Introduction to Regularity And Systems of PDE *

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Introduction

A partial differential equation is an equation satisfied by the partial derivatives of some unknown function which I assume to be complex valued and defined on some open subset U of $\mathbb{C}^n$. As an example, we may search for a complex valued function $y = y(u, v)$ defined on some domain $U \subset \mathbb{C}^2$ which satisfies the k-th order equation

$$\sum_{i+j\le k} a_{i,j} \frac{\partial^{i+j} y}{\partial^i u\, \partial^j v} = 0, \tag{1}$$

where the functions $a_{i,j} : U \to \mathbb{C}$, $i + j \le k$ are supposed to be sufficiently "nice". In fact, since the theory is only well developed for systems for which the a_{ij} are analytic functions of the coordinates u and v this will always be assumed in what follows.

Suppose that one tries to find a formal solution at a point $(u^o, v^o) \in U$:

$$y(u, v) = \sum \frac{y^{(s,t)}}{s!t!}(u - u^o)^s(v - v^o)^t, \quad y^{(s,t)} = \frac{\partial^{s+t} y}{\partial^s u \partial^t v}(u^o, v^o).$$

starting from the k-th order tail, where k is the order of the differential equation. These tails form a vector space of polynomials up to order k in the variables $u - u^o$ and $v - v^o$ and glue together to form a (trivial) vector bundle J_k over U, the space of k-jets of functions $U \to \mathbb{C}$. Any such polynomial $\sum y^{(i,j)} u^i v^j$ is determined by their coefficients $y^{(i,j)}$ which therefore can be considered themselves as giving coordinates on the fiber of J_k at (u_o, v_o). Then the equations (1) written as

$$\sum_{i+j\le k} a_{i,j} y^{(i,j)} = 0 \tag{2}$$

*These notes form the written and expanded version of talks given by Bernard Malgrange and myself in a seminar held at the University of Grenoble Oct–Dec. 2005. The results are not new but not easy to distill from existing literature, partly because notions from partial differential equations, differential geometry and algebraic geometry are needed and there is no uniformity when we go from one field to another. I hope that these notes will add to the comprehension of this intersecting cross-roads.

define conditions on k-th order tails to be the tail-end of a formal solution and the solutions to these equations are therefore called *k-th order jet of a solution*. In fact, giving such jets is the *same* as giving the system! The equation (2) defines a hypersurface inside the total space of the bundle J_k.

More generally, a subvariety Z_k of J_k corresponds to a system of partial differential equations, and a point $z \in Z_k$ represents a k-order jet of a solution of the system. The first step in solving the system consists of extending such a solution to the next order. This may or may not be possible; in general one has to restrict to a strictly smaller subvariety of Z_k. This variety is the projection under the natural map $J_{k+1} \to J_k$ of a subvariety $Z_{k+1} \subset J_{k+1}$ which corresponds to the $(k+1)$st order system obtained upon once formally differentiating the orginal set of differential equations. This system is the called the *first prolongation*. Iterating this procedure yields the higher order prolongations. The theory of formal solutions deals with the question of whether these higher order prolongations eventually gives a formal solution.

1 Linear Systems

Equation (1) is an example of a linear system. We have seen that we can equivalently give it by an equation (2). Let me explain how the general form of a linear system looks like. Let $\mathbf{x} = (x_1, \dots, x_n)$ be coordinates in an open set $U \subset \mathbb{C}^n$ and suppose that $y : U \to \mathbb{C}$ is a holomorphic function. It is convenient to use multi-index notation

$$\begin{array}{rclrclrcl} \underline{\alpha} & = & (\alpha_1, \dots, \alpha_n)\,, & |\underline{\alpha}| & = & \sum_{i=1}^n \alpha_i\,, & \underline{\alpha}! & = & \alpha_1! \cdots \alpha_n! \\ \mathbf{x}^{\underline{\alpha}} & = & x_1^{\alpha_1} \cdots x_n^{\alpha_n} & \partial^{\underline{\alpha}}\mathbf{x} & = & \partial^{\alpha_1}x_1 \cdots \partial^{\alpha_n}x_n\,, & \partial^{\underline{\alpha}}y(\mathbf{x}) & = & \dfrac{\partial^{|\underline{\alpha}|}y}{\partial^{\underline{\alpha}}\mathbf{x}}. \end{array}$$

The individual equations in a linear system $\mathbf{L} = (L_1, \dots, L_N) = \mathbf{0}$ can be written as

$$L_m \;=\; \sum_{\underline{\alpha}} L_m^{\underline{\alpha}}(\mathbf{x}) \partial^{\underline{\alpha}} y(\mathbf{x}) = 0. \tag{3}$$

If the functions $L_m^{\underline{\alpha}}(\mathbf{x})$ are constant one speaks of a linear system with *constant coefficients*. For the moment I don't assume this.

Fix $\mathbf{u}^o \in U$, and let $(J_k)_{\mathbf{u}^o}$ be the vector space of k-jets of functions at $\mathbf{u}^o$. This space can be identified with the space of polynomials of degree $\leq k$ in the variables $(x_1 - u_1^o, \dots, x_n - u_n^o)$. The collection of k-jets of functions $U \to \mathbb{C}$ then becomes identified with the product

$$\begin{array}{ccc} J_k & \simeq & U \times \prod_{|\underline{\alpha}| \leq k} \mathbb{C} \\ \sum_{|\alpha| \leq k} y^{\underline{\alpha}} \dfrac{(\mathbf{x} - \mathbf{u}^o)^{\underline{\alpha}}}{\underline{\alpha}!} & \mapsto & (\mathbf{u}^o, \cdots, y^{\underline{\alpha}}, \cdots). \end{array}$$

The linear system (3) defines the subvariety $Z_k \subset J_k$ given by the equations

$$L_m = \sum_{\underline{\alpha}} L_m^{\underline{\alpha}}(\mathbf{x}) y^{\underline{\alpha}} = 0,\ m = 1, \dots, N.$$

This subvariety represents the k-jets of possible solutions $y(\mathbf{x})$ to the system. It meets the fibers of the projection $J_k \to U$ in a linear subspace whose dimension may or may not vary with the fiber. [1]

There is another way to describe this. Note that the cotangent space bundle of U is trivialised by the coframe consisting of the differentials of the functions $\{x_1, \ldots, x_n\}$. At the point $\mathbf{u}^o$ these together with the constant function 1 gives a basis for the 1-jets at u^o of functions $U \to \mathbb{C}$. The value of the dual frame at $\mathbf{u}^o$ is thus the basis $\{y^1, \ldots, y^n\}$. The Fourier transform replaces the partial derivation $\dfrac{\partial}{\partial x_i}$ by multiplication with a variable ξ_i in the ring of functions in $(\xi_1, \ldots, \xi_n)$. Making this identification we thus have

$$\xi_i = y^i = \left.\frac{\partial}{\partial x_i}\right|_{\mathbf{u}^o}$$

so that

$$T_{U,\mathbf{u}^o} \simeq W := \mathbb{C}\xi_1 \oplus \cdots \oplus \mathbb{C}\xi_n. \tag{4}$$

Then under the Fourier transform $y^{\underline{\alpha}}$ gets identified with the polynomial

$$\xi^{\underline{\alpha}} = \xi_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

suggesting to use the corresponding graded ring of polynomials

$$\begin{aligned}
R &=: \mathbb{C}[\xi_1, \ldots, \xi_n] = \operatorname{Sym} W, \\
R_\ell &=: S^\ell W = \{\text{space of homogeneous polynomials of deg. } \ell\} \subset R, \\
R_{\leq \ell} &=: \bigoplus_{j \leq \ell} S^j W = \{\text{space of polynomials of deg. } \leq \ell\} \subset R.
\end{aligned}$$

Then at every point $\mathbf{u}^o \in U$ the Fourier transformation produces out of the system $\mathbf{L} = 0$ a $\mathbb{C}$-vector space

$$V_{\mathbf{L},\mathbf{u}^o} = \mathbb{C}F_1 + \cdots + \mathbb{C}F_N, \quad F_m = \sum_{\underline{\alpha}} L_m^{\underline{\alpha}}(\mathbf{u}^o)\xi^{\underline{\alpha}} \in R_{\leq k}. \tag{5}$$

The fiber of $J_k \to U$ at $\mathbf{u}^o$ is the vector space of polynomials in $(x_1 - u_1^o, \ldots, x_n - u_n^o)$. The coefficient $y^{\underline{\alpha}}$ of the monomial $(\mathbf{x} - \mathbf{u}^o)^{\underline{\alpha}}$ has been identified with the monomial $\xi^{\underline{\alpha}}$ and thus is a function on this vector space and taken together they give a dual basis for this fiber. Hence:

Lemma 1. [2] *Using* (4) *there is a canonical identification of the fiber of $J_k \to U$ at $\mathbf{u}^o$ and the dual $[R_{\leq k}]^*$. of $R_{\leq k}$. The fiber of $Z_k \to U$ at $\mathbf{u}^o$ is the* annihilator *$V_{\mathbf{L},\mathbf{u}^o}^\perp$ of $V_{\mathbf{L},\mathbf{u}^o}$ inside $[R_{\leq k}]^*$.*

[1] In the general case $Z_k \to U$ is locally a product of this sort but only when we leave out certain exceptional subvarieties from U.

[2] This anticipates a more general result (Lemma 18) to be proven later.

