on all of $H^*(X)$ and with these modifications we can get abelian varieties

$$J(H^{2k+1}(X), g_\omega, N_{d,k} *_{\mathbf{a}_\omega})$$

associated to all of odd degree cohomology.

4.2 Examples using K-theory

We can give a variant of the examples in § 4.1 using $K(X)$. Recall (see Appendix B), (9) that the Chern character:

$$\mathrm{ch} : K(X) \to H^{\mathrm{even}}(X; \mathbb{Q})$$

becomes an isomorphism after tensoring with $\mathbb{Q}$. So

$$\Lambda(X) = \mathrm{ch}(K(X)) \subset H^{\mathrm{even}}(X; \mathbb{Q})$$

is a *lattice*, i.e. a $\mathbb{Z}$-module of rank $\dim_{\mathbb{Q}} H^{\mathrm{even}}(X; \mathbb{Q})$. The intersection pairing as well as the two twisted pairings (6)(7) induce $\mathbb{Q}$-bilinear pairings on Λ. It can now happen that for specific $\mathbf{a}$ a twisted pairing becomes an integral pairing:

Definition 4.3. A *multiplier* is an element $\mathbf{a} \in H^{\mathrm{even}}(X)$ such that the pairings

$$\bullet_{\mathbf{a}}, \bullet^{\iota}_{\mathbf{a}} : \Lambda(X) \times \Lambda(X) \to \mathbb{Q}$$

are *integral.* If $\mathbf{a}_0 = 1$, such a multiplier is called *normalized.*

Now an interesting phenomenon occurs which makes use of the fact that for each element $\mathrm{ch}(a)$, $a \in K(X)$, the lowest order term turns out to be integral:

Lemma 4.4 ([AH3, 3.7])**.** *If $\mathbf{a} \in H^{\mathrm{even}}(X)$ is a normalized multiplier, the pairings $\bullet_{\mathbf{a}}$ and $\bullet^{\iota}_{\mathbf{a}}$ are unimodular on $\Lambda(X)$.*

Corollary 4.5. *Suppose* $\dim X \equiv 2 \bmod 4$, $\mathbf{a} \in H^*(X;\mathbb{Q})$ *a normalized multiplier and* $\mathbf{b} \in H^*(X,\mathbb{R})$ *invertible. Then (using the notation of Definition 3.5)*

$$J(\Lambda(X), b_{\mathbf{b}}^{(g)}, \bullet_{\mathbf{a}}^{\iota})$$

is a principally polarized Abelian variety.

Example 4.6. If X is spin, $\mathbf{a} = \hat{A}(X)$ is a normalized multiplier. Indeed, by the Index Theorem due to Atiyah and Singer (see B.9), for complex vector bundles E, F the rational number $\mathrm{ch}(E) \bullet_{\hat{A}(X)}^{\iota} \mathrm{ch}(F)$ is the index of the Dirac operator with values in $E \otimes \bar{F}$. The pairing being bilinear can be extended integrally to all of $K(X)$ and the result follows since $\hat{A}(X)$ is normalized: it starts of with $1 \in H^0(X)$. If one makes the choice

$$\mathbf{b} = 2\pi\sqrt{\hat{A}}$$

the principally polarized variety is the one from [Mo-Wi].

5 Special Theta Functions

5.1 Some reminders

Let $J = V/\Lambda$ be an abelian variety with principal polarization given by the unimodular integral and positive $(1,1)$-form ω. In this section let $g = \dim_{\mathbb{C}} V$ be the complex dimension of J. The set of line bundles L on J with $c_1(L) = \omega$ is a principal space under the Picard torus of J which is isomorphic to J. To single out a line bundle having ω as first Chern class one traditionally uses multipliers for ω:

Definition 5.1. A *multiplier for* ω is a function $\alpha : \Lambda \to \mathrm{U}(1)$ for which

$$\alpha(x+y) = (-1)^{\omega(x,y)} \cdot \alpha(x)\alpha(y). \tag{8}$$

A multiplier is entirely specified by its values on a symplectic basis and any value in $\mathrm{U}(1) \simeq S^1$ can be taken so that the above set is indeed a topological torus of dimension $2g$ as should be the case. Indeed, line bundles with given polarization ω are in 1-1 correspondence with multipliers for ω. See for example [Mumford, Chap I.2]. A choice of of symplectic basis $\mathsf{B} := \{e_1, \ldots, e_g, f_1, \ldots, f_g\}$ of Λ with respect to ω singles out a specific line bundle: the one for which $\alpha(b) = 1$ for all $b \in \mathsf{B}$. The corresponding multiplier takes its values in the the subgroup ± 1 of $\mathrm{U}(1)$. There are exactly

