CLASSIFICATION OF
COMPLEX ALGEBRAIC SURFACES

from the point of view of

MORI THEORY

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Preface

These notes are based on courses given in the fall of 1992 at the University of Leiden and in the spring of 1993 at the University of Grenoble. These courses were meant to elucidate the post-Mori point of view on classification theory of algebraic surfaces as briefly alluded to in [P].
The course in Leiden consisted in a full term course for students having mastered the first principles of algebraic geometry and contained much more material than presented here. The latter basically follows the short course on this subject given in Grenoble which was intended for a much more mature audience.
I profitted very much from the criticism of my students on previous drafts and from discussions and remarks from the audience in Grenoble. Especially I thank José Bertin, Jean-Pierre Demailly, Gerardo Gonzalez-Sprinberg, Robert Laterveer, Jan Nagel and Pieter Nijsse.

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1. Generalities on surfaces

In this section I recall a few basic tools for studying divisors and their linear systems on surfaces like Riemann-Roch, the Genus Formula and Kodaira-Vanishing.

Let S be an algebraic surface. For any line bundle $\mathcal{L}$ on S it is customary to put

$$h^i(\mathcal{L}) = \dim\, H^i(S, \mathcal{L})$$

and

$$\chi(\mathcal{L}) = h^0(\mathcal{L}) - h^1(\mathcal{L}) + h^2(\mathcal{L}) \quad \text{(Euler-Poincaré characteristic of } \mathcal{L}).$$

For a divisor D set

$$h^i(D) = h^i(\mathcal{O}(D)).$$

One may introduce intersection numbers (D, E) between divisors D and E either algebraically (see e.g.[Beau]) or topologically via the cup-product, the result is the same ([G-H]). These numbers only depend on the cohomology-classes of the divisors, or, what is the same, their first Chern classes.

Let me recall the basic tool, the Riemann-Roch theorem.

Theorem 1. (Riemann-Roch) *For any divisor D on S one has*

$$\chi(\mathcal{O}_S(D)) = \chi(\mathcal{O}_S) + \frac{1}{2}((D, D) - (D, K_S)).$$

Using Serre duality, one can rewrite the Riemann-Roch theorem as follows.

$$h^0(D) - h^1(D) + h^0(K_S - D) = \chi(\mathcal{O}_S) + \frac{1}{2}((D, D) - (D, K_S)).$$

From this way of writing Riemann-Roch one derives an inequality which will be used a lot in the sequel

$$h^0(D) + h^0(K_S - D) \geq \chi(\mathcal{O}_S) + \frac{1}{2}((D, D) - (D, K_S)).$$

The strong form of the Riemann-Roch theorem, also called Hirzebruch-Riemann-Roch theorem, expresses $\chi(\mathcal{O}_S)$ in topological invariants of S. For algebraic surfaces this goes back to Noether and therefore is called the Noether formula. It reads as follows

$$\chi(\mathcal{O}_S) = \frac{1}{12}((K_S, K_S) + e(S)).$$

That indeed the self intersection of K_S is a topological invariant follows since it equals the self-intersection of the first Chern class of the surface, which is a topological invariant. Let me refer to [G-H, Chapter 4.6] for a geometric proof of the Noether formula. For the general Riemann-Roch theorem for projective manifolds I refer to [Hir].

The Riemann-Roch theorem is also valid for line bundles $\mathcal{L}$ on any surface, even if these are not of the form $\mathcal{O}(D)$. Clearly, even to make sense of the Riemann-Roch formula, one needs to use here the topological definition of the intersection product.

Next, let me give a formula for the genus of an irreducible curve on a surface.

Lemma 2.

1. *For any effective divisor D on a surface S one has*

$$-\chi(\mathcal{O}_D) = \frac{1}{2}((D, D) + (D, K_S)).$$

2. (Genus formula) *For an irreducible curve C with genus $g = \dim\, H^1(\mathcal{O}_C)$ one has*

$$2g - 2 = (K_S, C) + (C, C).$$

Proof: There is an exact sequence

$$0 \to \mathcal{O}_S(-D) \to \mathcal{O}_S \to \mathcal{O}_D \to 0,$$

which implies $-\chi(\mathcal{O}_D) = -\chi(\mathcal{O}_S) + \chi(\mathcal{O}_S(-D)) = \frac{1}{2}((D,D)+(D,K_S))$ by Riemann-Roch. If D is irreducible, $h^0(\mathcal{O}_D) = 1$ and hence $2g(D) - 2 = -2\chi(\mathcal{O}_D)$. ∎

Remark 3. The genus g as defined above for a singular curve C is related to the genus $\tilde{g}$ of its normalisation $\nu : \tilde{C} \to C$ as follows. There is an exact sequence

$$0 \to \mathcal{O}_C \to \nu_* \mathcal{O}_{\tilde{C}} \to \Delta \to 0,$$

where $\Delta = \oplus \Delta_x$ is a sky-skraper sheaf concentrated in the singular points x of C. Taking the Euler-characteristics one gets

$$g(C) = g(\tilde{C}) + \sum_x \dim \Delta_x.$$

The important consequence is that

$$g(C) \geq g(\tilde{C}) \text{ with equality if and only if } C \text{ is smooth.}$$

A theorem which is often used in conjunction with the Riemann-Roch Theorem is the Kodaira Vanishing Theorem (see [We, Chapt. VI, §2]):

Theorem 4. (Kodaira Vanishing) *Let L be an ample line bundle on a projective manifold M then*

$$H^q(M, \mathcal{O}(K_M + L)) = 0 \quad \text{if } q > 0.$$

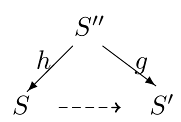
2. Birational geometry of surfaces

The results in this section deal with the study of birational maps between surfaces. It is classical that these can be described by means of blowings up of points. The latter are recognised by means of Castelnuovo's contraction criterion. These classical results are just stated without proof. The reader may consult [Beau]. The section ends by discussing the link with Mori-theory.

Theorem 1. *Any birational morphism between surfaces is the composition of a sequence of blowings up and isomorphisms.*

Since the indeterminacy locus of a rational map can be gotten rid of by successive blowing up points, one has the following structure theorem for biratonal maps.

Corollary 2. *Every birational map $S \dashrightarrow S'$ between surfaces fits into a commutative diagram*



with h and g a composition of blowings up and isomorphisms.

For the purpose of reducing the birational classification to a biregular classification, the previous theorem is important. One introduces the following basic definition, which underlines this.

Definition 3. *A surface S is minimal if every birational morphism $S \to S'$ is an isomorphism*

The previous theorem then shows that every surface can be mapped to a miminal surface by a birational morphism. Indeed, if S is not minimal, there is some surface S' and a birational morphism $S \to S'$ which, by the previous theorem is a sequence of blowings up and isomorphisms. Since under a blowing up the rank of the Néron-Severi-group increases by one, this process must terminate. It follows moreover, that on a non-minimal surface there must be exceptional curves for some σ-process. These are smooth rational curves with self intersection (-1). Let me call such curves (-1)-curves. These are always exceptional curves for a blowing up by the following theorem.

Theorem 4. (Castelnuovo's contraction criterion). *A smooth rational curve E on a surface S with $(E, E) = -1$ is the exceptional curve for a σ-process $S \to S'$.*

Corollary 5. *A surface is minimal if and only if it does not contain (-1)-curves.*

I end this section with a few remarks which are intended to illustrate the point of view of birational geometry since Mori theory came into existence.

As demonstrated previously, for surfaces there always exists some minimal model in the birational equivalence class of a given surface. In principle there could be many minimal models. It turns out that, with the exception of the ruled surfaces there is a unique minimal model up to isomorphism. By definition a *ruled surface* is any surface which admits a birational map onto $C \times \mathbb{P}^1$ with C a curve, so these are known from a birational point of view. However one also needs to know the distinct minimal ruled surfaces. The result is classical and will be described when stating the Enriques classification theorem.

In higher dimensions there need not exist a smooth minimal model. When the concept of minimal model is suitably modified, in order to have such a model one necessarily has to allow singularities in codimension ≥ 3. It turns out that you can only expect a minimal model if K, the canonical divisor is *nef* which means that K intersects non-negatively with any curve. Mori theory also shows that there is a basic distinction between the case K nef and K not nef. I shall illustrate this for surfaces.

