Random matrices and Brownian motion in gauge theory

Miguel Tierz
Einstein Institute of Mathematics,
The Hebrew University of Jerusalem

Marches aléatoires répulsives et matrices aléatoires, Les Houches
16 mai 2011 - 27 mai 2011.

tierz@math.huji.ac.il

Outline

- INTRODUCTION TO RANDOM MATRIX THEORY

Outline

- INTRODUCTION TO RANDOM MATRIX THEORY
    - Main definitions

Outline

- Introduction to random matrix theory
  - Main definitions
  - Orthogonal polynomials method

Outline

- Introduction to random matrix theory
  - Main definitions
  - Orthogonal polynomials method
  - Examples of random matrices with weakly confining potentials

Outline

- Introduction to random matrix theory
  - Main definitions
  - Orthogonal polynomials method
  - Examples of random matrices with weakly confining potentials
- the Stieltjes-Wigert random matrix model

Outline

- Introduction to random matrix theory
    - Main definitions
    - Orthogonal polynomials method
    - Examples of random matrices with weakly confining potentials
- the stieltjes-wigert random matrix model
    - Brief introduction to Chern-Simons (CS) theory

Outline

- Introduction to random matrix theory
  - Main definitions
  - Orthogonal polynomials method
  - Examples of random matrices with weakly confining potentials
- the Stieltjes-Wigert random matrix model
  - Brief introduction to Chern-Simons (CS) theory
  - Random matrix description of CS gauge theory

Outline

- INTRODUCTION TO RANDOM MATRIX THEORY
  - Main definitions
  - Orthogonal polynomials method
  - Examples of random matrices with weakly confining potentials
- THE STIELTJES-WIGERT RANDOM MATRIX MODEL
  - Brief introduction to Chern-Simons (CS) theory
  - Random matrix description of CS gauge theory
  - Partition function & other observables with SW polynomials

Outline

- Introduction to random matrix theory
  - Main definitions
  - Orthogonal polynomials method
  - Examples of random matrices with weakly confining potentials
- the Stieltjes-Wigert random matrix model
  - Brief introduction to Chern-Simons (CS) theory
  - Random matrix description of CS gauge theory
  - Partition function & other observables with SW polynomials
- brownian motion and measures on partitions

Outline

- Introduction to random matrix theory
  - Main definitions
  - Orthogonal polynomials method
  - Examples of random matrices with weakly confining potentials
- the Stieltjes-Wigert random matrix model
  - Brief introduction to Chern-Simons (CS) theory
  - Random matrix description of CS gauge theory
  - Partition function & other observables with SW polynomials
- Brownian motion and measures on partitions
- summary

- Let $H = (H_{jk})_{j,k=1}^{N}$ be a square $N \times N$ matrix with randomly distributed elements H_{jk}. This is a random matrix with respect to a probability distribution, defined by:

$$P_{\beta}^{(N)}(H) \propto \exp\left(-\beta \mathrm{Tr} \mathrm{V}(H)\right),$$

- The first and most studied ensembles are the Gaussian ensembles, $V(H) = H^2$. It can be shown that the previous expression is automatically restricted to the form

$$P(H) = \exp\left(-a\mathrm{Tr}H^2 + b\mathrm{Tr}H + c\right), \qquad a > 0,$$

  if one postulates statistical independence of the matrix elements H_{ij}. There are three different ensembles defined depending on the values of the parameter $\beta = 1, 2$ or 4.

Ensembles of random $N \times N$ matrices H are defined by the following demands:

1. The probability $P(H)d\left[H\right]$ is invariant under any transformation $H \rightarrow U^{-1}HU$, where U is either an orthogonal ($\beta = 1$), unitary ($\beta = 2$) or symplectic ($\beta = 4$) matrix. That is to say, if $H' = U^{-1}HU$ where U belongs to the unitary group $U(N;\beta)$, then $P(H')d\left[H'\right] = P(H)d\left[H\right]$.