2^{2g} such line bundles since one may choose $\alpha(b) \in \{\pm 1\}$ for every individual $b \in \mathsf{B}$ separately. In a more explicit fashion, take a holomorphic basis for V, or, equivalently, a basis for the space of holomorphic 1-forms on J chosen in such a way that the rows in the matrix

$$\Omega = (\mathbf{1}_g, Z), \quad {}^{\mathsf{T}}Z = Z, \, \mathrm{Im}(Z) > 0.$$

are the periods of this basis with respect to B. Choose $\theta \in \Lambda$ such that $\alpha(y) = (-1)^{\omega(\theta,y)}$ for all $y \in \Lambda$ which is possible since we have a symplectic basis. Then, setting

$$\begin{aligned} \Lambda_1 = \bigoplus \mathbb{Z}e_j \qquad & \Lambda_2 := \bigoplus \mathbb{Z}f_j \\ \theta = \theta_1 + \theta_2, \qquad & \theta_i \in \Lambda_i \\ u = \frac{1}{2}\theta_1 \bmod \Lambda_1 \in \frac{1}{2}\Lambda_1/\Lambda_1, \qquad & v = \frac{1}{2}\theta_2 \bmod \Lambda_2 \in \frac{1}{2}\Lambda_2/\Lambda_2 \end{aligned}$$

define

$$\Theta \begin{bmatrix} u \\ v \end{bmatrix} (z) := \sum_{x \in \Lambda_1 + u} \exp[\mathrm{i}\pi\langle x, Zx\rangle] \cdot \exp[2\pi\mathrm{i}\langle x, z + v\rangle].$$

It is the classical theta-function with *theta-characteristic* (u, v). Here $\langle x, y\rangle = {}^{\mathsf{T}}x \cdot y$ is the usual euclidean inner product on $\mathbb{C}^g$. It is the up to a multiplicative constant unique non-zero holomorphic section in the unique holomorphic line bundle on J with multiplier α.

5.2 Theta functions and Ramond-Ramond fields

Continue with the example 4.6 constructed from a compact spin manifold (X, g). So $\Lambda = \Lambda(X) \simeq K(X) \bmod \mathrm{tors}(K(X))$ and there is a natural unimodular symplectic form $\omega = \bullet^{\iota}_{\mathbf{a}}$ on it. One needs to specify a multiplier on this lattice. First extend ω to $K(X)$ by by defining $\omega(x, y) = \omega(\mathrm{ch}(x), \mathrm{ch}(y))$.

Next, one needs to make further assumptions. First of all, if $\dim(X) \equiv 2 \bmod 8$ by Prop. A.6 one has a homomorphism $j : \mathrm{KO}(X) \to \mathbb{Z}/2\mathbb{Z}$. If $x \in \mathrm{K}(X)$ is a virtual complex bundle $x \otimes \bar{x}$ is naturally an element of $\mathrm{KO}(X)$ and so we get a homomorphism

$$\alpha : K(X) \to \{\pm 1\}, \quad x \mapsto (-1)^{j(x \otimes \bar{x})}.$$

One can show that it satisfies the required transformation law (8) to make it a multiplier for the extended form ω. However, one needs it to descend to a multiplier on Λ and this is not automatic. But it is certainly the case if

$x \mapsto j(x \otimes \bar{x})$ is identically 1 on the torsion part of $K(X)$ (this is automatic if for instance $K(X)$ has no torsion). I can now formulate the main result of [Mo-Wi, §3] in mathematical terms:

Proposition 5.2 ([Mo-Wi, § 3.1])**.** *For a compact spin manifold* (X, g) *of dimension* $2 \bmod 8$ *consider the principally polarized abelian variety*

$$J(\Lambda(X), b^{(g)}_{2\pi\sqrt{\mathbf{a}}}, \omega = \bullet^{\iota}_{\mathbf{a}}),$$

where $\mathbf{a} = \hat{A}(X)$*. Under the above assumption, the map* $\alpha(x) = (-1)^{j(x \otimes \bar{x})}$ *is a multiplier for* ω *and hence defines a unique line bundle with first Chern class* ω*. Let* $\Theta \left[\begin{smallmatrix} u \\ v \end{smallmatrix} \right] (z)$ *be the corresponding normalized theta-function. Then the partition function for type II-A Ramond-Ramond fields on* X *is given by* $\Theta \left[\begin{smallmatrix} u \\ v \end{smallmatrix} \right] (0)/\Delta$ *where* Δ *is the determinant* (3) *for the non-zero modes on* X*.*

Appendices

A K-theory

The reader may consult the excellent introduction [At1]. It contains the article [At] on real K-theory. For a solid introduction to index theorems consult the appendices [Hir, §24-26] to Hirzebruch's classic.