Let us now discuss the notion of *prolongation* in this simplified setting. Suppose that $y(\mathbf{x})$ is a solution of (3). Then not only

$$g(\mathbf{x}) = f(\mathbf{x}, \cdots, \partial^{\underline{\alpha}} y(\mathbf{x}), \dots) = 0$$

but also $\dfrac{\partial g(\mathbf{x})}{\partial x_j} = 0$, for every $j = 1, \dots, n$. By the Chain Rule these new equations are obtained by substituting $y^{\underline{\beta}} = \partial^{\underline{\beta}} y(\mathbf{x})$ in the following system consisting of the nM equations

$$L^j_m \quad = \quad \textstyle\sum_{\underline{\alpha}} (L^{\underline{\alpha}}_m)' y^{\underline{\alpha}} + L^{\underline{\alpha}}_m y^{\underline{\alpha}+1_j} = 0,\ m = 1, \dots, N,\ j = 1, \dots, n. \qquad (6)$$

Here $\underline{\alpha} + 1_j = (\alpha_1, \dots, \alpha_j + 1, \dots, \alpha_n)$. The system obtained after adding these equations, denoted $\mathbf{L}^{(1)} = 0$, is called the *first prolongation* of the system $\mathbf{L} = \mathbf{0}$. It defines a subvariety $Z_{k+1} \subset J_{k+1}$. The *ℓ-th prolongation* $\mathbf{L}^{(\ell)} = \mathbf{0}$ is then defined by repeating this process ℓ times, obtaining $Z_{k+\ell} \subset J_{k+\ell}$.

Example 2. Let us assume that the coefficients are constants so that the first term in the equations (6) do not appear. So, if the Fourier transform of the system is given by a vector subspace $V \subset R_{\leq k}$, the first prolongation is corresponds to $V^{(1)}$, where

$$V^{(\ell)} \quad = \quad \sum_{|\underline{\beta}| \leq \ell} \xi^{\underline{\beta}} V. \qquad (7)$$

Prolongations come up naturally when solving a system by a formal power series $y(\mathbf{x}) := \displaystyle\sum_{|\underline{\gamma}| \geq 0} y^{\underline{\gamma}} \frac{(x - \mathbf{u}^o)^{\underline{\gamma}}}{\underline{\gamma}!}$ with given *initial condition* $\mathbf{u}^o_k = (\mathbf{u}^o, \cdots, \mathbf{u}^{\underline{\alpha}}, \cdots) \in p_k^{-1}(\mathbf{u}^o) \cap Z_k$. Points of Z_k represent all possible solutions up to order k. Adding the next term of the solution corresponds to a point in Z_{k+1} and if the projection of Z_{k+1} is strictly contained in Z_k there are obstructions to extending a given solution to the next order. Clearly, we can iterate the process of prolonging and define the ℓ-th prolongations which gives a subvariety $Z_{k+\ell} \subset J_{k+\ell}$. If at each stage the projections $Z_{m+1} \to Z_m$, $m = k, \dots, \ell - 1$ are surjective, there are no obstructions and we can use points in $Z_{k+\ell}$ to a find solution up to order $k + \ell$.

Example 3. In the case of constant coefficients, in terms of the associated vector spaces, the projection is given by the subspace $V_{\mathbf{L}^{(1)},\mathbf{u}^o} \cap R_{\leq k}$. Already in this situation this space might be different from $V_{\mathbf{L},\mathbf{u}^o}$. This does indeed happen. For example, consider on $\mathbb{C}^2$ with coordinates u, v the system $y_u = y_{vv} = 0$ with associated vector space having basis $\{\xi_1, \xi_2^2\}$. Its first prolongation has associated vector space having basis $\{\xi_1, \xi_1\xi_2, \xi_2^2, \xi_1^2, \xi_1\xi_2^2, \xi_1^3\}$ and in degree 2 two new generators presents itself. In terms of the jet spaces, the projection of the space Z_3 is contained in Z_2 and has codimension 2 in it given by the extra equations $y^{1,1} = y^{2,0} = 0$.

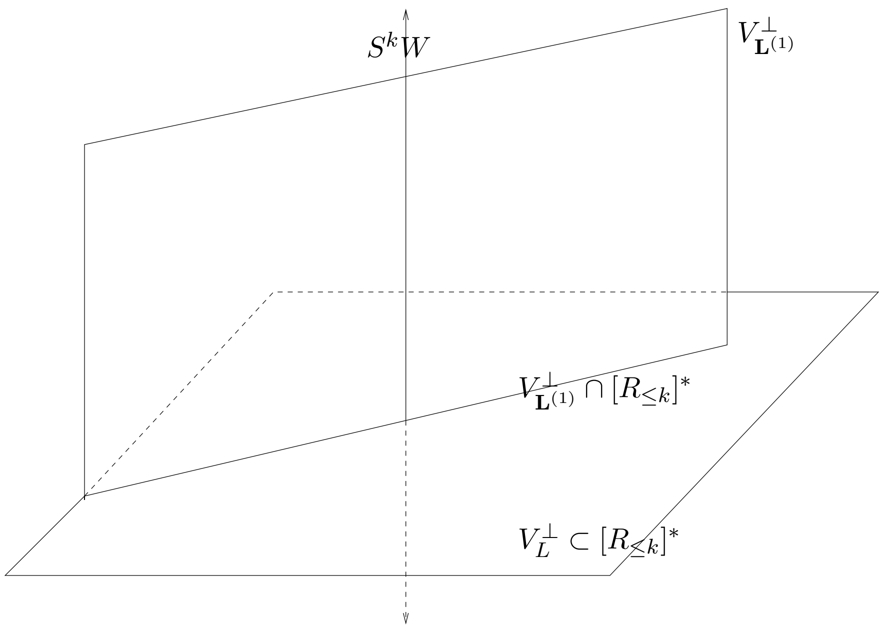

Figure 1: Fibers of the projection $Z_{k+1} \to Z_k$

To understand the prolongation from Z_m to Z_{m+1} from a geometric point of view, fix $\mathbf{u}^o \in U$ and a point $\mathbf{u}^o_m \in Z_m$ lying over $\mathbf{u}^o$. Compare the fiber $F_{m+1} \simeq \prod_{|\underline{\alpha}|\leq m} \mathbb{C}$ over $\mathbf{u}^o$ of the projection $J_{m+1} \to U$ with the fiber F_m of $J_m \to U$ over the same point. One sees that $F_m \subset F_{m+1}$ is the vector subspace given by the equations $y^{\underline{\alpha}} = 0$, $|\underline{\alpha}| = m+1$. The projection p_{m+1} restrict to a projection $q_{m+1} : F_{m+1} \to F_m$ and the fiber $p^{-1}_{m+1}(\mathbf{u}^o, \mathbf{u}^o_m)$ can be identified with the affine space $F_{\mathbf{u}^o_m} = \ker q_{m+1} + \mathbf{u}^0_m$. The subvarieties $Z_m \subset F_m$ define (vector) subspaces $Y_m \subset F_m$ and a point $\mathbf{u}^o_m$ is the projection of a point on Z_{m+1} precisely when the fiber $F_{\mathbf{u}^o_m}$ meets Y_{m+1}.

This can also be formulated in terms of the Fourier transform as follows. Let me simplify the notation by denoting the Fourier transform at the point $\mathbf{u}^o$ of the ℓ-fold prolonged system by $V^{(\ell)}$ (instead of $V_{\mathbf{L}^{(\ell)}}$). Since $Y_{k+\ell} \subset F_{k+\ell}$ is the annihilator of the subspace $V^{(\ell)} \subset R_{k+\ell}$ while $Y_{k+\ell+1} \subset R_{k+\ell+1}$ is the annihilator of the space $V^{(\ell+1)} \subset R_{k+\ell+1}$, the projected subspace $q_{k+\ell+1}Y_{k+\ell+1}$ is the annihilator of $V^{(\ell)} \cap R_{k+\ell} \subset R_{k+\ell}$. Working directly inside $Y_{k+\ell} = [R_{k+\ell}/V^{\ell}]^*$ this can be considered as a collection of "equations" $\left[V^{\ell+1)} \cap R_{k+\ell}/V\right]^{\perp}$. Evaluating this system in a point $\mathbf{u}^o_{k+\ell} \in Y_{k+\ell}$, one gets zero exactly when this point is the projection of a point of $Y_{k+\ell+1}$. Let me summarize what one has proved so far:

Proposition 4. *For all $m \geq k$, let $V^{(\ell)}$ be the Fourier transform at the point $\mathbf{u}^o$ of the ℓ-fold prolonged system. A point $(\mathbf{u}^o, \mathbf{u}^o_{k+\ell}) \in Z_{k+\ell}$ is in the* *projection of $Z_{k+\ell+1} \to Z_{k+\ell}$ precisely when* $\left[V^{(\ell+1)} \cap R_{\leq k+\ell}/V^{(\ell)}\right]^{\perp}\Big|_{u^o_{k+\ell}} = 0$.

This result gives the obstructions to formally extending an m-th order solution to an $(m+1)$-th order solution. If these obstructions are present, we need to restrict our initial condition further down to the image of Z_{m+1} inside Z_k.

Example 5. After a finite number of steps we might end up with the zero-section Z_ℓ which means that there are only polynomial solutions. This indeed is possible as shown by the system $y_u = y_{vv} = 0$. In this example Z_3 projects to a lower dimensional subvariety of Z_2 but for $k \geq 3$ the variety Z_{k+1} just consists of the 0-section of $J_{k+1} \to J_k$ and hence surjects onto Z_k.

One would like to construct some prolongation Z_ℓ such that *all* ℓ-th order solutions extend to a formal solution. This is equivalent to the statement that the projections $Z_{\ell+1} \to Z_\ell$ are all surjective from $\ell = m$ on. The smallest such ℓ is the *prolongation threshold* and the system then will be called *m-regular*. The original system of order k has to be replaced with the prolonged system which is of order m. By its very nature, the solutions for the prolonged system are the same as those for the original system. To prove that one has a finite prolongation threshold is not straightforward, even in the case of constant coefficients:

Example 3 (bis). Continue with Example 3. Let $V \subset R_{\leq k}$ be the Fourier transform of a given linear system with constant coefficients and let $V^{(q)}$ be the Fourier transform of the q-fold prolonged system. The system is m-regular precisely if

$$V^{(k+\ell+1)} \cap R_{\leq(k+\ell)} = V^{(k+\ell)} \text{ for all } \ell \geq m. \tag{8}$$

This poses a non-trivial algebraic problem. It will be addressed in § 3. The central feature here is that one only needs to prove (8) only for some ℓ which then automatically is an upper bound for the prolongation threshold.

