PART II

Mirror Symmetry and Calabi-Yau Manifolds

7. Introduction to mirror symmetry

Mirror symmetry is a phenomenon which has its origins in "physics". There is for the moment only a conjectural mathematical definition. The goal of this section is to suggest a rather incomplete definition and draw some mathematical consequences from it. The framework is the class of the so called Calabi-Yau manifolds; we describe these manifolds in detail.

§7.A. Motivation for mirror symmetry.

Mirror symmetry is a phenomenon which which has its origins in "physics" and as of today there is no precise mathematical definition. To raise the reader's curiosity, we give the definition which appear in the physics literature [**G-P**], [**C-O-G-P**], [**G**]. The mirror symmetry phenomenon has its origins in the study of super-conformal (2,2)-theories with central charge $c = 9$. The properties of the conformal fields of these theories are related to the geometry of non-linear sigma models on Calabi-Yau manifolds. A precise definition of these manifolds is delayed to the following section. Let us only say that these manifolds appear to ensure conformal invariance. In this framework, somewhat hostile for a mathematician, the physicists have constructed a remarkable correspondence between the abstract properties of conformal fields and the geometrical properties of the realizations in terms of sigma models. This correspondence in a natural way suggests that one should deform the complex structure and (or) the Kähler class on a Calabi-Yau manifold. These are the A- and B-models of the physicists [**G-P**]. The apparent asymmetry which is nothing but an ambiguity of sign, leads to geometrical models of definitively distinct flavor realizing the same conformal field theory. For X a Calabi-Yau manifold and T_X its holomorphic tangent bundle, the objects linked to the same theory $H^1(X, T_X)$ and $H^1(X, T_X^*) = H^1(X, \Omega_X^1)$ are totally different from the point of view of geometry. In fact, in §7.C we shall see that $\dim H^1(X, T_X)$ is the number of parameters for the complex structure, while $\dim H^1(X, T_X^*) = h^{1,1}(X)$ is the maximal number of Kähler classes on X.

This leads to postulating that Calabi-Yau threefolds (one restricts oneself to dimension three) have to come in pairs, say X and X^* which realize these two models and X^* is said to be *the mirror* of X (and vice-versa). There maybe is a more definitive definition in physics but it is difficult to assimilate it as such mathematically. It can be summarized into an identity of the form

$$Z = Z^*$$

between partition functions (Feynman integrals). The mathematical implications at the more naïve level of the Hodge numbers of X and X^* is the symmetry-relation

$$h^{2,1}(X^*) = h^{1,1}(X); \quad h^{1,1}(X^*) = h^{2,1}(X).$$

Of course this symmetry alone not is not sufficient to make X^* the mirror manifold of X. Between X and X^* exists a more profound relation which relates the space of deformations of the complex structure of X to the space of deformations of the Kähler class of X^* and vice-versa. This relation lies at the origin of the conjectural applications to enumerative geometry on X, as sketched in §10. In **[V]** the reader can find a more detailed explanation.

In the sequel, we shall make precise those aspects which are directly related to Hodge theory:

1. symmetry of Hodge numbers,
2. definition of the Yukawa coupling,
3. Use of the limit Hodge structure to study the asymptotical behavior of the Yukawa coupling.

§7.B. Construction of Calabi-Yau manifolds.

From the point of view of algebraic geometry, a Calabi-Yau manifold is a (complex) projective manifold V, such that the canonical sheaf K_V is trivial ($K_V = \Omega_V^n \cong \mathcal{O}_V$), and $h^{p,0} = 0$ for $p = 1, \dots, n-1$, ($n = \dim V$). The fundamental group $\pi_1(V)$ is often required to be finite, to avoid some marginal situations. In differential geometry these are the Kähler manifolds for which the Ricci curvature is zero ([**Dem**]) and which have holonomy group exactly $\mathrm{SU}(n)$. Directly related to this, there is the following classification result, due to several authors (see [**Beau**]): Let X be a compact Kähler manifold with first Chern class zero; there exists a non ramified finite covering $\widetilde{X} \to X$ such that $\widetilde{X}$ is isomorphic to a product $T \times \big(\prod_i V_i\big) \times \big(\prod_j W_j\big)$, where T is a complex torus, V_i is a simply connected Calabi-Yau manifold, and W_j is a symplectic manifold (there exists a 2-form holomorphic which is non-degenerate in any point). In the context of Calabi-Yau manifolds, the important theorem of Yau [**Y1**] (conjecture of Calabi) plays certainly a key role:

THEOREM. (Yau) *Let X a be Calabi-Yau manifold with Kähler metric g and Kähler form $\omega \in H^{1,1}(X)$. There exists a unique Kähler metric g_Y with Ricci curvature zero (Yau-metric) such that with ω_Y its associated form, $[\omega] = [\omega_Y] \in H^{1,1}(X)$.*

From now on, we only look at the case $n = 3$; observe that $h^{1,0} = 0$ implies $h^{2,0} = 0$, because by Serre duality $H^1(V, \mathcal{O}_V)$ is the dual of $H^2(V, \mathcal{O}_V)$, since $K_V \cong \mathcal{O}_V$.

It is easy to construct examples of Calabi-Yau threefolds. Let $H_1, \dots, H_r$ be hypersurfaces of $\mathbb{P}^N$ ($N = r+3$), of degrees respectively $d_1, \dots, d_r$ with $N+1 = \sum_i d_i$. If the intersection $V = \bigcap_{i=1}^r H_i$ is transversal, V is then smooth, and the adjunction formula shows that $K_V \cong \mathcal{O}_V$, thus V is Calabi-Yau (here $\pi_1(V) = 0$).

For example, you can take for V a hypersurface of degree 5 in $\mathbb{P}^4$ (quintic), an intersection of two cubic hypersurfaces in $\mathbb{P}^5$, of three quadrics in $\mathbb{P}^6$ (see §10 for these examples).

More generally $\mathbb{P}^N$ can be replaced by a product $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$ or by any other manifold, whose anti-canonical sheaf is ample (i.e. K_V^{-1} is ample). A hypersurface is then specified by its equation, i.e. a form of multidegree $d_i = (d_{1i}, \dots, d_{si})$. One forms a table

	H_1		H_r
n_1	d_{11}	$\cdots$	d_{1r}
n_2	d_{21}	$\cdots$	d_{2r}
$\vdots$	$\vdots$		$\vdots$
n_s	d_{s1}	$\cdots$	d_{sr}

and V is the intersection $V = H_1 \cap \cdots \cap H_r$, with $\dim V = 3$ if $\sum_i n_i = r + 3$. The condition $K_V \cong \mathcal{O}_V$ is equivalent to $\sum_{j=1}^{r} d_{ij} = n_i + 1$, $(i = 1, \ldots, s)$.

7.1. EXAMPLE. (Tian-Yau)

	H_1	H_2	H_3
3	3	0	1
3	0	3	1

You can take for example the complete intersection V of the hypersurfaces $\sum_{i=0}^{3} X_i^3 = 0$, $\sum_{i=0}^{3} Y_i^3 = 0$ and $\sum_{i=0}^{3} X_i Y_i$ in $\mathbb{P}^3 \times \mathbb{P}^3$, where (X_i), (Y_i) are homogeneous coordinates in the two copies of $\mathbb{P}^3$. Observe that in this example, circular permutation of the coordinates furnishes a free action of the group $G = \mathbb{Z}/3\mathbb{Z}$, and $W = V/G$ is then a Calabi-Yau threefold with Euler characteristic -6 ("model with generation number 3" thus "physically acceptable").

At this stage, the principal question can be summarized as follows. Given a Calabi-Yau threefold X, which geometrical construction gives the mirror threefold X^*, in fact a "candidate threefold"?

From arguments originating from physics it seems that X^* will often arise as a quotient of X by a finite group G of automorphisms of X, the group G acting trivially on $H^{3,0}(X)$ to ensure that a suitable desingularization of X/G will be Calabi-Yau; this is the orbifold method of the physicists. At this stage various difficulties appear; these are related to the singularities which result from the fixed points, because the action of G is not necessarily free. If $\hat{X}$ is a resolution of singularities of X/G, with $K_{\hat{X}} \cong \mathcal{O}_{\hat{X}}$ (one can prove that a such resolution exists in essentially all the cases [**B-M**]), there is the problem of computing the Hodge numbers $H^{p,q}(\hat{X})$, say from those of X and from data related to the action of G on X. For the Euler characteristic $\chi = \sum(-1)^{p+q}h^{p,q}$ there is the formula of Dixon-Vafa-Witten

$$\chi(\hat{X}) = \frac{1}{|G|} \sum_{gh=hg} \chi(X^g \cap X^h)$$

where the sum is taken over pairs g, h of elements of G which commute $(gh = hg)$ and $X^g = \{x \in X \mid g(x) = x\}$ is the manifold of fixed points of g. Observe that if $\hat{X}$ is the mirror of X, $\chi(\hat{X}) = -\chi(X)$. Remarkably enough, for Hodge numbers an analogous formula has been proposed by Batyrev and Zaslow [**Za**]. This conjectural formula is

$$h^{p,q}(\hat{X}) = \sum_{\{g\}} \dim \left(H^{p-f_g, q-f_g}(X^g)^{C(g)} \right)$$

where $C(g)$ is the commutator of g in G and $\{g\}$ the conjugation class of g. To define the integer f_g, consider the action of the automorphism g_x induced on the tangent space T_xX at $x \in X^g$. Since g induces the identity on $H^{3,0}$, the determinant of g_x is 1 and thus, if $e^{-2\pi\mathbf{i}\lambda_j}$, $0 < \lambda_j < 1$ are the eigenvalues of g_x on T_xX/T_xX^g, the normal space to X^g (these values are independent of the choice of the point $x \in X^g$), the sum $f_g = \sum_j \lambda_j$ is indeed an integer.

It is not hard to verify that the structure of the Hodge diamond of a Calabi-Yau threefold is preserved and thus that the numbers $h^{p,q}(\hat{X})$ are the Hodge numbers of a speculative Calabi-Yau threefold. This has been checked for Batyrev's construction of the mirror threefold by means of polyhedra.

To have more evidence that Calabi-Yau threefolds come in pairs (with maybe exceptions), more ways to construct Calabi-Yau threefolds are needed, because if X is a hypersurface of $\mathbb{P}^4$, there is little chance that $\hat{X}$ is also a hypersurface. Of course $\mathbb{P}^n$ or $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$ can be replaced by a projective space with weights, or by a product of such spaces. Consider a projective space with weights $\mathbb{P}^r(k_1, \ldots, k_{r+1})$, which is the algebraic manifold consisting of $(r+1)$-tuples $(z_1, \ldots, z_{r+1}) \in \mathbb{C}^{r+1} \setminus \{0\}$, modulo the equivalence relation

$$(z_1, \ldots, z_r) \sim (\lambda^{k_1} z_1, \ldots, \lambda^{k_{r+1}} z_{r+1}) \quad (\lambda \in \mathbb{C}^*).$$

The construction of the projective space $\mathbb{P}^r = \mathbb{P}^r(1, \ldots, 1)$ can be seen to generalize to this situation, but it can produce a singular variety. A hypersurface of degree d is the locus of zeroes of a quasi-homogeneous polynomial $P(z) = \sum_{i_1 k_1 + \cdots i_{r+1} k_{r+1} = d} c_{i_1 \cdots i_{r+1}} z_1^{i_1} \cdots z_{r+1}^{i_{r+1}}$. If P and its differential vanish simultaneously only at the origin, one says that P is *transversal*. Such a polynomial defines a smooth hypersurface and if $d = \sum k_i$ we get a Calabi-Yau hypersurface. We shall suppose that $r = 4$, to obtain a hypersurface of dimension 3. Experiment shows that in the list of the weights $\{k_i\}$ such that there exists a transversal quasi-homogeneous polynomial of degree $d = \sum k_i$, the distribution of Hodge numbers $(h^{1,1}, h^{2,1})$ is essentially symmetrical, i.e. in 90% of the cases, the pair $(h^{2,1}, h^{1,1})$ appears. The best way to explain the absence of complete symmetry is to invoke the construction of mirror symmetry by toric methods proposed by Batyrev [**Ba**]. Briefly the naïve duality $X \leftrightarrow X^*$ in the construction above coincides with the combinatorial duality between convex reflexive polyhedra which have the property explained in (loc. cit.). The polyhedron in question is the Newton polyhedron of the polynomial P. There exist combinatorial formulas for the Hodge numbers. The example of the quintic hypersurface can be treated via this process (see §10). The reader can consult [**H-L-T-Y**] for a detailed discussion on toric methods.

§7.C. Deformations.

The Hodge diamond of a Calabi-Yau threefold is

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{1,2} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

One has $h^{2,1} = \dim H^1(\Omega_V^2) = \dim H^1(V, T_V)$ because $T_V = T_V \otimes \Omega_V^3 \cong \Omega_V^2$. In section 3.C. (see 3.8) we have seen that this number yields the number of parameters for the complex structure, if there exists a versal deformation (with a non singular base).

Consider the Hodge structures on $H^3(V, \mathbb{R})$ polarized by the (skew and unimodular) intersection form. Then

$$H^3(X, \mathbb{C}) = H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3}$$

and

$$F^3 = H^{3,0}, \ F^2 = H^{3,0} \oplus H^{2,1}, \ F^1 = H^{3,0} \oplus H^{2,1} \oplus H^{1,2}.$$

Set $b = h^{2,1}$. Then F^2 is a totally isotropic subspace of $H^3(X, \mathbb{C})$, with respect to the skew form Q (cup product), and $F^1 = (F^3)^\perp$. The period domain for the Hodge structures of this type is of the form $D(b) = \mathrm{Sp}(2b+2, \mathbb{R})/U(1) \times U(b)$. This is a domain of dimension $\frac{1}{2}(b+1)(b+2)$.

In §3.C we have briefly looked at deformations and we saw that we can not in general expect that there exists a versal deformation with a non singular base. But for Calabi-Yau manifolds this is effectively the case by a theorem of Tian, Todorov and Bogomolov (see [**T**]):

7.2. THEOREM. *A Calabi-Yau manifold admits a locally universal deformation $X_s, s \in S$ over a smooth base S.*

Hence here $h^{2,1}$ is actually the number of effective parameters needed to describe the variation of the complex structure [**C-O**]. Assume now that the base S is simply connected. Then the period map is a holomorphic map $p : S \to D(b)$. It factors over $q : S \to \mathbb{P}^{2b+1}$, where $\mathbb{P}^{2b+1}$ is the projective space of lines in $H^{3,0}(X_s) \subset H^3(X_s, \mathbb{C})$ because p describes the position of $H^{3,0}(X_s) \oplus H^{2,1}(X_s)$ in the cohomology group $H^3(X_s)$ while q describes the position of $H^{3,0}(X_s)$.