Proposition 6. *If there exists a curve C on S with $(K_S, C) < 0$ and $(C, C) \geq 0$, all plurigenera of S are zero. If S is a surface with at least one non-vanishing plurigenus and C is a curve on S with $(K_S, C) < 0$, the curve C is an exceptional curve of the first kind, i.e. C is a smooth rational curve with $(C, C) = -1$.*

Proof: Let D be an effective pluricanonical divisor. Write it like $D = aC + R$. Since $(D, C) < 0$ the divisor D actually contains C, i.e. $a > 0$. Then $0 > m(K_S, C) = (D, C) = a(C, C) + (R, C) \geq a(C, C)$. Since this is ≥ 0 in the first case, one arrives at a contradiction: the plurigenera must all vanish. In the second case, if $(K_S, C) \leq -2$ the adjunction formula gives $(C, C) \geq 0$ and we again have a contradiction. So $(K_S, C) = -1$ and the adjunction formula shows that C is an exceptional curve of the first kind. ■

Recall that the Kodaira-dimension $\kappa(S)$ of S is equal to $-\infty$ means that all plurigenera of S vanish. This is for instance the case for rational and ruled surfaces. So using the notion of nef-ness and Kodaira-dimension there is a reformulation à la Mori for the previous Proposition.

Reformulation 7. *Suppose S is a surface whose canonical bundle is not nef. Then either S is not minimal or $\kappa(S) = -\infty$.*

Let me give a second illustration of the Mori-point of view with regards to the question of uniqueness of the minimal model.

Proposition 8. *Let S and S' be two surfaces and let $f : S \dashrightarrow S'$ be a birational map. If $K_{S'}$ is nef, f is a morphism. If moreover K_S is nef, f is an isomorphism.*

Proof: Let $\sigma : \tilde{X} \to X$ be the blow up of any surface with exceptional curve E and let $\tilde{C} \subset \tilde{X}$ an irreducible curve such that $C := \sigma(\tilde{C})$ is again a curve, one has

$$(K_{\tilde{X}}, \tilde{C}) = (\sigma^* K_X + E, \sigma^* C - mE) = (K_X, C) + m \geq (K_X, C).$$

So if K_X is nef there can be no curve $\tilde{C}$ on $\tilde{X}$ mapping to a curve on X and for which $(K_{\tilde{X}}, \tilde{C}) \leq -1$. Since any morphism is composed of blowings up this then also holds for an arbitrary morphism $X' \to X$.

Let me apply this in the present situation with $X = S'$. Resolve the points of indeterminacy of f. Choose a resolution where you need the minimal number of blowings up. One may suppose that one needs at least one blow up. Then the image $C = f(E)$ of the exceptional curve E of the last blow up must be a curve, which contradicts the preceding since $(K, E) = -1$ on the last blown up surface. So f is a morphism. Similarly, if K_S is nef, f^{-1} is a morphism and so f is an isomorphism. ■

3. Other concepts from algebraic geometry

In this section I gather a few technical concepts and theorems whose range is not restricted to the domain of surfaces: the concept of normalisation, the existence of a Stein factorisation and the notion of Kodaira dimension. No proofs are given, more details can be found in [Ue].

Let me start by recalling the notion of normalisation.

Definition 1. *Let X be a variety. A pair (X', f) consisting of a normal variety X' and a morphism $f : X' \to X$ is called a normalisation of X if f is finite and birational.*

Theorem 2. *For any projective algebraic variety X the normalisation exists as a projective algebraic variety. It is unique in the following sence. If $f'' : X'' \to X$ is another normalisation, there exists an isomorphism $\iota : X' \to X''$ with $f'' \circ \iota = f'$.*

Next, I recall a very useful construction, the Stein factorisation.

Theorem 3. (Stein factorisation) *Let $f : X \to Y$ be a surjective morphism between projective varieties. There exists a variety Y', a finite surjective morphism $g : Y' \to Y$ and a morphism $f' : X \to Y'$ with connected fibres such that the following diagram is commutative.*

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & {}_{f'}\searrow \quad \nearrow_{g} & \\ & Y'. & \end{array}$$

If X is normal, then so is Y'.

Next, I recall the notion of Kodaira-dimension.

Let X be a normal projective variety and $\mathcal{L} = \mathcal{O}_X(D)$ with D an effective divisor. Associated to D there is the ring

$$R(X, D) := \oplus_{k \geq 0} H^0(X, \mathcal{L}^{\otimes k})$$

and its homogeneous field of fractions

$$Q(X, D) := \{ \ \frac{s}{t} \mid s, t \in H^0(X, \mathcal{L}^{\otimes k}) \quad k \geq 0 \ \}.$$

Proposition 4. *$Q(X, D)$ is algebraically closed in $\mathbb{C}(X)$. In particular, its transcendence degree is finite and at most equal to* $\dim X$.

Definition 5. *The D-dimension $\kappa(D)$ is the transcendence degree of the field $Q(X, D)$. In the special case when D is a canonical divisor, $\kappa(D)$ is called the Kodaira dimension of X and denoted $\kappa(X)$.*

Remark 6. This definition can be extended to cases where D is not effective. If $H^0(X, \mathcal{L}^{\otimes k}) = 0$ for all $k \geq 0$ you simply set

$$\kappa(X, D) := -\infty.$$

Otherwise you introduce the set $\mathbb{N}(D) \subset \mathbb{N}$ of natural numbers k for which $\mathcal{L}^{\otimes k}$ does have sections and restrict the preceding discussion to the rational maps $\varphi_{\mathcal{L}^{\otimes k}}$ for $k \in \mathbb{N}(D)$. The definition is then easily modified.

Let me now relate the field $Q(X, D)$ to the geometry of the rational maps

$$\varphi_{\mathcal{L}^{\otimes k}} : X \dashrightarrow \mathbb{P}^{N_k}, \quad N_k + 1 = \dim\, H^0(\mathcal{L}^{\otimes k}).$$

Let W_k be the image of $\varphi_{\mathcal{L}^{\otimes k}}$, i.e., the closure in $\mathbb{P}^{N_k}$ of the image of the maximal subset of X on which $\varphi_{\mathcal{L}^{\otimes k}}$ is a morphism. In terms of a basis $\{ \ s_0, \ldots, s_{N_k} \ \}$ for the sections of $H^0(X, \mathcal{L}^{\otimes k})$ the function field of W_k is given by

$$\mathbb{C}(W_k) = \mathbb{C}(s_1/s_0, s_2/s_0, \ldots, s_{N_k}/s_0).$$

So the union of these fields is $Q(X, D)$.

The following proposition gives an equivalent definition for the Kodaira-dimension.

Proposition 7. $\kappa(X, D) = \max \dim W_k$.

Whereas this Proposition is fairly straightforward, the following characterisation of Kodaira-dimensions is much deeper.

Theorem 8. (Characterisation of the D- dimension) *Let X be a normal projective variety and let D be an effective divisor on X with D-dimension equal to κ. There exist positive numbers α and β such that for all sufficiently large k one has*

$$\alpha k^{\kappa} \leq \dim H^0(X, \mathcal{O}_X(kD)) \leq \beta k^{\kappa}.$$

Finally I state as a proposition how the the Kodaira dimension behaves under étale coverings. This will be needed later.

Proposition 9. *Let $f : X \to Y$ be an unramified covering between smooth projective varieties. Then $\kappa(X) = \kappa(Y)$.*

4. The Néron-Severi group

The purpose of this section is to rephrase well known theorems about ampleness of divisors, such as Nakai's criterion, in Mori's framework of the cone of nef-divisors.

The Néron-Severi group $\mathrm{NS}\, M$ of a smooth projective manifold of M is the group of isomorphism classes of divisors modulo homological equivalence, where two divisors are said to be homologically equivalent if they have the same first Chern class. The exponential sequence yields an exact sequence

$$0 \to \mathrm{NS}\, M \to H^2(M, \mathbb{Z}) \xrightarrow{k^*} H^2(\mathcal{O}_M).$$

Now look at the chain of inclusions $\mathbb{Z}_M \xrightarrow{i} \mathbb{C}_M \xrightarrow{j} \mathcal{O}_M$ inducing the triangle

$$\begin{array}{ccc} H^2(M,\mathbb{Z}) & \xrightarrow{k^*} & H^2(\mathcal{O}_M) \\ & {\scriptstyle i^*}\searrow \quad \nearrow{\scriptstyle j^*} & \\ & H^2(M,\mathbb{C}) & \end{array}$$

This diagram explains the meaning of the following classical theorem. For a proof, see e.g. [G-H] or [We].

Theorem 1. (Lefschetz' Theorem on (1,1)-classes) *The Néron-Severi group of a projective manifold consists precisely of the integral classes of Hodge type* $(1, 1)$.