Let me return to the general situation of possibly non-constant coefficients. The Fourier transform at $\mathbf{u}_o$ of the system is a finite dimensional vector subspace V of the polynomial ring R with generators in degrees $\leq k$, the rank of the system. It generates an ideal I_V which in general is not homogeneous. There is a canonical homogeneous ideal associated to I_V which turns out to be easier to handle:

Definition 6. The *symbol ideal* at $\mathbf{u}^o$ is the ideal generated by the homogeneous top-degree parts of the degree k generators of the Fourier transform of the system. I.e. the ideal generated by $s^k_{\mathbf{u}^o}(L_m) := \sum_{|\underline{\alpha}|=k} L^{\underline{\alpha}}_m(\mathbf{u}^o)\xi^{\underline{\alpha}}$.

Example 7. Consider on $U = \mathbb{C}^2$ the system $1 + y_u + 2y_{uv} = 0, y_v + y_{vv} = 0$ then its associated vector space V has basis $\{1 + \xi_1 + 2\xi_1\xi_2, \xi_2 + \xi_2^2\}$ while its symbol ideal is $(\xi_1\xi_2, \xi_2^2)$. In multi-index notation the system is given by $y^{0,0} + y^{1,0} + 2y^{1,1} = y^{0,1} + y^{0,2} = 0$. The standard monomial basis for $W := R_{\leq 2}$ is dual to the basis $\{1 = y^{0,0}, y^{1,0}, y^{0,1}, y^{2,0}, y^{1,1}, y^{0,2}\}$ for $\prod_{|\underline{\alpha}|\leq 2}\mathbb{C}$. In other words, every $y^{\underline{\alpha}}$ is a function on W and $Z_k = Y_k \times U$ where Y_k is the annihilator of V, i.e. the linear subspace of $\prod_{|\underline{\alpha}|\leq 2}\mathbb{C}$ given by the equations $y^{0,0} + y^{1,0} + 2y^{1,1} = y^{0,1} + y^{0,2} = 0$.

The notion of Castelnuovo-Mumford regularity for this ideal is just the tool to find a lower bound for the prolongation threshold. It is the subject of study of the next section.

2 Castelnuovo-Mumford Regularity

Consider the polynomial ring R which is the symmetric tensor algebra on a vector space W together with a given basis $\{\xi_1, \ldots, \xi_n\}$ for W. Then the *Koszul sequence* associated to these data is:

$$0 \to \Lambda^n W \otimes R \xrightarrow{d_n} \cdots W \otimes R \xrightarrow{d_1} R,$$

where $d_j : \Lambda^j W \otimes R \to \Lambda^{j-1} W \otimes R$ is given by $d_j(v_1 \wedge \cdots \wedge v_j \otimes F) = \sum_{k=1}^{j} (-1)^k (v_1 \cdots \widehat{v_j} \cdots \wedge v_j) \otimes v_k F$. Note that this sequence is *homogeneous* in that $d_j(\Lambda^j W) \otimes R_m \subset \Lambda^{j-1} W \otimes R_{m+1}$. It is not hard to show that it is exact. In fact, the evaluation map $R \to \mathbb{C}$ combined with the Koszul sequence gives a *free resolution* for $\mathbb{C}$ by R-modules. Tensoring the above Koszul sequence with any R-module M produces the Koszul sequence

$$K_\bullet(M) := [0 \to \Lambda^n W \otimes_R M \xrightarrow{d_n} \cdots W \otimes_R M \xrightarrow{d_1} M].$$

The sequence $K_\bullet(M)$ need not be exact: its homology gives $\mathrm{Tor}_p^R(\mathbb{C}\ , M)$. If, in addition, M is a *graded* R-module, for instance a homogeneous ideal in R or a quotient thereof, we have in fact

$$\begin{aligned} H_{p,q}(K_\bullet(M)) &= \mathrm{Tor}_p^R(\mathbb{C}, M)_{p+q} \\ &:= H\left(\Lambda^{p+1} W \otimes M_{q-1} \to \Lambda^p W \otimes M_q \to \Lambda^{p-1} W \otimes M_{q+1})\right). \end{aligned}$$

The notion of regularity uses these spaces:

Definition 8. The graded R-module M is *r-regular* if $H_{p,q}(K_\bullet(M)) = 0$ for all p and all $q \geq r$. The minimal such r is called the *regularity* of M, denoted $\operatorname{reg} M$. In other words, $K_\bullet(M)$ is exact on the submodule of M in degrees $\geq \operatorname{reg} M$ but not in smaller degrees.

Example 9. The module $M = R_{\geq k}$ has regularity $k+1$ since $H_{0,k}(M) \neq 0$ but $H_{p,q}(M) = 0$ for all p and $q \geq k+1$. If a module M is generated in degrees k and higher, the regularity of M must be $\geq k+1$ for the same reason: $H_{0,k}(M) \neq 0$.

Suppose that $M = M_k \oplus M_{k+1} \oplus \cdots$, and the regularity of M is $k + \ell$. Fix $m \geq k + \ell - 1$. Then on the one hand the regularity of $M' = M_{\geq m-1}$ is $\geq m$ and on the other hand since $H_{p,q}(M) = H_{p,q}(M')$ for $q \geq m - 1$, one has $H_{p,q}(M') = 0$ for $q \geq m$. It follows that the regularity of M' equals m.

Using the symmetry of Tor the cohomology groups of $H_{p,q}(K_\bullet(M))$ can also be found from a free resolution of the module M. In fact special type of resolutions can be used. To define them a special notation is needed: for any graded R-module M the notation $M(k)$ means that the degree ℓ part of $M(k)$ is the degree $k + \ell$-part of M. In other words, the degree has been shifted k places down. This is useful if one wants degree zero maps between graded modules. For instance a degree 1-map $f : M \to M$ is the same as a degree zero map $f : M(-1) \to M$. With this convention maps in complexes can always be assumed to have degree 0. Now, if $M = R^k$ is a free R-module, these maps are given as matrices of polynomials whose degrees are coded by writing such a map as $\bigoplus_i R(-s_i)^{a_i} \to \bigoplus_j R(-t_j)^{b_j}$. The matrix then consists of blocs of size $a_i \times b_j$ with only homogeneous polynomials of degree $s_i - t_j$.

Definition 10. Let M be a graded R-module. A *minimal free resolution* of M is an exact sequence

$$\begin{aligned} \cdots \to F^p \xrightarrow{d_p} \cdots \to F^1 \to F^0 \to M \to 0, \\ F^p \quad = \quad \bigoplus_q R(-q)^{n(p,q)}. \end{aligned} \tag{9}$$

Moreover, the matrices for the maps d_p do not contain constants.

Remark 11. In this set-up one sees that

$$\dim \operatorname{Tor}_p^R(\mathbb{C}, M)_{p+q} = n(p, p+q) \text{ in } (9).$$

It follows that M is r-regular if and only if $n(p, p+q) = 0$ for all $q \geq r$. This implies that in the matrices of a minimal free resolution only polynomials of degrees $1, \ldots, r$ can occur. However, one cannot read off the regularity from these degrees as shown by the following example.

Example 12. Consider the ideal $I = (\xi^3, \eta^3)$ in the ring $\mathbb{C}[\xi, \eta]$. Its minimal free resolution is

$$0 \to F^1 = R(-6) \xrightarrow{\begin{pmatrix} -\eta^3 \\ \xi^3 \end{pmatrix}} R^2(-3) = F^0 \to I \to 0.$$

hence the only non-zero $n(p, q)$ are the numbers $n(0, 0+3) = 2, n(1, 1+5) = 1$ and so the regularity is $5 + 1 = 6$ while only polynomials up to degree 3 occur.

3 Regularity and Linear Systems with Constant Coefficients

Let me continue with Example 12. The Fourier transform of the system $y_{uuu} = y_{vvv} = 0$ is given by $\mathbb{C}\xi^3 + \mathbb{C}\eta^3$ with symbol ideal $I = (\xi^3, \eta^3)$. The ideal $I^{(2)} = (\xi^5, \xi^4\eta, \xi^3\eta^2, \xi^2\eta^3, \xi\eta^4, \eta^5)$ equals $R_{\geq 5}$ and hence has regularity 6. The ideal $I^{(2)}$ is the symbol ideal of the system obtained after twice prolonging this system and hence its regularity is one more than the rank of the corresponding system. Indeed, this is generally true, it is just a translation of what has been observed in Example 9:

Fact 13. *If the regularity of the symbol ideal equals $k + \ell$, $\ell \geq 1$, after $m \geq \ell - 1$ prolongations its regularity becomes $k + \ell + m$.*

I use this remark in conjunction with the following result for which we first need a definition:

$$W \subset R_{\leq k} \text{ saturated in lower degrees} \iff \xi_j[W \cap R_{\leq (k-1)}] \subset W, \quad \forall\, j = 1, \dots, n.$$

Of course, if the above property holds, we also have

$$\xi^{\underline{\alpha}}[W \cap R_{\leq (k-\ell)}] \subset W, \quad \text{for all } \ell = 1, \dots, k \text{ and } \underline{\alpha} \text{ with } |\underline{\alpha}| = \ell. \tag{10}$$

Since taking one prolongation has the effect of adding all the missing generators in lower degrees, the Fourier transform of the once prolonged system is saturated in lower degrees and hence one may always assume from the start that this is the case. The central result is:

Proposition 14 (Malgrange)**.** *Assume that W is saturated in lower degrees. Let $\bar{I}_W$ be the ideal generated by the top degree homogeneous parts $\bar{F}$ of elements $F \in W$. Referring to* (7) *there is a natural* surjection

$$\tau : Z_{1,k}(K_\bullet(\bar{I}_W)) := \ker \left[R_1 \otimes [\bar{I}_W]_k \xrightarrow{d_1} [\bar{I}_W]_{k+1} \right] \to \frac{W^{(1)} \cap R_{\leq k}}{W}, \tag{11}$$

where d_1 is the Koszul map sending $F \otimes G$ to FG. This map factors over $H_{1,k}(K_\bullet(\bar{I}_W))$. In particular, if $\bar{I}_W$ is k-regular, $W = W^{(1)} \cap R_{\leq k}$.