Now we want to explain the theorem of Bryant and Griffiths [**B-G**]. The local system $\{H_3(X_s, \mathbb{Z})\}$ can be locally trivialized by means of a symplectic basis, i.e. a basis of 3-cycles $\{\gamma_i, \delta_j\}_{i,j=0,\dots,b}$ such that with respect to the intersection product

$$(\gamma_i, \delta_j) = \delta_{ij} \text{ and } (\gamma_i, \gamma_j) = (\delta_i, \delta_j) = 0.$$

The Poincaré dual basis $\{\alpha_i, \beta_j\}_{i,j=0,\dots,b}$ furnishes a trivialization of the local system $\mathbb{R}^3 f_*(\mathbb{Z})$. Let ω be a local section of $F^3 = f_*(\omega^3_{X/S})$ which trivializes this bundle. Consider now the periods of ω

$$\zeta_i(s) = \int_{\gamma_i} \omega(s), \ \xi_j(s) = \int_{\delta_j} \omega(s)$$

i.e.

$$\omega = \sum_i \zeta_i \alpha_i + \sum_j \xi_j \beta_j. \tag{6}$$

The 'partial period map' $q : S \to \mathbb{P}^{2b+1}$ can be described as $s \mapsto (\zeta_0(s), \dots, \zeta_b(s), \xi_0(s), \dots, \xi_b(s))$ and you can consider only 'half of it' $q' : S \to \mathbb{P}^b$ given by the γ-periods $s \mapsto (\zeta_0(s), \dots, \zeta_b(s))$

7.3 THEOREM (BRYANT-GRIFFITHS). *The map q' is an immersion so that the γ-periods $(\zeta_0, \dots, \zeta_b)$ serve as homogeneous parameters on S and the δ-periods ξ_b are holomorphic functions in $\zeta_0, \dots, \zeta_b$.*

SKETCH OF THE PROOF.

The proof is based on a reinterpretation of the period map q as a Legendre immersion. To be precise, a contact manifold is a pair $(M, \mathcal{L})$ with M a complex manifold of odd dimension $2m+1$ and $\mathcal{L} \subset \Omega^1$ a line subbundle of the cotangent bundle which is non-degenerate. This means that for any local section $\omega \neq 0$ of $\mathcal{L}$,

$$\omega \wedge (d\omega)^m \neq 0.$$

An associated Legendre manifold is an immersion $f : S \to M$ with $\dim S = m$ such that $f^*\omega = 0$ for any local section ω of $\mathcal{L}$.

If $H = H^3(X, \mathbb{C})$, the intersection form on H defines a contact structure on $\mathbb{P}(H)$ (X is a Calabi-Yau manifold of dimension 3, and $m = h^{2,1}$). In fact, we can suppose that a symplectic basis of H has been chosen. Let $\{p_1, \dots, p_{m+1}, q_1, \dots, q_{m+1}\}$ be the corresponding coordinate system. It suffices to specify a 1-form ω on any standard open subset of $\mathbb{P}(H)$, say on $U_i = \{p_i \neq 0\}$

$$\omega_i = -dq_i + \sum_{j \neq i} (q_j dp_j - p_j dq_j).$$

It can be checked easily that ω_i is a local basis on U_i of a subsheaf of $\Omega^1_{\mathbb{P}(H)}$ of rank one which is locally free and isomorphic to $\mathcal{O}(-2)$ (compare ω_j and ω_k on $U_j \cap U_k$). Since obviously $\omega \wedge (d\omega)^m \neq 0$, we have a contact structure on $\mathbb{P}(H)$. Now one shows that the period map is a Legendre immersion. It is an immersion, because dq, i.e. δ (§3.C) is injective. This is a simple consequence of the triviality of the canonical class. That it is also Legendre is simply a reformulation of the infinitesimal properties of the period map as developed in §3.C (compare also §10.A). To finish the proof, certain structure theorems on contact varieties from [**B-G**] are invoked.

□

The derivative dq being injective, the partials $\partial q / \partial \zeta_i$, $i = 0, \dots, b$ are independent, and thus the $\nabla_{\frac{\partial}{\partial \zeta_i}} \omega(s) \in F^2(X_s)$ give a basis.

We can now explicitly describe the period map $p : S \to D(b)$ as given by the matrix

$$\varpi = \Big(\int_{\gamma_k} \frac{\partial \omega(s)}{\partial \zeta_i}, \int_{\delta_k} \frac{\partial \omega(s)}{\partial \zeta_i} \Big).$$

This is a $(b+1)$ by $(2b+2)$ matrix which describes the position of $F^2(X_s)$ in $H^3(X_s)$. From the relation (6) above, it follows that $\varpi = [\mathbf{1}, \tau]$, with τ_{ij} symmetric. Since the form $-\mathbf{i}\omega \wedge \bar{\omega}$ as well as the forms $\mathbf{i}\alpha \wedge \bar{\alpha}$ for $\alpha \in H^{2,1}$ are positive, $\operatorname{Im} \tau$ has signature $(1, b)$. It can also be verified that a symplectic change of basis transforms τ into

$$\tau' = (A\tau + B)(C\tau + D)^{-1}, \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2b+2, \mathbb{Z})$$

as for Siegel's upper half space (§3.B).

8. Cohomology of hypersurfaces

Consider a smooth projective manifold P of dimension $n+1$ and a smooth hypersurface $X \subset P$. We want to relate the cohomology groups of $P \setminus X$ to the primitive cohomology groups of X, especially when $P = \mathbb{P}^{n+1}$ where rational forms having poles along X are used. This yields Griffiths' description ([**Grif2**] of the primitive cohomology of a hypersurface. This description has been generalized to complete intersections by Dimca [**Dim**] and by others.

§**8.A. Cohomology of the complement.** Recall the weak Lefschetz theorem implying that the cohomology of X differs from that of P only in rank n:

8.1. THEOREM (Lefschetz). *Let X be very ample and let $i : X \to P$ be the injection. Then*

$$i^* : H^m(P, \mathbb{C}) \to H^m(X, \mathbb{C}) \quad \text{is} \begin{cases} \text{an isomorphism if} & m \leq n-1 \\ \text{injective if} & m = n \end{cases}$$

We shall need the following consequence:

8.2. COROLLARY. *Let X be a very ample divisor. Let*

$$i_* : H^n(X, \mathbb{C}) \to H^{n+2}(P, \mathbb{C})$$

be the adjoint of $i^* : H^n(P, \mathbb{C}) \to H^n(X, \mathbb{C})$ with respect to cup product. Then i_* is surjective and the kernel is contained in the primitive cohomology $\mathrm{Prim}^n(X, \mathbb{C})$ *and* $\ker i_* = \mathrm{Prim}^n(X, \mathbb{C})$ *if* $\mathrm{Prim}^n(P, \mathbb{C}) = 0$.

PROOF. The first assertion is evident. The kernel consists of the classes $[\alpha]$ such that $\int_P \partial^n \alpha \wedge \beta = \int_P \alpha \wedge i^*\beta = 0$ for $[\beta] \in H^n(P, \mathbb{C})$. In particular $[\beta]$ can be taken of the form (Kähler class ω) $\wedge\, i^*$ (class of an $n-2$-form on P). But $i^* : H^{n-2}(P, \mathbb{C}) \to H^{n-2}(X, \mathbb{C})$ is an isomorphism (Lefschetz' theorem again) and thus $[\alpha] \wedge \omega = 0$, i.e. $[\alpha]$ is primitive.

□

The map i_* appearing in the Gysin sequence

$$\cdots \to H^{m-2}(X, \mathbb{Z}) \xrightarrow{i_*} H^m(P, \mathbb{Z}) \to H^m(P \setminus X, \mathbb{Z})$$
$$\xrightarrow{\partial} H^{m-1}(X, \mathbb{Z}) \xrightarrow{i_*} H^{m+1}(P, \mathbb{Z}) \cdots,$$

is obtained as follows. Let $T \subset P$ be a tubular neighborhood of X in P. As X is a retract of T, one has $H^k(T, \mathbb{Z}) \xrightarrow{\cong} H^k(X, \mathbb{Z})$, while $H^k(T, T \setminus X, \mathbb{Z}) \xrightarrow{\cong} H^{k-2}(X, \mathbb{Z})$ ('Thom isomorphism'). So, the inclusion $(T, T \setminus X) \to (P, P \setminus X)$ is an excision and thus $H^k(P, P \setminus X, \mathbb{Z}) \xrightarrow{\cong} H^k(T, T \setminus X, \mathbb{Z})$. The long exact sequence of the pair $(P, P \setminus X)$ yields then the Gysin sequence.

It suffices thus to calculate the pertinent part of the cohomology of $P \setminus X$. This computation is done using complexes of rational forms having only poles along X.

Recall the holomorphic Poincaré lemma (see §1)

$$\forall p \geq 1,\, d\alpha = 0,\, \alpha \in \Omega_P^p \quad \Longrightarrow \quad \alpha = d\beta,\, \beta \in \Omega_P^{p-1}.$$

This assertion is equivalent to the exactness of the complex $\Omega_P^\bullet$. This complex gives a resolution of the constant sheaf $\mathbb{C}_X$. The hypercohomology group $\mathbb{H}^m(\Omega_P^\bullet)$ is thus equal to $H^m(P,\mathbb{C})$. Analogously, $\Omega_{P\setminus X}^\bullet$ computes the cohomology of $P \setminus X$.

Passing to forms having poles, one puts

$$\begin{aligned}
\Omega_P^p(k) &:= \Omega_P^p \otimes_{\mathcal{O}_P} \mathcal{O}_P(kX) \quad \text{(sheaf of meromorphic } p\text{-forms} \\
&\qquad \text{with at most a pole of order } k \text{ along } X) \\
\mathcal{Z}_P^p(k) &: = \{\omega \in \Omega_P^p(k) \,|\, d\omega = 0\} \\
\Omega_P^p(*) &: = \text{sheaf of meromorphic } p\text{-forms with at worst poles along } X.
\end{aligned}$$

A but simple nevertheless central observation is

8.3. COMPUTATION. *Let $\alpha \in \mathcal{Z}_P^p(k), k \geq 2$. Then, if f is a local equation for X, one has*

$$\begin{aligned}
\alpha &= \frac{df \wedge \beta}{f^k} + \frac{\gamma}{f^{k-1}}, \quad \beta,\gamma \quad \text{holomorphic without } df \\
&= -\frac{1}{k-1} d\left(\frac{\beta}{f^{k-1}}\right) + \frac{\gamma + \frac{1}{k-1}d\beta}{f^{k-1}}.
\end{aligned}$$

In other words, if the pole order is ≥ 2 one can, at least locally, lower the order modulo exact forms.

Repeating this, we ultimately obtain a decomposition

$$\alpha = \beta \wedge \frac{df}{f} + \gamma,$$

with β and γ holomorphic. The *residue* of α is the form $\mathrm{res}(\alpha) = \beta|X$, defining a map

$$\mathrm{res} : \mathcal{Z}_P^p(1) \to \Omega_X^{p-1}.$$

The idea is to use these computations in De Rham cohomology, using C^∞-forms and partitions of unity to globalize. Start with a rational form on P of type $(n+1,0)$ and with at most a pole along X of order, say $\leq n+1-p$. Consider this form as a C^∞-form on $P \setminus X$ and then lower the pole order using the previous computation. This yields a closed C^∞-form of type $(n+1,0)+(n,1)$ because of $d\beta$ (β and γ don't necessarily stay holomorphic if one globalizes this using partitions of unity). After $n-p$ steps a closed form of type $(n+1,0)+\cdots(p+1,n-p)$ is obtained having a pole of order ≤ 1. Taking its residue, one finds a C^∞-form on X of type $(n,0)+\cdots+(p,n-p)$ which is closed. This form represents a class in $F^pH^n(X,\mathbb{C})$. It can be checked that this construction is well defined on the level of cohomology classes and that the map

$$\Gamma(\Omega_P^{n+1}(n-p+1)/d\Gamma(\Omega_P^n(n-p)) \to F^pH^n(X,\mathbb{C})$$

is injective and surjects onto the primitive part, at least in favorable cases such as $P = \mathbb{P}^{n+1}$.

We shall give another proof of this identification which remains in the framework of algebraic geometry. A version of the Poincaré lemma in the framework of forms with poles is now needed:

8.4. LEMMA.
i. *Assume $p \geq 1$. The complex (starting in degree p)*

$$\mathcal{P}_P^p := \{\Omega_P^p(1) \xrightarrow{d} \Omega_P^{p+1}(2) \xrightarrow{d} \cdots \Omega_P^{n+1}(n-p+2) \to 0\}$$

is exact and so gives a resolution of $\mathcal{Z}_P^p(1)$. Therefore

$$H^q(M, \mathcal{Z}^p(1)) = \mathbb{H}^{p+q}(M, \mathcal{P}_P^p).$$

ii. *The cohomology groups $H^q(\Omega^\bullet(*))$ of*

$$\Omega^\bullet(*) = \{\mathcal{O}_P(*) \to \Omega^1(*) \to \Omega^2(*) \to \dots\}$$

are zero for $q \geq 2$ while $H^0(\Omega^\bullet(*)) = \mathbb{C}_P$ and $H^1(\Omega^\bullet(*)) = \mathbb{C}_X$.

PROOF. The complex $\Omega_P^\bullet(*)$ coincides with $\Omega_P^\bullet$ outside of X and is exact on $P \setminus X$. Take a point $x \in X$ and a system of coordinates $f, x_1, \dots, x_n$ centered at x such that X is given by $f = 0$. Let $\alpha \in \Omega_P^p(k)$ with $k \geq 2$. In the chosen coordinates you write

$$\alpha = \frac{df \wedge \beta + \gamma}{f^k}, \quad \beta, \gamma \quad \text{holomorphic and without } df$$

The central computation shows that $\alpha \in \Omega^p(1)$ modulo $d\Omega^{p-1}(k-1)$. Such an element can be written

$$\alpha = \frac{df \wedge \beta}{f} + \gamma, \quad \beta, \gamma \quad \text{holomorphic and without } df.$$

The condition $d\alpha = 0$ implies that $d\beta = 0$, $d\gamma = 0$. Using the Poincaré lemma, you then write $\beta = d\sigma$, $\gamma = d\tau$ and thus

$$d\alpha = d\left(\frac{\sigma}{f}\right) + d\tau.$$

This shows i) and most of ii). It remains to verify that $H^0(\Omega^\bullet(*)) = \mathbb{C}_P$ and $H^1(\Omega^\bullet(*)) = \mathbb{C}_X$. The first assertion is immediate. The last assertion is shown by a local computation similar to the previous one, which we omit.

□

Now you pass to the subcomplex $\Omega_P^\bullet(\log X)$ of $\Omega^\bullet(*)$ formed by differential forms having logarithmic poles along X (**[Ill]**, §7). We shall prove that it is quasi-isomorphic to the full complex (and thus also computes the cohomology groups of $P \setminus X$). In the case at hand we can take as definition (loc. cit.):

$$\Omega_P^p(\log X) := \{\omega \in \Omega_P^p(1) \,|\, d\omega \in \Omega_P^{p+1}(1)\}.$$

The *residue* map

$$\text{res} : \Omega_P^p(\log X) \to \Omega_X^{p-1}$$

is defined as before. Locally, using coordinates $\{f, z_1, \dots, z_n\}$ such that X is given by $f = 0$, you write $\alpha = d\log f \wedge \beta$ and you put $\text{res}(\alpha) = \beta|_X$. This definition can be checked to be independent of the choice of coordinates and of the local equation

$f = 0$ of X. Thus this map is well defined. It appears in an exact sequence of complexes

$$(7) \qquad 0 \to \Omega_P^\bullet \to \Omega_P^\bullet(\log X) \xrightarrow{\text{res}} \Omega_X^\bullet[-1] \to 0.$$

This exact sequence shows for example that

$$\begin{aligned} H^0(\Omega^\bullet(\log X)) &= \mathbb{C}_Y = H^0(\Omega^\bullet(*X)), \\ H^1(\Omega^\bullet(\log X)) &= \mathbb{C}_X = H^1(\Omega^\bullet(*X)), \\ H^q(\Omega^\bullet(\log X)) &= 0 = H^q(\Omega^\bullet(*X)) \quad \text{for } q \geq 2, \end{aligned}$$

and thus $\Omega_P^\bullet(\log X)$ and $\Omega_P^\bullet(*)$ are quasi-isomorphic so that

$$\mathbb{H}^p(P, \Omega_P^\bullet(*)) = \mathbb{H}^p(\Omega_P^\bullet(\log X)) := \mathbb{H}^p$$

The long exact sequence in hypercohomology yields

$$\begin{aligned} \cdots \longrightarrow H^{m-2}(X, \mathbb{C}) &\xrightarrow{\partial} H^m(P, \mathbb{C}) \to \mathbb{H}^m \xrightarrow{\text{Res}} H^{m-1}(X, \mathbb{C}) \\ &\xrightarrow{\partial^m} H^{m+1}(P, \mathbb{C}) \to \cdots \end{aligned}$$

where Res = res* is induced by the 'residue'-map. We shall show that this sequence 'is' the Gysin sequence.