A.1 K-theory for complex bundles

We let X be a (topological) manifold and we let $K(X)$ be the Grothendieck group of *complex* vector bundles on X. Recall [AH1] that this is the free $\mathbb{Z}$-module generated by the isomorphism classes of complex vector bundles modulo the relations $E \oplus F - E - F$. It can be seen to be generated by *virtual bundles*, i.e. differences of the form $E - F$, where E and F are any two vector bundles. The tensor product on vector bundles is compatible with these relations and so $K(X)$ becomes a ring. If $f : X \to Y$ is continuous, pull back of bundles induced a ring homomorphism $f^* : K(Y) \to K(X)$.

The *suspension* SX is obtained from the product $S^1 \times X$ by identifying all points in the subspace $\{\mathbf{1}\} \times X$ where $\mathbf{1} = (1, 0) \in S^1 \subset \mathbb{R}^2$; n-fold iterated suspension is denoted $S^n X$. One defines

$$K^{-n}X := K(S^n X).$$

Bott's periodicity theorem [Bott] can be stated as $K^{-2}(X) \simeq K(X)$ which makes it possible to define $K^n X$ for all integers n. There are natural pairings

$K^n(X) \times K^m(X) \to K^{n+m}(X)$ compatible with the Bott periodicity making $\oplus_{n\in\mathbb{Z}} K^n(X)$ into a graded ring. In view of Bott's theorem the essential part of this ring is

$$K^*(X) = K(X) \oplus K^1(X)$$

with $\mathbb{Z}_2$-grading. The cohomology-ring can also be given a $\mathbb{Z}_2$-grading as $H^*(X) = H^{\text{even}} \oplus H^{\text{odd}}(X)$. The Chern character gives $\mathbb{Z}_2$-graded isomorphisms [AH2]

$$\text{ch} : K^*(X) \otimes \mathbb{Q} \xrightarrow{\sim} H^*(X; \mathbb{Q}). \tag{9}$$

One can also define relative K-theory groups $K^n(X, Y)$ where Y is a subset of X and these fit in exact sequences as for ordinary cohomology. One important fact is that one has a K-theoretic version of the *Thom isomorphism theorem*: Let $B(E)$, respectively $S(E)$ be the unit disk-bundle, unit sphere bundle associated to a hermitian vector bundle (E, g) of rank r. Then

$$K^*(B(E), S(E)) \simeq K^*(X).$$

A.2 The Index Theorem

Let X be a differentiable manifold, E, F two hermitian vector bundles, $\mathsf{D} : \Gamma(E) \to \Gamma(F)$ a differential operator with formal adjoint D^*. Recall that the *index* is given by

$$\text{ind}(\mathsf{D}) := \dim \ker \mathsf{D} - \dim \ker \mathsf{D}^*. \tag{10}$$

Its symbol defines an element $\sigma(\mathsf{D}) \in K(BX, SX)$ whose Chern character lands in $H^*(BX, SX) \simeq H_c^*(T^*X)$. Let $\pi : T^*X \to X$ be the natural projection. Since T^*X is a symplectic manifold and g a Riemannian metric on X, the bundle $T(T^*X)$ has a natural complex structure (Prop. 3.1) such that $\pi^*TX \otimes \mathbb{C} \simeq T(T^*X)$ and there is a Todd class $\text{td}(TX^*|BX) \in H^*(BX)$. Since $H^*(BX, SX)$ is a $H^*(BX)$-module the following formula makes sense; it defines the topological index.

$$\text{ind}_\tau(\mathsf{D}) := \int_{T^*X} \text{ch}(\sigma(\mathsf{D})) \cdot \text{td}(T^*X|BX) \tag{11}$$

One has:

Theorem A.1 ([AS1])**.** *Let X be a compact differentiable manifold and D an elliptic differential operator between complex vector bundles on X. Then the topological index* (11) *equals the (analytical) index* (10)*. In particular, it is an integer.*

The K-theoretic extension comes from the remark that the right hand side of (11) makes sense if we replace $\sigma(\mathsf{D})$ by any element $d \in K(BX, SX)$. This defines then the topological index

$$\operatorname{ind}_{\tau}(d) = \int_{T^*X} \operatorname{ch}(d) \cdot \operatorname{td}(T^*X|BX).$$

It can be shown that it also makes sense to speak of an *analytic index* $\operatorname{ind}(d)$ for such elements and that it is an integer:

Theorem A.2 ([AS2]). *For a compact differentiable manifold X, the two homomorphisms*

$$\operatorname{ind}, \operatorname{ind}_{\tau} : K(BX, SX) \to \mathbb{Q}$$

coincide and hence take values in $\mathbb{Z}$.