The next topic is the intersection form on the Néron-Severi group of a surface S. Assume that $S \subset \mathbb{P}^n$ and let ω the metric form belonging to the Fubini-Study metric. It is a $(1, 1)$)-form which is pointwise positive definite .

Define

$$H^2_{\mathrm{prim}}(S, \mathbb{Q}) := \{ \ [\alpha] \mid [\alpha \wedge \omega] = 0 \ \} = [\omega]^{\perp}$$

leading to the orthogonal direct sum decomposition

$$H^2(S, \mathbb{Q}) = \mathbb{Q} \cdot [\omega] \oplus H^2_{\mathrm{prim}}(S, \mathbb{Q}).$$

The classical Lefschetz decomposition (c.f. [G-H] or [We]) in the case of surfaces reads as follows.

Theorem 2. *The intersection product is negative definite on $H^2_{\mathrm{prim}}(S, \mathbb{R}) \cap H^{1,1}$.*

Corollary 3. (Algebraic Index Theorem) *The intersection pairing restricts non-degener-ately on* $\mathrm{NS}\, S$ mod torsion *and has signature* $(1, \rho - 1)$, *where $\rho = \operatorname{rank} \mathrm{NS}\, S$ is the Picard number.*

Proof: Note that $[\omega] \cdot [\omega] > 0$. Since by the theorem the intersection product is negative on $[\omega]^{\perp}$, the primitive part of the cohomology, the signature is $(1, h^{1,1} - 1)$ on $H^{1,1}$. So it either restricts non-degenerately with the stated signature or it is semi-negative (with rank one annihilator) on the Néron-Severi group. Since the latter always contains the class of an ample divisor this last possibility is excluded. ■

Remark 4. It follows that two divisors D and D' are *torsion equivalent*, i.e. homologically equivalent up to torsion if and only if they are *numerically equivalent*, i.e. $c_1(D) = c_1(D')$ mod torsion if and only if $(D, E) = (D', E)$ for all divisors E.

Remark 5. A very useful alternative formulation of the Algebraic Index Theorem runs as follows

> If D is a divisor with $(D, D) > 0$ and $(C, D) = 0$ then $(C, C) \leq 0$ with equality if and only if C is numerically equivalent to zero.

From the Algebraic Index Theorem it follows that the intersection pairing on the real vector space $N_{\mathbb{R}}(S) := \mathbb{NS}\, S \otimes \mathbb{R}$ has signature $(1, \rho - 1)$. Such quadratic forms have special properties. There is the *light cone* $x \cdot x = 0$ with disconnected interior $C^+(S) \amalg -C^+(S) = \{ x \in N_{\mathbb{R}}(S) \mid x \cdot x > 0 \}$. Each connected part is convex.

Recall that the dual of a cone C in a real vector space V with non-degenerate product is the cone

$$C^\vee := \{ y \in V \mid y \cdot x \geq 0 \quad \text{for all } x \in C \}.$$

If $x \neq 0$ is on the light cone and in the closure of $C^+(S)$, the dual of the half-ray $\mathbb{R}_{\geq 0} \cdot x$ is the half-space bounded by the hyperplane through this ray, tangent to the light cone and containing $C^+(S)$. The intersection of all such half spaces is the closure of $C^+(S)$. Using convexity it follows that the closed cone $\overline{C^+(S)}$ is self dual.

To study divisors inside the light cone, one uses Riemann-Roch.

Proposition 6. *If for a divisor D on a surface one has $(D, D) > 0$, then $(D, H) \neq 0$ for any ample divisor H. If $(D, H) > 0$ some positive multiple of D is effective and if $(D, H) < 0$, some negative multiple of D is effective.*

Proof: The first assertion follows from the Algebraic Index Theorem.

The Riemann-Roch inequality shows that $h^0(mD) + h^0(-mD + K_S) \geq \frac{1}{2} m^2 (D, D) +$ linear term in m. If $(D, H) > 0$, there can be no divisor in $|-mD + K_S|$ for m large and so $|mD|$ must contain effective divisors for m large enough. The proof of the second assertion is similar. ■

Since the effective divisors are all on the same side of the hyperplane defined by an ample divisor it follows from the preceding proposition that only one component of the interior of the light cone can contain effective divisors. Let me once and for all choose it to be $C^+(S)$. Let me also speak of $\mathbb{Q}$-*divisors* as a formal linear combination of irreducible curves with rational coefficients. Similarly one can speak of $\mathbb{Q}$-divisor classes, the rational points in $\mathbb{NS}(S) \otimes \mathbb{R}$. Such a class is called *effective* if a positive multiple can be represented by an effective divisor. Explicitly, a $\mathbb{Q}$-divisor class $[D]$ is effective if and only if there is an integer $n > 0$ such that there is an effective divisor numerically equivalent to nD. From the preceding Proposition it follows that for divisors with positive self-intersection in this definition one can replace "numerically equivalent" by "linearly equivalent", i.e. effectivity is a numerical property for divisors with positive self-intersection.

The preceding theorem now can be conveniently reformulated as follows.

Corollary 7. *The rational points in $C^+(S)$ are effective $\mathbb{Q}$-divisors.*

In general, there are more effective divisors in $\mathbb{NS}\, S$ spanning a convex cone $\mathrm{Ef}\, S$ in the real vector space spanned by divisors.

Let me consider the dual cone

$$\mathrm{Nef}\, S := \mathrm{Ef}\, S^\vee = \{ x \in \mathbb{N}_{\mathbb{R}}(S) \mid x \cdot e \geq 0 \quad \forall e \in \mathrm{Ef}\, S \}.$$

Its rational points are the classes of what are called nef-divisors ("numerically effective divisors").

Definition *A divisor D is nef if $(D, C) \geq 0$ for all irreducible curves C.*

The cone $\mathrm{Nef}\, S$ therefore is called the *nef-cone*.

Observation 8. *If for a divisor D one has $(D, C) \geq 0$ for all irreducible curves C then $(D, D) \geq 0$.*

Proof: One has $\mathrm{Nef}\, S = \mathrm{Ef}\, S^\vee \subset C^+(S)^\vee = \overline{C^+(S)}$. So $(D, D) \geq 0$ as desired. ■

Next, let me study the ample divisors. A central theorem about ample divisors is the following criterion (see [Ha] for a proof).

Theorem 9. (Nakai-Moishezon) *A divisor D on a surface S is ample if and only if $(D, D) > 0$ and $(D, C) > 0$ for all irreducible curves C.*

The following proposition is a useful consequence.

Proposition 10. *L is ample if and only if $(c_1(L), c) > 0$ for all $c \in \overline{\mathrm{Ef}\, S} \setminus \{ 0 \}$*

Proof: If L is ample, $(L, D) > 0$ for all effective divisors D and so $(c_1(L), c) \geq 0$ for all c in the closure of the effective cone. If $(c_1(L), c) = 0$ for some c in this closure and $c \neq 0$ one can find an effective $C' \in \mathrm{Pic}(S)$ with $(c, c_1(C')) < 0$ and then $(c_1(L^{\otimes n} \otimes C'), c) < 0$. On the other hand $L^{\otimes n} \otimes C'$ will be ample for n large enough by Nakai-Moishezon (for at worst finitely many of components D of C' you will have $(D, C') < 0$ and these can be taken care of by making n large enough). This is a contradiction and so $(c_1 L), c) > 0$.

Conversely, by the Nakai-Moishezon criterion, one only has to show that $(L, L) > 0$. Fix some ample line bundle H and consider the function $f(c) = (c_1(L), c)/(H, c)$ which is constant under homotheties and so to study its values one can restrict to the (compact) closure of $\mathrm{Ef}\, S$ in the unit ball with respect to some metric on the real vector space $N_{\mathbb{R}}(S)$. It has a positive (rational) maximum ϵ and so $(L - \frac{1}{2}\epsilon H, c) > 0$ for all $c \in \mathrm{Ef}\, S$ and in particular $L - \frac{1}{2}\epsilon H$ is nef and so has non-negative selfintersection. But then $(L, L) = (L - \frac{1}{2}\epsilon H, L - \frac{1}{2}\epsilon H) + \epsilon(H, L - \frac{1}{2}\epsilon H) + \frac{1}{4}\epsilon^2(H, H) > 0$. ■

Corollary 11. *The cone consisting of ample $\mathbb{Q}$-divisors forms an open subset in $\mathrm{NS}\, S \otimes \mathbb{Q}$ and its closure is the nef-cone.*

Proof: If H is ample and D any divisor $(H + tD, c) > 0$ for c in the closure $\mathcal{C}$ of $\mathrm{Ef}\, S$ in the unit ball in some metric on $N_{\mathbb{R}}(S)$ and for $|t| < t_0$ with t_0 the smaller of the minima of the two functions $f(c) = (-D, c)/(H, c)$ on $\mathcal{C} \cap \{(D, c) \leq 0\}$ and $g(c) = (H, c)/(D, c)$ on $\mathcal{C} \cap \{(D, c) \geq 0\}$. By the proposition $H + tD$ is ample for these values of t.