2. The matrix elements which are not related by the symmetry of the matrix are statistically independent (Gaussian ensembles)

- Diagonalization: for each matrix H there is a matrix U that maps it onto its eigenvalues. The Jacobian of the transformation is $J_\beta(\{x_i\}) = \prod_{i<j} |x_i - x_j|^\beta$. The resulting expression is

$$P(x_1, \ldots, x_N) = \mathrm{C_N} \prod_{i<j} |x_i - x_j|^\beta \prod_{i=1}^{N} \mathrm{e}^{-\frac{\beta}{2} V(x_i)}.$$

  The potential $V(x) = \log^2 x$ (log-normal weight function $\omega(x) = \mathrm{e}^{-\log^2 x}$) is at the center of most developments in this talk.

- The main relevant quantities are m-partial integrations over the previous N-dimensional probability density function

- A central and powerful result in random matrix theory is that m-point correlation function can be computed from the two-point kernel as follows (simplest case of a Hermitian ($\beta = 2$) ensemble)

$$R_m^{(N)}(x_1, ..., x_m) = \det\left(K_N\left(x_i, x_j\right)\right)_{1\leq i,j\leq m}$$

- Orthogonal polynomials method $\Longrightarrow$ explicit expressions for $K_N\left(x_i, x_j\right)$. Let $p_N(x) = c_N x^N + ...$ the *Nth* orthogonal polynomial associated to $\mathrm{e}^{-V(x)}$, the two-point kernel is

$$\begin{aligned} K_N(x,y) &= \mathrm{e}^{-\frac{V(x)+V(y)}{2}} \sum_{i=0}^{N-1} p_i(x)\, p_i(y) \\ &= \frac{c_{N-1}}{c_N} \frac{p_N(x)p_{N-1}(y) - p_{N-1}(x)p_N(y)}{x-y} \mathrm{e}^{-\frac{V(x)+V(y)}{2}} \end{aligned}$$

Examples of weakly confining potentials

- The Wigner-Dyson paradigm refers to strongly confining potentials like $V(x) = x^2$ (Gaussian), other classical ensembles (like Laguerre and Jacobi ensembles), or polynomial potentials.
- Models with growth $\lim_{x\to\infty} \omega(x) < \mathrm{e}^{-|x|}$ when $x \in (-\infty, \infty)$ or $\lim_{x\to\infty} \omega(x) < \mathrm{e}^{-\sqrt{x}}$ when $x \in (0, \infty)$ are weakly confining (moment problem).
- We study even weaker potentials that behave as $V(x) \sim k^2 \log^2 x$ for large x. These models have a two-point kernel ($q = \mathrm{e}^{-2a}$)

$$K(u-w) = \frac{a \sin(\pi(u-w))}{\pi \sinh(a(u-w))}$$

- We consider Chern-Simons theory on a three-manifold M and for a gauge group G, with action

$$S(A) = \frac{k}{4\pi} \int_M \mathrm{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right),$$

where A is a connection on M.

- Witten showed in 1989, that the partition function of Chern-Simons theory

$$Z_k(M) = \int \mathcal{D}A e^{iS_{\mathrm{CS}}(A)},$$

defines a topological invariant.

- Chern-Simons theory is of interest in the study of topological strings, the fractional quantum Hall effect, ...
- The partition function of CS theory on certain manifolds has simple expressions (M. Mariño, Comm. Math. Phys. 253, 25 (2004)).The simplest case is S^3 and gauge group $U(N)$

$$Z_{\mathrm{CS}}(S^3) = \int_{-\infty}^{\infty} \prod_i \frac{du_i}{2\pi} \mathrm{e}^{-\frac{u_i^2}{2g_s}} \prod_{i<j} \left(2 \sinh \left(\frac{u_i - u_j}{2} \right) \right)^2$$

- Thus, we have N-dimensional integral expressions for Chern-Simons partition functions whose expression resemble that of random matrix theory.