This version has a relative form for differentiable locally trivial fibrations $f : X \to T$. The starting observation is the fact that $K(\text{point}) = \mathbb{Z}$ so that the integer $\operatorname{ind}(\mathsf{D})$ is just the K-theoretic difference of the vector spaces $\ker \mathsf{D} - \ker \mathsf{D}^*$. For a family over T this pointwise construction gives a difference of complex bundles on T and hence an element of $K(T)$. For the topological index one has to replace T^*X by the relative cotangent bundle $T^*(X/T)$ and one gets:

Theorem A.3 ([AS3]). *For a differentiable family $f : X \to T$ of compact differentiable manifolds, the two homomorphisms*

$$\operatorname{ind}, \operatorname{ind}_{\tau} : K(B(X/T), S(X/T)) \to K(T) \otimes \mathbb{Q}$$

coincide and hence take values in $K(T)$.

A.3 K-theory of real vector bundles

The Grothendieck group of real vector bundles on a manifold X is denoted by $\mathrm{KO}(X)$. As in the complex case one sets $\mathrm{KO}^{-n}(X) = \mathrm{KO}(S^n X)$ and now there is periodicity of order 8.

We shall be interested in real Riemannian vector bundles E that come with a so called *spin^c-structure.* By definition this means that there is a class $c \in H^2(X)$ such that its mod 2 reduction is the Stiefel-Whitney class $w_2(E)$. They behave cohomologically as complex bundles.

Example A.4. An oriented Riemannian vector bundle E of rank r is a spin bundle if and only if the structure group $\mathrm{SO}(r)$ can be reduced to $\mathrm{Spin}(r)$. It is well-known that this is equivalent to $w_2(E) = 0$ and independent of the metric or orientation. In particular, spin bundles have a spin^c-structure and the tangent bundle of a spin manifold admits a spin^c-structure.

There is still another K-group defined for pairs (X, ι) where X is a manifold and ι is an involution. See [At]. One defines $\mathrm{KR}(X)$ as the K-group for complex bundles E on X admitting involutions covering ι and which are $\mathbb{C}$ anti-linear on the fibres. Again, there is a Bott-periodicity result, namely $\mathrm{KR}^*(X) \simeq \mathrm{KR}^{*+8}(X)$. The standard example is the total space of a *real* vector bundle E with involution ι given by id on the fibres. This gives back $KO(X)$. Another example is the Thom space (BV, SV) of a Riemannian vector bundle V. Here the involution is the antipodal map. This space figures in a very general form of the Thom isomorphism theorem which can be deduced from [At2, Theorem 6.2]. We explain the latter theorem in a simplified situation. Let X be a compact differentiable manifold, G a compact Lie-group acting trivially on X and suppose we have a group homomorphism $\rho : G \to \mathrm{Spin}^c(8r)$. Moreover let V be a vector bundle on X of rank $8r$ with spin^c-structure. Put a G-module structure on V through ρ. Then there is a natural isomorphism

$$\varphi : \mathrm{KR}(X) \to \mathrm{KR}((BV, SV) \times_G X).$$

Specialize this to the case where X is a spin manifold of dimension $(8r - m)$ so that TX gets a spin^c-structure, let $G = \mathrm{Spin}^c(8r-m)$ and let $\rho : \mathrm{Spin}^c(8r-m) \hookrightarrow \mathrm{Spin}^c(8r)$ the embedding. Put $V = TX \oplus \mathbb{R}^m$. Then $(BV, SV) \times_G X = (BX, SX) \oplus (B^m, S^m)$ where the involution on the second summand is not the identity *but the antipodal map.* Applying the periodicity [At, Theorem 2.3] we deduce:

Theorem A.5 (Thom isomorphism theorem)**.** *Suppose X is a spin manifold of dimension $(8r - m)$. There is a natural isomorphism*

$$\varphi : \mathrm{KO}(X) = \mathrm{KR}(X) \xrightarrow{\sim} \mathrm{KR}^m(B(X), S(X)).$$

Now it is time to pass to index theory. It can be shown that the symbol of a real elliptic operator belongs to $\mathrm{KR}(BX, SX)$ where one complexifies the operator first; the involution covering ι comes then from complex conjugation. So, by construction, there is a forgetful map $\mathrm{KR}(BX, SX) \to \mathrm{K}(X)$ and the index theorem for complex bundles can be applied, but this gives nothing extra. However, for families $X \to T$ the situation becomes different. The (analytic) index can be extended to a homomorphism

$$\mathrm{ind} : \mathrm{KR}(B(X/T), S(X/T)) \to \mathrm{KO}(T).$$

covering the complex index map. But since the covering maps are in general not injective one gets extra information from the Index theorem for families of real elliptic operators [AS4]. It states that (complexified) analytic index equals an explicit expression in terms of Chern classes and which can be called the topological index, ind_τ.