First we have to relate ∂^m and i_*. A computation in local coordinates that we omit shows that

$$(8) \qquad \partial^m : H^{m-1}(X, \mathbb{C}) \to H^{m+1}(P, \mathbb{C})$$

is the adjoint (with respect to cup product) of

$$i^* : H^{2n-m+1}(P, \mathbb{C}) \to H^{2n-m+1}(X, \mathbb{C}).$$

Next, we note that there is a natural map

$$j : \mathbb{H}^m = \mathbb{H}^m(\Omega_P^\bullet(\log X)) \to \mathbb{H}^m(\Omega_{P \setminus X}^\bullet) = H^m(P \setminus X, \mathbb{C})$$

which commutes with the two restriction maps $H^m(P, \mathbb{C}) \to \mathbb{H}^m(\Omega_P^\bullet(\log X))$ and $H^m(P, \mathbb{C}) \to H^m(P \setminus X, \mathbb{C})$. Thus, in the ladder with exact rows

$$\begin{array}{ccccccc} \cdots \longrightarrow H^{m-2}(X, \mathbb{C}) & \xrightarrow{\partial} & H^m(P, \mathbb{C}) & \to & \mathbb{H}^m & \xrightarrow{\text{Res}} \\ \| & & \| & & \downarrow j & \\ \cdots \longrightarrow H^{m-2}(X, \mathbb{C}) & \xrightarrow{i_*} & H^m(P, \mathbb{C}) & \to & H^m(P \setminus X, \mathbb{C}) & \xrightarrow{\partial} \end{array}$$

$$\begin{array}{ccccc} \xrightarrow{\text{Res}} & H^{m-1}(X, \mathbb{C}) & \xrightarrow{\partial^m} & H^{m+1}(P, \mathbb{C}) \cdots \longrightarrow \\ & \| & & \| \\ \xrightarrow{\partial} & H^{m-1}(X, \mathbb{C}) & \xrightarrow{i_*} & H^{m+1}(P, \mathbb{C}) \cdots \longrightarrow \end{array}$$

the two first squares commute as well as the last. Then j is injective and thus an isomorphism. To express j, consider the spectral sequence $E_1^{p,q} = H^q(\Omega_P^p(*X)) \Longrightarrow \mathbb{H}^{p+q}$. Then $E_2^{m,0}$ = closed m-forms modulo exact forms and, using the natural map $E_2^{m,0} \to \mathbb{H}^m$, one considers a closed m-form as representing a cohomology class on $P \setminus X$. It is easily verified that

$$\partial : H^m(P \setminus X, \mathbb{Q}) \to H^{m-1}(X, \mathbb{Q})$$

is the transpose of the 'tube' map

$$\tau : H_{m-1}(X,\mathbb{Q}) \xrightarrow{\cong} H_{m+1}(T, T\setminus X,\mathbb{Q}) \xrightarrow{\partial} H_m(T\setminus X,\mathbb{Q}) \to H_m(P\setminus X,\mathbb{Q}).$$

(Intuitively, the tube map associates to a cycle the tube above this cycle in the complement of X in P). Next, for $\gamma \in H_m(X,\mathbb{Z})$, $\omega \in H^{m+1}(P\setminus X,\mathbb{C})$ there is the 'residue formula':

$$\int_\gamma \operatorname{res}(\omega) = \frac{1}{2\pi i}\int_{\tau(\gamma)} \omega \tag{9}$$

and thus $\operatorname{res} = \frac{1}{2\pi \mathbf{i}}\partial$: the third square of the diagram commutes (up to multiplication with $\frac{1}{2\pi \mathbf{i}}$).

8.5. Proposition. *Let X be a very ample divisor. Then,* $\operatorname{Res} : H^{n+1}(P\setminus X,\mathbb{C}) \to H^n(X,\mathbb{C})$ *is always injective. If* $\operatorname{Prim}^n(P,\mathbb{C}) = 0$*, then the image is the primitive part of* $H^n(X,\mathbb{C})$.

Proof. By the Lefschetz theorem, $i^* : H^{n+1}(P,\mathbb{C}) \to H^{n+1}(X,P)$ is an isomorphism and therefore the adjoint $\partial^{n-1} : H^{n-1}(X,\mathbb{C}) \to H^{n+1}(P,\mathbb{C})$ is also a isomorphism and thus $\operatorname{Res}^n : H^{n+1}(P\setminus X,\mathbb{C}) \to H^n(X,\mathbb{C})$ is injective. By (8), the image of this map can be identified with the kernel of $i_* : H^n(X,\mathbb{C}) \to H^{n+2}(P,\mathbb{C})$ which (if we assume $\operatorname{Prim}^n(P,\mathbb{C}) = 0$) also consists of primitive classes (by Corollary 2).

□

Fixing a degree in (7), the long sequence in cohomology reads

$$\cdots \longrightarrow H^{q-1}(\Omega_X^{p-1}) \xrightarrow{\partial^{q-1,p-1}} H^q(\Omega_P^p) \longrightarrow H^q(\Omega_P^p(\log X)) \longrightarrow$$
$$\longrightarrow H^q(\Omega_X^{p-1}) \xrightarrow{\partial^{p,q-1}} H^{q+1}(\Omega_P^p) \longrightarrow \cdots .$$

The map i^* preserves the Hodge decomposition and hence the adjoint i_* is a homomorphism of degree $(1,1)$. Thus by Corollary 2 and Proposition 5, $\partial^{q-1,p-1}$ is an isomorphism and $\partial^{p,q-1}$ is surjective whenever $p+q = n+1$. The same argument as used in the proof of Corollary 3 then shows

8.6. Corollary. *In the situation of the previous proposition there is a decomposition*

$$H^{n+1}(P\setminus X,\mathbb{C}) = \bigoplus_{p+q=n+1} H^q(\Omega_P^p(\log X))$$

and the residue map induces an isomorphism

$$H^q(\Omega_P^p(\log X)) \xrightarrow{\sim} \operatorname{Prim}^{p-1,q}(X).$$

§8B. The pole order filtration and the Hodge filtration.

As in the compact case (§1 or [**Dem**], §9) the naïve filtration F can be introduced on the complexes $\Omega^\bullet(*)$ and $\mathcal{P}_P^k$. The induced filtration on hypercohomology will also be denoted by F. The hypercohomology spectral sequence in this case reads

$$H^q(P,\Omega_P^p(*)) \Longrightarrow H^{p+q}(P\setminus X,\mathbb{C})$$

but this sequence does not in general degenerate. The Hodge filtration F in this situation can be found from the subcomplex $\Omega^p_P(\log X)$ of $\Omega^\bullet(*)$. It can be see directly that

$$\ker(d : \Omega^p(\log X) \to \Omega^{p+1}(\log X)) = \ker(d : \Omega^p_P(1) \to \Omega^{p+1}_P(2))$$

and hence

$$\begin{aligned} F^p H^{p+q}(P \setminus X, \mathbb{C}) &= F^p \mathbb{H}^{p+q}(\Omega^\bullet(\log X)) \\ &= i^p_* \mathbb{H}^{p+q}(F^p(\Omega^\bullet(\log X))) = i^p_* H^q(P, \mathcal{Z}^p(1)). \end{aligned}$$

8.7. LEMMA. *If $H^a(P, \Omega^b_P(c)) = 0$ for all $a, b, c > 0$, then $H^q(P, \mathcal{Z}^p_P(1)) = \Gamma(P, \Omega^{n+1}_P(q+2))/d\Gamma(\Omega^n_P(q+1))$ where $p + q = n + 1$.*

PROOF. As in the classical sheaf theoretical proof of the De Rham theorem (see [**God**]), the conditions of the lemma imply that

$$H^q(P, \mathcal{Z}^p_P(1)) = H^q(\Gamma(P, \mathcal{P}^\bullet)),$$

where the complex $\Gamma(P, \mathcal{P}^\bullet))$ is considered as a complex beginning in degree zero.

□

8.8. COROLLARY. *In the situation of the previous lemma*

$$F^{p+1} H^{n+1}(P \setminus X, \mathbb{C}) = H^{n-p}(P, \mathcal{Z}^p_P(1)) = \Gamma(\Omega^{n+1}_P(n-p+1))/d\Gamma(\Omega^n_P(n-p))$$

Combining this result with Corollary 5 yields:

8.9. THEOREM. *Let P be a projective manifold of dimension $n+1$ and let $X \subset P$ be a smooth hypersurface cut out by a very ample divisor. Suppose that $\mathrm{Prim}^n(P, \mathbb{C}) = 0$ and that $H^a(\Omega^b_P(c)) = 0$ for all $a, b, c > 0$. Then the 'Residue' map induces an isomorphism*

$$F^{p+1} H^{n+1}(P \setminus X, \mathbb{C}) = \Gamma(\Omega^{n+1}_P(n-p+1))/d\Gamma(\Omega^n_P(n-p)) \to F^p \,\mathrm{Prim}^n(X, \mathbb{C}).$$

Now, let $X_f \subset \mathbb{P}^{n+1}$ be a smooth hypersurface given by a homogeneous polynomial f of degree d in homogeneous coordinates $Z_0, \ldots, Z_{n+1}$ of $\mathbb{P}^{n+1}$. The only interesting cohomology group of $\mathbb{P}^{n+1} \setminus X_f$ is the group in dimension $n+1$. The conditions of the theorem are verified (Bott's vanishing theorem [**Bott**]) and we get Griffiths' result:

8.10. THEOREM ([**Grif2**]). *The residue map induces an isomorphism from the subspace of the De Rham group $H^{n+1}_{\mathrm{DR}}(\mathbb{P}^n \setminus X_f)$ spanned by the classes of forms having a pole of order $\leq n-p+1$ onto the p-th part F^p of the Hodge filtration on $\mathrm{Prim}^n(X_f)$.*

In particular, each rational $n+1$-form with at most a pole along X_f must be cohomologous to a form having a pole of order at most $n+1$, because $F^0 = H^n(X_f)$. Indeed, Griffiths gives a formula to lower the pole order by adding exact forms. To explain this you must know how to write the $n+1$ rational forms on $\mathbb{P}^{n+1}$ having at most a pole of order k. By a direct computation in affine coordinates such a form can be seen to be expressible as

$$\frac{A}{f^k}\Omega,$$

where

$$\Omega = \sum_j (-1)^j Z_j dZ_0 \wedge \ldots \widehat{dZ_j} \ldots\ldots \wedge dZ_{n+1} \quad \text{and where} \quad \deg A + n + 2 = kd.$$

So, a rational n-form with pole along X_f can be written

$$\varphi = \frac{1}{f^{k-1}} \sum_{i<j} (-1)^{i+j} [Z_i A_j - Z_j A_i] dZ_0 \wedge \ldots \widehat{dZ_i} \wedge \ldots \wedge \widehat{dZ_j} \wedge \ldots \wedge dZ_{n+1}$$

and thus:

8.11. LEMMA. *Let $A_0, \ldots, A_{n+1}$ be polynomials of order $(k-1)d - n - 1$. Then*

$$\frac{(k-1)\sum_{j=0}^{n+1} A_j \frac{\partial f}{\partial Z_j}}{f^k}\Omega \equiv \frac{\sum_{j=0}^{n+1} \frac{\partial A_j}{\partial Z_j}}{f^{k-1}}\Omega + d\varphi \tag{10}$$

8.12. COROLLARY. *Let $J_f \subset \mathbb{C}[Z_0, \ldots, Z_{n+1}]$ be the Jacobi ideal of f i.e. the ideal generated by $\partial f / \partial Z_j$, $j = 0, \ldots, n+1$. The residue map induces an isomorphism*

$$(\mathbb{C}[Z_0, \ldots, Z_{n+1}]/J_f)^{d(n+1-p)-(n+2)} \xrightarrow{\cong} \mathrm{Prim}^{p,n-p}(X_f).$$

PROOF. Theorem 10 implies that there is a surjection

$$(\mathbb{C}[Z_0, \ldots, Z_{n+1}])^{d(n+1-p)-(n+2)} \to F^p/F^{p+1} = \mathrm{Prim}^{p,n-p}(X_f)$$

with kernel consisting of polynomials A coming from forms of type $d\varphi+$ (forms having order of pole $\leq n-p$) and because of the Lemma these are exactly the polynomials of the form $A' + fB$ where $A' \in J_f$. The Euler identity $\sum_j Z_j \frac{\partial f}{\partial Z_j} = \deg(f) f$ shows that $f \in J_f$ and the Corollary follows.

□

9. Picard-Fuchs equations

The goal of this section is to define the Picard-Fuchs equation, and for a family of projective manifolds with one parameter to explain the relation with the Gauss-Manin connection. We determine this equation in some examples. The last example will used in §10 to find the q-expansion related to mirror symmetry. We also explain how to compute the local monodromy for this example.

Assume in the sequel that S is a smooth complex algebraic curve , $S = \overline{S} \setminus T$, where $\overline{S}$ is a smooth compact curve and T a finite number of points. Let $\underline{V}_S$ be a local system on S let ∇ be the flat Gauss-Manin connection on the associated bundle $\mathcal{V} = \underline{V}_S \otimes \mathcal{O}_S$ defined by (see §2)

$$\nabla(v \otimes f) = v \otimes df.$$

On $\mathcal{V}^\vee$, the dual of $\mathcal{V}$ there is a natural connection $\nabla^\vee$ defined by

$$d\langle \nu, v \rangle = \langle \nabla^\vee \nu, v \rangle + \langle \nu, \nabla v \rangle,$$

where v is a local holomorphic section of $\mathcal{V}$ and ν a local section of $\mathcal{V}^\vee$ (See [**Dem**]).