- Ingredients: 1) Change of variables $e^{u_i} = x_i$ 2) Symmetry of the log-normal $\omega(xq) = \sqrt{q}x\omega(x)$ (when $\omega(x) = e^{-\log^2 x_i/2g_s}$), then

$$Z(S^3) = \int \prod_{i=1}^{N} \frac{du_i}{2\pi} e^{-\frac{u_i^2}{2g_s}} \prod_{i<j} \left(2\sinh\left(\frac{u_i - u_j}{2}\right)\right)^2$$

$$= (2\pi)^{-N} e^{\frac{-N^3 g_s}{2}} \int \prod_{i=1}^{N} dx_i e^{-\frac{\log^2(x_i)}{2g_s}} \prod_{i<j} (x_i - x_j)^2 .$$

- Last expression is the Stieltjes-Wigert matrix model. For the partition function computation, we actually only need the leading coefficients, $p_i(x) = a_i x^i + \ldots$ which are

$$a_j = q^{(j+1/2)^2} \left\{ (1-q) \ldots (1-q^j) \right\}^{-1/2} .$$

- The partition function in terms of the orthogonal polynomials is:

$$
\begin{aligned}
Z &= \int \dots \int \prod_{i=1}^{N} \omega(x_i)\, dx_i \prod_{i<l} (x_i - x_l)^2 \\
&= \frac{N!}{\prod_{i=0}^{N-1} a_i^2} = N! a_0^{-2N} \prod_{i=1}^{N-1} \left(\left(\frac{a_{i-1}}{a_i} \right)^2 \right)^{N-i} .
\end{aligned}
$$

- Using the coefficients, we have $\left(\frac{a_{j-1}}{a_j}\right)^2 = q^{-4j}\left(1-q^j\right)$ and $a_0 = q^{1/4}$, leads to

$$
Z_{\mathrm{SW}} = N! q^{-\frac{1}{6}N(2N-1)(2N+1)} \prod_{j=1}^{N-1} \left(1-q^j\right)^{N-j}
$$

- Then, the N-dimensional integral is computed explicitly:

$$Z_{\text{sinh}} = \left(\frac{g_s}{2\pi}\right)^{N/2} N! \mathrm{e}^{\frac{1}{6} g_s N \left(N^2-1\right)} \prod_{j=1}^{N-1} \left(1-q^j\right)^{N-j},$$

and transforming the product term and identifying $g_s = \frac{2\pi i}{k+N}$ (coupling constant with CS parameter) we finally find:

$$Z\left(S^3\right) = \mathrm{e}^{\frac{1}{4} i\pi N^2} \left(k+N\right)^{-N/2} \prod_{j=1}^{N-1} \left(2 \sin \frac{\pi j}{k+N}\right)^{N-j}.$$

- More analytical computations done. The Chern-Simons invariant of the unknot are quantum dimensions. We showed

$$\begin{aligned} < \quad \mathfrak{s}_\lambda(M) >_w &= \int [dM] \mathfrak{s}_\lambda(M) \mathrm{e}^{-\frac{1}{2g_s}\mathrm{Tr}(\log M)^2} \\ &= \quad q^{-n|\lambda|-\frac{1}{2}C_\lambda^{U(n)}} \mathcal{D}_\lambda . \end{aligned}$$

where $C_\lambda^{U(n)}$ is the Casimir of $U(N)$ and the last term are the quantum dimensions:

$$\mathcal{D}_\lambda \equiv \prod_{x\in\lambda} \frac{\lfloor n+c(x) \rfloor}{\lfloor h(x) \rfloor},$$

where for each box of the diagram $h(x) \equiv \lambda_i + \lambda_j' - i - j + 1$ is the hook-length and $c(x) \equiv j - i$ the content of x.