Conversely, by the Proposition, one has $(a, c') \geq 0$ for all $c' \in \overline{\mathrm{Ef}\, S}$. But this is the case precisely when $(a, c) \geq 0$ for all $c \in \mathrm{Ef}\, S$, i.e. when a is nef. ■

5. Rationality theorem and applications

In this section I prove, following [Wi], the rationality theorem for the Néron-Severi group of surfaces and give a few striking applications to classification theory of surfaces such as Castelnuovo's rationality theorem

Let me recall that the Néron-Severi group $\mathrm{NS}\, S$ is the group of divisor classes modulo homological equivalence on S. The cup product on the real vector space $\mathrm{NS}\, S \otimes \mathbb{R}$ makes it into a self dual vector space. So you may view a divisor either as giving a class in $\mathrm{NS}\, S$ or as giving a hyperplane in $\mathrm{NS}\, S \otimes \mathbb{R}$. One has the real cone $\mathrm{Ef}\, S$ of effective divisors (with real coefficients) whose dual is called the cone of nef-divisors and denoted by $\mathrm{Nef}\, S$. So a divisor D is nef if and only if the cone $\mathrm{Ef}\, S$ is on the non-negative side of the hyperplane which D defines. The cone Nef is a closed cone whose integral points in the interior are the classes of the ample divisors. So H is ample if and only if $(H, H) > 0$ and $\mathrm{Ef}\, S \setminus 0$ is on the positive side of the hyperplane defined by H. If some D is not nef the hyperplane it defines will have some part of $\mathrm{Ef}\, S$ on its negative side and in the pencil $H + sD$ there will be a smallest value for wich the resulting hyperplane no longer has Ef on the positive side. The rationality theorem says that for $D = K_S$ this happens for a rational value. This theorem has surprizingly many consequences for the classification of surfaces as you will see.

Theorem 1. (Rationality theorem) *Let S be a surface and let H be very ample on S. Assume that K_S is not nef. Then there is a rational number b such that the hyperplane corresponding to $H + bK_S$ touches the cone* $\mathrm{Ef}\, S$.

Proof: (See [Wi]) Introduce

$$b := \sup\{ \ t \in \mathbb{R} \mid H_t = H + tK_S \text{ is nef } \ \}.$$

Set

$$P(v, u) := \chi(vH + uK_S).$$

By Riemann-Roch this is a quadratic polynomial in v, u. If u and v are positive integers with $(u - 1)/v < b$ the divisor $vH + (u - 1)K_S$ is ample and so by Kodaira Vanishing (Appendix A3) $H^i(vH + uK_S) = 0$ for $i = 1, 2$. It follows that $P(v, u) \geq 0$.

Assume now that b is irrational. Number theory ([HW, Theorem 167]) implies that b can be approximated by rational numbers of the form p/q, p and q arbitrarily large integers in such a way that

$$p/q - 1/(3q) < b < p/q.$$

The polynomial $P(kq, kp)$ is quadratic in k. If it is identically zero, $P(v, u)$ must be divisible by $(vp - uq)$. Taking p and q sufficiently large one may assume that this is not the case. For $k = 1, 2, 3$ the numbers $v = kq$ and $u = kp$ satisfy $(u - 1)/v < b$ and hence $P(kq, kp) \geq 0$ for these three values of k. Since a quadratic polynomial has at most two zeroes, it follows that for at least one pair of positive integers (v, u) with $t_0 := u/v > b$ one has $\dim H^0(vH + uK_S) > 0$. So there is an effective divisor (with coefficients in $\mathbb{Q}$) $L := H_{t_0} = \sum a_j \Gamma_j$, $a_j > 0$. Now H_{t_0} is not nef. Since L is effective, it can only be negative on the Γ_j. But then one can subtract off a rational multiple of K_S from H_{t_0} to get H_b and so b would be rational contradicting our assumption. ■

Let me give a first application.

Proposition 2. *A minimal algebraic surface with K not nef is either a geometrically ruled surface or $\mathbb{P}^2$.*

Proof: Let me first look at the positive half ray in $\mathrm{NS}\, S \otimes \mathbb{Q}$ spanned by $-K_S$. There are two possibilities. The first possibility is that all ample classes of S are on this line and hence $-K_S$ is ample and $\mathrm{Pic}\, S$ has rank 1. Kodaira-Vanishing implies that $h^0(K_S) = h^1(K_S) = 0$ and so $p_g = q = 0$. It follows that $\mathrm{Pic}\, S \xrightarrow{\cong} H^2(S, \mathbb{Z})$ has rank one. Moreover $b_2 = 1$ and $b_1 = 0$ imply that $e(S) = 3$ and by Noether's Formula one has $(K, K) + 3 = 12(1 - q + p_g) = 12$ and so $(K, K) = 9$. Next, take an ample generator H of $\mathrm{Pic}\, S$ mod torsion and apply Riemann-Roch to H. Note that since $H - K_S$ is ample, Kodaira-Vanishing gives that $h^1(H) = 0 = h^2(H)$ and one finds $h^0(H) = \frac{1}{2}(H, H - K_S) + 1 = 3$. Indeed, since $(K, K) = 9$, K must be numerically equivalent to $-3H$. One gets a dominant rational map $f : S \dashrightarrow \mathbb{P}^2$ which maps H to the class of a line. Now $(H, H) = \frac{1}{9}(K_S, K_S) = 1$ implies that $|H|$ can have no fixed points and that f is birational (why?). Now f cannot contract any curves to points, since $\mathrm{Pic}\, S$ has rank 1. From the discussion about birational geometry it follows that f must be biregular and so S is isomorphic to $\mathbb{P}^2$.

So one may assume that there exists an ample H such that its class in $\mathrm{NS}\, S \otimes \mathbb{Q}$ does not belong to the positive half-ray spanned by $-K_S$. Now apply the rationality theorem to H and K_S.

Clearing denominators one finds a divisor

$$L = vH + uK_S,\ b = u/v = \sup\{\ t \in \mathbb{R} \mid H_t = H + tK_S \text{ is nef }\ \}.$$

Now L belongs to the closure of the nef-cone, which- as shown before- is itself closed. So L is a nef divisor and so in particular, $(L, L) \geq 0$ (see Observation 4.8). If you subtract any positive rational multiple of K_S from L you come into the interior of the nef-cone, which is the ample cone. So $mL - K_S$ is ample for all $m \geq 1$. Serre duality implies that $\dim H^2(mL) = \dim H^0(-(mL - K_S)) = 0$ and so by Riemann-Roch

$$\dim H^0(mL) \geq \chi(mL) = \chi(S) + \frac{1}{2}(mL, mL - K_S).$$

One can distinguish two cases, namely $(L, L) > 0$ or $(L, L) = 0$.

i) $(L, L) > 0$. Since L is nef, for any effective divisor, one has $(L, D) \geq 0$. The equality sign can be excluded as follows. Any irreducible curve D for which $(L, D) = 0$ must be an exceptional curve of the first kind. Indeed, from the definition of L one sees that $(K_S, D) < 0$, while the Algebraic Index Theorem applied to L and D shows that $(D, D) < 0$. In combination with the adjunction formula this shows that D has to be an exceptional curve of the first kind. By assumption these don't exist and so $(L, D) > 0$ for all curves D and so, by the Nakai-Moishezon criterion, L is ample, which is impossible by construction (L is on the boundary of the nef-cone).

ii) $(L, L) = 0$. Since L is nef one has $(L, H) \geq 0$, and if $(L, H) = 0$ an application of the Algebraic Index Theorem shows that L is numerically trivial. In this last case, the class of H in $\mathrm{NS}\, S \otimes \mathbb{Q}$ would be on the positive half-ray spanned by the class of $-K_S$, which has been excluded. So $(L, H) > 0$. From $0 = 1/v(L, L) = (L, H + bK_S)$ one infers that $(L, K_S) < 0$ and so $\dim H^0(mL)$ grows like a linear function of m. You may replace L by mL and assume that $\dim |L| \geq 1$. Now write $L = L' + L_{\text{fixed}}$, where L_{fixed} is the fixed part of $|L|$. I claim that L' is still nef and that still $(L', L') = 0$. The first is clear since L' moves in a linear system. So $(L', L) \geq 0$ and $(L', L_{\text{fixed}}) \geq 0$. From

$$0 = (L, L) = (L', L) + (L_{\text{fixed}}, L) \geq 0$$

one infers $(L', L) = (L_{\text{fixed}}, L) = 0$, while

$$0 = (L', L) = (L', L') + (L', L_{\text{fixed}}) \geq 0$$

implies that $(L', L') = 0$. Moreover, for every irreducible component D of $|L'|$ the equality $(L, L') = 0$ implies that $(L, D) = 0$ and since $(L', L_{\text{fixed}}) = 0$ one also has $(D, L_{\text{fixed}}) = 0$. So $(L', D) = 0$ and from this you easily see that $(D, D) \leq 0$. By definition of L from the equality $(L, D) = 0$ one concludes that $(D, K_S) < 0$. The Adjunction Formula then implies that D is a smooth rational curve with $(D, D) = 0$.