Let $S_o \subset \overline{S}$ be an affine Zariski open set over which there is a trivialization

$$\mathcal{V}^\vee|S_o \xrightarrow{\cong} \mathcal{O}_{S_o}^{\oplus r} \qquad (r = \text{rang } \underline{V}_S).$$

An affine coordinate s induces a vector vector field d/ds on S_o and by composing the connection $\nabla^\vee$ on $\mathcal{V}^\vee|S_o$ and the contraction with d/ds yields the endomorphism

$$D : \mathcal{V}^\vee|S_o \to \mathcal{V}^\vee|S_o.$$

If α is a meromorphic section of $\mathcal{V}^\vee$ without poles in the open set S_o, using the trivialization, the sections $\alpha, D\alpha, D^2\alpha, \dots, D^r\alpha$ viewed as contained in $\mathbb{C}(S)^r \supset \Gamma(S_o, \mathcal{O}^{\oplus r})$ are dependent over the field $\mathbb{C}(S)$. There is a minimal value p such that $\alpha, D\alpha, \dots, D^p\alpha$ are dependent and, replacing D by d/ds, there results a differential equation (normalized by the fact that the coefficient of $(\frac{d}{dt})^p$ is one)

$$(d/dt)^p + A_{p-1}(s)(d/dt)^{p-1} + \cdots + A_0(s) = 0.$$

The solutions form the local system $\mathrm{Sol}(D)$ and for each flat section v of $\underline{V}_S$ the function $\langle \alpha, v\rangle$ is a solution of $D = 0$. In fact, $d\langle\alpha, v\rangle = \langle \nabla^\vee\alpha, v\rangle$ gives $\big((d/dt)^p + A_{p-1}(s)(d/dt)^{p-1} + \cdots + A_0(s)\big)\langle\alpha, v\rangle = \langle(\nabla^\vee)^p\alpha + A_{p-1}(s)(\nabla^\vee)^{p-1}\alpha + \cdots + A_0(s)\alpha, v\rangle = 0$.

There results a surjective homomorphism of local systems

$$\underline{V}_S \to \mathrm{Sol}(D)$$

which is an isomorphism if $p = r$. In this case α is called a *cyclic section.*

9.1. Example. The local system coming from the homology of the fibers of an algebraic family $f : X \to S$. For $\underline{V}_S$ you take the local system whose fiber above $s \in S$ is the homology group $H_n(X_s, \mathbb{C})$ of the fiber $X_s = f^{-1}(s)$ in dimension $n = \dim X_s$.

The pairing given by integration over n-cycles

$$\underline{V}_S \times R^n f_*\mathbb{C} \to \mathbb{C}$$
$$(\gamma, [\omega]) \mapsto \int_\gamma \omega$$

makes $\underline{V}_S$ the dual of $R^n f_*\mathbb{C}$, the local system which has for fiber above s the cohomology group $H^n(X_s, \mathbb{C})$ (see §1).

We know that the bundle $\mathcal{V}^\vee = R^n f_*\mathbb{C} \otimes \mathcal{O}_S$ supports a variation of Hodge structure and the subbundle $\mathcal{F}^n$ is the subbundle of classes of relative n-forms. On each fiber these give the holomorphic n-forms. A meromorphic section $\omega(s)$ of $\mathcal{V}^\vee$, holomorphic on S and belonging to $\mathcal{F}^n$, is the same as a family of holomorphic forms depending meromorphically on s. In this case, the differential equation associated to *the cohomology class* $[\omega(s)]$ is called the *Picard-Fuchs equation.* The preceding discussion implies that its solutions are given by the periods $\int_\gamma \omega(s)$, $\gamma \in H_n(X_s, \mathbb{C})$ provided that one considers γ as a (multi-valued) flat section of the local system $R_n f_*\mathbb{C}$.

9.2. REMARK. The section $[\omega(s)]$ is not necessarily cyclic. However, it will be cyclic for the local subsystem $\underline{V}_S^\vee$ of $R^n f_*\mathbb{C}$ generated by $[\omega(s)]$. The (classical) monodromy of this differential equation coincides with the monodromy of this subsystem. In fact, $V_{S,s}$ is orthogonal (with respect to intersection between n-cycles) to the annihilator of $\underline{V}_{S,s}^\vee$, the smallest subspace of $H^n(X_s,\mathbb{C})$ containing $[\omega]$ and stable under monodromy. In particular $\int_\gamma \omega = 0$ for $\gamma \in V_{S,s}$ implies that $\gamma = 0$. In other words, analytical continuation of the local solutions $\int_\gamma \omega$ yields solutions of the form $\int_{\gamma'} \omega$ (classical monodromy) where γ' is obtained from γ through the monodromy of the system V_S.

Now let s be a coordinate around one of the points $t \in T$. Introduce

$$\Theta := s\frac{d}{ds}$$

so that the Picard-Fuchs equation now reads

$$[\Theta^p + B_{p-1}(s)\Theta^{p-1} + \cdots + B_0(s)]\phi = 0. \qquad (*))$$

9.3. LEMMA-DEFINITION ([**Del**]). *The functions $B_j(s)$ are holomorphic around each of the points $t \in T$. The point t is called a regular singular point.*

This implies that in this case the connection ∇ can be extended to a connection with logarithmic poles on T

$$\overline{\nabla} : \mathcal{V} \to \mathcal{V} \otimes \Omega^1(\log T).$$

See §8 after Lemma 8.4 for the definition of the sheaf $\Omega^1(\log T)$. Note that if the dimension is 1, $\Omega^1(\log T) = \Omega^1(T)$ is, locally around a point in T, generated by ds/s. The operator Θ corresponds to $\overline{\nabla}_{s\frac{d}{ds}}$.

The equation $(*)$ is equivalent to a system

$$\Theta X(s) = A(s)X(s)$$

where (f being a searched for solution of the equation)

$$X(s) = \begin{pmatrix} f \\ \Theta f \\ \vdots \\ \Theta^{p-1} f \end{pmatrix}$$

and

$$A(s) = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots \; 0 & & 1 \\ -B_0(s) & \cdots & -B_{p-1}(s) & -B_p(s) \end{pmatrix}$$

The matrix $A(0)$ is called the *residue* of the connection and is denoted

$$\operatorname{Res}(\nabla) := A(0).$$

9.4. LEMMA ([**C-L**]). *Assuming that for all distinct eigenvalues λ and μ of* $\operatorname{Res}(\nabla)$ *one has $\lambda - \mu \notin \mathbb{Z}$, the monodromy around of t is given by* $\mathbf{e}^{2\pi \mathbf{i} \operatorname{Res}(\nabla)}$.

In particular we find:

9.5. COROLLARY. *If $B_j(0) = 0$, $j = 0, \dots, p$ the local monodromy around t is $\mathbf{e}^{2\pi \mathbf{i} N}$ where N is the nilpotent matrix*

$$N = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & \\ \vdots & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

To apply this corollary in the situation of a family of hypersurfaces $X_{f(s)}$ in $\mathbb{P}^{n+1}$ with equation $f(s)(Z_0, \dots, Z_{n+1}) = 0$, let us complete the discussion of the previous section. Assume that $\dim(S) = 1$. Let s be a local parameter on S and let $\Omega(s) = h(s)\Omega$ be a rational $n+1$-form on $\mathbb{P}^{n+1}$ which depends holomorphically on s. The effect of the flat Gauss-Manin connection is described by

$$\text{(11)} \qquad \mathrm{Res}_{X_{f(s)}} \left[\frac{d^k}{ds^k} \Omega(s) \right] = \left[\nabla^k_{d/ds} \, \mathrm{res}_{X_{f(s)}} \Omega(s) \right],$$

where $[\alpha]$ denotes the cohomology class of a form α. This formula can easily be deduced from the formula §8(9).

9.6. EXAMPLE. Consider the family of elliptic curves (Hesse family)

$$f(u) := Z_0^3 + Z_1^3 + Z_2^3 - 3uZ_0Z_1Z_2$$

above $\mathbb{P}^1 \setminus \{\infty, 1, \rho, \rho^2\}$ where $\rho = \mathbf{e}^{\frac{2\pi \mathbf{i}}{3}}$. For $u = \infty$ the curve degenerates into three lines and we shall study the situation around this point. We shall first determine the differential equation associated to the holomorphic forms $\omega(u)$ of the family $f(u) = 0$. Write

$$(12)_\ell \qquad \Omega_\ell(u) := \frac{(-1)^{\ell-1}(\ell-1)!\, u^\ell \left(\prod Z_j^{\ell-1}\right)}{f(u)^\ell} \Omega, \quad \ell = 1, \dots.$$

Note that $\mathrm{res}(\Omega_1(s)) = \omega(s)$ is a holomorphic form on $X_{f(s)}$ and thanks to formula (11) we have

$$(11)_{\mathrm{bis}} \qquad \left(u\frac{d}{du} \right)^k \Omega_1(u) = \nabla^k_{u\frac{d}{du}} \omega(u) \quad \text{mod exact forms.}$$

Computations give

$$(Z_0Z_1Z_2)^2(1-u^3) = \sum_{k=0}^{2} A_k \frac{\partial f}{\partial Z_k}$$

where

$$A_0 = \frac{1}{3} u Z_0 Z_1^3$$
$$A_1 = \frac{1}{3} u^2 Z_0 Z_1^2 Z_2$$
$$A_2 = \frac{1}{3} Z_0^2 Z_1^2$$

Using formula (10) (see Lemma 8.11) we find that $\Omega_3(u) \equiv P\Omega$ modulo exact forms, where

$$P = \frac{u^3}{1-u^3} \cdot \frac{\frac{1}{3}uZ_1^3 + \frac{2}{3}u^2 Z_0Z_1Z_2}{f^2}.$$

Since $P\Omega$ and $\Omega_2(u)$ have a pole of order two, Corollary 8.12 shows that there exists a function $\varphi(u)$ such that $P\Omega - \varphi(u)\Omega_2(u) = \frac{q}{f^2}\Omega$ with $q \in J_f$. In fact, we find that

$$\frac{u^3}{1-u^3} \cdot (\frac{1}{3}uZ_1^3 + \frac{2}{3}u^2 Z_0 Z_1 Z_2) + \frac{u^3}{1-u^3}(-u^2 Z_0 Z_1 Z_2) = \frac{1}{9}\frac{u^3}{1-u^3}uZ_1\frac{\partial f}{\partial Z_1} \in J_f$$

and another application of (10) yields that

$$\Omega_3(u) + \frac{u^3}{1-u^3}\Omega_2(u) - \frac{1}{9}\frac{u^3}{1-u^3}\Omega_1(u) = 0 \quad \text{mod exact forms.}$$

Note now that $\mathbb{Z}/3\mathbb{Z}$ acts: $\rho \cdot (Z_0, Z_1, Z_2, u) = (Z_0, Z_1, \rho Z_2, \rho^2 u)$ and thus the fibers above u, ρu and $\rho^2 u$ are isomorphic. It is thus natural pass to the parameter

$$s = u^{-3}.$$

Then

$$\Theta = -\frac{1}{3}u\frac{d}{du} = s\frac{d}{ds}$$

and, using $\Theta\Omega_k = (-k/3)\Omega_k + \Omega_{k+1}$, $k = 1, 2$, we get (always modulo exact forms)

$$[\Theta^2 + B_1\Theta + B_0]\Omega_1(u) = 0$$

where

$$B_0 = \frac{2}{9}\frac{s}{s-1}$$
$$B_1 = \frac{s}{s-1}.$$

This equation is equivalent to the system

$$\Theta\begin{pmatrix}\Omega_1\\ \Theta\Omega_1\end{pmatrix} = A(s)\begin{pmatrix}\Omega_1\\ \Theta\Omega_1\end{pmatrix}$$

where

$$A(s) = \begin{pmatrix}0 & 1\\ -B_0 & -B_1\end{pmatrix}.$$

Using formula $(11)_{\text{bis}}$ we find that the 1-forms $\omega(s)$ on the family of elliptic curves satisfies the same system of equations. This system is equivalent to the Picard-Fuchs equation.

Corollary 5 yields: $A(0) = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$ and then the local monodromy operator is $\begin{pmatrix}1 & 2\pi\mathbf{i}\\ 0 & 1\end{pmatrix}$.

9.7. EXAMPLE. In this example a family of Calabi-Yau threefolds is considered (See [**Mor2**] for details)

$$f(s) = Z_0^5 + Z_1^5 + Z_2^5 + Z_3^5 + Z_4^5 - 5uZ_0Z_1Z_2Z_3Z_4, \quad s = u^{-5}.$$

By a computation identical to that of the previous example ($\Theta = s\frac{d}{ds}$) we find

$$[\Theta^4 + B_3\Theta^3 + B_2\Theta^2 + B_1\Theta + B_0]\varphi = 0$$

with coefficients

$$\begin{aligned} B_0 &= \frac{24}{625} \cdot \frac{s}{s-1} \\ B_1 &= \frac{2}{5} \cdot \frac{s}{s-1} \\ B_2 &= \frac{7}{5} \cdot \frac{s}{s-1} \\ B_3 &= 2 \cdot \frac{s}{s-1}. \end{aligned}$$

and the matrix $A(s)$ of Θ with respect to $\{\omega_1, \Theta\omega_1, \Theta^2\omega_1, \Theta^3\omega_1\}$ is equal to

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -B_0 & -B_1 & -B_2 & -B_3 \end{pmatrix}$$

Here the local monodromy is $\mathbf{e}^{2\pi \mathbf{i} N}$ where $N = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. For the sequel we need a holomorphic solution around the point $s = 0$. To obtain it, note that you can rewrite the differential equation as

$$[\Theta^4 - s(\Theta + 5^{-1})(\Theta + 2 \cdot 5^{-1})(\Theta + 3 \cdot 5^{-1})(\Theta + 4 \cdot 5^{-1})]\varphi = 0$$

(multiply by $(1 - s)$) and then, since the relations

$$(n+1)^4 a_{n+1} = (n + 5^{-1})(n + 2 \cdot 5^{-1})(n + 3 \cdot 5^{-1})(n + 4 \cdot 5^{-1}) a_n$$

admit a solution $a_n = \dfrac{(5n)!}{5^{5n}(n!)^5}$, we get a holomorphic solution

$$f_0(s) = \sum_{n \geq 0} \frac{(5n)!}{(n!)^5} \left(\frac{s}{5^5}\right)^n. \tag{13}$$

This is the unique holomorphic solution around $s = 0$ with $f_0(0) = 1$.

The reader could complete these examples by treating the intermediate case of Fermat quartic (K3-surface).

10. Calabi-Yau threefolds and mirror symmetry

We continue the discussion of §7 by considering the universal family of a Calabi-Yau manifold of dimension 3 and its infinitesimal variation which leads to the Yukawa coupling. We show that for a 1-dimensional base the Yukawa coupling satisfies a differential equation of order 1 whose coefficients are linked to those of the Picard-Fuchs equation, which is an equation of order 4. The search for a canonical coordinate q leads to the limit mixed Hodge structure. In the last subsection we come back to example 9.7 and we discuss the prediction resulting from mirror symmetry: the coefficients of the q-expansion of the Yukawa coupling, properly normalized, are directly related to the numbers of rational curves on the generic member of the mirror family (conjecturally the family of quintics hypersurfaces in $\mathbb{P}^4$).

§10.A. The Yukawa coupling.

