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ON ARAKELOV'S FINITENESS THEOREM FOR HIGHER DIMENSIONAL VARIETIES

1. Introduction.

Fix a smooth complex-projective curve B and a subset S of B consisting of finitely many points and consider families over $B\backslash S$ (i.e. proper smooth morphisms $X \to B\backslash S$) whose fibres belong to some pre-selected class $\underline{V}$ of smooth projective varieties:

$$\underline{V}(B,S) = \{\text{families over } B\backslash S \text{ with fibres in } \underline{V}\}/\text{isomorphisms} .$$

If $\underline{V}$ contains infinitely many non-isomorphic varieties, $\underline{V}(B,S)$ is infinite. So if one wants a finiteness statement, one should put further restrictions on the sort of families one considers. A celebrated theorem of Arakelov implies finiteness for $\underline{V} = \{\text{curves of genus } g\}$, $g \geqslant 2$, provided one restricts oneself to families that do not become trivial after a finite base change (cf. [A]). Recently, Faltings considered $\underline{V} = \{\text{principally polarized abelian varieties of dimension } g\}$ and he introduced the following condition:

$(*)_f$ Every skew-symmetric endomorphism of $R^1 f_* \mathbb{Z}$ is of type $(0,0)$.

In [F] Faltings shows

(1) $\{f \in \underline{V}(B,S);\ (*)_f \text{ holds}\}$ is finite,

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(2) $\{f \in \underline{V}(B,S);\ (*)_f$ is not true$\}$ is either empty or contains a positive dimensional component, smooth at f. Moreover there is at least one example where $(*)_f$ is not satisfied.

Since $(*)_f$ is a property of the variation of Hodge structure underlying $R^1 f_* \mathbb{Z}$, one expects a purely Hodge-theoretic proof for these results. In section 2 I sketch how to achieve this by replacing Faltings argument for the boundedness statement by a Hodge-theoretic proof. I should remark that as a consequence the same proof is valid also in case one does not have a *principal* polarisation. See also [Z] for a different proof.

After having communicated my results to Deligne, I received his recent preprint [D2] in which the same bound as the one in Proposition 2 can be found. The main result of [D2] is entirely in the spirit of this note. It is the following beautiful finiteness result:

THEOREM (Deligne). *Fix a smooth complex algebraic variety S, a point $0 \in S$ and an integer N. There are at most finitely many isomorphism classes of representations*

$$\rho : \pi_1(S,0) \to \operatorname{Aut} H^*(X_0, \mathbb{Q}) \ , \quad \dim H^*(X_0, \mathbb{Q}) = N$$

arizing from algebraic families $\{X_t\}_{t \in S}$.

This theorem has been proved independently by C. Simpson [SI].

The investigations leading to the results exposed on the conference were very much inspired by questions of Phil Griffiths and by reading Faltings' article. Finally I want to express my thanks to Joseph Steenbrink, Gerd Faltings and in particular to Pierre Deligne for many helpful remarks.

2. A Hodge-theoretic proof of Faltings theorem.

The setting is as follows. Over $B^0 = B \backslash S$ one has a local system $\mathbb{V}^0$ of free $\mathbb{Z}$-modules of rank $2g$ equipped with a non-degenerate integral skew form $\langle\ ,\ \rangle$ and a holomorphic subbundle F^0 of $V^0 = \mathbb{V}^0 \otimes \mathcal{O}_{B^0}$ such that $V^0 = F' \oplus \overline{F'}$, F' is totally isotropic with respect to $\langle\ ,\ \rangle$ and finally $i\langle f, \bar{f}\rangle > 0$ for all nonzero local holomorphic sections f of F'. The pair $(\mathbb{V}^0, F^0)$ is called a *variation of weight one Hodge structure* (underlying $\mathbb{V}^0$).

Since the period space in this case is nothing but the Siegel upper half space $h_g = \{Z \in M_g(C)\ Z = Z^t, \operatorname{Im} Z > 0\}$ the period map is a holomorphic map

$$\phi^0 : B^0 \to A_g := \Gamma_g \backslash h_g ,$$

where Γ_g is a suitable symplectic group. Giving a locally liftable holomorphic map as ϕ^0 is equivalent to giving a variation of weight one Hodge structure up to Γ_g-action and one says that ϕ^0 satisfies $(*)_{\phi^0}$ if the corresponding variation satisfies the equivalent of

$(*)_{\phi^0}$: Every skew-symmetric endomorphism of $\mathbb{V}^0$ is of type $(0,0)$.