In the special case of a product family $X \times S^m \to S^m$ with X a spin manifold of dimension $8r+m$, these two maps together with the above Thom isomorphism theorem (note the change of sign!) induce a commutative diagram

$$\begin{tikzcd}
\mathrm{KR}((BX, SX) \times S^m) \arrow[r] \arrow[d] & \mathrm{KO}(S^m) \arrow[d] \\
\mathrm{KR}^{-m}(BX, SX) \arrow[r] \arrow[d, equal] & \mathrm{KO}^{-m}(\mathrm{point}) \arrow[d, equal] \\
\mathrm{KO}(X) \arrow[r] & \mathrm{KO}^{-m}(\mathrm{point}).
\end{tikzcd}$$

In particular, if $d = 8r + 2$ and $m = 2$ Bott periodicity gives two maps

$$\mathrm{ind}, \mathrm{ind}_\tau : \mathrm{KO}(X) \to \mathrm{KO}^{-2}(\mathrm{point}) = \mathbb{Z}/2\mathbb{Z}.$$

These are equal and called the *mod-2-index* for a family over S^2. This can in particular be applied to real bundles E; the Dirac operator $\not{D}_E$ on X with values in E has index 0 (see Cor. B.10 in Appendix B) and a priori one does not expect information. But from such E one can canonically construct a family of Diracs depending on a complex parameter which then extends to the Riemann sphere S^2; hence, the above considerations with $m = 2$ apply. It turns out (see [AS4] for details) that the analytic index is the mod-2 dimension of the bundle $\ker(\not{D}_E)$ and the topological index comes from the Gysin map associated to $X \to$ point. Recall at this point that for any map $f : X \to Y$ between compact *spin* manifolds, there are Gysin maps

$$f_! : \mathrm{KO}^*(X) \to \mathrm{KO}^{*-c}(Y), \quad c = \dim X - \dim Y.$$

The upshot is

Proposition A.6. *Let X be a compact spin manifold of dimension* 2 mod 8*. Let $a_X : X \to$ point be the natural map. For $x \in \mathrm{KO}(X)$, let $j(x)$ be the mod-*2 *index of the Dirac operator with values in x. Then there we have an equality*

$$j = (a_X)_! : \mathrm{KO}(X) \to \mathrm{KO}^{-2}(\mathrm{point}) = \mathbb{Z}/2.$$

B Dirac operators

For this section the reader may consult [B-G-V, Chapters 3,4].

B.1 Clifford algebras

Let k be a field of characteristic $\neq 2$ and let $V = (V, q)$ be a (finite dimensional) k-inner product space. Its tensor algebra $\mathsf{T}V$ gets a natural k-inner product, also denoted by q. The unit $1 \in k$ serves as a unit in $\mathsf{T}V$. Its *Clifford algebra* is the following quotient algebra of dimension 2^n where $n = \dim V$:

$$\mathsf{C}(V) = \mathsf{C}(V, q) := \mathsf{T}V/\text{ideal generated by } \{x \otimes x + q(x, x) \cdot 1 \mid x \in V\}.$$

There is a natural map $c : V \to \mathsf{C}(V)$, $x \mapsto$class of x. The induced action $c(x) : \mathsf{C}(V) \to \mathsf{C}(V)$ is called the *Clifford action.* One easily shows that the pair $(\mathsf{C}(V), c)$ satisfies the following property: It is the unique pair (C, c) consisting of a k-algebra C with unit together with a k-linear map $c : V \to C$ such that[3]

$$c(x) \cdot c(y) + c(y) \cdot c(x) = -2q(x, y) \cdot 1 \tag{12}$$

which is universal with respect to this property.