Let us consider a family $f : X \to S$ of Calabi-Yau threefolds and the VHS defined by the cohomology groups $\{H^3(X_s, \mathbb{C})\}$. Since $h^{3,0} = 1$, locally around $s_0 \in S$ we can suppose that $\mathcal{F}^3 = f_*(\Omega^3_{X/S})$ is trivial. Choose a (relative) holomorphic 3-form ω such that $\omega(s) \neq 0$ for s near s_0. Trivialize the vector bundle $\mathcal{H}^3(X/S)$ by means of flat sections ($\nabla\alpha = 0$). Let $\tau_1, \dots, \tau_{2b+2}$ be such a trivialization (here $b = h^{2,1}(X_s)$). Consider $\{\tau_i\}$ as the dual basis of a (constant) homology basis $\{\gamma_i\}$. Recall that the Hodge-Riemann form (see §3.A) is given by $Q(\alpha, \beta) = -\int_{X_s} \alpha \wedge \beta, \quad (k = n = 3)$ and thus

$$f_i := Q(\tau_i, \omega) = -\int_{\gamma_i} \omega$$

is a holomorphic function in the neighborhood of s_0 we consider. These are the periods of ω. Relative to the chosen basis ω decomposes as

$$\omega = \sum_{i=1}^{2b+2} \alpha_i \tau_i \quad (\alpha_i \text{ holomorphic at } s_0).$$

Since $\nabla \tau_i = 0$,

$$\nabla\omega = \sum_{i=1}^{2b+2} d\alpha_i \otimes \tau_i.$$

If $t_1, \dots, t_r$ are local coordinates around s_0,

$$\nabla_{\partial/\partial t_\alpha} = \sum_i \frac{\partial \alpha_i}{\partial t_\alpha} \tau_i.$$

Note that Griffiths' transversality property with respect to the Hodge filtration $\{\mathcal{F}^p\}_{0 \le p \le 4}$, gives

$$\frac{\partial\omega}{\partial t_\alpha} := \nabla_{\partial/\partial t_\alpha} \omega \in \mathcal{F}^2 \text{ and } \frac{\partial^2\omega}{\partial t_\alpha \partial t_\beta} \in \mathcal{F}^1.$$

Hence

$$Q\Big(\omega, \frac{\partial\omega}{\partial t_\alpha}\Big) = Q\Big(\omega, \frac{\partial^2\omega}{\partial t_\alpha \partial t_\beta}\Big) = 0.$$

However

$$Q\Big(\omega, \frac{\partial^3\omega}{\partial t_\alpha \partial t_\beta \partial t_\gamma}\Big) = \int_{X_t} \omega \wedge \frac{\partial^3\omega}{\partial t_\alpha \partial t_\beta \partial t_\gamma}$$

is in general different from zero. We shall prove that this function represents the linear map δ (see formula (5) in §3.C) associated to the infinitesimal variation. We have seen in §2.D that the differential of the period map is given by

$$\sigma : T_{S,s_0} \longrightarrow \bigoplus \mathrm{Hom}(H^{p,q}, H^{p-1,q+1})$$

where $\sigma(\partial/\partial t)$ acts via cup product with $\rho(\partial/\partial t)$, image of $\partial/\partial t$ by the Kodaira-Spencer map $\rho : T_{S,s_0} \to H^1(T_{X_{s_0}})$. The bundle $\mathcal{F}^3$ is trivialized by the form ω

and in our case the formula (5) reads

$$\begin{aligned}\sigma : \operatorname{Sym}^3 T_{S,s_o} &\longrightarrow \operatorname{Hom}(H^{3,0}(X_{s_0}), H^{0,3}(X_{s_0})) = \\ &= \operatorname{Hom}(\mathbb{C}\cdot\omega(s_0), \mathbb{C}\cdot\overline{\omega}(s_0)) \\ \partial/\partial t_\alpha \otimes \partial/\partial t_\beta \otimes \partial/\partial t_\gamma &\longmapsto \{\omega(s_0) \mapsto \frac{\partial^3\omega}{\partial t_\alpha \partial t_\beta \partial t_\gamma}(s_0)\}.\end{aligned}$$

Therefore, we can write

$$\sigma(\partial/\partial t_\alpha \otimes \partial/\partial t_\beta \otimes \partial/\partial t_\gamma)(\omega(s_0)) = \int_{X_{s_0}} \omega \wedge \frac{\partial^3\omega}{\partial t_\alpha \partial t_\beta \partial t_\gamma}.$$

Thus the infinitesimal variation of Hodge structure furnishes the invariants

$$\kappa_{\alpha\beta\gamma} := \int_{X_{s_0}} \omega \wedge \frac{\partial^3\omega}{\partial t_\alpha \partial t_\beta \partial t_\gamma}$$

in this context called *Yukawa coupling* ([**Mor1**], [**C-O**], [**H**]). The associated invariant tensor is

$$\kappa = \sum_{\alpha,\beta,\gamma} \kappa_{\alpha\beta\gamma} dt_\alpha \otimes t_\beta \otimes t_\gamma \in \operatorname{Sym}^3(\Omega^1_S).$$

If $\dim S = 1$, t is a local coordinate around s_0, we write

$$\kappa_{ttt} = \int_{X_t} \omega \wedge \frac{d^3\omega}{dt^3}$$

which is a holomorphic function of t (in a neighborhood of s_0) and the invariant tensor is

$$\kappa = \kappa_{ttt}(dt)^{\otimes 3} \in (\Omega^1_S)^{\otimes 3}.$$

If in addition $f : X \to S$ is a versal family (see §3.C), we have $\dim H^1(T_{X_{s_0}}) = 1 = \dim H^{2,1}(X_{s_0})$ and thus $H^3(X_{s_0})$ is of dimension 4. The versality implies that the Kodaira-Spencer map is an isomorphism and thus that the three maps $H^{k,3-k} \to H^{k-1,4-k}$, $k = 1, 2, 3$ are isomorphisms (these spaces have dimension 1 and the maps are obtained by taking cup product with the Kodaira-Spencer class $\rho(\partial/\partial t)$). Thus in this case $\kappa_{ttt} \neq 0$ and the sections $\left\{\frac{d^i\omega}{dt^i}\right\}_{i=0,1,2,3}$ form a basis of the bundle $\mathcal{H}^3(X/S)$ in a neighborhood of s_0. Hence a linear dependence relation

$$\frac{d^4\omega}{dt^4} = \sum_{i=0}^{3} A_i(t)\frac{d^i\omega}{dt^i} \tag{14}$$

which is the Picard-Fuchs equation. If α is a flat section of $\mathcal{H}^3(X/S)$, the period $\varpi = Q(\alpha, \omega) = \int_\gamma \omega$ (α is the class Poincaré dual to the cycle γ), satisfies the same equation (14). Now we can differentiate under the sum because Q is constant. Since Q is skew-symmetric, we have $Q(\frac{d^2\omega}{dt^2}, \frac{d^2\omega}{dt^2}) = 0$ and differentiating the relation $Q(\omega, \frac{d^2\omega}{dt^2}) = 0$ twice gives

$$\frac{d}{dt}Q\Big(\omega, \frac{d^3\omega}{dt^3}\Big) = -Q\Big(\frac{d\omega}{dt}, \frac{d^3\omega}{dt^3}\Big) - Q\Big(\frac{d^2\omega}{dt^2}, \frac{d^2\omega}{dt^2}\Big) = -Q\Big(\frac{d\omega}{dt}, \frac{d^3\omega}{dt^3}\Big),$$

but also

$$\frac{d\kappa_{ttt}}{dt} = Q\Big(\frac{d\omega}{dt}, \frac{d^3\omega}{dt^3}\Big) + Q\big(\omega, \frac{d^4\omega}{dt^4}\big) = Q\Big(\frac{d\omega}{dt}, \frac{d^3\omega}{dt^3}\Big) + A_3\big(\omega, \frac{d^3\omega}{dt^3}\big),$$

and so, adding these two equations we get

$$\frac{d\kappa_{ttt}}{dt} = \frac{1}{2}A_3\kappa.$$

A solution, unique up to a multiplicative constant, is given by

$$\kappa_{ttt} = \mathbf{e}^{\frac{1}{2}\int A_3(t)dt}. \tag{15}$$

Let us note that under our assumption, the differential equation $\nabla\alpha = 0$, is a linear system which is equivalent to the 4-th order equation (14). This explains that the local information about the monodromy for κ_{ttt} can be deduced from the explicit computation of the Picard-Fuchs equation.

Finally a few words about Picard-Fuchs equations having regular singular points. Assumes that s is a local coordinate around such a point and we write

$$\kappa = \kappa_{sss}\left(\frac{ds}{s}\right)^{\otimes 3}$$

and, as usual,

$$\Theta = s\frac{d}{ds}.$$

Now, to find κ_{sss} we solve the equation

$$s\frac{d\kappa}{ds} = -\frac{1}{2}B_3\kappa$$

where B_3 is the coefficient of Θ in the Picard-Fuchs equation $\Theta^4 + B_3\Theta^3 + B_2\Theta^2 + \cdots = 0$.

10.1. EXAMPLE. In example 9.7 from the previous section, using the coordinate s we find

$$\kappa_{sss} = C_1\frac{1}{s-1} \quad C_1\text{=integration constant .}$$

We discuss the possible normalizations of the Yukawa coupling in the case of a parameter s. Apply first the classical result (see [**Ince**]):

10.2 THEOREM. *Let there be given a differential equation of order* ≥ 2 *on a disk around of* 0 *having a regular singularity at* 0. *Assume that the local monodromy* T *around of* 0 *has exactly one Jordan block for the eigenvalue 1 of size* ≥ 2. *Then there exists a solution* f_0 *which is regular and univalent around* 0. *Moreover, there exists a local solution* f_1 *around* 0, *independent from* f_0 *such that* $g(s) = 2\pi\mathbf{i}f_1(s) - \log(s)\cdot f_0(s)$ *is univalent. The solution* f_0 *is unique up to a multiplicative constant and the solution* f_1 *is unique up to a multiple of* f_0.

If $f_0 \neq 0$, f_0 can be normalized: $f_0(0) = 1$ and then f_1 by $g(0) = 0$. You can always replace s by another coordinate $w(s)$; from $\kappa_{sss}(ds/s)^{\otimes 3} = \kappa_{www}(dw/w)^{\otimes 3}$, you find that κ gets multiplied by $(w/s)(ds/dw)^3$.

We want to find a 'normalized' coordinate q in the disk. As a first step, consider the multi-valued function

$$\tau(s) = f_1(s)/f_0(s)$$

as a uniform parameter on the Poincaré upper half plane $\mathfrak{h}$. When s turns once around $s = 0$, the parameter τ changes into $\tau + 1$, and this determines τ up to an additive constant; this comes from the fact that we can replace f_1 by $f_1 + \frac{1}{2\pi\mathbf{i}} \cdot \log c_2 \cdot f_0$, since the point $s = 0$ does not have any intrinsic meaning on $\mathfrak{h}$. Thus the parameter

$$q = \exp\left(2\pi\mathbf{i}\frac{f_1(s)}{f_0(s)}\right) = s\exp\left(\frac{g}{f_0}\right)$$

on the punctured d disk is well defined up to a multiplicative constant $c_2 \in \mathbb{C}^*$.

Next, we want to normalize $\kappa_{\tau\tau\tau}$. First, observe that κ depends on the choice of the relative 3-form ω. If ω is transformed into $k(s)\omega$, κ_{sss} is transformed into $k(s)^2\kappa_{sss}$. Note that the solution f_0 is of the form $f_0 = \int_\gamma \omega$ for a 3-cycle γ, which is invariant under local monodromy. Such a cycle γ is unique up to a multiplicative constant. Thus, the 3-form $\tilde{\omega} = f_0(s)^{-1}\omega = \omega / \int_\gamma \omega$ is a holomorphic 3-form $\tilde{\omega}(s)$ such that there exists an invariant cycle γ in $H_3(X_s, \mathbb{C})$ with $\int_\gamma \tilde{\omega} = 1$. So $\tilde{\omega}$ is unique up to a multiplicative constant. Conclusion: with this normalization we have

$$\kappa = c_1 \frac{\exp\left(-\frac{1}{2}\int B_3(s)\frac{ds}{s}\right)}{f_0(s)^2}\left(\frac{1}{2\pi\mathbf{i}}\frac{ds}{s}\right)^{\otimes 3}, \quad c_1 \in \mathbb{C}^*.$$

Next note that $\kappa_{\tau\tau\tau}(d\tau)^3 = \kappa_{sss}\left(\frac{1}{2\pi\mathbf{i}} \cdot \frac{dq}{q}\right)^3$ is periodic in τ and thus there exists a q-expansion, where

$$q := \mathbf{e}^{2\pi\mathbf{i}\tau(s)}.$$

One has

$$\kappa = c_1 \cdot \left(\sum_{j=0}^{\infty} \kappa_j \left(\frac{q}{c_2}\right)^j\right) \cdot \left(\frac{dq}{2\pi\mathbf{i}q}\right)^{\otimes 3} \tag{16}$$

It can be seen easily (cf. [**Mor2**]) that the coefficients κ_j are rational numbers if the coefficients B_j, of the Picard-Fuchs equation, can be written as a series with rational coefficients.

Recall that this computation is done under the crucial assumption that $f_0(0) = \int_\gamma \omega(0) \neq 0$. We shall verify it in the following subsection.

EXAMPLE. Consider example 10.1. Here $f_0(0) = 1$ (see Example 9.7) and observe that the assumption concerning $B_j(s)$ holds. Here you get

$$\kappa_{sss} = \frac{c_1}{(s-1)f_0(s)^2}.$$

10.3 REMARKS.
I. In connection with the preceding computations, recall the theorem of Bryant and Griffiths (Theorem 7.3). From the assumption that the family $f : X \to S$ is the universal deformation of $X_0 = f^{-1}(0)$ for which the Kodaira-Spencer map is an isomorphism for any $s \in S$, we have $\dim(S) = h^{2,1} = b$. By the theorem of Bogomolov and Tian (see §7.C), S is smooth. We assume that S is isomorphic to a disk of dimension b. Trivialize the local system $\{H_3(X_s, \mathbb{Z})\}$ by means of a symplectic basis $\{\gamma_i, \delta_j\}_{i,j=0,\dots,b}$. Let ω be a local section of $F^3 = f_*(\omega^3_{X/S})$ which trivializes this bundle. The theorem of Bryant and Griffiths says that the γ-periods $\zeta_i(s) = \int_{\gamma_i} \omega(s)$ can serve as homogeneous coordinates on S (see §7).

LEMMA. *With* $\xi_j(s) = \int_{\delta_j} \omega(s)$ *we get the relations*

$$\xi_i = \sum_j \zeta_j \frac{\partial \xi_i}{\partial \zeta_j}$$

and $\{\xi_i\}$ *is the gradient of a holomorphic function* G *which is homogeneous of degree* 2 *in the variables* $\zeta_0, \dots, \zeta_b$.

PROOF. As before, there are the relations

$$\int_X \omega \wedge \frac{\partial \omega}{\partial \zeta_i} = \int_{X_s} \omega \wedge \frac{\partial^2 \omega}{\partial \zeta_i \partial \zeta_j} = 0\ .$$

If we replace ω by the expansion $\omega = \sum \zeta_i \alpha_i + \sum \xi_j \beta_j$ (see the §7.C for the notation) and if you keep account of the fact that $\{\alpha_i\}$ and $\{\beta_j\}$ are constants sections, the announced relations follow. These relations imply

$$2\xi_i = \frac{\partial}{\partial \zeta_i}\Big(\sum_k \zeta_k \xi_k\Big)$$

hence if $G(\zeta) = \frac{1}{2}\big(\sum_k \zeta_k \xi_k\big)$, $\xi_i = \frac{\partial G}{\partial \zeta_i}$.