For technical reasons one has to assume that all the local monodromies of $\mathbb{V}^0$ around points of S are unipotent. One can always achieve this by passing to a finite ramified covering of B and given (B,S) for the various local systems over $B\backslash S$ one only needs a finite collection of ramified covers of B, so for finiteness one may indeed make this assumption. Now $\mathbb{V}^0 \otimes \mathcal{O}_{B^0}$ has a canonical extension to a locally free sheaf V on B and F' extends to a algebraic subbubdle F of V in such a way that the Hodge-metric $i\langle v,\overline{w}\rangle$ on F' has at most logarithmic singularities near S. It follows that ϕ^0 extends to a holomorphic map

$$\phi : B \to A_g^* = \text{Satake-compactification of } A_g .$$

If one starts off with a family $f : X \to B\backslash S$ of polarized abelian varieties of dimension g, $\mathbb{V}^0 = R^1 f_* \mathbb{Z}$ underlies the obvious variation of weight one Hodge structure and if all local monodromies are unipotent the assignment $f \to$ period map of f induces a map from $\underline{V}(B,S)$ to the set $\mathrm{Mor}^\#(B^0, A_g)$ of locally liftable morphisms $B^0 \to A_g$ (i.e. locally factoring over the Siegel upper half space h_g).

This map has finite fibres and hence to prove Faltings' theorem it suffices to show

PROPOSITION 1. *The set of* $\phi^0 \in \mathrm{Mor}^\#(B^0, A_g)$ *such that* $(*)_{\phi^0}$ *is verified, is finite.*

The idea how to prove this is due to Arakelov (see [M, Lecture II, appendix] for a nice outline of Arakelov's proof). One first observes that $\mathrm{Mor}^\#(B^0, A_g) = \{\phi : B \to A_g^* ;\ \phi^{-1}(A_g^* \backslash A_g) \subset S\}$, hence has the structure of a scheme. Now the proof continues as follows

(1) First, one shows that for a suitable ample line bundle L on A_g^* there is a universal bound on $\deg \phi^* L$, hence $\mathrm{Mor}^\#(B^0, A_g)$ has finitely many irreducible components. *(Boundedness).*

(2) Next, one computes the tangent space to $\mathrm{Mor}^{\#}(B^0, A_g)$ at ϕ^0 and shows that $(*)_{\phi^0}$ implies that this tangent space is reduced to (0). *(Rigidity).*

Let me proceed by giving some details of these two steps.

(1) *Boundedness.* Since $\mathfrak{h}_g$ is contained (in a natural way) in the Grassmannian of g-planes in $\mathbb{C}^{2g}$ one has a Γ_g-invariant tautological subbundle $\mathcal{W}_g$ on $\mathfrak{h}_g$. Its determinant extends to a "line-bundle" $\mathcal{L} \in \mathrm{Pic}(A_g^*) \otimes \mathbb{Q}$ which is well known to be ample. Clearly $\phi^* \mathcal{L} = \det \mathcal{F}$ and the bound follows from

PROPOSITION 2. $\deg \phi^* \mathcal{L} \leqslant g(q - 1 + \frac{1}{2} \# S)$, $(q = \text{genus } B)$ whenever ϕ is non-constant.

Sketch of proof. The Gauss-Manin connection induces a homomorphism

$$\kappa : \mathcal{F} \to \mathcal{F}^v \otimes \Omega_B^1(S)$$

between $\mathcal{O}_B$-modules of rank g. If $\mathcal{G} = (\mathcal{F}/\mathrm{Ker}\,\kappa)$, then κ induces a homomorphism of maximal rank $k = \mathrm{rank}_x \kappa$, x generic:

$$\bar{\kappa} : \mathcal{G} \to \mathcal{G}^v \otimes \Omega_B^1(S)$$

and hence a non-zero morphism between the determinant line-bundles

$$\det \bar{\kappa} : \Lambda^k \mathcal{G} \to (\Lambda^k \mathcal{G}^v) \otimes \Omega_B^1(S)^{\otimes k} .$$