The same reasoning applies to *any* linear subsystem of $|L|$ which has no fixed part. You can for instance take a one-dimensional subsystem of $|L|$, take off the fixed part and end up with a pencil $\mathbb{P}$ without fixed components and with $(F, F) = 0$ for every $F \in \mathbb{P}$. By the preceding discussion every irreducible component of a member of $|F|$ is a smooth rational curve.

Since $(F, F) = 0$ there can be no fixed points and so one gets a morphism $f : S \to \mathbb{P}^1$. Now by taking the Stein factorization of f (see §3) one obtains a fibration $f' : S \to C$ of S onto a curve whose fibres are smooth rational curves. So S is a geometrically ruled surface. ■

Corollary 3. (Uniqueness of Minimal Model) *If S, S' are two minimal surfaces which are not ruled then any birational map $f : S' \to S$ is an isomorphism. In particular, any surface which is not ruled or rational has a unique minimal model.*

Proof: This follows from the previous theorem and Proposition 2.8 ■

Let me give an application of which the full strength will be shown in the next sections.

Proposition 4. *Let K_S be nef. There are the following possibilities for S.*

1. *$(K_S, K_S) > 0$. Then $P_m \geq \frac{1}{2}m(m-1)(K_S, K_S) + 1 - q + p_g$ for $m \geq 2$ and always $P_2 > 0$.*
2. *$(K_S, K_S) = 0$, $q = 0$. Then $P_2 > 0$.*
3. *$(K_S, K_S) = 0$, $p_g > 0$ and $q > 0$.*
4. *$(K_S, K_S) = 0$, $p_g = 0$, $q = 1$ and $b_2 = 2$.*

Proof: Observe that nefness of K_S implies that $(K_S, K_S) \geq 0$. Now you only have to prove the following three assertions:

i. If $(K, K) > 0$ the stated bound for P_m is valid and $P_m > 0$.
ii. If $(K, K) = 0$, $p_g = q = 0$ implies $P_2 > 0$.
iii. If $(K, K) = 0$, $p_g = 0$ and $q > 0$ one has $b_2 = 2$.

If $p_g > 0$, clearly $P_m > 0$ for all $m \geq 1$ so to, prove that $P_2 > 0$ it suffices to look at the case $p_g = 0$.

So let me first consider the case $p_g = 0$. Noether's formula in this case reads

$$12(1 - q) = (2 - 4q + b_2) + (K_S, K_S).$$

So $b_2 = 10 - 8q - (K_S, K_S) \geq 1$ implies that $q \leq 1$.

If $q = 1$ and $(K_S, K_S) = 0$ one must have $b_2 = 2$ and this is case 4. This already proves iii.

In the remaining cases one either has $q = 1$, $(K, K) \geq 1$ or $q = 0$ which makes the right hand side of the Riemann-Roch inequality for mK_S positive in all cases:

$$h^0(mK_S) + h^0(-(m-1)K_S) \geq \frac{1}{2}m(m-1)(K_S, K_S) + 1 - q + p_g.$$

In particular, $P_m \geq \frac{1}{2}m(m-1)(K_S, K_S) + 1 - q + p_g$ as soon as $H^0(-((m-1)K_S)) = 0$. Therefore, to prove i. and ii. I only need to see that $H^0(-((m-1)K_S)) = 0$ if $m \geq 2$. This is an immediate consequence of the following Lemma. ■

Lemma 5. *Let L be a nef line bundle on a surface S such that L^{-1} has a section. Then L is trivial.*

Proof: Suppose L (and hence L^{-1}) is not trivial. Then there would exist a section of L^{-1} vanishing along a divisor and any curve C transversal to this divisor would satisfy $-(L, C) > 0$ which contradicts the nefness of L. ■

Corollary 6. (Castelnuovo's Rationality Criterion) *A surface is rational if and only if $P_2 = q = 0$.*

6. Statement of the Enriques Classification Theorem

In this section I give many examples of all possible classes of surfaces, and at the same time introduce the basic vocabulary for the Enriques classification, which then is stated at the end of this section, deferring proofs to two final sections

Let me first introduce a number of useful concepts and illustrate these by examples.

Definition 1.
1. *A surface S is called a ruled surface if it is birationally isomorphic to $C \times \mathbb{P}^1$ where C is a smooth curve. If $C = \mathbb{P}^1$ one calls S rational.*
2. *A surface S is called geometrically ruled if there is a morphism $p : S \to C$ of maximal rank onto a smooth curve with fibres $\mathbb{P}^1$.*

Two remarks are in order. First, a surface is rational if and only if it is isomorphic to $\mathbb{P}^2$ since, as one can easily verify, $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ are birationally isomorphic. Secondly, it is by no means clear that a geometrically ruled surface is actually ruled. This however is the case. In fact a little more is true.

Proposition 2. *If $f : S \to C$ is a surjective morphism of a surface S onto a curve C for which $c \in C$ is a regular value and whose fibre at c is isomorphic to $\mathbb{P}^1$, then there is a Zariski-open neighbourhood U of c in C such that $f^{-1}U$ is isomorphic to $U \times \mathbb{P}^1$ in a fibre preserving manner.*

For a proof of this proposition and the following results concerning ruled surfaces see [Beau].

Proposition 3. *A minimal model of a non-rational ruled surface is geometrically ruled. Every geometrically ruled surface $S \to C$ is isomorphic to the $\mathbb{P}^1$-bundle associated to some rank two vector bundle E on C. Two bundles $\mathbb{P}(E)$ and $\mathbb{P}(E')$ are isomorphic if and only if $E' \cong E \otimes L$ for some line bundle L on C.*

The rational ruled surfaces are those which are fibred over $\mathbb{P}^1$. The minimal models are to be found among the *Hirzebruch surfaces*

$$\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)).$$

Indeed, one has:

Proposition 4. *A geometrically ruled surface over $\mathbb{P}^1$ is a Hirzebruch surface. Any two Hirzebruch surfaces are pairwise non-isomorphic. For $n \neq 1$ these are precisely the minimal geometrically ruled surfaces with $q = 0$.*

Definition 5. *Let S be a surface and C a smooth (projective) curve. A morphism $f : S \to C$ is called a fibration if f is surjective and has connected fibres. If the generic fibre (which is a smooth projective curve) has genus g, the fibration f is called a genus-g fibration. A genus-1 fibration is also called an elliptic fibration. Any surface admitting an elliptic fibration is called an elliptic surface.*

Examples 6.

1. The easiest example is of course a product $C \times F$ of two smooth curves, which is a fibration in two ways. Let me calculate the invariants. One easily sees that $q(C \times F) = g(C) + g(F)$, the sum of the genera of the factors and that $p_g(C \times F) = g(C) \cdot g(F)$. In a similar way one finds $P_n(C \times F) = P_n(C) \cdot P_n(F)$. So this gives examples of Kodaira-dimensions $-\infty$ (one of the factors $\mathbb{P}^1$), 0 (both factors elliptic), 1 (one factor elliptic and one of genus ≥ 2) or 2 (both factors of genus ≥ 2).

2. Another type are the fibre bundles over a smooth curve C with fibre F and structure group $\operatorname{Aut} F$, the group of biholomorphic automorphisms of F. You construct them by covering C by Zariski-open sets U_j and glueing the union $U_j \times F$ by means of transition functions $U_i \cap U_j \to \operatorname{Aut} F$.

Concrete examples are the quotients of a product of two curves, $C' \times F'$ by a finite group G, where G is a group of automorphisms of C' and F'. Even more concretely, one may take $G = \mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z}$ as a subgroup of translations of some elliptic curve F' and construct a ramified Galois-cover $C' \to \mathbb{P}^1$ ramified in three points with covering group G. (I leave the construction of such coverings as an exercise.) This yields $C' \times F'/G$, at the same time a fibration over $\mathbb{P}^1$ with fibre F' and a fibration over F'/G with fibres C'.