□

II. An elementary computation leads to the following expression for the Yukawa coupling

$$\kappa_{ijk} = \int_{X_s} \omega \wedge \frac{\partial^3 \omega}{\partial \zeta_i \partial \zeta_j \partial \zeta_k} = \frac{\partial^3 G}{\partial \zeta_i \partial z_j \partial z_k}.$$

III. On the local moduli space S a Kähler metric (in fact a Hodge metric ([**Dem**]) can be defined, by its local potential. The Riemann relations show that

$$\mathbf{i} \int \omega \wedge \overline{\omega} = \mathbf{i}\Big(\sum_a \overline{\zeta}_a \frac{\partial G}{\partial \zeta_a} - \zeta_a \frac{\partial \overline{G}}{\partial \overline{\zeta}_a}\Big) > 0.$$

Set $\kappa = -\log\big(\mathbf{i}\int \omega \wedge \overline{\omega}\big)$. Then the metric (called Weil-Peterson metric, [**T**]) on S is defined locally by

$$g_{i\overline{j}} = \frac{\partial^2 \kappa}{\partial \zeta_i \partial \overline{\zeta}_j}.$$

The form of this potential κ shows the special character of this metric (see the next Remark IV). If you identify $T_s S$ and $H^{2,1}(X_s)$ using $\Omega^3_{X_s} \cong \mathcal{O}_{X_s}$, the Weil-Peterson metric is the same as

$$\langle \psi, \phi \rangle_{\mathrm{WP}} = \int_{X_s} \psi \wedge *\bar{\phi}.$$

There is a precise relation with the period map q introduced above. From the Riemann relations R1 and R2 of §3.A, the line $H^{3,0}(X)$ belongs to an open subset of a complex quadric $Q \subset \mathbb{P}^{2b+1}$. On the restriction of the tautological bundle of $\mathbb{P}^{2b+1}$ to Q the Hodge-form induces clearly a hermitian metric. If ω is its Chern form, it can be shown [**T**] that the Kähler form ω_{WP} of the Weil-Peterson metric coincides with the inverse image of ω.

IV. The previous geometric considerations are carried out on the space of parameters for the infinitesimal complex structures $H^{2,1}$. It is not a priori obvious that similar constructions exist for the space of parameters for the Kähler classes, which we define now. If $J \in H^{1,1}(X,\mathbb{R})$ is the Kähler form of a Kähler metric on X, then J is positive and, in particular, for any algebraic curve $C \subset X$,

$$\int_{[C]} J > 0.$$

In the real vector space $H^{1,1}(X,\mathbb{R}) = H^2(X,\mathbb{R})$ these inequalities define an open cone $K(X)$, called the Kähler cone. The *complexified Kähler cone* is

$$CK(X) = \{B + \mathbf{i}J \mid B, J \in K(X)\}.$$

It is also important to consider the closed cone $\overline{K(X)}$.

There exist no variation of Hodge structures supported on the complexified Kähler cone, and thus there is not an evident counterpart to the Yukawa coupling. We only have the topological triple product given by intersection of $(1,1)$-forms $\kappa(\rho,\sigma,\tau) = \int \rho \wedge \sigma \wedge \tau$. It is already remarkable that in this dual situation, the "geometry" of the moduli space of complex structures subsists formally [**C-O**], reinforcing the hypothesis of symmetry between the two types of deformations. The differential geometric properties described above have been formalized under the name of "special geometry" [**Str**]. We refer to this article for a precise definition. The study of this "geometry" on the complexified Kähler cone is at the heart of mirror symmetry. A precise mathematical definition can be considered as being equivalent to the existence of a variation of Hodge structures over the complexified Kähler cone, leading to a triple pairing which "corrects" in a certain sense the pairing κ above, and which under the duality between X and X^*, plays the role of the previous variation for X^*. For a more precise formulation the reader may consult [**Mor4**], [**G**]. We only want to retain from this discussion that Hodge theory is certainly at the basis of a rigorous formulation of the principle of symmetry. See also §11 for a discussion going in this direction.

§10.B. Mathematical Normalization.

We need information on the asymptotic behavior of the periods, and of the Yukawa coupling. This comes from a general study of the asymptotic behavior of a variation of Hodge structure (singularities of the period map).

Let there be given a family of Calabi-Yau threefolds. For simplicity assume that $h^{2,1} = 1$ and that $\dim S = 1$ (we saw that $X \to S$ is universal at any point $s \in S$ (Theorem 7.2). Assume that $S = \overline{S} \setminus \{b_1, \dots, b_r\}$ with $\overline{S}$ is a complete non-singular curve so that above the points $\{b_i\}$ the family possibly has singular fibers (see the example of the quintic family and its mirror family in §7.A). We analyze the behavior of the variation of Hodge structure carried by $\mathcal{H}^3(X/S) = \mathbb{R}^3 f_*(\omega^\bullet_{X/S})$ when the parameter s approaches a singular point. At such a point b_i, we have seen that $\mathcal{H}^3(X/S)$ admits a privileged extension (over a parametrized disk of center b_i), and that the fibers of the Hodge flag $\mathcal{F}^p$ $(p = 0,1,2,3)$ extend as subbundles $\overline{\mathcal{F}}^p$ of the privileged extension $\overline{\mathcal{H}}^3(X/S)$. We make now an assumption [**Mor1**] which is verified in the example of interest.

10.4. ASSUMPTION. (See Theorem 4.1) The local monodromy operator T at b_i is maximally unipotent. I.e. $(T-1)^3 \neq 0$ and thus $N = \log T$ has only one Jordan block $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

With the help of this assumption it can be checked immediately that the filtration $W_\bullet$ is of the form

$$W_0 = W_1 = \ker(N),\ W_2 = W_3 = \ker(N^2),\ W_4 = W_5 = \ker(N^3).$$

The Hodge structure on $\mathrm{Gr}^W_{2\ell}$ ($\ell = 0, 1, 2, 3$) reduces to $\mathrm{Gr}^W_{2\ell} = I^{\ell,\ell}$. In particular $I^{a,b} = 0$ if $a \neq b$ and

$$W_\ell = \bigoplus_{a+b\leq \ell} I^{a,b},\ F^p_\infty = \bigoplus_{a\geq p} I^{a,b}$$

Recall that the bundle $\mathcal{H}^3(X/S)$ is trivialized on Δ^* and extended to Δ. If α is the value of a section of this bundle at $s_0 \in \Delta^*$, and if $\alpha(s)$ is its (multi-valued) continuation by parallel transport by the Gauss-Manin connection, the extended horizontal section is $\alpha^*(s) = \exp\big((\frac{\log s}{2\pi i}N[\alpha(s)]\big)$. In particular, if $\alpha \in W_0$, $T(\alpha) = \alpha$, we have $\alpha^*(s) = \alpha(s)$. Likewise, with $\beta \in \mathcal{H}^3(X/S)_{s_0}$, define $\beta^*(s)$. Since Q is flat, in the trivialized bundle $\overline{\mathcal{H}}^3$ this means that $Q(\alpha^*(s), \beta^*(s)) = Q(\alpha, \beta) = \text{const}$. Because $\omega(s)$, a section which trivializes $\tilde{F}^3$, is a linear combination of sections of the form $\beta^*(s)$ with holomorphic coordinates, $Q(\alpha(s), \omega(s)) = Q(\alpha^*(s), \omega(s))$ is a holomorphic function on Δ. If α is the dual class of the cycle γ_0, this function represents the period

$$f_0(s) := \int_{\gamma_0} \omega(s).$$

Let us show that $f_0(0) \neq 0$. If not, we have $Q(\alpha(0), \omega(0)) = 0$ in the fiber of $\overline{\mathcal{H}}^3$ at $s = 0$. But $\omega(0) \in \overline{\mathcal{F}}^3(0)$, and $\alpha(0) \in W_0$. Now the weight filtration is self dual with respect to Q (since $N \in \mathfrak{g}_{\mathbb{Q}}$), i.e. $W_\ell^\perp = W_{6-\ell-2}$. Thus $\omega(0) \in \overline{\mathcal{F}}^3(0) \cap W_4 = 0$. This shows that we must have $f_0(0) \neq 0$.

Let us discuss now the choice of an intrinsic coordinate on Δ^*. Let $\beta \in W_2 = \ker(N^2)$ be linearly independent from α. There is an integer λ such that $N(\beta) = \lambda\alpha$. Let $\beta^*(s)$ be the canonical (horizontal) extension of β to $\overline{\mathcal{H}}^3$. Then

$$\begin{aligned}\beta^*(s) &= \exp\Big(-\frac{\log s}{2\pi\mathbf{i}}N\Big)\beta(s)\\ &= \beta(s) - \frac{\log s}{2\pi\mathbf{i}}\lambda\alpha^*(s)\ .\end{aligned}$$

Thus

$$\begin{aligned}f_1(s) &= \int_{\gamma_1} \omega(s) \text{ (if } \beta \text{ is the class dual to } \gamma_1)\\ &= \frac{\log s}{2\pi\mathbf{i}}\lambda f_0(s) + Q(\beta^*(s), \omega(s))\end{aligned}$$

and $Q(\beta^*(s), \omega(s))$ is holomorphic on Δ. Thus:

$$\tau = \frac{\lambda^{-1}\int_{\gamma_1}\omega}{\int_{\gamma_0}\omega}$$

is a parameter on $\mathfrak{h}$ and

$$q = \exp(2\pi\mathbf{i}\tau)$$

a parameter on Δ.

Observe that τ, being the quotient of two periods, does not depend on the section ω. If $\{\alpha', \beta'\}$ is another choice, leading to the periods $\{\omega_0', \omega_1'\}$ and to the parameters t', q', we have $a, b, c \in \mathbb{C}$, $ac \neq 0$, with

$$\alpha' = a\alpha, \ \beta' = b\alpha + c\beta$$

thus

$$N\beta' = \lambda'\alpha' \text{ avec } \lambda' = \frac{c\lambda}{a}.$$

Hence

$$\tau' = \tau + \frac{b}{c\lambda} \text{ et } q' = \exp\big(2\pi\mathbf{i}\frac{b}{c\lambda}\big)q\ .$$

Relating this to the discussion in §10.A we observe that the constant c_2 gets identified with $\exp(2\pi\mathbf{i}\frac{b}{c\lambda})$.

These remarks being made, we certainly need the integral structure in order to normalize the periods to obtain a "canonical" coordinate on the disk. Denote by L the integral lattice ($L = (\mathcal{H}_\mathbb{Z})_{s_0}$) in H and recall that $T \in \mathrm{Aut}_\mathbb{Z}(L, Q)$. Then $L \cap W_0$ is of rank one and so you can take for α a generator of this group. Then $T = \exp(N) = 1 + N$ on $W_2 = \ker(N^2)$, and hence $N = T - 1$ is integral on $W_2 \cap L$, i.e. $N(W_2 \cap L) \subseteq L \cap W_0$. So we can choose a basis of the rank 2 group $W_2 \cap L$ of the form $\{\alpha, \beta\}$, and $N(\beta) \in N\alpha$, let $N(\beta) = m\alpha$ with $m \geq 1$. Another basis of this type is $\alpha' = \pm\alpha$, $\beta' = \pm\beta + \ell\alpha$ ($\ell \in \mathbb{Z}$).

Concluding, the parameter q obtained by this normalization is defined up to an mth root of unity, and if $m = 1$ (the monodromy is "small": dixit Morrison), q is then determined uniquely. In this case following Morrison Mor1 we say that q is the *canonical parameter* around the singularity. Summarizing, we have shown:

10.5 Proposition. (Mathematical normalization) *Let $f : X \to \Delta$ be a one-parameter degeneration of Calabi-Yau threefolds with $h^{2,1} = 1$. Let ω be a nowhere zero section of $\mathcal{F}^3$ on Δ^*. Suppose also that the local monodromy of the local system of cohomology in dimension 3 is unipotent of index 4. Put $N = \log T$. Fix $s_o \in \Delta$, fix a generator α of $H^3(X_{s_o}, \mathbb{Z}) \cap \mathrm{Ker}\, N$ and a basis $\{\alpha, \beta\}$ of $H^3(X_{s_o}, \mathbb{Z}) \cap \mathrm{Ker}\, N^2$ so that $N\beta = m\alpha, m \in \mathbb{Z}_{>0}$. Let $\gamma_0, \gamma_1 \in H_3(X_{s_o}, \mathbb{Z})$ be the dual classes. Then the function*

$$q(s) = \exp\big(\frac{2\pi\mathbf{i}}{m}\frac{\int_{\gamma_1}\omega(s)}{\int_{\gamma_0}\omega(s)}\big)$$

is well-defined up to an m-th root of unity.

10.6 Example. **[Hu]** The situation is analogous to the case of genus 1 curves. Consider the family of genus 1 curves $y^2 = x(x-1)(x-\lambda)$, $\lambda \neq 0, 1$ (*Legendre form*) ; $\omega = \frac{dx}{2y}$ defines a section of the Hodge bundle $\mathcal{F}^1$.

The two periods are given (with respect to a basis of $H_1(X_\lambda, \mathbb{Z})$) by

$$\omega_1 = \int_1^\infty \frac{dx}{\sqrt{x(x-1)(x-\lambda)}} \text{ and } \omega_2 = \int_{-\infty}^0 \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}.$$

Express now ω_1 and ω_2 as a function of λ, by means of the hyper-geometric series

$$F(\lambda) = {}_2F_1\big(\frac{1}{2}, \frac{1}{2}, 1; \lambda\big) = \sum_{n=0}^{\infty} \binom{-1/2}{n}^2 \lambda^n$$

which is convergent for $|\lambda| < 1$. Then a classical result says that $\omega_1 = \pi F(\lambda)$, $\omega_2 = \mathbf{i}\pi F(1-\lambda)$ ($|\lambda| < 1$) and that these are the two independent solutions of the Picard-Fuchs equations, which are the hyper-geometric differential equations

$$s(1-s)f''(s) + (1-2s)f'(s) - \frac{1}{4}f(s) = 0 \ .$$

This is indeed the start of the Gauss-Manin connection!

We return to the Yukawa coupling. If s is a local coordinate in the disk Δ with center the singular point b_i (here $s(b_i) = 0$), and if ω is a local section of $\overline{\mathcal{F}}^3$ which trivializes this line bundle on Δ, the (non-normalized) Yukawa coupling has been defined as the function on Δ^* given by

$$\kappa_{sss} = Q\Big(\omega, \frac{d^3\omega}{ds^3}\Big) \ .$$

The function κ_{sss} depends on the coordinate s, as well as on the local section ω of $\overline{\mathcal{F}}^3$ on Δ. Passing from ω to $f\omega$ ($f(0) \neq 0$), transforms κ_{sss} into $f^2\kappa_{sss}$.

For a section ω of $\overline{\mathcal{F}}^3$ which is a local basis at $s = 0$, the normalized period $f_0 = \int_{\gamma_0} \omega$ is then defined up to sign. Normalize the form by replacing ω by ω/f_0, and now $f_0(0) = 1$. Then the Yukawa coupling κ_{ttt} is normalized, and thus is a function defined intrinsically on Δ^* ; we shall call it the *mathematically normalized Yukawa coupling*.

We do not pursue the computations of this mathematical normalization in the examples, because it is easier to normalize the two constants c_1 and c_2 introduced in §10.A. We shall take this route in the following subsection (see Conjecture 10.7).

