Degeneration of the Leray spectral sequence for certain geometric quotients

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Abstract

We prove that the Leray spectral sequence in rational cohomology for the quotient map $U_{n,d} \to U_{n,d}/G$ where $U_{n,d}$ is the affine variety of equations for smooth hypersurfaces of degree d in $\mathbb{P}^n(\mathbb{C})$ and G is the general linear group, degenerates at E_2.

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1 Introduction

We consider an affine complex algebraic group G which acts on a smooth algebraic variety X. Assume that a geometric quotient $f : X \to Y$ for the

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action of G on X exists (cf. [11, Sect. 0.1]). We want to give geometric conditions ensuring that the Leray spectral sequence degenerates at $E_2^{p,q} = H^p(Y, R^q f_* \mathbb{Q})$.

The cohomology ring of G is well known ([8], [2]). It is an exterior algebra with exactly one generator η_i in certain odd degrees $2r_i - 1$, $i = 1, \ldots, r = r(G)$, the *rank* of G. So, if G acts with finite stabilizers and the Leray spectral sequence for f degenerates at E_2, knowing the cohomology of the source X is equivalent to knowing that of the target Y. As an example of how this could be used, we point out that for any group G acting with finite stabilizers on a topological space X the equivariant cohomology $H_G^\bullet(X, \mathbb{Q})$ equals $H^\bullet(X/G, \mathbb{Q})$ ([3, §1, Remark 2]) and the former can often be calculated group theoretically. See [3] for examples. So, in these cases one knows $H^\bullet(X, \mathbb{Q})$.

We prove a general result (Theorem 3) giving sufficient geometric conditions for this to happen. These turn out to be satisfied for the group $\mathrm{GL}_{n+1}(\mathbb{C})$ acting on the affine variety $U_{n,d}$ of those homogeneous polynomials of degree d in $(n+1)$ variables which give smooth hypersurfaces in $\mathbb{P}^n$:

Theorem 1 *Let $d \geq 3$. Then the Leray spectral sequence in rational cohomology for the quotient map $U_{n,d} \to M_{n,d} := U_{n,d}/G$, where $G = \mathrm{GL}_{n+1}(\mathbb{C})$, degenerates at E_2.*

Examples

1. By results of Vassiliev [12] the map $H^*(U_{n,d}; \mathbb{Q}) \to H^*(\mathrm{GL}_{n+1}(\mathbb{C}); \mathbb{Q})$ is an isomorphism in the cases $(n, d) = (2, 3), (3, 3)$. Moreover Gorinov [6] has proved the same result for the cases $(n, d) = (4, 3), (2, 5)$. It follows that $M_{n,d}$ has the rational cohomology of a point in these cases.

2. For the case $(n, d) = (2, 4)$ it follows from [12] and Theorem 2 that the space $M_{2,4}$ has a cohomology group of dimension 1 in degrees 0 and 6 and has zero rational cohomology in other degrees. This agrees with a result of Looijenga [10] about the Poincaré-Serre polynomial of $M_{2,4}$:

$$H^6(M_{2,4}; \mathbb{Q}) \simeq \mathbb{Q}(-6)$$

and the other cohomology groups are those of a point.

Remark In [1] there is a description of $M_{3,3}$ using periods of threefolds. This moduli space turns out to be a certain explicitly described open subset of the quotient of complex hyperbolic 4-space by a certain discrete group. From this description it is quite unexpected that $M_{3,3}$ has the rational cohomology of a point. It is an interesting question to calculate the cohomology of the various compactifications of $M_{3,3}$ studied in loc. cit.

2 Generalizing the Leray-Hirsch theorem

The proof of the Leray-Hirsch theorem as given in [9, p. 229] is valid for a locally trivial fibration $p : M \to B$. For cohomology with *rational* coefficients, the same proof applies to a slightly more general situation:

Definition A continuous map $p : M \to B$ is a locally trivial fibration, say with fibre F, in the *orbifold sense* if for every $b \in B$ there exists a neigbourhood V_b, a topological space U_b, and a topological group G_b such that

1. G_b acts on U_b and on F; the action on F is by homeomorphisms homotopic to the identity;
2. V_b is homeomorphic to U_b/G_b;
3. $p^{-1}V_b$ is homeomorphic to the quotient of $U_b \times F$ by the product action of G_b.