Proof. As in the proposition, let me use overlines to denote the highest degree homogeneous part of any polynomial. A one-cycle in the Koszul complex at degree k corresponds to an n-tuple of polynomials $F_1, \dots, F_n$ in W such that $\sum_j \xi_j \overline{F_j} = 0$. This means precisely that $\sum_j \xi_j F_j \in W^{(1)} \cap R_{\leq k}$. This element is not uniquely determined by $(F_1, \dots, F_n)$, since it is determined by the homogeneous parts $\overline{F_j}$ of the F_j of degree k. So any other n-tuple of polynomials in W with the same leading terms give an element which differs from $\sum \xi_j F_j$ by an element in $\sum \xi_j [W \cap R_{(k-1)}]$, which by saturation belongs to W. So the map τ defined by $(F_1, \ldots, F_n) \mapsto \sum \xi_i F_i$ mod W is well defined and clearly surjective.

It remains to show that it is zero on co-boundaries, i.e. if one has $\overline{F_j} = \sum_j \xi_i \xi_j \overline{G_{ij}}$ where $G_{ij} \in R_{\leq k-2} \cap W$. We may assume that $G_{ij} = -G_{ji}$. Now consider $H_j = F_j - \sum_i \xi_i G_{ij} \in R_{\leq k-1}$. Since W is saturated, $H_j \in W$ and, again by saturation, it follows that $\sum_j \xi_j H_j = \sum \xi_j F_j \in W \cap R_{\leq k}$ so that $\tau(\sum \xi_j \otimes F_j) = 0$ as desired. $\square$

Corollary 15. *Let V be the Fourier transform of a k-th order linear system with constant coefficients. After finitely many prolongations the system becomes formally solvable.*

Proof. The symbol ideal I_V has regularity $k+\ell$, $\ell \geq 1$ since it is generated in degrees k. Put $W = V^{(\ell)}$, a subspace of $R_{\leq (k+\ell)}$ saturated in lower degrees (since $\ell \geq 1$). The associated ideal $\bar{I}_W$ is *not* the symbol ideal of W, but strictly larger since it has generators in degrees k. In fact $\bar{I}_W = I_V$ by construction and hence has regularity $k + \ell$. So the above Lemma can be applied. It follows that the desired condition (8) for formal solvability is verified for the ℓ-fold prolonged system. $\square$

4 From a General Analytic System of PDE to Jets

The general problem is to solve a system of differential equations of order k given by

$$f_m(\mathbf{x}, \cdots, y^{\underline{\alpha}}, \cdots) = 0, \quad m = 1, \ldots, N, \, |\underline{\alpha}| \leq k. \tag{12}$$

Again $y : U \to \mathbb{C}$ is an unknown function[3] and as before the functions f_j are functions defined on $J_k = U \times \prod_{|\underline{\alpha}| \leq k} \mathbb{C}$. The system (12) is called an *analytic system* if these functions are all holomorphic in both the variables x and $y^{\underline{\alpha}}$. In other words, an analytic system of PDE can be interpreted as being given by an ideal

$$\mathfrak{I}_k = (f_1, \ldots, f_N), \quad f_j \text{ holomorphic on } J_k.$$

The locus of zeros is denoted by Z_k. As before, a function $y : U \to \mathbb{C}$ is a *solution* $y(\mathbf{x})$ for the system (12) (valid on U) if (12) is identically true when we substitute for $y^{\underline{\alpha}}$ the $\underline{\alpha}$-th derivative of $y(\mathbf{x})$.

This can be viewed more intrinsically as follows. Consider the projection $p = p_0 : J_0 = U \times \mathbb{C} \to U = J_{-1}$. A function $\mathbf{x} \mapsto y(\mathbf{x})$ is the same as a *section* $s(\mathbf{x}) = (\mathbf{x}, y(\mathbf{x}))$ for the projection, and one has

Lemma 16. *Points of J_k correspond one to one to k-jets of sections of the bundle $p : U \times \mathbb{C} \to U$. More precisely, given $\mathbf{u}^o \in U$, the fiber of $J_k \to U$ over $\mathbf{u}^o$ consists of k-jets at $\mathbf{u}^o$ of sections for p.*

[3]We may also consider functions $y : U \to \mathbb{C}^r$; the theory changes very little, roughly we have to take factors $\mathbb{C}^r$ instead of $\mathbb{C}$ in the definition of the jet-spaces J_k below.

One word of explanation. The Taylor expansion for the function $y(\mathbf{x})$ at $\mathbf{u}^o$ can be written

$$\begin{aligned} y(x) &= \sum_{0\le|\underline{\alpha}|\le k} y^{\underline{\alpha}}(\mathbf{u}^o)\frac{[\mathbf{x}-\mathbf{u}^o]^{\underline{\alpha}}}{\underline{\alpha}!} + \text{higher order terms} \\ &= j^k_{\mathbf{u}^o}(y) + \text{higher order terms}. \end{aligned} \tag{13}$$

The degree $\le k$-part $j^k_{\mathbf{u}^o}y)$ is called the *k-jet* of s (or of y) at $\mathbf{u}^o$. A jet at $\mathbf{u}^o$ gives a point in $\prod_{|\underline{\alpha}|\le k}\mathbb{C}^{\underline{\alpha}}$. Conversely, a point with coordinates $(\mathbf{u}^o,\cdots,y^{\underline{\alpha}},\cdots)$ then corresponds to the k-jet $\sum_{0\le|\underline{\alpha}|\le k} y^{\underline{\alpha}}\frac{(\mathbf{x}-\mathbf{u}^o)^{\underline{\alpha}}}{\underline{\alpha}!}$. It is important to observe that *any section determines a k-jet at every point, but a given k-jet need not come from a section.*

In this set-up, a *solution* to (12) valid on U is a section $s(\mathbf{x})$ whose k-jet $j^k s$ at any point $\mathbf{u}^o\in U$ satisfies the equations $f_m=0$, $m=1,\dots,N$.

5 Jet Spaces

In this section I briefly review how the formalism of jet-spaces can be set up invariantly. See [Saun, §4 and §6].

The point of departure is a surjective submersive holomorphic map $p: Y\to X$. In the previous section $X=U$, $Y=U\times\mathbb{C}$ and p is the projection and the coordinates $(x_1,\dots,x_,y)$ respected the projection. In the general set-up the implicit function theorem shows that locally Y is biholomorphic to a product $U\times V$ where U is open on X and p becomes the projection $U\times V\to U$. For simplicity one may assume that U, respectively $U\times V$ is a coordinate patch on X and Y respectively. Such coordinate patches give by definition *p-adapted coordinates* $(x_1,\dots,x_n,y_1,\dots,y_r)$ where the x_j are coordinates on $U\subset X$ and the y_j are the "vertical" coordinates. In the previous section we had $r=1$ and for simplicity of notation we continue to assume this. Take any section $s:X\to Y$ for p. Suppose we take p-adapted coordinates $(x_1,\dots,x_n,y)$ at a given point $\mathbf{y}_0=s(\mathbf{u}^o)$ so that we may write and $s(\mathbf{x})=(\mathbf{x},y(\mathbf{x}))$. Now, and this is the crucial point, *the value at this point of the partial derivative $\partial^\alpha y(x_1,\dots,x_n)$ is independent of the choice of adapted coordinates.* This requires a small calculation involving changes of adapted coordinate sets. This will be omitted since a similar calculation will be done below. This remark makes it possible to define:

Definition 17. Two germs of sections at $\mathbf{u}^o$ are *m-jet equivalent* if their partial derivatives up to order m in some adapted (and hence any) set of coordinates are the same. An *m-jet* $j^m_{\mathbf{u}^o}(s)$ of a germ of a section s at $\mathbf{u}^o$ is the equivalence class of s under m-jet equivalence.

The m-jets at a given point naturally form a finite dimensional vector space. Indeed, on any fixed adapted coordinate patch with coordinates

$(x_1, \ldots, x_n, y)$ for every multi-index $\underline{\alpha}$ with $|\underline{\alpha} \leq k$ new coordinates $y^{\underline{\alpha}}$ can be introduced as in the previous section. This proves also that the collection of all m-jets forms a complex manifold J_m, the total space of a bundle with fibers vector spaces isomorphic to $\prod_{|\underline{\alpha}| \leq m} \mathbb{C}$. More precisely:

Lemma 18. *Put $p = p_0$ and $J_0 = Y$; for $0 \leq \ell < k$ set $p_\ell^k = p_\ell \circ \cdots \circ p_k : J_k \to J_\ell$. The bundles $J_{\ell+1} \to J_\ell$ are affine bundles whose associated vector bundle on J_ℓ is*

$$T_\ell := (p_\ell^0)^* S^\ell T_X^* \otimes (p_\ell^1)^* T_{Y/X}.$$

Proof. Rather than giving the proof in general, let me give the crucial calculation for $\ell = 0$. So I have to show how the vector bundle $p^* T_X^* \otimes T_{Y/X}$ acts fibrewise on $J_1 \to Y$. This will be done in adapted coordinates $(x_1, \ldots, x_n, y)$ on Y and using the associated coordinates $(x_1, \ldots, x_n, y, y^{1_1}, \ldots, y^{1_n})$ on J_1. Here 1_k is shorthand for the row of length n having zeros everywhere except, on the k-th place, which shows a 1. Let

$$\sum \xi_j \left(dx_j \otimes \frac{\partial}{\partial y} \right) \in p^* T_X^* \otimes T_{Y/X}.$$