§10.C. Relation to the number of rational curves in some examples.
The applications to enumerative geometry ("prediction formulas") are based on the precise sense that we should attribute to the corrections ("instanton corrections") to the topological triple product κ (remark IV of §10.A) which are related to "the action" for the sigma models supported on Calabi-Yau threefolds [**F-G**], [**G**]. More precisely, the integral Z^* of §7.A admits an expansion to which the morphisms of $\mathbb{P}^1$ to the Calabi-Yau threefold contribute. See in particular [**P**], §5.6 for an explicit statement.

In the sequel we merely observe the internal coherence of these expansions in few examples, particularly the one from [**C-O-G-P**].

Let T be the open subset of $\mathbb{P}(\mathrm{Sym}^5\mathbb{C}^5))$ parametrizing the nonsingular hypersurfaces of degree 5 in $\mathbb{P}^4$ and let Y_t, $t \in T$ be the corresponding tautological family. This family is a family of Calabi-Yau threefolds with $\dim H^1(T_{Y_t}) = \dim T - \dim \mathrm{PGL}(5) = 101$ and $h^{1,1}(Y_t) = 1$. Mirror symmetry predicts the existence of a family X_s, $s \in S$ with $\dim S = \dim H^1(T_{X_s}) = 1$ and $h^{1,1}(X_s) = 101$. The candidate proposed for X_s is a suitable resolution of singularities of the quotient of the family

$$f(s) = Z_0^5 + Z_1^5 + Z_2^5 + Z_3^5 + Z_4^5 - 5tZ_0Z_1Z_2Z_3Z_4, \quad s = t^{-5}$$

by the group

$$G = \big\{(a_0, a_1, a_2, a_3, a_4) \in \mu_5^5 |\, a_0a_1a_2a_3a_4 = 1\big\}$$

where μ_5 is the group of the 5-th roots of unity. In fact we have studied this family in the preceding sections (the example 9.7) and the classes of the forms $\mathrm{res}(\Omega_j)$, $j = 1, 2, 3, 4$ constitute a basis of the G-invariant part of the cohomology, and thus gives a basis for $H^3(X_s, \mathbb{C})$. The Picard-Fuchs equation we found is the equation for ω_1, residue of Ω_1, considered as 3-form holomorphic on X_s. Mirror symmetry predicts in addition that the Yukawa coupling, properly normalized, admits a q-expansion $\sum a_d q^d$ such that the coefficients a_d determine the numbers n_d of rational curves of degree d on the generic member of the family Y_t. Here q is the canonical parameter of §10.B.

Unfortunately this number is not a priori finite. In fact, there exist Calabi-Yau threefolds with an infinite number of rational curves of fixed degree. For example, consider a covering double of $\mathbb{P}^3$ ramified along a surface S of degree 8. There is a family of dimension ≥ 1 of rational curves having for image a line three times tangent to the surface S (it is one condition for a line to be tangent to a surface). In spite of this, Clemens' conjecture says that on a general quintic there are only a finite number of rational curves of a given degree. But if you do not want to assume this conjecture, you need to find some interpretation for the numbers n_d. A suggestion is to interpret these in the framework of symplectic geometry as the Gromov-Witten invariants for rational curves of degree d. But this is another history for which [**Mor3**], [**D-S**] can be consulted. This being said, there is the

10.7. Conjecture. *If, in the formula (16) of §10.A, you choose $c_1 = -5$ and $c_2 = 5^{-5}$ and write*

$$\kappa_{\tau\tau\tau} = n_0 + \sum_{d=1}^{\infty} \frac{n_d d^3 q^d}{1 - q^d} \tag{17}$$

then $n_0 = 5$ and for $d \geq 1$, n_d is the Gromov-Witten invariant for rational curves of degree d on a generic hypersurface in $\mathbb{P}^4$ of degree 5. This number coincides with the number of rational curves of degree d if Clemens' conjecture is true.[1]

This prediction has been verified for $d \leq 3$. See [**Mor2**] for references. Here is the table of the numbers n_d for $d \leq 10$:

1	2875
2	609250
3	317206375
4	242467530000
5	229305888887625
6	248249742118022000
7	295091050570845659250
8	375632160937476603550000
9	503840510416985243645106250
10	704288164978454686113488249750

10.8. Other examples. See [**L-T**] and [**B-S**], §5 for details. The only complete intersections of $\mathbb{P}^{3+r}$ defined by degrees $d_1, \ldots, d_r$ giving a Calabi-Yau threefold are those with degrees $(3,3), (2,4), (2,2,2,2)$ and $(2,2,3)$. For these examples $h^{1,1} = 1$ and there is a natural construction for the (conjectural) mirror family (see

[1] Translator's note : There are new computations showing that this is not true, see [**Co-K**].

§7.B). First define the Laurent polynomials $f_j(u, X)$ in the variables X_j:

$(3,3)$	$f_1 = 1 - (u_1X_1 + u_2X_2 + u_3X_3)$ $f_2 = 1 - (u_4X_4 + u_5X_5 + u_6(X_1 \cdots X_5)^{-1})$
$(2,4)$	$f_1 = 1 - (u_1X_1 + u_2X_2)$ $f_2 = 1 - (u_3X_3 + u_4X_4 + u_5X_5 + u_6(X_1 \cdots X_5)^{-1})$
$(2,2,2,2)$	$f_1 = 1 - (u_1X_1 + u_2X_2)$ $f_2 = 1 - (u_3X_3 + u_4X_4))$ $f_3 = 1 - (u_5X_5 + u_6X_6$ $f_4 = 1 - (u_7X_7 + u_8(X_1 \cdots X_7)^{-1})$
$(2,2,3)$	$f_1 = 1 - (u_1X_1 + u_2X_2)$ $f_2 = 1 - (u_3X_3 + u_4X_4)$ $f_3 = 1 - (u_5X_5 + u_6X_6 + u_7(X_1 \cdots X_6)^{-1})$.

These equations define a family Y_z of complete intersections in the algebraic torus $(\mathbb{C}^*)^{3+r}$ parametrized by $z = \prod u_j$. There exists a smooth compactification of $\cup Y_z$ having as fibers Calabi-Yau threefolds. For this family one can compute the Picard-Fuchs equation explicitly:

$$\Theta^4 - \mu z(\Theta + \alpha_1)(\Theta + \alpha_2)(\Theta + \alpha_3)(\Theta + \alpha_4) = 0$$

where $\Theta = z\frac{\partial}{\partial z}$ and the coefficients $\mu, (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ are given in the following table.

$(3,3)$	$\mu = 3^6$	$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (1/3, 1/3, 2/3, 2/3)$
$(2,4)$	$\mu = 2^{10}$	$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (1/4, 2/4, 2/4, 3/4)$
$(2,2,2,2)$	$\mu = 2^8$	$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (1/4, 1/4, 1/4, 1/4)$
$(2,2,3)$	$\mu = 2^4 3^3$	$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (1/3, 1/2, 1/2, 2/3)$

The normalized Yukawa coupling for these four examples can then be computed and yields the Gromov-Witten invariants in each case.

degree	intersection type $= (3,3)$	intersection type $= (2,4)$
1	1053	1280
2	52812	92288
3	6424326	15655168
4	1139448384	3883902528
5	249787892583	1190923282176
6	62660964509532	417874605342336
7	17256453900822009	160964588281789696
8	5088842568426162960	66392895625625639488
9	1581250717976557887945	28855060316616488359936
10	512045241907209106828608	13069047760169269024822656

degree	intersection type $= (2,2,2,2)$	intersection type $= (2,2,3)$
1	512	720
2	9728	22428
3	416256	1611504
4	25703936	168199200
5	1957983744	21676931712
6	170535923200	3195557904564
7	16300354777600	517064870788848
8	1668063096387072	89580965599606752
9	179845756064329728	16352303769375910848
10	20206497983891554816	3110686153486233022944

Recent articles (Ellingsrud, Libgober) confirm these numbers, at least in small degree.

11 Relation with mixed Hodge theory

In this section we explain how mixed Hodge theory makes it possible to formulate an interesting aspect of mirror-symmetry.

Recall briefly some basic notions which complete the definitions of §4.

11.1. DEFINITION. Let $H_{\mathbb{R}}$ be a real finite dimensional vector space and set $H = H_{\mathbb{R}} \otimes \mathbb{C}$. A real mixed Hodge structure on H consists of an increasing filtration $W_\bullet$ of H defined on $H_{\mathbb{R}}$ and a decreasing filtration $F^\bullet$ of H such that on Gr_ℓ^W $F^\bullet$ induces a Hodge structure of weight ℓ.

11.2. EXAMPLE. A. Let M be a compact Kähler manifold of dimension d. Take $H = \sum_p H^p(M,\mathbb{C})$, $W_\ell = \bigoplus_{a\geq \ell} H^a(M,\mathbb{R})$.
B. Let M_t be a family of Kähler manifolds on a punctured disk. Assume that the monodromy on $H^d(M_t)$ is unipotent. Then $N := \log T$ satisfies $N^{d+1} = 0$ and there exists a unique filtration $0 \subset W_0 \subset W_1 \ldots \subset W_{2d-1} \subset W_{2d}$ on $H^d(M_t,\mathbb{R})$ such that $NW_\ell \subset W_{\ell-2}$ and N^ℓ induce an isomorphism between $\mathrm{Gr}^W_{d+\ell}$ and $\mathrm{Gr}^W_{d-\ell}$ (see [**S**] for details). We have introduced (§4) the filtration $F^\bullet_\infty$ on $H^d(M_t)$. $W_\bullet$ and $F^\bullet_\infty$ define a mixed Hodge structure. See [**S**].

In example B even more is true:

1. the polarization form Q on $H^d(M_t)$ is such that
$$Q(Nu,v) + Q(u,Nv) = 0.$$
2. $Q(F^p, F^{d-p+1}) = 0$;
3. There is a Lefschetz decomposition $\mathrm{Gr}^W_{d+\ell} = \bigoplus_{j\geq 0} N^j(P_{\ell+2j})$ where
$$P_\ell = \ker N^{\ell+1} : \mathrm{Gr}^W_{d+\ell} \to \mathrm{Gr}^W_{d-\ell-2}$$

such that $Q(-, N^\ell -)$ polarizes the Hodge structure of weight $d+\ell$ on $\mathrm{Gr}^W_{d+\ell}$. In this case we say that N *polarizes the mixed Hodge structure.*

Recall that the classical Lefschetz-decomposition states that multiplication with the Kähler class furnishes an operator L with $L^{d+1} = 0$ such that the kernel of L^d

admits a polarization. But you cannot use L to polarize the mixed Hodge structure of Example A since L is of weight $(1,1)$. Instead you need to take the adjoint Λ (see [**Dem**], §6A). It can be verified [**C-K**] that now indeed Lefschetz theory can be expressed by saying that the mixed Hodge structure of example A is polarized by Λ provided you use the quadratic form $Q(a,b) = (-1)^{\frac{1}{2}p(p-1)} \int_M a \cup b$, $a \in H^p$, $b \in H^{2d-p}$.

There is an inverse of the nilpotent orbit theorem saying that, given a mixed Hodge structure $(F^\bullet, W_\bullet)$ on H polarized by N with $N^{d+1} = 0$, the filtration

$$F^\bullet_{\text{new}} := \exp\left(\frac{-\log s}{2\pi\mathbf{i}} N\right) F^\bullet$$

is a pure Hodge structure of weight d for s small. You even obtain a variation of Hodge structure of polarized by Q. See [**C-K-S**].

Applying this to example A gives:

$$h^{d-q,q}_{\text{new}} = \sum_a h^{a,q}$$

We shall see that this idea leads to a duality on the level of variations of Hodge structure which is related to mirror symmetry.

For Calabi-Yau manifolds M of dimension 3 the Hodge diamond is as follows (see §7)

$$\begin{array}{ccccccc} & & & h^{3,3}=1 & & & \\ & & h^{3,2}=0 & & h^{2,3}=0 & & \\ & h^{3,1}=0 & & h^{2,2}=a & & h^{1,3}=0 & \\ h^{3,0}=1 & & h^{1,2}=b & & h^{2,1}=b & & h^{0,3}=1 \\ & h^{2,0}=0 & & h^{1,1}=a & & h^{0,2}=0 & \\ & & h^{1,0}=0 & & h^{0,1}=0 & & \\ & & & h^{0,0}=1 & & & \end{array}$$

and the new variation of Hodge structure has Hodge numbers

$$h^{3,0}_{\text{new}} = 2 = h^{0,3}_{\text{new}}, \quad h^{2,1}_{\text{new}} = h^{1,2}_{\text{new}} = a + b = h^{1,1} + h^{1,2}.$$

This variation can be regarded as follows. The choice of a Kähler class determines a one parameter-variation Hodge structure of weight 3 and Hodge numbers $(2, a+b, a+b, 2)$. This structure is direct sum of a variation with Hodge numbers $(1,a,a,1)$, the part which comes from the even cohomology of M, and a constant variation with Hodge numbers $(1,b,b,1)$ coming from the odd cohomology. A priori the variation depends on the chosen parameter.

EXAMPLE 11.3. Consider the case $a = 1$. Let $H^+(M) = H^0 \oplus H^2 \oplus H^4 \oplus H^6 = \bigoplus_{k=0}^{3} \mathbb{Z} f_k$ be the even cohomology with f_i the positive generator of $H^{2i}(M)$. With the parameter $q(s) = \dfrac{\log(s)}{2\pi\mathbf{i}}$ on Δ^* the connection of the new variation in term of the basis $\{f_0, f_1, f_2, f_3\}$ can be written as

$$\nabla = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \dfrac{dq}{q} & 0 & 0 & 0 \\ 0 & \deg(M)\dfrac{dq}{q} & 0 & 0 \\ 0 & 0 & \dfrac{dq}{q} & 0 \end{pmatrix}$$

In the mirror symmetry story, the preceding variation must be modified so that the numbers of rational curves in any degree appear ("quantum deformation or instanton corrections"). The physicist have proposed to use the flat connection related to the "A-model", which in terms of the basis $\{f_0, f_1, f_2, f_3\}$ is given by

$$\nabla_A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{dq}{q} & 0 & 0 & 0 \\ 0 & K(q)\frac{dq}{q} & 0 & 0 \\ 0 & 0 & \frac{dq}{q} & 0 \end{pmatrix} \tag{18}$$

where

$$K(q) = \deg(M) + \sum_{d=1}^{\infty} n_d \frac{d^3 q^d}{1-q^d},$$

with n_d ($d \geq 1$) the number of rational curves of degree d on M (or the Gromov-Witten invariant if you wish) and so ∇_A is entirely defined in terms of the geometry on M.

This construction generalizes to several parameter-degenerations. See **[C-K-S]**. Then, using a system of a generators for the Kähler cone of M, yields a variation of Hodge structure which depend on a parameters, sum of a variation with Hodge numbers $(1, a, a, 1)$ and a constant variation $\mathcal{V}_2(M)$ with Hodge numbers $(1, b, b, 1)$.