Proposition-Definition B.1. *A* $\mathsf{C}(V)$-Clifford module *A is a k-algebra equipped with a* Clifford action *of V, i.e. a linear map $c : V \to A$ satisfying* (12)*. If the Clifford-action is q-skew-adjoint, one says that A is* self-adjoint*:*

$$q(v \cdot x, y) + q(x, v \cdot y) = 0, \quad \text{for all } v \in V,\ x, y \in A.$$

The Clifford algebra is a twisted version of the exterior algebra:

Lemma B.2. $\mathsf{C}(V)$*, as a vector space is isomorphic to ΛV, the exterior algebra. The Clifford-action on ΛV is given by*

$$c(x)\alpha = x \wedge \alpha - \iota(x)\alpha$$

where the linear map $\iota(x)$ is the contraction with the co-vector $[- \mapsto q(x, -)]$. This makes ΛV into a self-adjoint Clifford module. The bigrading given by odd and even degree in the exterior product descends:

$$\mathsf{C}^+(V) := c\left(\Lambda^{\text{even}} V\right), \quad \mathsf{C}^-(V) := c\left(\Lambda^{\text{odd}} V\right).$$

Proof: Since $\iota(x)$ is the q-adjoint of the map $\alpha \mapsto x \wedge \alpha$ clearly $\Lambda V$is a self-adjoint Clifford module. To see that one gets an isomorphism, note that

$$\sigma : \mathsf{C}(V) \xrightarrow{\sim} \Lambda V, \quad a \mapsto c(a) \cdot \mathbf{1}. \tag{13}$$

---

[3] The multiplication in $\mathsf{C}(V)$ is written with dots.

is a bijective k-linear map whose inverse

$$c : \Lambda V \to \mathsf{C}(V)$$

can be explicitly given as follows: let $\{e_1, \ldots, e_n\}$ be an orthogonal basis for V, then send $e_{i_1} \wedge \cdots \wedge e_{i_k}$ to the element $e_{i_1} \cdots e_{i_k}$. $\square$

Clifford modules are in general *not* representation spaces of the orthogonal group $\mathrm{O}(V, q)$ but they are representations of the *spin group* $\mathrm{Spin}(V, q)$, i.e. an unramified double cover[4] of the orthogonal group $\mathrm{SO}(V, q)$. For self-adjoint Clifford modules this representation descends to a representation of $\mathrm{SO}(V, q)$.

Example B.3. Let V be even dimensional euclidean oriented space. Then

$$\mathrm{Spin}(V) = \{x_1 \cdot \cdots \cdot x_{2k} \in \mathsf{C}^+(V) \mid \|x_j\| = 1, j = 1, \ldots, k\}.$$

Continue with the case $\dim V = n = 2m$, an even number. Suppose $k = \mathbb{R}$. Extend q bilinearly to $V_{\mathbb{C}} = V \otimes \mathbb{C}$. Then there exist maximal isotropic subspaces $H \subset V_{\mathbb{C}}$ of complex dimension m. If V is oriented with orthonormal oriented basis $\{e_1, \ldots, e_{2m}\}$ we can take for H the subspace spanned by $e_{2k-1} + \mathrm{i}e_{2k}$, $k = 1, \ldots, m$. Such H is an *oriented* maximal isotropic subspace.

Example B.4. Let V itself be a complex hermitian vector space with complex structure $J \in \mathrm{End}(V)$. Then V is an even dimensional real euclidian space and $H = V^{1,0}$, the eigenspace for J and eigenvalue i.

Now introduce the *spinor spaces*

$$\mathsf{S} = \mathsf{S}(V) := \Lambda H, \quad \mathsf{S}^+(V) = \Lambda^{\mathrm{even}} H, \quad \mathsf{S}^-(V) = \Lambda^{\mathrm{odd}} H.$$

The first, S, is clearly a complex Clifford module through the usual Clifford action given by Lemma B.2. It turns out to be an irreducible complex spinor representation. On the other hand, the two spinor spaces $\mathsf{S}^{\pm}(V)$ can be shown to be irreducible as *real* representations of the spinor group. They are called the *half spinor representations.*

The metric on the spinor space coming from the metric on H induced by the hermitian form $(x, y) \mapsto q(x, \bar{y})$ makes $\mathsf{S}^+(V)$ and $\mathsf{S}^-(V)$ orthogonal to each other and $\mathsf{S}(V)$ is a self-adjoint Clifford-module. One has:

Proposition B.5. *Every complex $\mathsf{C}(V)$-module E is of the form $\mathsf{S}(V) \otimes W$ where the* twisting space $W = \mathrm{Hom}_{\mathsf{C}(V)}(\mathsf{S}(V), E)$ *is a complex vector space with trivial $\mathsf{C}(V)$-action.*

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[4] Provided $\dim V > 1$.

The spinor space is a $\mathbb{Z}_2$-graded complex $\mathsf{C}(V)$-module. This is also the case for general Clifford modules, but here one has to consider how the *chirality operator*

$$\gamma := \mathrm{i}^m e_1 \cdots e_{2m} \in \mathsf{C}(V).$$

acts:

$$E^{\pm} := \{e \in E \mid \gamma \cdot v = \pm v\}.$$

This is compatible with action of γ on $\mathsf{S}(V)$ since it turns out that $\gamma = \pm\mathbf{1}$ on $\mathsf{S}^{\pm}(V)$. In particular, one has a $\mathbb{Z}_2$-graded action of $\mathsf{C}(V)$ on E.