To compute the invariants of $S = C' \times F'/G$ note that

$$\begin{aligned} H^0(S, \Omega^1) &= H^0(C' \times F', \Omega^1)^G = H^0(C', \Omega^1)^G \oplus H^0(F', \Omega^1)^G \\ H^0(S, K_S^{\otimes m}) &= H^0(C' \times F', K_{C' \times F'}^{\otimes m})^G = H^0(C', K_{C'}^{\otimes m})^G \otimes H^0(F', K_{F'}^{\otimes m})^G. \end{aligned}$$

Now one uses a special case of Hurwitz formula for mappings $f : C' \to C$ between curves. Recall that locally f is given by $z \mapsto w = z^e$ and e is the ramification index and it equal to 1 except for finitely many points, the ramification points R_j with corresponding e_j. The divisor $R = \sum_j (e_j - 1)R_j$ is the ramification-divisor and Hurwitz formula (see e.g. [Ha] or [Beau]) states

$$K_{C'} = f^* K_C \otimes \mathcal{O}(R).$$

If C is the quotient of C' by a group G acting on C', the group-action forces the ramification to be the same on all points of a fibre of $C' \to C = C'/G$. So $R = \sum (e_k - 1) f^{-1}(Q_k)$, where $f^{-1}Q_k$ is a complete fibre above Q_j. Now $f^*Q_k = e_k f^{-1}Q_k$, and hence $R = \sum_k (1 - \frac{1}{e_k})Q_k$. It follows that

$$K_{C'}^{\otimes m} = f^*\Big(K_C^{\otimes m} \otimes \sum_k (1 - \frac{1}{e_k}) \cdot mQ_k\Big).$$

To compute $H^0(C', K_C^{\otimes m})^G$ note that any G-invariant m-canonical holomorphic form comes from an m-canonical meromorphic form on C and the preceding formula then shows that

$$H^0(C', K_{C'}^{\otimes m})^G = H^0(K_C^{\otimes m} \otimes \sum_k [(1 - \frac{1}{e_k}) \cdot m] Q_k),$$

where $[s]$ means the integral part of the number s. For simplicity, let me write

$$R_m(C', G) = \sum_k [(1 - \frac{1}{e_k}) \cdot m] Q_k).$$

Combining the formulas yields

$$\begin{aligned} q(S) =& g(C'/G) + g(F'/G) \\ p_g(S) =& g(C'/G) \cdot g(F'/G) \\ P_m(S) =& h^0(C'/G, K_{C'/G}^{\otimes m} \otimes \mathcal{O}(R_m(C', G))) \cdot h^0(F'/G, K_{F'/G}^{\otimes m} \otimes \mathcal{O}(R_m(F', G))). \end{aligned}$$

3. A special case of the previous case form the so-called bi-elliptic surfaces.

Definition 7. *A surface $S = E \times F/G$, where E and F are elliptic curves, G a group of translations of E acting on F in such a way that $p_g(S) = 0$ is called bi-elliptic.*

By the previous calculation $p_g(S) = 0$ if and only if F/G is a rational curve. It is relatively simple to classify the possibilities for G and F (any E will work). Since G is a translation subgroup of E it must be abelian and as a transformation group of F it then is the direct product $T \times A$ of its subgroup T of translations and the subgroup A consisting of automorphisms of F preserving the origin. Since the product is direct, the points of T must be invariant under A, which strongly restricts the possible T. Furthermore, since F/G is rational, G cannot consist of translations of F only, and so A must be cyclic of order 2, 3, 4 or 6. Since G is a group of translations of E it is either cyclic or a direct sum of two cyclic groups. From these remarks the following list of possibilities is almost immediate:

1a. $G = \mathbb{Z}/2\mathbb{Z}$ with generator acting as the canonical involution $x \mapsto -x$ on F.
1b. $G = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ with one generator acting as in 1 a., while the other generator acts as translation over a point of order 2.
2a. $F = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}i)$ and $G = \mathbb{Z}/4\mathbb{Z}$, the generator acting as multiplication by i.
2b. $F = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}i)$ and $G = \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, one generator acting as before, the other by translation over a point of order 2.
3a. $F = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\rho)$, $\rho = e^{2\pi i/3}$ and $G = \mathbb{Z}/3\mathbb{Z}$, the generator acting as multiplication by ρ.
3b. $F = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\rho)$, $\rho = e^{2\pi i/3}$ and $G = \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$, one generator acting as multiplication by ρ, the other by translation over $(1 - \rho)/3$.
4. $F = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\rho)$, $\rho = e^{2\pi i/3}$ and $G = \mathbb{Z}/6\mathbb{Z}$, with generator acting as multiplication by $-\rho$.

The formulas established in Example 2 can be used to show that first of all $p_g = 0$ and $q = 1$. Then one sees that $P_m \leq 1$ and that $K^{\otimes m} = \mathcal{O}$ where $m = 2$ (in case 1a,b), $m = 4$ (in case 2a,b), $m = 3$ (in case 3a,b), $m = 6$ (in case 4). So the Kodaira-dimension is 0 in all cases.

4. Take a fixed point free linear system $|D|$ on a curve C. Let p and q be the two projections of $C \times \mathbb{P}^2$ onto C and $\mathbb{P}^2$ and consider a generic member S of the the linear system $|p^*D \otimes q^*(3H)|$. By Bertini, S will be smooth. The projection p induces a fibration $S \to C$ with fibres plane cubic curves, i.e. this is an elliptic fibration. The canonical bundle formula shows that $K_S = p^*(K_C \otimes D)$ and hence the Kodaira dimension is 1 whenever $\deg D > -\deg K_C = 2 - 2g(C)$.

Definition 8.

1. *A surface with $q = 0$ and trivial canonical bundle is called a K3-surface.*
2. *A surface with $q = 0, p_g = 0$ and $K^{\otimes 2}$ trivial is called an Enriques surface,*
3. *A complex 2-torus which admits an embedding into a projective space is called an Abelian surface,*
4. *A surface is a surface of general type if its Kodaira-dimension is 2.*

Examples 9.

1. Let me consider complete intersections S of multidegrees $d_1, d_2, \ldots, d_n$ in $\mathbb{P}^{n+2}$. The canonical bundle formula shows that $K_S = \mathcal{O}_S(d_1 + d_2 + \ldots + d_n - n - 3)$. It is easy to see that the only combinations giving trivial K_S are $d_1 = 4$. By Lefschetz theorem on hyperplane sections, S is simply connected so that $q(S) = 0$. So there are three types of K3-surfaces which are complete intersections.

2. To find an Enriques surface S one first observes that K_S gives an element of order exactly 2 in the Picard group (since $p_g = 0$, the canonical bundle cannot be trivial) and so there exists an unramified Galois cover $\tilde{S}$ of degree 2 with $K_{\tilde{S}}$ trivial. Indeed, one may take

$$\tilde{S} = \{ \ s \in \mathcal{K}_S \mid s^{\otimes 2} = 1 \ \},$$

where one considers the total space of K_S and 1 is the global section corresponding with the constant section 1 of the trivial bundle. Noether's Formula expresses $\chi(\mathcal{O}(S))$ as a linear combination of $e(S)$ and (K_S, K_S). The Euler number gets multiplied by the degree of the cover, while the selfintersection number of K_S of course also gets multiplied by the degree. So $\chi(\mathcal{O}_{\tilde{S}}) = 2\chi(\mathcal{O}_S) = 2$ and hence $q(\tilde{S}) = 0$. Conversely, any K3-surface $\tilde{S}$ with a fixed point-free involution i yields an Enriques surface. This one sees as follows. Let $q : \tilde{S} \to \tilde{S}/i = S$ be the natural degree 2 cover. Then for any divisor D on S one has $q_* q^* D = 2D$. This is clear for irreducible curves and it then follows by linearity. In particular $q_* q^* K_S = 2K_S$, but $q^* K_S = K_{\tilde{S}}$ as locally any holomorphic 2-form on $\tilde{S}$ is a lift of a holomorphic 2-form on S and so $2K_S$ is trivial.