- The density of states can be computed exactly with the orthogonal polynomials using $\rho\left(x\right)=\mathrm{e}^{-V(x)}\sum_{n=0}^{N-1}P_{n}\left(x\right)^{2}$. In the case $q\longrightarrow 1$, the model tends to a Gaussian ensemble and the corresponding density of states tends to the well-known semi-circle law.

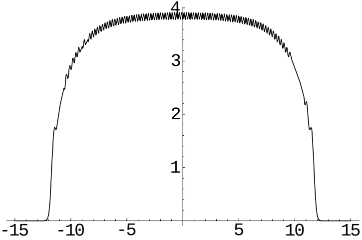
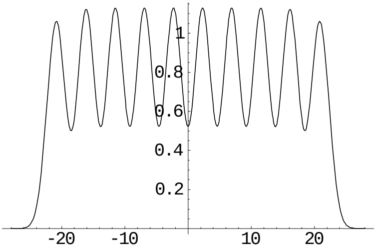
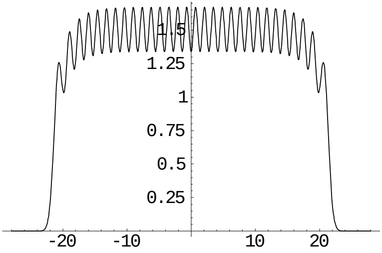


Figure: $q=0.9$ and $N=100$

- However, in contrast with ordinary (e.g. Gaussian) random matrix ensembles, the density of states shows a crystalline, oscillatory pattern.



$N = 10$ and $q = 0.3$



$N = 30$ and $q = 0.5$

Biorthogonal Stieltjes-Wigert model

Y.D and MT. (J. Math. Phys. 48, 023507 (2007)), MT (J. Math. Phys. 51, 063509 (2010))

- We also have studied a biorthogonal version of the Stieltjes-Wigert model

$$Z^{P,Q} = \int \prod_i \frac{\mathrm{d}w_i}{2\pi} \mathrm{e}^{-\kappa^2 P^2 \log^2 x_i} \prod_{i<j} (x_i - x_j)(x_i^{P/Q} - x_j^{P/Q}),$$

  that appears when the manifold is a lens space, instead of S^3.
- For this we had to find out the biorthogonal version of the Stieltjes-Wigert polynomials

$$\begin{aligned} \int Y_n(x,k)\, x^{\theta j} \omega(x)\, dx &= \alpha_n^{(\theta)} \delta_{n,j}, \\ \int Z_n(x,k)\, x^{j} \omega(x)\, dx &= \beta_n^{(\theta)} \delta_{n,j}. \end{aligned}$$

- 1D Brownian motion: the transition probability density is Gaussian and satisfies Heat equation (it is a diffusion process):

$$p_t(x,y) = \frac{1}{\sqrt{2\pi Dt}} \exp(-(x-y)^2/2Dt)$$

$$\frac{\partial p_t(x,y)}{\partial t} = D\frac{\partial^2 p_t(x,y)}{\partial x^2}$$

- Model of N vicious walkers (*Walks, Walls, and Melting, J. Stat. Phys. 34, 667, (1984)*):

$$p_{t,N}(\lambda,\mu) = \frac{1}{(2\pi t)^{N/2}} \mathrm{e}^{-\frac{|\lambda|^2+|\mu|^2}{2t}} \det|\mathrm{e}^{\lambda_i\mu_j/t}|_{1\leq i<j\leq N} .$$

- We consider now initial and final boundary conditions, $\mu = \lambda$, and an equal spacing condition, that is, $\lambda_{0j} - \lambda_{0,j+1} = a$, where a is the initial and final spacing between two neighboring movers and compute the probability of a reunion:

$$p_{t,N}(\lambda_0, \lambda_0) = \frac{1}{(2\pi t)^{N/2}} \prod_{k=1}^{N} (1 - \mathrm{e}^{-ka^2/t})^{N-k}$$