The first variation should be modified as follows. Let f_0 be the positive generator of $H^0(M)$, f_2 the dual generator of $H^4(M)$, $\{f_1^1, \ldots, f_1^a\}$ an integral basis of $H^2(M)$, $\{f_2^1, \ldots, f_2^a\}$ the dual basis of $H^4(M)$, and let finally $q_1, \cdots, q_a$ be parameters in $(\Delta^*)^a$. Set

$$K_{ijk} := f_1^i \cdot f_1^j \cdot f_1^k + \sum_{\eta} n_{ijk}(\eta) \frac{q^\eta}{1-q^\eta}$$

where $\eta \in H^4(M)$ runs over the classes of rational curves on M, and where $n_{ijk}(\eta)$ is the Gromov-Witten invariant (see **[D-S]**. Let us only say that $n_{ijk}(\eta)$ is the number of pseudo-holomorphic curves $f : \mathbb{P}^1 \to M$ of class η such that $f(0) \in D_j, f(1) \in D_j, f(\infty) \in D_k$ where D_i, D_j, D_k are effective divisors which represent the classes f_1^i, f_1^j, f_1^k) and where one puts $q^\eta = q_1^{c_1} \cdots q_a^{c_a}$, $c_i = \eta \cdot f_1^i$. The connection ∇_A is then given by

$$\nabla_A f_0 = \sum_{i=1}^{a} f_1^i \otimes \frac{dq_i}{q_i};$$

$$\nabla_A f_1^k = \sum_{i,j=1}^{a} K_{ijk} f_2^j \otimes \frac{dq_i}{q_i}, \quad k = 1, \ldots, a;$$

$$\nabla_A f_2 = 0.$$

See **[B-S]**, §3.1 for details. Let us call this variation $\mathcal{V}_1(M)$.

Mirror symmetry predicts that there exists a versal family of mirror Calabi-Yau threefolds with $h^{2,1} = a$ and $h^{1,1} = b$. It seems natural to conjecture that the variation $\mathcal{V}_2(M)$ coincides with the variation given by the third cohomology group of the mirror family, at least if this family is restricted to an open coordinate neighborhood with suitable coordinates.

Visibly, this construction is asymmetric in a and b. To restore the symmetry, you need a versal family $M_t, t \in T$ with $\dim T = b = H^{1,2}(M_t)$, and then you consider the complexified Kähler cone (see Remark 10.3 IV) $CK(M_t)$ on each fiber M_t which yields a manifold $\hat{T}$ of dimension $a+b$ parameters. It is a bundle over T, the fiber above t being $CK(M_t)$. The variations $\mathcal{V}_1(M_t)$ glue together to a variation $\mathcal{V}_1$ over $\hat{T}$. The variations $\mathcal{V}_2(M_t)$ also glue together to a variation $\mathcal{V}_2$ over $\hat{T}$. Mirror symmetry predicts the existence of a universal family $N_s, s \in S$, $\dim S = a$, $h^{1,1}(N_s) = b$ and you get as before two variations $\mathcal{W}_1$, with Hodge numbers $(1,a,a,1)$ and $\mathcal{W}_2$ with Hodge numbers $(1,b,b,1)$ over a manifold $\hat{S}$ fibered over S with fiber above s equal to $CM(N_s)$.

Now mirror symmetry can be formulated in terms of variations of Hodge structure:

CONJECTURE. *Let $\{M_t\}, t \in T$ be a versal family of Calabi-Yau threefolds and $\hat{T}$ be the union of the complexified Kähler cones of all the fibers M_t. Let $\mathcal{V}_1$ be the variation of Hodge structure over $\hat{T}$ coming from the even cohomology of the fibers M_t (the " quantum deformation" of the nilpotent orbit introduced above) and let $\mathcal{V}_2$ be the variation over $\hat{T}$ coming from the odd cohomology. There exists a versal family M_t^*, $t \in \hat{T}^*$ of Calabi-Yau threefolds with $H^{2,1}(M_t^*) = H^{1,1}(M_t)$; $H^{1,1}(M_t^*) = H^{2,1}(M_t)$ and an isomorphism $\hat{T} \xrightarrow{\cong} \hat{T}^*$ exchanging the two types of variations $\mathcal{V}_1$ and $\mathcal{V}_2$.*

In this formulation there is a problem due to the fact that the first variation depends on the choice of parameters while the second does not. Here we not will discuss this problem in general, but rather we will regard the case $b=1$ in some more detail, the case of a versal family with one parameter s. We assume that the base of the variation (a quasi-projective curve) admits a compactification with only one point around which the local monodromy T is maximally unipotent. Let

$$0 \subset W_0 = W_1 \subset W_2 = W_3 \subset W_4 = W_5 \subset W_6$$

be the weight filtration. Let $\{\alpha_0, \alpha_1\}$ be a basis of W_2 such that $N\alpha_0 = 0$ and $N\alpha_1 = \alpha_0$ with $N = \log T$. Complete this to an adapted symplectic basis $\{\alpha_0, \alpha_1, \beta_1, \beta_0\}$, i.e. $Q(\alpha_0, \beta_0) = Q(\alpha_1, \beta_1) = 1$, $Q(\alpha_0, \alpha_1) = Q(\alpha_0, \beta_1) = Q(\alpha_1, \beta_0) = 0$ and $N\beta_1 = k\alpha_1, N\beta_0 = -\beta_1$. Suppose moreover that $k=1$, which is the case for the quintic hypersurface in §10.C (it is implicit in the calculations of [**Mor1**] appendix A, C).

We know that the filtration $F^\bullet_\infty$ induces a pure structure of weight $2j$ on $\mathrm{Gr}_{2j}(W)$, $j = 0,1,2,3$ and thus necessarily β_0 is of type $(3,3)$ and we have $F^3_\infty = \mathbb{C}\beta_0$ because $\dim F^3_\infty = 1$. Also, β_1 is of type $(2,2)$ and thus $F^2_\infty = \mathbb{C}\beta_1 + F^3_\infty$. Similarly you find that $F^1_\infty = \mathbb{C}\alpha_1 + F^2_\infty$. Since we may write $F^\bullet(s) = X(s)F^\bullet_\infty$ where $X(s) = e^{Y(s)}$, $Y(s) = -(\log s/2\pi\mathbf{i})\, N \in \bigoplus_{r<0} \mathfrak{g}^{r,-r}$, with respect to the basis $\{\beta_0, \beta_1, \alpha_1, \alpha_0\}$ we have

$$X(s) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ f(s) & 1 & 0 & 0 \\ * & g(s) & 1 & 0 \\ * & * & f(s) & 1 \end{pmatrix}.$$

Let $\{\omega_0, \omega_1, \nu_1, \nu_0\}$ be the basis of $H^3(X_s, \mathbb{C})$ which you get in this way. It is adapted to the new Hodge filtration

$$\begin{pmatrix} \omega_0 \\ \omega_1 \\ \nu_1 \\ \nu_0 \end{pmatrix} = \begin{pmatrix} 1 & f(s) & * & * \\ 0 & 1 & g(s) & * \\ 0 & 0 & 1 & f(s) \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \alpha_1 \\ \alpha_0 \end{pmatrix}.$$

Apply now the Gauss-Manin connection. Using the above expression, Griffiths' transversality yields

$$\nabla \omega_0 = f'(s)\omega_1 \cdot ds, \quad \nabla \omega_1 = g'(s)\nu_1 \cdot ds, \quad \nabla \nu_1 = f'(s)\nu_0 \cdot ds$$

and thus you find back the Yukawa coupling

$$\kappa_{sss} = f'(s)^2 g'(s).$$

As in §11 take $\tau = Q(\omega_0, \alpha_1) = f(s)$ as the canonical parameter and $q = \exp 2\pi \mathbf{i} \tau$. Thus, with the coordinate q you find

$$\begin{pmatrix} \nabla \omega_0 \\ \nabla \omega_1 \\ \nabla \nu_1 \\ \nabla \nu_0 \end{pmatrix} = \frac{1}{2\pi \mathbf{i}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ \dfrac{dq}{q} & 0 & 0 & 0 \\ 0 & 2\pi \mathbf{i} q \cdot \dfrac{dg}{dq}\dfrac{dq}{q} & 0 & 0 \\ 0 & 0 & \dfrac{dq}{q} & 0 \end{pmatrix} \begin{pmatrix} \omega_0 \\ \omega_1 \\ \nu_1 \\ \nu_0 \end{pmatrix}.$$

Let us summarize:

PROPOSITION 11.4. *Let $f : X \to \Delta$ be a one-parameter degeneration of 3-dimensional Calabi-Yau threefolds with $h^{2,1} = 1$. Suppose that the Hodge bundle $\mathcal{F}^3$ on Δ^* is trivialized by ω_0. Let $\{\omega_0, \omega_1\}$ be a basis of $\mathcal{F}^2$. Suppose moreover that the local monodromy of the local system formed by the cohomology in dimension 3 is unipotent of index 4 and that there is an adapted symplectic basis $\{\alpha_0, \alpha_1, \beta_1, \beta_0\}$. Then, putting*

$$f(s) = Q(\omega_0, \alpha_1),$$
$$g(s) = Q(\omega_1, \beta_1),$$

the canonical parameter is

$$q = \exp 2\pi \mathbf{i}(f(s))$$

and the (normalized) Yukawa coupling is

$$\kappa = 2\pi \mathbf{i} q \cdot \frac{dg}{dq} \left(\frac{dq}{2\pi \mathbf{i} q} \right)^{\otimes 3} \tag{19}$$

Finally, we shall discuss a few related results by Deligne [**Del6**] without giving proofs. The central notion is that of an extension of mixed Hodge structures, introduced by Carlson [**Ca**]. Let us just give an example to illustrate this notion and refer to loc. cit. for the details.

EXAMPLE. Let $\mathbb{Z}(-k)$ be the Hodge structure of dimension 1 which is pure of type (k,k), $k \in \mathbb{Z}$ and given by the lattice $(2\pi\mathbf{i})^k\mathbb{Z} \subset \mathbb{C}$ (Tate structure). An extension of $\mathbb{Z}(-1)$ by $\mathbb{Z}(0)$ is an exact sequence

$$0 \to \mathbb{Z}(0) \xrightarrow{\alpha} H \xrightarrow{\beta} \mathbb{Z}(-1) \to 0$$

of *mixed* Hodge structures. Such an extension is classified by a non-zero complex number q. Concretely, $H_{\mathbb{C}} = \mathbb{C}^2$ admits a basis $\{e_0, e_1\}$ such that $\alpha(1) = e_1$, $\beta(e_0) = 2\pi\mathbf{i}$. And $H_{\mathbb{Z}}$ has a basis $\{f_0 = e_0 + \dfrac{\log q}{2\pi\mathbf{i}} e_1, f_1 = e_1\}$. The choice of the branch of $\log q$ is immaterial, a different branch leading to $\{f_0 + kf_1, f_1\}$, $k \in \mathbb{Z}$, another basis of $H_{\mathbb{Z}}$. The Hodge and weight filtrations are given by $W_2 = \mathbb{Q}e_1$, $W_4 = H_{\mathbb{Q}}$, $F^0 = F^1 = \mathbb{C}e_0$, $F^2 = 0$.

In the sequel we need a version with parameters and so the natural context is that of *variations* of mixed Hodge structure over a basis S. See [**B-Z**], §7 for the definition. For a rough comprehension of what follows the next example however suffices.

EXAMPLE. Let $S = \Delta^*$ with coordinate s. An extension of the constant "variation" $\mathbb{Z}(-1)$ by $\mathbb{Z}(0)$ is completely determined by $q(s)$, a function which is meromorphic on Δ, holomorphic and everywhere non-zero on Δ^* and of order $m \in \mathbb{Z}$. The integral structure is given by the basis $\{f_0 = e_0 + \dfrac{\log q(s)}{2\pi\mathbf{i}} e_1, f_1 = e_1\}$ with corresponding connection $\nabla e_0 = -\dfrac{dq(s)}{2\pi\mathbf{i}q(s)} e_1, \nabla e_1 = 0$. The local monodromy T satisfied $Te_0 = e_0 + me_1, Te_1 = e_1$ and so $Ne_0 = me_1$, $Ne_1 = 0$ ($N = \log T$). Here also the weight and Hodge filtrations are given by $W_2 = \mathbb{Q}e_1$, $W_4 = H_{\mathbb{Q}}$, $F^0 = F^1 = \mathbb{C}e_0$, $F^2 = 0$.

In our situation, the fact that Gr_W^{2k} is of rank one (and thus pure of type (k,k) implies that for each point s near the privileged point, the filtration $F_s^\bullet$ together with the weight filtration give a mixed Hodge structure with $h^{0,0} = h^{1,1} = h^{2,2} = h^{3,3} = 1$. The mixed Hodge structure can be described as in the example by an iterated extension of Tate structures $\mathbb{Z}(-3)$, $\mathbb{Z}(-2)$, $\mathbb{Z}(-1)$ and $\mathbb{Z}(0)$. Let $\{e_0, e_1, e_2, e_3\}$ be a symplectic basis adapted to the weight filtration $0 \subset W_0 = W_2 \subset W_2 = W_3 \subset W_4 = W_5 \subset W_6$ such that $\{e_3\}$ is a basis of F^3, $\{e_3, e_2\}$ of F^2 and $\{e_3, e_2, e_1\}$ of F^1. The extension classes are then given by $q = \exp(2\pi\mathbf{i}f)$ (the canonical parameter), $q_2 = \exp(2\pi\mathbf{i}g)$ (the function related to the Yukawa coupling via (18) above) and $q_3 = q$ by "duality". The underlying lattice is based by $\{e_0, e_0 + f(s)e_1, e_1 + \dfrac{g(s)}{2\pi\mathbf{i}} \cdot e_2, e_2 + \dfrac{f(s)}{(2\pi\mathbf{i})^2} \cdot e_3\}$.

Because

$$\kappa_{\tau\tau\tau} = q\frac{\partial}{\partial q} \log q_2,$$

the expansion of $\kappa_{\tau\tau\tau}$ (see (17)) is equivalent to an infinite product expansion

$$q_2 = q^{n_0} \prod_{d \geq 1} (1 - q^d)^{-n_d d^2},$$

giving an interpretation of (17) purely in terms of mixed Hodge structures. Let M^* the generic member of the mirror family M_t^* and let $H^+(M^*) = H^0 \oplus H^2 \oplus H^4 \oplus H^6 = \bigoplus_{k=0}^3 \mathbb{Z}f_k$ be the even cohomology. The constant "variation" on $H^+(M^*) \times$

Δ^* can be modified using the nilpotent orbit associated to Λ as explained in example 11.2. This gives an iterated extension of Tate structures $\mathbb{Z}(-3), \mathbb{Z}(-2), \mathbb{Z}(-1)$ and $\mathbb{Z}(0)$ with extension classes q, $\deg(N)q$, q which is not an interesting variation: the flat connection can be written in terms of f_k as in example 11.3 and and it needs to be corrected in the same manner as before by equation (18) where now

$$K(q) = \deg(M^*) + \sum_{d=1}^{\infty} n_d \frac{d^3 q^d}{1 - q^d},$$

with n_d ($d \geq 1$) the number of rational curves of degree d on M^* (or the corresponding Gromov-Witten invariant). So this new connection ∇_A is entirely determined in terms of the geometry of the mirror. For the corrected variation the extension classes are q, $K(q)$ and q. So, comparing this with (19), you see that the mirror symmetry conjecture can be reformulated as follows.

FINAL CONJECTURE. *For every $q \in \Delta^*$, the mixed Hodge structure on $H^+(M^*) \times \{q\}$ coincides with Deligne's mixed Hodge structure from* [**Del6**] *on $H^3(M_q)$.*