B.2 Dirac operators and spin structures

This can be globalized to the framework of vector bundles on a compact Riemannian manifold (X, g). The cotangent bundle T^*X is a metric bundle. The Clifford algebras $\mathsf{C}(T^*_x)$, $x \in X$ glue together to give the *Clifford algebra* $\mathsf{C}(X)$. The Riemannian structure defines a unique g-connection on TX (and on T^*X) without torsion, the *Levi Civita connection.* Both g and this connection extend to ΛT^*X and thus by Lemma B.2 produces a g-connection and an induced Levi-Civita connection ∇^{LC} on the Clifford-algebra. The Clifford algebra has a self-adjoint Clifford action. More generally one defines:

Definition B.6. A *Clifford bundle* is a triple (E, h, ∇) consisting of a $\mathbb{Z}_2$-graded complex $\mathsf{C}(X)$-module $E = E^+ \oplus E^-$ equipped with a hermitian metric h and an h-connection ∇ such that
i) The Clifford action on the module E is graded, E^+ and E^- are mutually h-orthogonal and the action is self-adjoint with respect to h, i.e.

$$h(c(\alpha)\sigma, \tau) + h(\sigma, c(\alpha)\tau) = 0,$$

for all differentiable (local) sections σ, τ of E and differential 1-forms α;
ii) The connection is compatible with the Levi-Civita connection in the sense that for any local vector field ξ one has

$$\nabla_\xi(c(\alpha)\sigma) = c(\nabla^{\mathrm{LC}}_\xi \alpha)\sigma + c(\alpha)(\nabla_\xi \sigma),$$

for all differentiable (local) sections σ of E, and differentiable 1-forms α.

The *Dirac operator* $\not{D}_E$, a first order differential operator on sections of E is defined as follows:

$$\not{D}_E : \Gamma(E) \xrightarrow{\nabla} \Gamma(E \otimes_{\mathbb{C}} (T^*X_{\mathbb{C}})) \xrightarrow{c} \Gamma(E).$$

It sends sections in $E^{\pm}$ to sections in $E^{\mp}$.

Example B.7. The bundle $\mathsf{C}(X)_{\mathbb{C}} = \mathsf{C}(X) \otimes \mathbb{C}$ with its hermitian extension of g and Levi-Civita connection ∇^{LC} is a Clifford bundle with Dirac operator $d + d^*$.

More examples of Dirac operators come from spin-bundles.

Definition B.8. A *spin structure* on a manifold X is a $\mathrm{Spin}(n)$-principal bundle $\mathrm{Spin}(X)$ on X such that

$$T^*X \simeq \mathrm{Spin}(X) \times_{\mathrm{Spin}(n)} \mathbb{R}^n.$$

In particular, X is an oriented Riemannian manifold of dimension n. If the dimension of X is even, by Example A.4 X has a spin structure if and only if its second Stiefel-Whitney class $w_2(X) \in H^2(X, \mathbb{Z}_2)$ vanishes.

Next, introduce the *spinor bundle*

$$\mathcal{S} = \mathrm{Spin}(X) \times_{\mathrm{Spin}(n)} \mathsf{S}$$

It is a Clifford bundle when equipped with the Levi-Civita connection ∇^{LC} coming from restricting the usual Levi-Civita connection to the subbundle $\mathcal{S}$ of the complexified Clifford algebra $\mathsf{C}(X)_{\mathbb{C}}$. The associated Dirac operator is called *the Dirac operator* $\not{D}_X$ of the spin-manifold X.

Let W be any hermitian vector bundle on X with h-connection ∇. The twisted bundles $E = W \otimes \mathcal{S}$ have a product hermitian structure and a natural product connection which is compatible with this metric. All Clifford bundles are of this form. The associated Dirac-operator is called the *Dirac operator with coefficients in E.*

B.3 The index theorem

Let E be a Clifford bundle and $\not{D}_E : \Gamma(E) \to \Gamma(E)$ be the Dirac operator

$$\not{D}_E := \begin{pmatrix} 0 & \not{D}_E^- \\ \not{D}_E^+ & 0 \end{pmatrix}.$$

Its index is

$$\mathrm{ind}(\not{D}_E) = \dim\ker(\not{D}_E^+) - \dim\mathrm{coker}(\not{D}_E^+) = \dim\ker(\not{D}_E^+) - \dim\ker(\not{D}_E^-).$$

The Atiyah-Singer index theorem for the Dirac operator $\not{D}_E$ is stated using the $\hat{A}$-genus of the underlying differentiable manifold X. To explain it, one needs the *Pontrjagin classes* of X. More generally, for a vector bundle F over X these are related to its Chern classes as follows:

$$p_i(F) := (-1)^i c_{2i}(F \otimes \mathbb{C}) \in H^{4i}(X), i = 1, \dots, m = \operatorname{rank}(F).$$

The Pontrjagin classes of X are those of TX. Usually one associates to these the polynomial $p(F) := 1 + p_1(F)z + \cdots + p_m(F)z^m$.