In this setting, composing the natural quotient map $F \to F/G_b$ with the homeomorphism $(F/G_b) \xrightarrow{\sim} p^{-1}b$ and the inclusion $p^{-1}b \hookrightarrow X$, defines the *orbifold fibre inclusion* $r_b : F \to X$.

Indeed, in this setting the proof as given in loc. cit. applies starting from the observation that over the rationals we still have graded isomorphisms (replacement of the Künneth formula)

$$\begin{aligned} H^\bullet(p^{-1}V_b;\mathbb{Q}) &\cong H^\bullet(U_b \times F;\mathbb{Q})^{G_b} \cong H^\bullet(U_b;\mathbb{Q})^{G_b} \otimes H^\bullet(F;\mathbb{Q})^{G_b} \\ &\cong H^\bullet(V_b;\mathbb{Q}) \otimes H^\bullet(F;\mathbb{Q}), \end{aligned}$$

because $g \in G_b$ acts trivially on $H^q(F;\mathbb{Q})$ since it is homotopic to the identity by assumption.

We thus arrive at:

Theorem 2 *Let $p : M \to B$ be a fibration which is locally trivial in the orbifold sense. Suppose that for all $q \geq 0$ there exist classes $e_1^{(q)}, \ldots, e_{n(q)}^{(q)} \in H^q(M;\mathbb{Q})$ that restrict to a basis for $H^q(F;\mathbb{Q})$ under the map induced by the orbifold fibre inclusion $r_b : F \to M$. The map $a \otimes r_b^*(e_i) \mapsto p^*a \cup e_i$, $a \in H^\bullet(B;\mathbb{Q})$ extends linearly to a graded linear isomorphism*

$$H^\bullet(B;\mathbb{Q}) \otimes H^\bullet(F;\mathbb{Q}) \xrightarrow{\sim} H^\bullet(M;\mathbb{Q}).$$

Example Let $\phi : X \to Y$ be a geometric quotient for G. Suppose that G is connected and that for all $x \in X$, the identity component of the stabiliser S_x of x is contractible (e.g. when S_x is finite). For $y \in Y$ we take for U_y any open slice for the action of G through $x \in \phi^{-1}y$, i.e. a contractible submanifold through x which intersects Gx transversally at x. Then, if gx is any other point in same orbit, gU_y is a slice through gx and $gS_xg^{-1} = S_{gx}$ so that for all $g \in G$, the quotient gU_y/S_{gx} gives the same neighbourhood V_y of y . We have $(U_y \times G)/S_x = \phi^{-1}(V_y)$. The assumption that G is connected implies that multiplication by $g \in G$ is homotopic to the identity in G. So ϕ is indeed locally trivial in the orbifold sense (with typical fibre G).

We study this example in more detail in the next section.

3 The case of a geometric quotient for a reductive group

We assume that G is a reductive complex affine group, that V is a representation space for G and that X is an affine G-invariant open subset of V such that the action of G on X is closed. Let $\Sigma = V \setminus X$. For $x \in X$ the orbit map is denoted as follows

$$\begin{aligned} o_x : G &\to X \\ g &\mapsto g(x), \end{aligned}$$

and the geometric quotient (which exists in this case, cf. [11, p. 30]) by

$$\phi : X \longrightarrow Y = X/G.$$

Recall that $H^\bullet(G)$ is an exterior algebra freely generated by classes $\eta_i \in H^{2r_i-1}(G)$. Note also that V being a vector space, we have isomorphisms

$$H^{2r_i-1}(X) \xrightarrow{\sim} H^{2r_i}_\Sigma(V).$$

We can now apply the variant of the Leray-Hirsch theorem as stated in the previous section to the geometric quotient ϕ and we obtain:

Theorem 3 *Suppose that there are schemes $Y_i \subset \Sigma$ of pure codimension r_i in V whose fundamental classes map to a non-zero multiple of η_i under the composition*

$$H^{2r_i}_{Y_i}(V) \to H^{2r_i}_{\Sigma}(V) \xrightarrow{\sim} H^{2r_i-1}(X) \xrightarrow{o_x^*} H^{2r_i-1}(G).$$

Denote the image of $[Y_i]$ in $H^\bullet(X;\mathbb{Q})$ by y_i; then the map $a \otimes \eta_i \mapsto \phi^* a \cup y_i$, $a \in H^\bullet(X/G;\mathbb{Q})$ extends to an isomorphism of graded $\mathbb{Q}$-vector spaces

$$H^\bullet(X/G;\mathbb{Q}) \otimes H^\bullet(G;\mathbb{Q}) \xrightarrow{\sim} H^\bullet(X;\mathbb{Q}).$$

4 Properties of fundamental classes

We collect some facts on fundamental classes that we need later on. We refer to [4] for the cohomology-version and [5] for the Chow-version.

1. For any connected submanifold Z of pure codimension c in a complex algebraic manifold X, its fundamental class $[Z] \in H^{2c}_Z(X)(c)$ is the image of $1 \in H^0(Z)$ under the Thom-isomorphism $H^\bullet(Z) \xrightarrow{\sim} H^\bullet_Z(X)[2c](c)$. For Z an irreducible subvariety, one still has a fundamental class as above, since restriction to the smooth part of Z induces isomorphisms between the relevant cohomology groups with support in Z, respectively the smooth part of Z. If $Z = \sum_i n_i Z_i$ is a cycle of codimension c (with Z_i irreducible), with support $|Z|$, there is a cycle class $[Z] \in H^{2c}_{|Z|}(X)(c)$. More generally still, one may assume Z to be a complex subscheme of pure codimension c with irreducible components Z_i of multiplicity n_i in Z and define the fundamental class to be the fundamental class of the associated cycle $\sum_i n_i Z_i$. There are natural maps $H^\bullet_{Z_i} \to H^\bullet_{|Z|}$ and if we identify $[Z_i]$ with their images under these maps we have the equality

$$[Z] = \sum_i n_i [Z_i].$$

2. The fundamental classes behave functorially as follows. Let $f: X \to Y$ be a holomorphic map between complex algebraic manifolds, $Z \subset X$, $W \subset Y$ subschemes such that Z is contained in the scheme-theoretic inverse image $f^{-1}W$. Then f induces $H^\bullet_W(Y) \to H^\bullet_Z(X)$ and if moreover $Z = f^{-1}W$ has

the same codimension c as W, then $f^*[W] = [Z]$. In particular, if W is irreducible and the cycle associated to $Z = f^{-1}W$ is $\sum n_i Z_i$, we find

$$f^*[W] = [f^{-1}W] = \sum n_i[Z_i] \in H^{2c}_{|Z|}(X)(c).$$

3. We can refine the fundamental class of Z, a purely c-codimensional subscheme of X to a class in the Chow group $A_{n-c}(X)$, $n = \dim(X)$. The Chow group $A_{n-c}(Z)$ is generated by the Chow cycle classes $[Z_i]$ of the irreducible components of Z. If the generic point of Z_i has multiplicity n_i then the fundamental class of Z is given by

$$[Z] = \sum n_i[Z_i] \in A_{n-c}(Z).$$

There is a push forward map

$$A_\bullet(Z) \to A_\bullet(X)$$

and a cycle class map

$$A_k(Z) \to H^{\mathrm{BM}}_{2k}(Z)(-k)$$

sending the Chow cycle of Z to $[Z]$. Composing this map with Poincaré-duality for Borel-Moore homology, which reads

$$H^{\mathrm{BM}}_\ell(Z) \xrightarrow{\sim} H^{2n-\ell}_Z(n)$$

and taking $\ell = 2k$, we obtain the cycle class map

$$A_k(Z) \to H^{2n-2k}_Z(X)(n-k).$$

Abusing notation, we denote the Chow cycle also by $[Z]$. This is especially useful if Z is the scheme of zeros of a section s of a vector bundle E over X. In fact, if $s : E \to X$ is the zero-section with image, say $\{0\}$, there is a Gysin isomorphism $s^* : A_\bullet(E) \to A_\bullet(X)[-r]$ with the property