- If we choose $a^2 = 1$ and identify $-\frac{1}{t} = g_s = \frac{2\pi i}{k+N}$ then

$$Z_{\mathrm{CS}}(S^3) = \mathrm{e}^{\frac{\pi i}{2} N^2} q^{-\frac{1}{12} N(N^2-1)} p_{t,N}(\lambda_0, \lambda_0),$$

where the label 0 refers to the Weyl vector $\lambda_0 = \rho$

- By imposing the same initial and final conditions we are dealing with N non-intersecting Brownian motions on the surface of a cylinder.
- Other invariants are obtained with other boundary conditions:

$$\begin{aligned} p_{t,r}(\lambda,\rho) &= \frac{1}{(2\pi t)^{r/2}}\,\mathrm{e}^{-\frac{|\lambda|^2+|\rho|^2}{2t}} \prod_{\alpha>0} 2\sinh\frac{(\alpha,\lambda)}{2t} \\ &= \left\langle W_\lambda\,(\mathrm{unknot})\right\rangle, \end{aligned}$$

and $p_{t,r}(\lambda,\mu)$ corresponds to the Hopf link invariant $\left\langle W_{\mu\lambda}\,(\mathrm{Hopf})\right\rangle$.

Measures on partitions and 2d Yang-Mills theory

R.J. Szabo and MT (arXiv:1005.5643, arXiv:1102.3640)

- One can write 2d Yang-Mills theory observables in terms of measures on partitions (Schur measure and z-measure). Ex: Chiral 2d YM on S^2

$$Z_{YM}^{+}(S^2, SU(N)) = \sum_{\lambda} q^{|\lambda|} (\dim \lambda)^2 .$$

- A unitary random matrix ensemble can be given (alternative to the Meixner ensemble)

$$Z_{YM}^{+} = \int dU \det(1 + \sqrt{q} U^{-1})^N \det\left(1 + \sqrt{q} U\right)^N .$$

- This leads to connections with Painlevé V equation and also to interpretations in terms of stochastic models.

Conclusions and Outlook

- Random matrix models with a $V(x) = \log^2 x$ potential can be solved exactly, giving CS observables. Schur polynomials and combinatorics also needed in general.

Conclusions and Outlook

- Random matrix models with a $V(x) = \log^2 x$ potential can be solved exactly, giving CS observables. Schur polynomials and combinatorics also needed in general.
- The models behave differently from the better known classical random matrix ensembles (e.g. Gaussian models).

Conclusions and Outlook

- Random matrix models with a $V(x) = \log^2 x$ potential can be solved exactly, giving CS observables. Schur polynomials and combinatorics also needed in general.
- The models behave differently from the better known classical random matrix ensembles (e.g. Gaussian models).
- Chern-Simons theory and 2D Yang-Mills theory can also be related to Brownian motion on a Weyl chamber

Conclusions and Outlook

- The study of measures on partitions (Plancherel measure, z-measure, Schur measure) is relevant in 2d Yang-Mills theory, leading to a discrete matrix model and, in some cases, alternatively, to unitary matrix models.

Conclusions and Outlook

- The study of measures on partitions (Plancherel measure, z-measure, Schur measure) is relevant in 2d Yang-Mills theory, leading to a discrete matrix model and, in some cases, alternatively, to unitary matrix models.
- The unitary matrix model is useful to establish connections with Painlevé equations and also with stochastic models (like lock-step vicious walkers, corner growth model, ...)

Conclusions and Outlook

- The study of measures on partitions (Plancherel measure, z-measure, Schur measure) is relevant in 2d Yang-Mills theory, leading to a discrete matrix model and, in some cases, alternatively, to unitary matrix models.
- The unitary matrix model is useful to establish connections with Painlevé equations and also with stochastic models (like lock-step vicious walkers, corner growth model, ...)
- Main idea: Several "simple" gauge theories (e.g. topological) in low dimensions (3d and 2d) can be solved in terms of random matrices and the observables of the theory have combinatorial and stochastic interpretations.