It operates on a point with coordinates $(a_1, \ldots, a_n, b, b^{1_1}, \ldots, b^{1_n})$ by sending it to $(a_1, \ldots, a_n, b, b^{1_1} + \xi_1, \ldots, b^{1_n} + \xi_j)$. This is compatible with changes in adapted coordinates as one can check. Start with a 1-jet $j(s) := j^1_{\mathbf{u}^o}(s)$ and let $(\tilde{x}_1, \ldots \tilde{x}_n, \widetilde{y^{1_1}}, \ldots, \widetilde{y^{1_n}})$ be a new set of adapted coordinates. Then

$$\begin{aligned} \widetilde{y^{1_j}}[j(s)] &= \left. \frac{\partial [y \circ s]}{\partial \tilde{x}_j} \right|_{\mathbf{u}^o} \\ &= \sum_k \left[\left. \frac{\partial \tilde{y}}{\partial x_j} \right|_{s(\mathbf{u}^o)} + \left. \frac{\partial \tilde{y}}{\partial y} \right|_{s(\mathbf{u}^o)} y^{1_k}[(j(s))] \right] \left. \frac{\partial x_k}{\partial \tilde{x}_j} \right|_{\mathbf{u}^o}. \end{aligned}$$

This shows once again the affine nature of the bundle and that the linear part changes as a section of the bundle $p^* T_X^* \otimes T_{Y/X}$, as claimed. □

The associated vector bundle can de described alternatively as follows. The vector bundle T_{J_{m+1}/J_m} is intrinsically a vector bundle on J_{m+1}: the fiber at $\mathbf{u}_{m+1}$ consists of vectors tangent to J_{m+1} at $\mathbf{u}_{m+1}$ which project to zero under $p_{m+1} : J_{m+1} \to J_m$. However, for all points on the fiber F of p_{m+1} through $\mathbf{u}_{m+1}$ the vector space $T_{\mathbf{u}_{m+1}}$ is the same space: since F is an affine space its tangent space at each point is the vector space associated to the affine space F. The following result then follows by inspecting what happens under coordinate changes.

Lemma 19. *One has $T_{J_{m+1}/J_m} = p_{m+1}^* T_m$ where the vector bundle T_m on J_m is the vector bundle associated to the affine bundle $p_{m+1} : J_{m+1} \to J_m$. Hence*

$$T_{J_{m+1}/J_m} = (p_{m+1}^0)^* S^m T_X^* \otimes (p_{m+1}^1)^* T_{Y/X}$$

6 Prolongations and Formal Derivatives

Let me first introduce for any holomorphic function g on J_k a *formal derivative* in the x_j-direction as follows

$$D_j g := \frac{\partial g}{\partial x_j} + \sum_{|\underline{\alpha}| \leq k} \frac{\partial g}{\partial y^{\underline{\alpha}}} y^{\underline{\alpha}+1_j}, \tag{14}$$

where I recall that $\underline{\alpha} + 1_j = (\alpha_1, \dots, \alpha_j + 1, \dots, \alpha_n)$ and hence the left hand side has to be interpreted as a holomorphic function on J_{k+1} which requires that g be considered as a function on J_{k+1} via the projection p_{k+1}. These formal derivatives come up naturally when substituting a k-th order solution and derivating once the equations of the system:

$$\left.\begin{array}{rcl} \dfrac{\partial g(\mathbf{x}, \dots, y^{\underline{\alpha}}(\mathbf{x}), \dots)}{\partial x_j} & = & \dfrac{\partial g}{\partial x_j}(\mathbf{x}, \dots, y^{\underline{\alpha}}(\mathbf{x}), \dots) + \\ & & + \displaystyle\sum_{|\underline{\alpha}| \leq k} \frac{\partial g}{\partial y^{\underline{\alpha}}}(\mathbf{x}, \dots, y^{\underline{\alpha}}(\mathbf{x}), \dots) y^{\underline{\alpha}+1_j}(\mathbf{x}) \\ & = & D_j g(\mathbf{x}, \dots, y^{\underline{\alpha}}(\mathbf{x}), \dots). \end{array}\right\} \tag{15}$$

From this the following is obvious:

Lemma 20. *If $y(\mathbf{x})$ is a solution of the system* (12)*, then it is also a solution of the system $D^j f_m = 0$, $m = 1, \dots, N$, $j = 1, \dots, n$.*

This motivates to introduce:

Definition 21. The *first prolongation* of the system (12) is given by the system with equations

$$\begin{array}{rcll} f_m \circ p^k_{k+1} & = & 0, & m = 1, \dots, N \\ D_j f_m & = & 0, & j = 1, \dots, n,\ m = 1, \dots, N. \end{array}$$

Its associated ideal and variety are denoted by $\mathfrak{J}_{k+1}$ and Z_{k+1} respectively.

Clearly, this procedure can be iterated to obtain prolongations $Z_{k+\ell}$ for any $\ell \in \mathbb{N}$ defined by the ideals $\mathfrak{J}_{k+\ell}$. Its equations involve the higher formal derivatives

$$D^{\underline{\beta}} = D_1^{\beta_1} \circ \dots \circ D_n^{\beta_n}$$

which by (15) come up in the Taylor expansion of the composite functions $\mathbf{x} \mapsto g(j^k_{\mathbf{u}^o} y)$ (see also (13)):

$$g(j^k_{\mathbf{u}^o} y) = \sum D^{\underline{\beta}} g(\mathbf{u}^o, \dots, y^{\underline{\alpha}}(\mathbf{u}^o), \dots) \frac{(\mathbf{x} - \mathbf{u}^o)^{\underline{\beta}}}{\underline{\beta}!}. \tag{16}$$

This ties in with the method of formally solving (12). As in the linear case, one tries a solution

$$y(\mathbf{x}) := \sum_{|\underline{\gamma}|\geq 0} y^{\underline{\gamma}} \frac{(x-\mathbf{u}^o)^{\underline{\gamma}}}{\underline{\gamma}!} \tag{17}$$

obeying the *initial condition*

$$\mathbf{u}^k = (\mathbf{u}^o, \cdots, \mathbf{u}^{\underline{\alpha}}, \cdots) \in p_k^{-1}(\mathbf{u}^o) \cap Z_k. \tag{18}$$

The new conditions are $D^{\underline{\beta}} g(\mathbf{u}^o, \ldots, \mathbf{u}^{\underline{\alpha}}, \ldots) = 0$ which explicitly read (using (15) repeatedly):

$$\partial^{\underline{\beta}} f_j \Big|_{\mathbf{u}_k} + \sum_{|\underline{\alpha}|\leq k} \frac{\partial^{|\underline{\beta}|} f_j}{\partial y^{\underline{\beta}}}\Bigg|_{\mathbf{u}_k} y^{\underline{\alpha}+\underline{\beta}} = 0, \quad j = 1, \ldots, N, \quad |\underline{\beta}| \geq 1. \tag{19}$$

For $|\underline{\beta}| = 1$ these give the extra equations needed to define the first prolongation Z_{k+1}. The old constants $\mathbf{u}^{\underline{\alpha}}$ for $|\underline{\alpha}| \leq k$ still might be present in the new equations. Eliminating the new constants $y^{\underline{\alpha}+1_j}$ from the new equations yields equations for the image of the projection $p_{k+1}Z_{k+1}$ inside Z_k. These equations are the *new integrability conditions* for the first prolongation. If these don't come up, i.e. when $p_{k+1}Z_{k+1} = Z_k$ we say that there are *no* $(k+1)$*-st order obstructions* to formal solutions. In that case, writing $\mathbf{u}^{\underline{\gamma}}$ instead of $y^{\underline{\gamma}}$ for the new indices, $y_{\leq k+1} := \sum_{0\leq|\underline{\gamma}|\leq k+1} \mathbf{u}^{\underline{\gamma}} \frac{(\mathbf{x}-\mathbf{u}^o)^{\underline{\gamma}}}{\underline{\gamma}!}$ is called a *formal solution up to order* $k+1$. By construction the point $\mathbf{u}_{k+1} := (\mathbf{u}^o, \ldots, \mathbf{u}^{\underline{\alpha}}, \ldots)$, $|\underline{\alpha}| \leq k+1$ belongs to Z_{k+1} and $p_{k+1}\mathbf{u}_{k+1} = \mathbf{u}_k$. Clearly this procedure can be iterated and yields *obstructions to* m*-th order prolongation*. When these are absent for $m = 1, \ldots, \ell$ there exist *formal solutions up to order* $k+\ell$

$$y_{\leq k+\ell} := \sum_{0\leq|\underline{\gamma}|\leq k+\ell} \mathbf{u}^{\underline{\gamma}} \frac{(\mathbf{x}-\mathbf{u}^o)^{\underline{\gamma}}}{\underline{\gamma}!}$$

and points

$$\mathbf{u}_{k+\ell} := (\mathbf{u}^o, \ldots, \mathbf{u}^{\underline{\alpha}}, \ldots) \in Z_{k+\ell}, \quad |\underline{\alpha}| \leq k+\ell \tag{20}$$

mapping to $\mathbf{u}_k$ under the projections $Z_{k+\ell} \to Z_k$. Clearly, a solution gives formal solutions up to every order. Conversely, a formal power series

$$\sum_{|\underline{\gamma}|\geq 0} y^{\underline{\gamma}} \frac{\mathbf{t}^{\underline{\gamma}}}{\underline{\gamma}!} \in \mathbb{C}[[t_1, \ldots, t_n]], \quad \mathbf{t} = \mathbf{x} - \mathbf{u}^o,$$

whose truncation to every finite order $k+\ell$ gives a formal solution is called *a formal solution.* If the above formal series converges, it follows from what has been said so far that it is an actual solution.