$$A_n(E) \ni [\{0\}] \xmapsto{s^*} c_r(X) \in A_{n-r}(X).$$

See [5, Example 3.3.2]. This Gysin map is in fact the inverse of the isomorphism

$$\pi^* : A_{n-r}(X) \xrightarrow{\sim} A_n(E).$$

5 The cohomology ring of the general linear group

We turn to $G = G_n = \mathrm{GL}_n(\mathbb{C})$, $n \geq 1$. In this case, by [2], $H^\bullet(G)$ is the exterior algebra with generators $\eta_\ell^{(n)}$in all odd degrees $2\ell - 1$, $\ell = 1, \ldots, n$. In other words $r_1 = 1, r_2 = 2, \ldots, r_n = n$. Since G_n is contained in the vector space $M_n = \mathrm{Mat}_n(\mathbb{C})$, we have an identification of mixed Hodge structures

$$H^\bullet(G) \xrightarrow{\sim} H^\bullet_{D_n}(M_n)[1],$$

where

$$D_n = \{A \in M_n \mid \det(A) = 0\} = M_n \setminus G_n,$$

and so $\eta_\ell^{(n)}$ corresponds to some class in $H^{2\ell}_{D_n}(M_n)$. The goal is to find explicit descriptions of this class as fundamental class of the subvariety $D_{n,\ell} \subset D_n$ to be defined below. This will turn out to be essential for the next section. We are going to show this by first defining classes $\eta_\ell^{(n)}$ that clearly have this property. Then we prove that these classes do generate $H^\bullet(G)$ as an exterior algebra.

We introduce the following notation:

- $D_{n,\ell} \subset D_n$: the subvariety consisting of those matrices for which the first $n + 1 - \ell$ columns are linearly dependent. Note that $D_{n,\ell}$ has codimension ℓ in M_n.
- $\tilde{D}_n = \{(A, p) \in D_n \times \mathbb{P}^{n-1}(\mathbb{C}) \mid [p] \subset \mathrm{Ker}(A)\}$ (where $[p]$ stands for the line in $\mathbb{C}^n$ corresponding to p) and $\pi_n : \tilde{D}_n \to D_n$ is the projection to the first factor.
- $Q_n = \{(x, y) \in \mathbb{C}^n \times \mathbb{C}^n \mid x \bullet y = 1\}$.
- $\alpha_n : M_{n-1} \to M_n$ is the inclusion which maps a matrix A to $\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$.
- h: the hyperplane class in $H^2(\mathbb{P}^n(\mathbb{C}))$.

Note that the projection to the second factor turns $\tilde{D}_n$ into a vector bundle of rank $n^2 - n$ over $\mathbb{P}^{n-1}(\mathbb{C})$, so $\tilde{D}_n$ is smooth and π_n is a resolution of singularities of D_n.

Lemma 4 *Let X be a smooth variety, $D \subset X$ a subvariety of codimension k and $\pi : \tilde{D} \to D$ a resolution of singularities. Then there are natural Gysin maps $\beta_\ell : H^{\ell-2k}(\tilde{D})(-k) \to H^\ell_D(X)$ which are morphisms of mixed Hodge structures.*

Proof Let $n = \dim(X)$. As $\tilde{D}$ is smooth, cup product with the fundamental class $[\tilde{D}]$ induces an isomorphism

$$H^{\ell-2k}(\tilde{D})(-k) \to H^{\mathrm{BM}}_{2n-\ell}(\tilde{D})(-n).$$

As Borel-Moore homology is covariant for proper morphisms we have natural maps

$$H^{\mathrm{BM}}_{2n-\ell}(\tilde{D})(-n) \to H^{\mathrm{BM}}_{2n-\ell}(D)(-n).$$