Now you construct a fixed point free-involution on a suitable K3 which is an intersection of three quadrics in $\mathbb{P}^5$. Let $X_0, \ldots, X_5$ be homogeneous coordinates on $\mathbb{P}^5$ and consider the intersection $\tilde{S}$ of three quadrics $Q'_j(X_0, X_1, X_2) + Q''_j(X_3, X_4, X_5)$, $j = 1, 2, 3$. For generic choices of Q'_j and Q''_j this intersection $\tilde{S}$ will be a smooth surface. The involution ι given by $(X_0, X_1, X_2, X_3, X_4, X_5) \mapsto (X_0, X_1, X_2, -X_3, -X_4, -X_5)$ has two planes of fixed points: the planes $P_1 = \{ \ X_0 = X_1 = X_2 = 0 \ \}$ and $P_2 = \{ \ X_3 = X_4 = X_5 = 0 \ \}$. They miss $\tilde{S}$ precisely if the three quadrics Q'_1, Q'_2, Q'_3, resp. Q''_1, Q''_2, Q''_3 (considered as quadrics in P_2, resp. P_1) have no point in common. For generic choices of Q'_j and Q''_j this will be the case. So then $\tilde{S}/\iota$ will be an Enriques surface.

After these preparations let me state the Enriques Classification theorem.

Theorem 10. (Enriques Classification) *Let S be a minimal algebraic surface. Then S belongs to one of the following non-overlapping classes:*

1. *($\kappa = -\infty$, $q = 0$) $S = \mathbb{P}^2$, $S = \mathbb{F}_n$, ($n = 0, 2, 3, \ldots$).*
2. *($\kappa = -\infty$, $q > 0$) S a geometrically ruled, surface over a curve of genus > 0.*
3. *($\kappa = 0$, $q = 2, p_g = 1$) S is an Abelian surface,*
4. *($\kappa = 0$, $q = 1, p_g = 0$) S is bi-elliptic,*
5. *($\kappa = 0$, $q = 0, p_g = 1$) S is K3,*
6. *($\kappa = 0$, $q = 0, p_g = 0$) S is Enriques,*
7. *($\kappa = 1$) S is minimal elliptic but NOT $\kappa = 0$ or $\kappa = -\infty$,*
8. *($\kappa = 2$) S is of general type.*

7. Proof of the classification: first reduction

An important part of the classification theorem rests on the following proposition which deals with Case 4. of Proposition 5.4. The proof of this proposition is very technical and will be dealt with in the following sections. In this section, the proof of the Kodaira classification will be reduced it.

Proposition 1. *Suppose S is a surface with K_S nef and $(K_S, K_S) = 0$, $q = 1$ and $p_g = 0$. Then $\kappa(S) = 0$ or 1 and $\kappa(S) = 0$ if and only if S is bi-elliptic.*

Let me give two immediate consequences of this proposition: the characterisation of $\kappa = -\infty$-surfaces and the characterisation of the minimal rational surfaces.

Corollary 2. *For a minimal surface K_S is nef if and only if $\kappa \geq 0$. In particular a surface is ruled (or rational) if and only if $\kappa = -\infty$.*

Proof: If K_S is nef Proposition 5.4 combined with the previous proposition shows that either $P_2 > 0$ so that $\kappa \geq 0$ or the surface is bielliptic and then also $\kappa \geq 0$. Conversely, if K_S is nef, the surface must be minimal by Reformulation 2.7.

This proves the first part of the corollary.

For the second part, one may assume that S is minimal and from the first part one may assume that K_S is not nef. But then S is geometrically ruled or $S = \mathbb{P}_2$. ■

Corollary 3. *Let S be a minimal rational surface. Then $S \cong \mathbb{P}^2$ or $S \cong \mathbb{F}_n$, $n \neq 1$.*

Proof: Since S is rational, $\kappa(S) = -\infty$. So, by Proposition 5.2 S is the projective plane or S is geometrically ruled. In the latter case, since $q(S) = 0$, by Proposition 6.4 the surface S must be a Hirzebruch surface. ■

Next, let me continue the proof of the classification theorem by considering the case of an elliptic fibration.

Theorem 4. *Suppose that S is a surface with K_S nef and $(K_S, K_S) = 0$. Then $\kappa(S) = 0$ or 1. In the last case S admits the structure of an elliptic fibration.*

Proof: By Proposition 7.2 one has $\kappa(S) \geq 0$. Assume that $\kappa(S) \geq 1$. Then for n large enough $|nK_S|$ is at least 1-dimensional. Let D_{f}, resp $|D|$ be the fixed part, resp. the variable part of this linear system.

Claim $(D, D) = (K_S, D) = 0,$

Proof: (of Claim) One has $0 = n(K_S, K_S) = (D_{\mathrm{f}}, K_S) + (D, K_S)$ and since each term is ≥ 0 by nefness of K_S these must vanish. Now $0 = n(D, K_S) = (D, D) + (D, D_{\mathrm{f}})$ and again, each term is non-negative, since D moves and so $(D, D) = 0 = (D, D_{\mathrm{f}})$. ■

The claim implies that the rational map $f = \varphi_{nK_S}$ is a morphism and that $f : S \to C$ maps every divisor $D \in |D|$ to a point and so C is a curve. This is true for all n large enough so that $|nK_S|$ is at most 1-dimensional and hence $\kappa(S) = 1$ in this case. If D is a smooth fibre of f, the adjunction formula says that the connected components are elliptic curves and so, taking the Stein factorisation of f, one obtains an elliptic fibration. ■

Finally, consider the case of Kodaira dimension 0.

Proposition 5. *Suppose that K_S is nef, that $(K_S, K_S) = 0$ and that $\kappa(S) = 0$. Then S is bi-elliptic, an abelian surface, a K3-surface or an Enriques surface.*

Proof: By Proposition 5.4 and Proposition 7.1 you only have to consider the cases $p_g(S) = 0 = q(S)$ and the case $p_g(S) > 0$ (and hence $p_g = 1$). Moreover, if $p_g = 0$ one must have $P_2 = 1$, again by 5.4. Let me first deal with this case. I claim that $P_3 = 0$. If not, then $P_3 = 1$. Let $D_2 \in |2K_S|$ and $D_3 \in |3K_S|$. So $3D_2$ and $2D_3$ are both divisors in $|6K_S|$. Since $P_6 \leq 1$ you must have $3D_2 = 2D_3$ and there must be an effective divisor D with $D_2 = 2D$ and $D_3 = 3D$. Necessarily $D = D_3 - D_2 \in |K_S|$, but $p_g = 0$. So indeed $P_3 = 0$.

Now apply the Riemann-Roch inequality to $3K_S$. One has

$$h^0(3K_S) + h^0(-2K_S) \geq 1$$

and hence $h^0(-2K_S) \geq 1$. Since $P_2 = h^0(2K_S) = 1$ this is only possible if $2K_S$ is trivial. It follows that S is an Enriques surface.

I next suppose that $p_g = 1$. Consider the Noether formula in this case. It reads as follows.

$$12(2 - q(S)) = e(S) = 2 - 4q(S) + b_2$$

and hence $b_2 = 22 - 8q(S)$. So $q(S) = 0,\ 1,\ 2$.

In the first case you have a K3-surface. Indeed the Riemann-Roch inequality applied to $2K_S$ yields $h^0(2K_S) + h^0(-K_S) \geq 2$. In a similar way as in the previous case, I infer from this that K_S is trivial.

I shall exclude the possibility $q(S) = 1$ and show that S is a torus in the remaining case.

Since $q(S) > 0$, you can find a non-trivial line bundle $\mathcal{O}_S(\tau)$ with $\mathcal{O}_S(2\tau) = \mathcal{O}_S$ (any non-trivial 2-torsion point of the torus $\mathrm{Pic}^0(S)$ gives such a line bundle). If $q = 1$, the Riemann-Roch inequality reads

$$h^0(\mathcal{O}_S(\tau)) + h^0(\mathcal{O}_S(K_S - \tau)) \geq 1$$

and hence $h^0(\mathcal{O}_S(K_S - \tau) \geq 1$. Take $D \in |K_S - \tau|$ and let K be any canonical divisor. One has $2D = 2K$ since $P_2 = 1$ and hence $D = K$, contradicting the fact that $\mathcal{O}_S(\tau) \not\cong \mathcal{O}_S$.

In the second case, you first look at the possible components of the canonical divisor $K = \sum_j m_j C_j$. Since K_S is nef and $(K_S, K_S) = 0$ you find $(K_S, C_i) = 0$. Writing down

$$0 = (K, C_j) = m_j(C_j, C_j) + \sum_{i \neq j} m_i(C_i, C_j)$$

you see that either $(C_j, C_j) = -2$ and C_j is a smooth rational curve, or you have $(C_j, C_j) = 0$ but also $(C_j, C_i) = 0$ for all $i \neq j$. So you can partition the connected components of $\cup C_i$ into two types: unions of smooth rational curves on the one hand and smooth elliptic curves or rational curves with one node on the other hand.