7 Geometric Interpretation

In § 5 I showed that the bundle $J_m \to J_0 = U \times \mathbb{C}$ is an affine bundle with fiber $\prod_{|\underline{\alpha}| \le m} \mathbb{C}$ and that the successive projections $p_m : J_m \to J_{m-1}$ are affine bundles with fibers $\prod_{|\underline{\alpha}|=m} \mathbb{C}$ with associated vector bundle T_{m-1} for which $T_{J_m/J_{m-1}} = p_m^* T_{m-1}$.

The method of step-by-step constructing a solution leads naturally to the study of *relative differentials* δg of holomorphic functions g on J_m. The ordinary differentials dg are sections of the cotangent bundle Ω_{J_m}. Relative differentials are sections of the relative cotangent bundle

$$\Omega_{J_m/J_{m-1}} := \Omega_{J_m}/p^*\Omega_{J_{m-1}}.$$

By way of explanation: the relative tangent bundle of p_m is the kernel of the surjection $T_{J_m} \to p_m^* T_{J_{m-1}}$ and the relative cotangent bundle is the dual, hence equals the image $\Omega_{J_m}/p^*\Omega_{J_{m-1}}$. The relative differential of g then is calculated pointwise: at a point $\mathbf{b} \in J_k$ we should take the image of $dg_{\mathbf{b}}$ in the fiber at $\mathbf{b}$ of this quotient bundle.

Observe also that $\Omega_{J_m/J_{m-1}}$ is the trivial bundle over J_m with basis $\delta y^{\underline{\alpha}}$ with $|\underline{\alpha}| = m$ and one may write[4]

$$\delta g = \sum_{|\underline{\alpha}|=k} g_{\underline{\alpha}} \delta y^{\underline{\alpha}}, \quad g_\alpha \in \mathcal{O}(J_k).$$

The process of going from J_m to J_{m+1} is described by the operators

$$\begin{array}{rcl} \xi_j : p_{m+1}^* \Omega_{J_m/J_{m-1}} & \to & \Omega_{J_{m+1}/J_m} \\ \delta y^{\underline{\alpha}} & \mapsto & \delta y^{\underline{\alpha}+1_j}. \end{array} \tag{21}$$

Let me formulate this in a more canonical way. Dualizing Lemma 19 gives

$$\Omega_{J_{m+1}/J_m} = (p_{m+1}^0)^* S^m T_U \otimes (p_{m+1}^1)^* \Omega_{U\times\mathbb{C}/U}$$

so that one may write $\delta y^{\underline{\alpha}} = (p_{m+1}^0)^* \partial^{\underline{\alpha}} \otimes dy$, where

$$\partial^{\underline{\alpha}} = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} \in S^m T_U.$$

The above map (21) becomes multiplication

$$(p_{m+1}^0)^* S^{m-1} T_U \otimes (p_{m+1}^1)^* \Omega_{U\times\mathbb{C}/U} \xrightarrow{\xi_j} (p_{m+1}^0)^* S^m T_U \otimes (p_{m+1}^1)^* \Omega_{U\times\mathbb{C}/U}$$

induced by the bundle homomorphism

$$\begin{array}{rcl} S^{m-1} T_U & \to & S^m T_U \\ \partial^{\underline{\alpha}} & \mapsto & \partial^{\underline{\alpha}+1_j}. \end{array}$$

[4]On a complex analytic space X the sheaf of germs of holomorphic functions on X is denoted as usual by $\mathcal{O}_X$. Then $\mathcal{O}(X)$ denotes the ring of all holomorphic functions on X.

This comes from multiplication by an element $\xi_j \in T_U = U \times \mathbb{C}^n$.

Starting with the rank one free module $\Omega_{U\times\mathbb{C}/U} = \mathcal{O}_U[\delta y]$ one can iterate this: for every multi-index $\underline{\alpha}$ of length ℓ one gets

$$\delta y^{\underline{\alpha}} = \xi^{\underline{\alpha}} \delta y \in \mathcal{O}_{J_\ell}[\delta y][\xi_1, \dots, \xi_n]_\ell,$$

where the right hand side means polynomials of degree ℓ in the variables $\xi_1, \dots, \xi_n$. More intrinsically, as noticed before, the right hand side should be replaced by

$$(p_\ell^1)^* \Omega_{U\times\mathbb{C}/U} \otimes (p_\ell^0)^* S^\ell T_U.$$

For any holomorphic function g on J_m its relative differential can be written as

$$\delta g = \sum \frac{\partial g}{\partial y^{\underline{\alpha}}} \delta y^{\underline{\alpha}} = \sum \frac{\partial g}{\partial y^{\underline{\alpha}}} \xi^{\underline{\alpha}} \delta y.$$

The easily verified but crucial formula relating all this with the formal derivative is

$$\delta(D_j g) = \xi_j \delta(g), \tag{22}$$

which may be rephrased as:

> *On differentials on J_m, the formal derivative D_j corresponds to multiplication with ξ_j in the sheaf of rings $\mathcal{O}_{J_m}[\xi_1, \dots, \xi_n]$.*

Now to symbols! Loosely speaking passing to a symbol means that one only pays attention to highest degrees of truncated formal solutions. More precisely:

Definition 22.

1. Let $g \in \mathcal{O}_{J_m}$. The m-th order *symbol* of g is the class of δg in $\Omega_{J_m/J_{m-1}} \simeq \mathcal{O}_{J_m}[\xi_1, \dots, \xi_n]_m$ where the right hand side are the homogeneous expressions in the ξ_j of degree m. The symbol of g as viewed in $\mathcal{O}_{J_m}[\xi_1, \dots, \xi_n]_m$ will be denoted by $s^m(g)$.

2. For every $g \in \mathfrak{I}_m$, the ideal defining the $(m-k)$-th prolongation of (12) its symbol $\bar{s}^m(g)$ is defined to be the class of $s^m(g)$ modulo the ideal $\mathfrak{I}_m$, i.e. $\bar{s}^m(g) \in \mathcal{O}_{Z_m}[\xi_1, \dots, \xi_n]_m$. The submodule $\mathcal{N}_m \subset \mathcal{O}_{Z_m}[\xi_1, \dots, \xi_n]_m$ generated by the symbols $\bar{s}^m(g)$, $g \in \mathfrak{I}_m$ is called the *m-th order symbol* of $\mathfrak{I}_m$; the ideal $\mathcal{I}_m \subset \mathcal{O}_{Z_m}[\xi_1, \dots, \xi_n]$ its *symbol* and the quotient $\mathcal{M}_m = \mathcal{O}_{Z_m}[\xi_1, \dots, \xi_n]/\mathcal{I}_m$ its *characteristic module.*

Remark 23. Another way of obtaining the symbol $\bar{s}^m g$ is by first taking the class dg of g in $\mathfrak{I}_m/\mathfrak{I}_m^2$. Note that the totality of these classes generate the conormal sheaf of Z_m which is a subsheaf of $\Omega_{J_m} \otimes \mathcal{O}_{Z_m}$. In general it is a coherent $\mathcal{O}_{Z_m}$-module, but if Z_m is a manifold it is locally free (the dual of the conormal bundle of Z_m in J_m). Its quotient is the coherent $\mathcal{O}_Z$-module Ω_{Z_m} which in the smooth case is the locally free sheaf of holomorphic 1-forms on Z_m. Dividing out the conormal sheaf by $p_{m-1}^* \Omega_{J_{m-1}} \otimes \mathcal{O}_{Z_m}$ gives the coherent $\mathcal{O}_{Z_m}$-module $\mathcal{N}_m$. Finally, the symbol $\bar{s}^m g$ of g is the class of dg in this $\mathcal{O}_{Z_m}$-module.

As to the characteristic module, note that dividing out Ω_{Z_m} by $p^*_{m-1}\Omega_{J_{m-1}} \otimes \mathcal{O}_{Z_m}$ gives the relative cotangent sheaf $\Omega_{Z_m/J_{m-1}}$, and so

$$\mathcal{M}_m = \Omega_{Z_m/J_{m-1}}. \tag{23}$$

I now relate this to the method of formal integration explained above. For any solution (17) first consider its $(k+\ell)$-th order part

$$y_{k+\ell} = \sum_{|\underline{\gamma}|=k+\ell} y^{\underline{\gamma}} \frac{(\mathbf{x}-\mathbf{u}^o)^{\underline{\gamma}}}{\underline{\gamma}!}.$$

Since the coefficients $y^{\underline{\gamma}}$ in this expression give a point in the fiber at the point $\mathbf{u}_{k+\ell}$ (20) of the vector bundle $T_{J_{k+\ell}/J_{k+\ell-1}}$ one has

$$y_{k+\ell} \in [T_{J_{k+\ell}/J_{k+\ell-1}}]_{\mathbf{u}_{k+\ell}} \simeq \mathbb{C}[\mathbf{x}-\mathbf{u}^o]_{k+\ell}.$$

Dually

$$[\Omega_{J_{k+\ell}/J_{k+\ell-1}}]_{\mathbf{u}_{k+\ell}} \simeq \mathbb{C}[\xi_1, \dots, \xi_n]_{k+\ell},$$

where one pairs $y_{k+\ell}$ and $\eta = \sum a_{\underline{\beta}} \xi^{\underline{\beta}}$ as follows: $\langle \eta, y_{k+\ell} \rangle = \sum a_{\underline{\beta}} y^{\underline{\beta}}$ which means precisely that $\xi^{\underline{\beta}}$ acts as the partial derivative $\frac{\partial^{|\underline{\beta}|}}{\partial x^{\underline{\beta}}}$ on the function $y_{k+\ell}$ at the point $\mathbf{u}^o$. So this duality corresponds precisely to the *Fourier transformation*:

Lemma 24. *The highest order term $y_{k+\ell}$ in a formal solution up to order $k+\ell$ at a point $\mathbf{u}_{k+\ell} \in Z_{k+\ell}$ corresponds via the Fourier transform to a $(k+\ell)$-th order symbol:*

$$s^{k+\ell}(y_{\le(k+\ell)}) \in [\mathcal{I}_{k+\ell}]_{\mathbf{u}_{k+\ell}} \subset [\mathcal{O}_{Z_{k+\ell}}]_{\mathbf{u}_{k+\ell}}[\xi_1 \dots, \xi_n]_{k+\ell}.$$

Let me continue the geometric study of the jet bundles. The prolongations give rise to subvarieties $Z_{k+\ell} \subset J_{k+\ell}$ and the successive projections $p_{k+\ell+1}$ by definition map $Z_{k+\ell+1}$ to $Z_{k+\ell}$.