Because X is smooth, Poincaré-duality for Borel Moore homology gives an isomorphism of mixed Hodge structures

$$H^{\mathrm{BM}}_{2n-\ell}(D)(-n) \simeq H^\ell_D(X).$$

by [5, Sect. 19.1]. The map β_ℓ is obtained as the composition of these maps. □

Let us apply this to the situation of $\tilde{D}_n \to D_n \hookrightarrow M_n$. We obtain maps

$$\beta^{(n)}_\ell : H^{2\ell-2}(\mathbb{P}^{n-1}(\mathbb{C}))(-1) \to H^{2\ell}_{D_n}(M_n)) \simeq H^{2\ell-1}(G_n)$$

and define for $\ell = 1, \ldots, n$:

$$\eta^{(n)}_\ell := \beta^{(n)}_\ell \left(\frac{h^{\ell-1}}{2\pi i} \right) \in H^{2\ell-1}(G_n).$$

We observe that the class in $H^{2\ell}_{D_n}(M_n)$ corresponding to $\eta^{(n)}_\ell$ is indeed the fundamental class of $D_{n,\ell} \subset D_n$.

Lemma 5 *The map $\alpha : M_{n-1} \to M_n$ maps D_{n-1} and G_{n-1} to D_n and G_n respectively and $\alpha^*(\eta^{(n)}_\ell) = \eta^{(n-1)}_\ell$ for $\ell = 1, \ldots, n-1$ while $\alpha^*(\eta^{(n)}_n) = 0$.*

Proof Observe that $\alpha^{-1}(D_{n,\ell}) = D_{n-1,\ell}$. One checks that this holds not only set theoretically, but even as schemes. Then the lemma follows from property 2) from section 4.

Because the classes $\eta_\ell^{(n)}$ are of odd degree, they have square zero and anti-commute, so we have a homomorphism of graded algebras

$$R_n : \Lambda(z_1, \ldots, z_n) \to H^*(G_n).$$

Here $\Lambda(z_1, \ldots, z_n)$ is the exterior algebra on n generators $z_1, \ldots, z_n$ with z_i of degree $2i - 1$, and $R_n(z_\ell) = \eta_\ell^{(n)}$.

Theorem 6 *The map R_n is an isomorphism. Moreover, the generators $\eta_\ell^n \in H^{2\ell-1}(G_n)$ have pure type (ℓ, ℓ) and map to the fundamental classes $D_{n,\ell}$ under the identification $H^{2\ell-1}(G_n) \simeq H^{2\ell}_{D_n}(M_n)$.*

Proof By induction on n. For $n = 1$ everything is clear. Suppose the map R_{n-1} is an isomorphism. We consider the map

$$\rho : G_n \to Q_n,\ \rho(g) = (g(e_1), {}^t g^{-1}(e_1)).$$

This is the orbit map of a transitive action of G_n on Q_n and $\alpha(G_{n-1})$ is the isotropy subgroup of $(e_1, e_1) \in Q_n$. Therefore, ρ is also the quotient map for the action of G_{n-1} on G_n by left translation via α. As the classes $\eta_\ell^{(n-1)}$ generate the cohomology ring of G_{n-1} and are images of classes on G_n, the restriction maps $\alpha^* : H^i(G_n) \to H^i(G_{n-1})$ are surjective. Hence by Theorem 2 we have an isomorphism

$$H^*(Q_n) \otimes H^*(G_{n-1}) \simeq H^*(G_n).$$

The variety Q_n is homotopy equivalent to a sphere of dimension $2n - 1$ (in fa ct to its subvariety consisting of pairs (x, y) with $y = \overline{x}$). Moreover, a generator of $H^{2n-1}(Q_n)$ is mapped to a non-zero multiple of $\eta_n^{(n)}$ by the map ρ^*. This implies the surjectivity and hence bijectivity of R_n. $\square$

Remark For any Lie group G, the map $g \mapsto g^{-1}$ induces multiplication by -1 on the Lie algebra, hence on $H^k(G)$ it induces multiplication by $(-1)^k$. The involution $\sigma : G_n \to G_n$ given by $\sigma(g) = {}^t g^{-1}$ has $\sigma^*(\eta_n^{(n)}) = (-1)^n \eta_n^{(n)}$. Indeed, if we let $\sigma : Q_n \to Q_n$ be given by $\sigma(x, y) = (y, x)$ then ρ becomes equivariant, and it is an easy exercise to see that $\sigma^* = (-1)^n$ on $H^{2n-1}(Q_n)$. We conclude that transposition τ on G_n induces $\tau^*(\eta_n^{(n)}) = (-1)^{n-1}\eta_n^{(n)}$. As the inclusion $G_{n-1} \to G_n$ commutes with transposition, we conclude that $\tau^*(\eta_\ell^{(n)}) = (-1)^{\ell-1}\eta_\ell^{(n)}$ for all $\ell \leq n$.