This is only possible if $\deg \mathcal{G} \leqslant \deg \mathcal{G}^v + k \deg \Omega_B^1(S)$, i.e.

$$(*) \qquad \deg \mathcal{G} \leqslant k(q - 1 + 1/2\ \# S) .$$

The curvature of the metric connection vanishes on the subbundle $\ker \kappa$ of $\mathcal{F}$. Since the curvature decreases on subbundles, it follows that

$$\deg \mathcal{F} \leqslant \deg \mathcal{G} .$$

Combining this with $(*)$ yields the desired bound.

(2) *Rigidity.* I sketch Faltings' argument, for the sake of completeness. One shows easily that

$$T_{\phi^0}(\mathrm{Mor}^{\#}(B^0, A_g)) = \mathrm{Hom}_B(\mathrm{Sym}^2 \mathcal{F}, \mathcal{O}_B)$$

and an essential fact to use now is

THEOREM [G], [S]. *The curvature matrix of* $\mathcal{F}$ *(and hence of* $\mathrm{Sym}^2\mathcal{F}$*) is non-negative on* B^0 *and has at most logarithmic growth near* S.

A standard inequality then yields:

COROLLARY 1. $\mathrm{Hom}_B(\mathrm{Sym}^2\mathcal{F}, \mathcal{O}_B) = \{$flat sections of $\mathrm{Sym}^2\mathcal{F}^v$ with constant Hodge norm$\}$.

Since any global section t of $\mathrm{Sym}^2\mathcal{F}^v$ induces one of $\mathrm{Hom}_B(\mathcal{F}, \mathcal{F}^v) \cong$ $\cong \mathrm{Hom}_B(\mathcal{F}, \overline{\mathcal{F}})$ (via the Hodge metric), one gets a global endomorphism τ of $\mathcal{V}$ by defining τ to be zero on $\overline{\mathcal{F}}$. If t is flat, actually $\tau \in \mathrm{End}\,\mathbb{V} \otimes \mathbb{C}$ and t being symmetric implies τ is anti-symmetric, hence:

COROLLARY 2. $T_\phi(\mathrm{Mor}^\#(B^0, A_g)) \cong \{\tau \in \mathrm{End}\,\mathbb{V} \otimes \mathbb{C};\ \tau$ anti-symmetric and τ of type $(-1,1)\}$.

It follows that $(*)_{\phi^0}$ implies $T_\phi(\mathrm{Mor}^\#(B^0, A_g)) = \{0\}$.

3. Discussion for Higher weights.

Arakelov's procedure, as sketched in section 2 gives several problems, which I'11 briefly comment upon.

1) If $D = G/V$ is a classifying space of Hodge structures of weight m and given type, the quotient $G_{\mathbb{Z}}/D$ in general is not quasi-projective. But nevertheless one can prove that

$$\mathrm{Mor}^\#(B^0, G_{\mathbb{Z}}/D) = \{\phi^0 : B^0 \to G_{\mathbb{Z}}/D,\ \phi^0 \text{ locally liftable with horizontable lifts}\}$$

is a complex variety. Indeed, fix a monodromy representation

$$\rho : \pi_1(B^0) \to G_{\mathbb{Z}}$$

and let $\mathbb{V}_{\mathbb{Z}}$ be the resulting local system on B^0. The variations of weight m Hodge structure (polarizable as usual) on $\mathcal{V}^0 = \mathbb{V}_{\mathbb{Z}} \otimes \mathcal{O}_{B^0}$ are parametrized in a natural way by the points of a (possibly empty) period domain. This follows from [D2], Proposition 1.13. Combined with Deligne's finiteness statement (cf. Introduction) this shows:

THEOREM. *$\mathrm{Mor}^{\#}(B^0, G_{\mathbb{Z}}/D)$ is a complex finite-dimensional variety with at most finitely many components.*

Below I discuss an alternate proof of this theorem which then (obviously) would reprove Deligne's finiteness.