It is not at all clear a priori that the tower of varieties $Z_{k+\ell}, \quad k \in \mathbb{N}$ eventually stabilizes. One is not only adding new equations but also new variables. So one needs an analog of the Hilbert basis theorem in this setting which states that the ascending chain of ideals which is being constructed in fact stabilizes, and, moreover, if the ideal is not the entire ring (which is the case here) then the ideal really defines a non-empty variety. In the algebraic setting this is not too hard. It follows from Ritt's analog of the Hilbert basis theorem [Ritt] in the setting of differential ideals. In the analytic case this is much harder. See for instance [Malg05, Chap. V].

It turns out that the considerations in the next section show that one does not need the entire strength of Ritt's result. It will be shown that one can get by assuming just the following crucial

Fact 25. *Restricting U to a Zariski dense open subset, for some sufficiently large $m \in \mathbb{N}$ the projection $Z_{m+1} \to Z_m$ is surjective.*

8 Enter: Regularity Conditions

The situation and notation is as before. By Fact 25, one may assume that after possibly replacing U by a Zariski-open dense subset, for some $m \geq k$ the projection $Z_{m+1} \to Z_m$ is surjective. By Remark 23, at any given point $\mathbf{u}_{m+1} \in Z_{m+1}$ one has $[\Omega_{Z_{m+1}/Z_m}]_{\mathbf{u}_{m+1}} = [\mathcal{M}_{m+1}]_{\mathbf{u}_{m+1}}$. Hence

Lemma 26.
$$\Omega_{Z_{m+1}/Z_m} = \mathcal{M}_{m+1}.$$
If $\mathcal{M}_{m+1}$ is locally free, p_{m+1} exhibits Z_{m+1} as an affine bundle over its image and so, if Z_m is smooth, also $p_{m+1}^{-1}Z_{m+1}$ is smooth. In particular, since $p_{m+1} : Z_{m+1} \to Z_m$ is onto, *Z_{m+1} is smooth.*

The previous lemma summarizes the geometric content of the regularity conditions:

Definition 27. A system of partial differential equations is *m-regular* if

1. the m-fold prolongation defines a smooth variety $Z_m \subset J_m$, the characteristic modules $\mathcal{M}_m$ and $\mathcal{M}_{m+1}$ for the m-th and $(m+1)$-th prolongations respectively, are locally free; and the projection $Z_{m+1} \to Z_m$ is surjective;

2. for every $\mathbf{u}_m \in Z_m$ the module $[\mathcal{M}_m]_{\mathbf{u}_m}$ is m-regular in the sense of Castelnuovo-Mumford.[5]

If $m = k$ the system is called *regular*.

The crucial result is:

Theorem 28. *If a system of partial differential equations is regular, all prolongations are regular.*

Corollary 29. *A regular system has formal solutions with any given initial condition given by a point on the variety the system defines.*

Remark 30. 1) If a system is not regular, it is still true that the symbol ideal $\mathcal{I}_{k+\ell}$ for the ℓ fold prolonged system will be $(k+\ell)$-regular for some $\ell \geq 0$ and by Example 9, the ideal $\mathcal{I}_{k+\ell}$ will be $(k+\ell)$-regular. The geometric conditions 1 in the definition of regularity are true over a Zariski-open subset of Z_k and similarly over a Zariski-open subset of $Z_{k+\ell}$. This can be achieved upon replacing Z_k by the smaller subset Z_k^ℓ which is the image of the projection of $Z_{k+\ell}$ to Z_k. This corresponds to replacing the original system by an equivalent system. Now for general points in Z_k^ℓ this system is formally solvable: just prolong ℓ times to obtain a regular system and apply the preceding Corollary.

[5]Equivalently: the symbol ideal $\mathcal{I}_m$ is $(m+1)$-regular.

2) By a long series of estimates one can show that the formal solutions one constructs with the above methods are in fact convergent. The result is the *Cartan-Kähler theorem* stating that the formal solution for regular systems are convergent, hence regular systems have analytic solutions. See [Malg05, Chap III]

Start of the proof theorem 28. By definition of the formal derivatives, D_j corresponds to multiplication with ξ_j on the level of differentials. So the ideal $\mathcal{I}_{k+1}$ is generated as ideal inside $\mathcal{O}_{J_{k+1}}[\xi_1, \ldots, \xi_n]$ by $p_{k+1}^{-1}\mathcal{I}_k$ and the ξ_j i.e. $\mathcal{I}_{k+1} = p_{k+1}^{-1}\mathcal{I}_k \cdot \mathcal{O}_{Z_{k+1}}[\xi_1, \ldots, \xi_n]_{\geq k+1}$. In other words, at a given point $\mathbf{u}_{k+1} \in Z_{k+1}$ the corresponding ideal $\mathcal{I}' = [\mathcal{I}_{k+1}]_{\mathbf{u}_{k+1}} \subset \mathbb{C}[\xi_1, \ldots, \xi_n]$ is obtained from the ideal $\mathcal{I} = [\mathcal{I}_k]_{\mathbf{u}_k} \subset \mathbb{C}[\xi_1, \ldots, \xi_n]$ by multiplying the generators by some ξ_j, i.e. $\mathcal{I}' = \mathcal{I}\xi_1 + \cdots \mathcal{I}\xi_n$. By Example 9 if $\mathcal{I}$ is $(k-1)$-regular then $\mathcal{I}'$ is k-regular. For the characteristic modules this means that if $\mathcal{M}_k$ is k-regular then $\mathcal{M}_{k+1}$ is $(k+1)$-regular.

Note that by Lemma 26 and the other assumptions Z_{k+1} is smooth. In fact, one has

Claim. $\mathcal{M}_{k+\ell}$ is locally free for all $\ell \geq 2$.

I give the proof from [Malg05, Prop. II, 2.4]. Note that by assumption $\mathcal{M}_k$ and $\mathcal{M}_{k+1}$ are locally free. Let $W = \mathbb{C}\xi_1 \oplus \cdots \oplus \mathbb{C}\xi_n$. Consider the symbol ideal of the ℓ-fold prolongation $\mathcal{I}_\ell$ at a point $\mathbf{u}_{k+\ell} \in Z_{k+\ell}$ lying over $\mathbf{u}_k \in Z_k$. It is a homogeneous ideal in the polynomial ring $\mathbb{C}[\xi_1, \ldots, \xi_n]$, depending on the point $\mathbf{u}_{k+\ell}$ and is equal to the symbol ideal at the point $\mathbf{u}_k$ of the original system in degrees $\geq k + \ell$ and is zero below this degree. Now choose a point $a \in Z_{k+\ell+1}$ over it and $b \in Z_{k+\ell+2}$ over a. Abusing notations, consider $\mathcal{I}_\ell$ at b as well. This can be compared with the ideals $\mathcal{I}_{\ell+1}$ and $\mathcal{I}_{\ell+2}$ at this point through the Koszul type sequence

$$\Lambda^2 W \otimes [\mathcal{I}_\ell]_b \xrightarrow{d} W \otimes [\mathcal{I}_{\ell+1}]_b \xrightarrow{d} [\mathcal{I}_{\ell+2}]_b \to 0. \tag{24}$$

This sequence is indeed the homogeneous strand of the Koszul sequence for the ideal $\mathcal{I}_k$ at the point $\mathbf{u}_k$ in degree $k+\ell+2$. The last map in this sequence is surjective since the symbol ideal $\mathcal{I}_k$ is generated in degree k. Exactness at the middle is a translation of $H_{1,k+\ell+1}(\mathcal{I}_k) = 0$ which is a consequence of $(k+1)$-regularity of $\mathcal{I}_k$.

Suppose that $F(u) : V \to W$ is a linear map between vector spaces which depends analytically on some parameter $u \in U$. Since the locus in U where F has rank $\leq k$ is closed, the function $u \mapsto \dim \operatorname{Im}(F(u))$ is lower semi-continuous and the function $u \mapsto \dim \ker(F(u)$ is upper semi-continuous. Apply these remarks to the first and the last map figuring in (24), making use of the observation that it suffices to show that $d_m = \dim[\mathcal{I}_m]_b$ is locally constant on Z_m for all $m \geq k+2$. Indeed, by induction this can be assumed for target and source of the first map and hence the dimension of the image of the first map is lower semi-continuous. The kernel of the last map is the same as the kernel of the multiplication map $W \otimes [\mathcal{I}_{\ell+1}]_b \to \mathbb{C}[\xi_1, \ldots, \xi_n]$ which is a map between locally constant dimensional spaces. So the dimension of these kernels behaves in an upper-semi continuous fashion. Hence this dimension is locally constant and hence also $\dim[\mathcal{I}_{\ell+2}]_b$ is locally constant. The Claim follows.

Suppose that one can show that $Z_{k+2} \to Z_{k+1}$ is surjective, then, by the previous Claim and Lemma 26 Z_{k+2} is smooth. This forms the start of an obvious inductive procedure. So one needs to prove the following

Claim. $Z_{k+2} \to Z_{k+1}$ is surjective.