More generally, start with any formal power series $p(z) = 1 + p_1 z + p_2 z^2 + \cdots \in 1 + \mathbb{Z}[z]$ whose m-th order truncation has a formal factorization

$$1 + p_1 z + \cdots + p_m z^m = (1 + \beta_1 z) \cdots (1 + \beta_m z).$$

Such a power series can be coupled to any other formal power-series

$$q(z) = 1 + q_1 z + q_2 z^2 + \cdots \in \mathbb{Q}[[z]]$$

to produce an associated formal series in z whose coefficients are universal polynomials in the p_i. This goes as follows. For some fixed m write down the m-fold product series

$$\begin{aligned} q(\beta_1 z) \cdot q(\beta_2 z) \cdot \ \dots \ \cdot q(\beta_z) &= 1 + Q_1(p_1)z + Q_2(p_1, p_2)z^2 + \cdots \\ &= 1 + q_1 p_1 z + \left[(-2q_2 + q_1^2)p_2 - q_1 p_1^2\right] z^2 + \cdots , \end{aligned}$$

where by definition the Q_j are the coefficients of z^j. These turn out to be universal polynomials of total degree j in the first j "variables" $p_1, \dots, p_j$ with coefficients expressible in the coefficients of the formal series $q(z)$. To find these, one calculates successively, setting $m = 1$, $m = 2$ etc.

The particular choice

$$q(z) = \frac{\frac{1}{2}z}{\sinh \frac{1}{2}z} = 1 - \frac{1}{2^2} \cdot \frac{1}{6}z + \frac{1}{2^4} \cdot \frac{7}{360}z^2 + \cdots$$

leads to the $\hat{A}$-sequence

$$\hat{A}(z, p_1, p_2, \dots)) = 1 - \frac{1}{2^2} \cdot \frac{1}{6}p_1 z + \frac{1}{2^4}\left(-\frac{1}{90}p_2 + \frac{7}{360}p_1^2\right) z^2 + \cdots$$

Now one writes $p_i(F)$ for p_i, bringing back the Pontrjagin classes of F. Stop at $m = \operatorname{rank}(F)$ and set $z = 1$. This gives the *$\hat{A}$-genus*

$$\hat{A}(F) := \hat{A}(1, p_1(F), \dots, p_m(F)) \in H^{4*}(X; \mathbb{Q}).$$

In particular

$$\hat{A}(X) := \hat{A}(TX).$$

For any complex vector bundle F one also has the Chern character $\mathrm{ch}(F)$. It is defined as follows. Write formally $1 + c_1(F)x + \cdots c_m(F)x^m = (1 + \gamma_1 x) \cdots (1 + \gamma_m x)$ and evaluate

$$\mathrm{ch}(F) = \sum e^{\gamma_i} = m + c_1(F) + \frac{1}{2}(c_1^2(F) - c_2(F)) + \cdots \in H^{2*}(X; \mathbb{Q}). \quad (14)$$

Now we have

Theorem B.9 (Atiyah-Singer index theorem, [AS1]). *Let X be a manifold with a spin structure, E a complex bundle on X. Let $\not{D}_E$ be the Dirac operator with coefficients in E. We have*

$$\mathrm{ind}(\not{D}_E) = \int_X \hat{A}(X)\mathrm{ch}(E).$$

This is a special case of the general index theorem, discussed in Appendix A, Theorem A.3.

Let us see what this gives if E is real. Since $c_i(\bar{E}) = (-1)^i c_i(E)$ we see from (14) that the complex conjugation fixes the terms in $\mathrm{ch}(E)$ of degree divisible by 4 while it acts as minus the identity on the other terms. So the Atiyah-Singer index theorem implies

Corollary B.10. *If* $\dim X \equiv 2 \mod 4$ *one has* $\mathrm{ind}(\not{D}_{\bar{E}}) = -\mathrm{ind}(\not{D}_E)$. *In particular, if E is real, the index of the Dirac operator with values in E. vanishes.*

It is exactly at this point that one passes to real operators and real K-theory to extract finer invariants. See Theorem A.6.

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