2) Let L^0 be the "linebundle" on $G_{\mathbb{Z}}/D$ arizing from the determinant of the Hodge bundle of level m on D. The linebundle $(\phi^0)^* L^0$ on B^0 extends. It is the determinant of the canonical extension F^m of the Hodge bundle of level m on B^0 (I am assuming unipotent local monodromy along S).

PROPOSITION 3. *One has the inequalitites*

$$0 \leqslant \deg F^m \leqslant h^{m,0} \cdot m \cdot (g-1+\frac{1}{2}\ \# S)$$

where $h^{m,0} = \mathrm{rank}\ F^m$.

The first inequality is in [P]. The second inequality can be proved as in the proof of Proposition 2. Originally I had a cruder bound, but Deligne showed me how to refine my argument as to obtain the preceding bound.

One hopes to use bounds like these to reprove the theorem in 3.1. Alternatively one could make use of the distance decreasing property of the period maps to show that $\mathrm{Mor}^{\#}(B^0, G_{\mathbb{Z}}/D)$ has compact closure in the compactification resulting from describing this set as a period domain.

3) The tangent space at ϕ to $\mathrm{Mor}^{\#}(B^0, G_{\mathbb{Z}}/D)$ can be identified with the subspace of $\mathrm{End}_B(V)$ consisting of endomorphisms which relative to the Hodge-decomposition of V decompose as

$$\begin{pmatrix} 0 & & & 0 \\ Z_m & & & \\ 0 & Z_{m-1} & & \\ & 0 & & 0 \\ 0 & 0 & 0 & Z_1 \end{pmatrix}, \quad \begin{array}{l} Z_j \in \mathrm{Hom}_B(F^j/F^{j+1}, F^{j-1}/F^j) \\ \langle Z_j v, w\rangle + \langle v, Z_{m-j+1} w\rangle = 0 \\ \forall v \in F^j/F^{j+1},\ w \in F^{m-j+1}/F^{m-j+2} \end{array}$$

The curvature conditions only give that $Z = Z_1 \circ \ldots \circ Z_m \in \mathrm{Hom}_B(S^2 F^m, 0)$ is a global flat section of constant Hodge length. It can be considered as an endomorphism of $\mathbb{V} \otimes \mathbb{C}$ of type $(-m, m)$ with $\langle Zv, w\rangle = (-1)^m \langle v, Zw\rangle$, $\forall v, w \in \mathbb{V} \otimes \mathbb{C}$. Consequently, one has:

LEMMA. *Using the previous notations, if*

$$(*)_{\phi^0} : [U \in \operatorname{End} \mathbb{V},\ U \text{ is } (-1)^m\text{-symmetric}]^{(-m,m)} = 0$$

then $Z = 0$.

This seems not sufficient to conclude that $Z_i = 0 \ \forall i$, even in the case $m = 2$. If $m = 2$, $h^{2,0} = 1$ the condition $(*)_{\phi^0}$ is equivalent to ϕ^0 being constant. The result here is a follows:

PROPOSITION. *There are only finitely non-trivial variations of weight 2 Hodge structures with* $h^{2,0} = 1$ *and for which* $\operatorname{Hom}_B(\mathcal{F}^2, \mathcal{F}^1/\mathcal{F}^2) = 0$.

It seems hard to decide when the condition of the previous proposition holds.

The condition $(*)_{\phi^0}$ in general certainly is not equivalent to ϕ^0 being constant, even in the case $m = 1$, where Deligne has given arithmetic conditions on $\operatorname{End} \mathbb{V}$ garanteeing $(*)_{\phi^0}$ (cf. [D, Prop. 4.4.1]).

4) In [SI] the following immediate generalization of Faltings' results is proved:

THEOREM (Simpson). *There are at most finitely many (polarizable) variations of Hodge structure of weight* m *on* $\mathbb{V}$ *having the property that* $\operatorname{End} \mathbb{V}$ *has pure type* $(0,0)$.

It seems hard to find conditions on $\operatorname{End} \mathbb{V}$ as in [D] implying that $\operatorname{End} \mathbb{V}$ admits only those variations of Hodge structure. In particular it is not clear whether $(*)_{\phi^0}$ implies Simpson's property, even for weight 2, $h^{2,0} = 1$ as in 3).

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