6 Moduli of smooth hypersurfaces

We let $\Pi_{n,d} = \mathbb{C}[x_0, \dots, x_n]_d$ denote the vector space of homogeneous polynomials of degree d in $n+1$ variables over $\mathbb{C}$. We let

$$\Sigma_{n,d} = \{f \in \Pi_{n,d} \mid f \text{ has a critical point outside } 0\}.$$

There exists an irreducible polynomial Δ in the coefficients of $f \in \Pi_{n,d}$ such that $f \in \Sigma_{n,d}$ if and only if $\Delta(f) = 0$. Moreover, Δ is homogeneous of degree $(n+1)(d-1)^n$.

We let $U_{n,d} = \Pi_{n,d} \setminus \Sigma_{n,d}$. The group $\mathrm{GL}_{n+1}(\mathbb{C})$ acts on $U_{n,d}$. For $d \leq 2$ or $d = 3, n = 1$ it acts transitively, but in the remaining cases it acts with finite isotropy groups, so the action is closed. As $U_{n,d}$ is affine, by [11, p. 30] we have a geometric quotient $M_{n,d}$ which is a coarse moduli space for non-singular projective hypersurfaces of degree d in $\mathbb{P}^n(\mathbb{C})$. In our situation we fix a particular $f = f_{n,d} \in U_{n,d}$, the Fermat hypersurface:

$$f_{n,d} = x_0^d + \cdots + x_n^d,$$

and the orbit map then extends to a map

$$\begin{aligned} r_n : M_{n+1} &\to \Pi_{n,d} \\ A &\mapsto f_{n,d} \circ A. \end{aligned}$$

It induces maps for cohomology with supports:

$$H^{2\ell}_{\Sigma_{n,d}}(\Pi_{n,d}) \xrightarrow{r_n^*} H^{2\ell}_{D_{n+1}}(M_{n+1}).$$

Let $e_0, \dots, e_n$ denote the standard basis vectors of $\mathbb{C}^{n+1}$. Define for $\ell = 1, \dots, n+1$

$$\Sigma^{(\ell)}_{n,d} = \{f \in \Pi_{n,d} \mid V(f)^{\mathrm{sing}} \cap \mathbb{P}[e_0, \dots, e_{n-\ell+1}] \neq \emptyset\}.$$

Then $\Sigma^{(\ell)}_{n,d} \subset \Sigma_{n,d}$ has codimension ℓ in $\Pi_{n,d}$. Below we shall prove:

Lemma 7 *The class $r_n^*([\Sigma^{(\ell)}_{n,d}])$ is a non-zero multiple of $[D_{n+1,\ell}]$.*

Recall from the previous section that $[D_{n+1,\ell}]$ corresponds to the generator $\eta^{(n)}_\ell \in H^{2\ell-2}(G)$ and we now apply Theorem 3 to deduce:

Theorem 8 *Let $d \geq 3$. Then the Leray spectral sequence in rational cohomology for the quotient map $U_{n,d} \to M_{n,d}$ degenerates at E_2.*