I now need the construction and properties of the Albanese map. See e.g. [Beauville] for details.

The Albanese torus is defined by

$$\mathrm{Alb}(S) = H^0(\Omega^1_S)^* / \operatorname{im} H_1(S, \mathbb{Z}),$$

where $\gamma \in H_1(S, \mathbb{Z})$ is mapped to the functional on $H^0(\Omega^1_S)$ given by integration over γ. Fixing a point $x_0 \in S$ and choosing any path from x_0 to x, integration along this path gives a well defined element $\alpha(x) \in \mathrm{Alb}(S)$. This gives then a holomorphic map, the *Albanese map*

$$\alpha : S \to \mathrm{Alb}(S).$$

It either maps to a curve $C \subset \mathrm{Alb}\, S$ or it maps onto the (two-dimensional) Albanese. In the first case, since $q(S) = 2$ the curve C is a genus 2 curve because of the following result.

Lemma 6. *If the image of the Albanese map $X \to \mathrm{Alb}\, X$ is a curve C, the fibres are connected. Moreover, C is smooth and has genus $q(S)$.*

Let $f : S \to C$ be the resulting fibration. By the preceding analysis, every connected component D of the canonical divisor K is either rational or elliptic. Since such curves cannot map onto a curve of genus 2 these must be contained in some fibre F of f, say over $c \in C$.

Now the following lemma, useful in many situations, can be applied. For a proof I refer to [Beau].

Lemma 7. (Zariski's Lemma) *Let $f : S \to C$ be a fibration of a surface S to a curve C and let $F = \sum_i m_i C_i$ be a fibre, where C_i is irreducible. Let $D = \sum_i r_i C_i$ be a $\mathbb{Q}$-divisor. Then $(D, D) \leq 0$ and equality holds if and only if $D = rF$ for some $r \in \mathbb{Q}$.*

It follows that $D = a/b \cdot F$ with a, b positive integers. Then $bD = f^*(a[c])$ and hence $h^0(ndD)$ and $h^0(ndK_S)$ grow indefinitely when n tends to infinity. This contradicts $\kappa(S) = 0$. The possibility that K_S is trivial is left. In this case, simply take an unramified cover $C' \to C$ of degree ≥ 2 and pull back your fibration. You get an unramified cover $S' \to S$ of degree ≥ 2 and $K_{S'}$ is still trivial, $\chi(\mathcal{O}_{S'}) = 0$ and hence $q(S') = 2$ by what we have seen. But $q(S') \geq q(C') \geq 3$, a contradiction.

There remains the case that $\mathrm{Alb}\, S$ is a 2-torus and that the Albanese maps surjectively onto it. It is an elementary fact that in this case $\alpha^* : H^2(\mathrm{Alb}\, S) \to H^2(S)$ is injective (dually: every 2-cycle on $\mathrm{Alb}\, X$ is homologous to a cycle which lifts to a 2-cycle on X). Since $b_2(S) = 6$ this then is an isomorphism and so no fundamental cohomology-class of a curve maps to zero. In particular, the Albanese map must be a finite morphism. So, if D is a connected component of the canonical divisor it cannot map to a point and hence it must be an elliptic curve E which maps to an elliptic curve $E' \subset \mathrm{Alb}\, S$. Now form the quotient elliptic curve $E'' = \mathrm{Alb}\, S/E'$ and consider the surjective morphism $S \to E''$. The Stein-factorisation then yields an elliptic fibration and D is contained in a fibre. By Zariski's lemma it follows that D is a rational multiple of a fibre and as before one concludes that $\kappa(S) = 1$. It follows that the only possibility is that K_S is trivial, but then, by the well-known behaviour of canonical divisor under coverings (the canonical divisor 'upstairs' is the pull back of the canonical divisor 'downstairs' plus the ramification divisor) I conclude that the Albanese map is a finite unramified covering and therefore S itself is a torus. ∎

8. The Enriques classification: the final step

In this section a sketch of the proof of the technical proposition 7.1 is provided.

I start by stating some preliminaries. The proofs are of a very different level of sophistication. First a rather easy fact about the topology of fibrations. See [Beau] for a proof.

Proposition 1. *Let S be a surface with K_S nef and let $f : S \to C$ be a fibration onto a curve. Let $\delta(f) \subset C$ be the (finite) set of critical values of f, i.e. $c \in \delta(f)$ if and only if at some $s \in f^{-1}(c)$ the map $df(s)$ vanishes. Let F be a smooth fibre and let $F_c = f^{-1}(c)$. One has*

$$e(S) = e(C)e(F) + \sum_{b \in \delta(f)} (e(F_b) - e(F)).$$

Furthermore, $e(F_b) - e(F) \geq 0$ with equality if and only if F_b supports a smooth elliptic curve.

Next, I need the much more involved canonical bundle formula for elliptic surfaces. A complete proof can be found in [BPV].

Theorem 2. *Let S be surface with K_S nef and let $f : S \to C$ be an elliptic fibration. Let F_i, $i = 1, \dots, m$ be the multiple fibres and let m_i be the multiplicity of $F_i = m_i F_i'$. One has*

$$K_S = f^* L + \sum_{i=1}^{m} (m_i - 1) F_i'$$

with L a divisor on C of degree $\chi(\mathcal{O}_S) - \chi(\mathcal{O}_C)$.

Finally, some subtle results about moduli of curves are needed. To explain them, I introduce the following notion.

Definition 3. *A fibration $f : X \to Y$ between projective manifolds is called isotrivial if there exists a finite unramified covering $g : Y' \to Y$ such that the pull back $f' : X' = X \times_Y Y' \to Y'$ of f is isomorphic to a product-fibration $X' \cong Y' \times F$, for some projective manifold F.*

Example 4. Let G be a finite group which is the quotient of $\pi_1(Y)$ and which acts on a manifold F. Let $g : Y' \to Y$ be the covering defined by G and consider the product action of G on $Y' \times F$. The quotient manifold $(Y' \times F)/G$ admits an isotrivial fibration onto Y. Conversely, one can show that any fibre bundle $X \to Y$ such that the fibre F has a finite group of automorphisms arises in this way.

The required result then is:

Proposition 5. *Suppose that $f : S \to C$ is a fibration of a surface onto a curve of genus 0 or 1 and suppose that f has everywhere maximal rank. Then f is isotrivial.*

A reduction of the proof of this result to 'standard' facts about moduli of algebraic curves can be found in [Beau].

Now, finally, the proof of Proposition 7.1 can be given. Let me recall it before giving the proof.

Proposition 6. *Suppose S is a surface with K_S nef and $(K_S, K_S) = 0$, $q = 1$ and $p_g = 0$. Then $\kappa(S) = 0$ or 1 and $\kappa(S) = 0$ if and only if S is bielliptic.*

Proof: Since $q(S) = 1$, the Albanese of S is an elliptic curve C and by Lemma 7.6 the Albanese mapping $\alpha : S \to C = \operatorname{Alb} S$ has connected fibres. Recall (Proposition 5.4) that $b_2(S) = 2$ and hence $e(S) = 2 - 2b_1(S) + b_2(S) = 0$. Now apply the topological lemma 8.1 to conclude that $\alpha : S \to C = \operatorname{Alb} S$ is either a genus g fibration with $g \geq 2$ and α everywhere of maximal rank, or an elliptic fibration with only smooth fibres, some of which are possibly multiple. In the first case, apply Proposition 8.5 to conclude that $\alpha : S \to C = \operatorname{Alb} S$ is isotrivial, so that there exists a finite unramified covering $\hat{S} \to S$ which is a product. By Proposition 3.9 the Kodaira-dimension does not change under finite unramified covers and so $\kappa(S) = 1$ in this case. So S is not bielliptic. If $\alpha : S \to C = \operatorname{Alb} S$ is elliptic and has multiple fibres, an application of the elliptic bundle formula 8.2 shows that $\kappa(S) = 1$ in this case too. There remains the possibility that $\alpha : S \to C = \operatorname{Alb} S$ is an elliptic fibration which is everywhere of maximal rank and then, again by Proposition 8.5 one has an isotrivial fibration. There exists therefore an elliptic curve F, an elliptic curve E, a group of automorphisms G of F acting as translations on E such that $S = E \times F/G$. Since $p_g(S) = 0$, by definition S is bi-elliptic. So only in the case $\kappa(S) = 0$ you get a bi-elliptic surface, in all other cases $\kappa(S) = 1$.

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