The idea is to extend the proof of Corollary 15 to the general case. An essential ingredient in the proof of Corollary 15 is the saturatedness in lower degrees. Note that by formula (14) the formal derivatives of elements in the ideal of the m fold prolongation are linear expressions in the $y^{\underline{\beta}}$ with $|\underline{\beta}| = m+1$, but if the system is not a linear system with constant coefficient, it may also contain expressions in the variables $y^{\underline{\beta}}$ with $|\underline{\beta}| \leq m$. The algebra works only well if the latter part is absent. To work one's way around it, note that if $f \in \mathcal{O}_{J_m}$ is the lift to J_m of a function g on J_{m-1} one has $D_i f = D_i g$. In particular, if $g \in \mathfrak{J}_{m-1}$ and $m \geq k+1$, by definition $\mathfrak{J}_m$ contains such prolongations $D_i g$. This remark will be crucial when proving the following Lemma.

Lemma 31. *Suppose that the derivatives of $f \in [\mathfrak{J}_{k+1}]_a$ with respect to $y^{\underline{\alpha}}$, $|\underline{\alpha}| = k+1$ belong to $[\mathfrak{J}_{k+1}]_a$, then $D_j f$ belongs to $[\mathfrak{J}_{k+1} \otimes \mathcal{O}_{J_{k+2}}]_a$.*

Proof. Since $q : Z_{k+1} \to Z_k$ is a submersion, one may choose coordinates (v_i, v'_j) on $\prod_{|\underline{\alpha}| \leq k} \mathbb{C}$ and (y_i, y'_j) on $\prod_{|\underline{\alpha}|=k+1} \mathbb{C}$ so that Z_{k+1} locally at a given point a has equation $v_i = y_j = 0$ and q is given by $(u, v', y') \mapsto (u, v')$. The function f can be written $f = g + h$ where $g \in [\mathfrak{J}_k]_a$ and $h = \sum_{|\underline{\beta}|=k+1} f_{\underline{\beta}}(u, v, v') y^{\underline{\beta}} +$ (terms involving higher powers of the y_j). By the previous remark $D_j g \in [\mathfrak{J}_{k+1}]_a$. By assumption $f_{\underline{\beta}} = \dfrac{\partial f}{\partial \underline{\beta}} \in [\mathfrak{J}_{k+1}]_a$ and so the terms in $D_j f$ coming from the terms of order 1 in the y_i belong to $\mathfrak{J}_{k+1} \otimes \mathcal{O}_{J_{k+2},a}$. Finally, the higher order terms of f belonging to $[\mathfrak{J}^2_{k+1}]_a$ have their formal derivatives automatically in $\mathfrak{J}_{k+1} \otimes \mathcal{O}_{J_{k+2},a}$ and hence also $D_j f$ belongs to $\mathfrak{J}_{k+1} \otimes \mathcal{O}_{J_{k+2},a}$. $\square$

As will be demonstrated in a moment, this lemma makes it possible to define a *refined* symbolic derivation

$$\bar{D}_j : \mathcal{I}_{k+1} \to \mathcal{O}_{Z_{k+1}} \otimes \mathcal{O}_{J_{k+2}}.$$

Suppose that $F \in [\mathcal{I}_{k+1}]_a$ has a representative $\sum_j h_j \delta g_j$ with $h_j \in [\mathcal{O}_{Z_{k+1}}]_a$ and $g_j \in [\mathfrak{J}_{k+1}]_a$. Note that F is also represented by $g = \delta(\sum_j h_j g_j) \in [\mathfrak{J}_{k+1}]_a$. If $g' \in [\mathfrak{J}_{k+1}]_a$ is still another such representative, $\delta(g - g')$ represents

0 in $[\mathfrak{J}_{k+1}]_a$ which means that the partial derivatives of $g - g'$ with respect to the variables $y^{\underline{\beta}}$, $|\underline{\beta}| = k+1$ are in the ideal $[\mathfrak{J}_{k+1}]_a$. So Lemma 31 can be applied and shows that $D_j(g - g')$ in fact belongs to the ideal of Z_{k+1} in J_{k+2}. So $\bar{D}_j F := D_j g = D_j g'$ is well defined as an element of $\mathcal{O}_{Z_{k+1}} \otimes \mathcal{O}_{J_{k+2}}$.

Next consider the Koszul sequence (24) for $\ell = 0$. For simplicity, write $\mathcal{J}$ for the ideal $[\mathcal{I}_k]_a$ and write $\mathcal{J}_m$ for the degree m-polynomials in $\mathcal{J}$:

$$\Lambda^2 W \otimes \mathcal{J}_k \xrightarrow{d} W \otimes \mathcal{J}_{k+1} \xrightarrow{d} \mathcal{J}_{k+2}.$$

The Koszul map $d : W \otimes \mathcal{J}_k \xrightarrow{d} \mathcal{J}_{k+1}$ is given by $F = \sum \xi_j \otimes F_j \mapsto \sum \xi_j F_j$. Suppose that $F_j = \delta(f_j)$. By (22) one has $0 = \sum \xi_j \delta(f_j) = \delta(\sum D_j f_j)$, i.e. the symbol of $\sum D_j f_j$ is zero which means that $\sum \bar{D}_j F_j$ has no terms involving $y^{\underline{\beta}}$ for which $|\underline{\beta}| = k+2$, i.e. the latter belongs to $\mathcal{O}_{Z_{k+1}}$ and defines what is called the *torsion* :

$$\tau : \ker d \to \mathcal{O}_{Z_{k+1}}, \quad \tau(\sum \xi_j \otimes F_j) = \sum \bar{D}_j F_j \tag{25}$$

This is the general form of the map constructed in Prop. 14. In fact, one has:

Lemma 32. *The point $a \in Z_{k+1}$ is the image of a point in Z_{k+2} if and only the torsion morphism τ vanishes at the point a.*

Proof. Let me give the simple proof from [Malg05, p. 38]. Let $a \in Z_{k+1}$. For any $f \in \mathcal{O}_a(J_{k+1})$ be in the ideal of Z_{k+1} write $D_j g = D'_j g + D''_j g$ where

$$D''_j g = \frac{\partial g}{\partial y^{\underline{\alpha}}} y^{\underline{\alpha}+1_j}.$$

Take any system of generators $(f_1, \ldots, f_N)$ at a of the ideal defining Z_{k+1}. To find $b \in Z_{k+2}$ projecting to a one needs to solve the linear system in the variables $y^{\underline{\beta}}$ given by $D'_j f_m(a) + D''_j f_m(a) = 0$, $j = 1, \ldots, n$, $m = 1, \ldots, N$. Write this system as $AY = B$ (with $B \neq 0$) where Y is the column vector of the $y^{\underline{\beta}}$, AY represents the linear form $D'_j f_m(a)$ and B the length N vector $D''_j f_m$. Such a system has a solution if and only if $b \in \mathrm{Im}(A) = [\ker {}^T A]^\perp$. Dually, this means that $\ker {}^T A$ should be contained in the hyperplane $B^\perp$, i.e. for any row vector z with $zA = 0$ one should have $zB = 0$. Translating this back in the present situation one finds that it suffices to show that any relation $\sum_{jm} c_{jm} D''_j f_m(a) = 0$ should imply $\sum_{jm} c_{jm} D'_j f_m(a) = 0$.

Put $g_j = \sum_m c_{jm} f_m \in \mathcal{I}_a$ and $F = \sum \xi_j \otimes \delta g_j \mod \mathcal{I}_a$. Then the relation $\sum_{jm} c_{jm} D''_j f_m(a) = 0$ just means that $dF|_a = 0$ and hence F belongs to the kernel of the Koszul map d at a and then the second relation $\sum_{jm} c_{jm} D'_j f_m(a) = 0$ means that $\tau(F)|_a = 0$. So a solution to the system exists over a if and only if the torsion morphism at that point vanishes as desired. $\square$

Finally one invokes a result which in fact generalizes Prop. 14:

Lemma 33. [Malg05, p. 39] *The map τ factors over the image of d, i.e. at every point $\mathbf{u}_{k+1} \in Z_{k+1}$ there is a well defined map*

$$\tau|_{\mathbf{u}_{k+1}} : H_{1,k+1}(\mathcal{J}) = \ker d/\mathrm{im}\ d \to \mathbb{C}.$$

Proof. Let $G = \sum_{i<j} \xi_i \xi_j \otimes \delta g_{ij} \in \Lambda^2 W \otimes \mathcal{J}_{k+1}$ with $g_{ij} \in [\mathfrak{J}_{k+1}]_a$. Then $dG = \sum \xi_i \otimes \delta[D_j g_{ij}] - \xi_j \otimes \delta[D_i g_{ij}]$. Now, by definition (see (25)) one has $\tau(dG) = \sum \bar{D}_i(\delta[D_j g_{ij}] - \sum \bar{D}_j \delta[D_i g_{ij}]$, but $\bar{D}_\ell(\delta(g)$ is represented by $D_\ell(g))$ (by the definition of $\bar{D}_\ell$) and so $\tau(dG)$ is represented by $\sum D_i D_j g_{ij} - \sum D_j D_i g_{ij} = 0$ □

End of the proof of Theorem 28. Since $\mathcal{M}_k$ is k-regular, the ideal $\mathcal{J}$ is $(k+1)$-regular which means that $H_{p,q}(\mathcal{J}) = 0$ for all p and all $q \geq k+1$. So lemma 33 implies that the torsion vanishes everywhere on Z_{k+1} and hence by lemma 32 every point is in the image of the projection p_{k+2}. Next continue the induction procedure with the higher order prolongations. □

References

[L-S] Lambe, L. and W. Seiler: Differential equations, Spencer cohomology, and computing resolutions. Georgian Mathematical Journal, 9 (2002) 723–772.

[Malg05] Malgrange, B.: *Systèmes différentiels involutifs*, Panoramas et synthèses - Société mathématique de France (2006).

[Ritt] Ritt, J. F.: *Differential Algebra*, Coll. publ. **33**, Am. Math. Soc. (1966).

[Saun] Saunders, D. J.: *The Geometry of Jet Bundles*, London Math. Soc. Lect. Notes **142**, Cambridge University Press, 1989.