Let us proceed to give a **proof of Lemma** 7. We want to do this by induction on n, so we fix an embedding $\iota : \Pi_{n-1,d} \hookrightarrow \Pi_{n,d}$ by posing

$$\iota(h) = x_0^d + h(x_1, \ldots, x_n).$$

Note that $\iota(f_{n-1,d}) = f_{n,d}$ and that $\iota(\Pi_{n-1,d}) \cap \Sigma_{n,d} = \iota(\Sigma_{n-1,d})$. The intersection multiplicity however is equal to $d-1$. Indeed, the multiplicity of a stratum of the discriminant corresponding to hypersurfaces with isolated singularities is equal to the sum of their Milnor numbers, and adding the term x_0^d multiplies the Milnor numbers by $d-1$. We obtain a commutative diagram

$$\begin{array}{ccc} M_n & \xrightarrow{r_{n-1}} & \Pi_{n-1,d} \\ \downarrow \alpha & & \downarrow \iota \\ M_{n+1} & \xrightarrow{r_n} & \Pi_{n,d}. \end{array}$$

We have a corresponding diagram in cohomology with supports

$$\begin{array}{ccc} H^{2\ell}_{\Sigma_{n,d}}(\Pi_{n,d}) & \xrightarrow{r_n^*} & H^{2\ell}_{D_{n+1}}(M_{n+1}) \\ \downarrow \iota^* & & \downarrow \alpha^* \\ H^{2\ell}_{\Sigma_{n-1,d}}(\Pi_{n-1,d}) & \xrightarrow{r_n^*} & H^{2\ell}_{D_n}(M_n) \end{array}$$

Observe that $\iota^*([\Sigma^{(\ell)}_{n,d}]) = \nu^{(\ell)}_{n,d}[\Sigma^{(\ell)}_{n-1,d}]$ where $\nu^{(\ell)}_{n,d}$ is the intersection multiplicity of $\Sigma^{(\ell)}_{n,d}$ with $\iota(\Pi_{n-1,d})$ in $\Pi_{n,d}$. In particular, $\nu^{(\ell)}_{n,d}$ is a positive integer.

We can now prove the lemma by induction on n using the above diagram, provided we check the case $\ell = n+1$ for each n.

The variety $S = \Sigma^{(n+1)}_{n,d}$ is the linear space of all polynomials singular at e_0. Its pre-image under r_n has two irreducible components: one consists of the matrices whose first column is zero, i.e. with $A(e_0) = 0$; this component is exactly $T_1 = D_{n+1,n+1}$. The other component, T_2, which has the same dimension, is the closure of

$$\{A \in M_{n+1} \mid 0 \neq A(e_0) \in V(f_{n,d}),\ \mathrm{Im}(A) = T_{A(e_0)}V(f_{n,d})\}.$$

The component T_2 has multiplicity one, whereas T_1 has multiplicity $d(d-1)^n$. We have the commutative diagram

$$\begin{array}{ccc} H^{2n+2}_S(\Pi_{n,d}) & \to & H^{2n+2}_{\Sigma_{n,d}}(\Pi_{n,d}) \\ \downarrow r_n^* & & \downarrow \\ H^{2n+2}_{T_1 \cup T_2}(M_{n+1}) & \to & H^{2n+2}_{D_{n+1}}(M_{n+1}) \end{array}$$

and therefore

$$r_n^*([S]) = d(d-1)^n[T_1] + [T_2] \tag{1}$$

by Property 1) in Sect. 4.
Claim: *We have*

$$[T_2] = (-1)^n(1-(1-d)^n)[T_1] \text{ in } H^{2n+2}_{D_{n+1}}(M_{n+1}).$$

Combining the Claim with (1) we find:

$$r_n^*[S] = d(d-1)^n[T_1] + [T_2] = ((d-1)^{n+1} + (-1)^n)[T_1] \neq 0,$$

which proves the Lemma.

It remains to prove the Claim. Let T_2' denote the image of T_2 under the transposition map τ. Then

$$[T_2] = (-1)^n[T_2'] \tag{2}$$

in $H^{2n+2}_{D_{n+1}}(M_{n+1})$ by the Remark at the end of Sect. 5. Let $\tilde{T}_1 = T_1 \times \{e_0\} \subset \tilde{D}_{n+1}$.

Write $X = V(f_{n,d}) \subset \mathbb{P}^n$ and let $\gamma : X \to \mathbb{P}^n$ be the Gauss map, which associates to a point $p \in X$ the coordinates of its tangent hyperplane, i.e. $\gamma(p) = \nabla f_{n,d}(p)$.

The space $\tilde{D}_{n+1}$ is the total space of a vector bundle E over $\mathbb{P}^n$ of rank $r = n(n+1)$. Let

$$\tilde{T} := \{(A,p) \in M_{n+1} \times X \mid (df_0)_p \circ^t A = 0 \text{ and } {}^tA(e_0) = p\}.$$

Then $\tilde{T}$ is the total space of a vector bundle F over X of rank $r - n + 1$ which is a subbundle of $\gamma^*(E)$, because $(A,p) \in \tilde{T}$ implies that $A(\gamma(p)) = 0$. The projection of $\tilde{T}$ in M_{n+1} is precisely T_2'.

We will carry out our calculations in Chow groups instead of cohomology groups, using property 3) in Sect. 4. Consider the diagram

$$\begin{array}{ccccc} F & \hookrightarrow & \gamma^*(E) & \stackrel{\tilde{\gamma}}{\to} & E \\ \downarrow & & \downarrow \pi' & & \downarrow \pi \\ X & = & X & \stackrel{\gamma}{\to} & \mathbb{P}^n \end{array}$$

We let s be the 0-section of E, and s' that of γ^*E and recall from Sect. 4 3) that these induce Gysin maps in Chow groups.

The strategy is to compare the classes $\tilde{T_1}$ and $\tilde{T}$ by pushing them to $\mathbb{P}^n$. We get two 0-cycles on $\mathbb{P}_n$ whose degrees we compare. Clearly $\deg s^*[\tilde{T_1}] = 1$ and so it suffices to calculate the degree of

$$s^*\tilde{\gamma}_*([F]) \in A_0(\mathbb{P}^n).$$

By [5, Proposition 1.7] we find that

$$\tilde{\gamma}_*\pi'^*\alpha = \pi^*\gamma_*\alpha \in A_{i+r}(E)$$

for any $\alpha \in A_i(X)$. Applying this to $\alpha = s'^*[F]$ we find

$$\tilde{\gamma}_*[F] = \pi^*\gamma_* s'^*[F].$$

Next, applying s^* to both sides and using that the Gysin map s^* is in fact the inverse of the isomorphism induced by the bundle projection $\pi : E \to \mathbb{P}^n$, and similarly for s', we get

$$s^*\tilde{\gamma}_*[F] = \gamma_* s'^*[F].$$

We next compute $s'^*[F] \in A_0(X)$. By [5, Example 3.3.2] applied to the vector bundle $\gamma^*(E)/F$ we get

$$s'^*[F] = c_{n-1}(\gamma^*(E)/F).$$

On $\mathbb{P}^n$ we have the exact sequence

$$0 \to E \to \mathcal{O}^{(n+1)^2} \to \mathcal{O}(1)^{n+1} \to 0$$

showing that $c(E) = (1+h)^{-n-1}$ where $h = c_1(\mathcal{O}(1))$. As $\gamma^*\mathcal{O}(1) = \mathcal{O}_X(d-1)$ we get

$$c(\gamma^*E) = (1 + (d-1)h_X)^{-n-1}$$

where $h_X = c_1(\mathcal{O}_X(1))$. For the bundle F we have the exact sequences

$$0 \to F \to \mathcal{O}_X^{(n+1)^2} \to Q_X \oplus \mathcal{O}_X(d-1)^n \to 0$$

$$0 \to \mathcal{O}_X(-1) \to \mathcal{O}_X^{n+1} \to Q_X \to 0$$

so Q_X is the restriction of the universal quotient bundle to X. Hence we find

$$c(F) = (1 + (d-1)h_X)^{-n} c(Q_X)^{-1} = (1 + (d-1)h_X)^{-n}(1 - h_X)^{-1}$$

so

$$c(\gamma^* E/F) = (1 + (d-1)h_X)^{-1}(1 - h_X)^{-1}.$$

We find

$$c_{n-1}(\gamma^* E/F) = \left(\frac{1-(1-d)^n}{d}\right) h_X^{n-1}$$

which has degree equal to $1-(1-d)^n$. Combining this with (2), the Claim then follows. □

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