Calcul formel
et
Mathématiques
avec
la HP49G
en mode algébrique

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Foreword

Sometimes, I am asked this question: why put symbolic manipulation capabilities into a calculator, while specialized computer software for this is now either cheap, or available for free?

In my opinion, a calculator is the most effective way to integrate calculation aids with the teaching of mathematics, because it is both easy to carry out with you and to use in a classroom.

However, using a symbolic manipulation software package is not as simple as its interface could suggest... Therefore, having an adequate documentation for it is important. The HP49G user guide describes the computer algebra system very briefly, so this manual is its essential complement:

it describes the HP49G from the point of view of someone that “wants to do maths”.

In fact, readers interested in maths can read only this book as well, because the author starts with an introduction to the calculator, describes in more detail the computer algebra system commands arranged by purpose (the index allows the reader to find all commands in alphabetical order), and then focuses on programming the calculator in algebraic mode.

Each command is demonstrated by an example, and many of them are leveraged to solve a “baccalaureat” problem. The programming section has several programs, arithmetic programs in particular.

Briefly, this is the manual I should have written if I had enough patience! I thank Renée for carrying it out.

Bernard Parisse
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Author of the Computer Algebra System of the HP49G

1 Getting started

1.1 Introduction

1.1.1 Turning on the calculator

Press the ON key.

When the calculator is turned on, the same key exits an application: it acts as either EXIT or CANCEL.

To turn off the calculator, press red-shift and then ON.

If the calculator does not respond in despite of several ON (CANCEL), press both ON and F3 simultaneously to reinitialize it.

1.1.2 What am I looking at?

From top to bottom, you can see:

1. the screen
   1.a the calculator status
   1.b the calculation history
1. The screen:

1.a The calculator status describes the current calculator modes:

- **RAD** if the calculator is working in radians, **DEG** if it is working in degrees.
- **XYZ** shows that rectangular coordinates are in use.
- **HEX** shows that binary integers prefixed by # are displayed in base 16.
- **R** if the calculator is in **REAL** mode, **C** if it is in **COMPLEX** mode.
- **=** if the calculator is in **EXACT** mode (symbolic calculations), ~ if the calculator is in **APPROXIMATE** mode (numeric calculations).
- ‘X’ denotes the name of the current variable stored in VX: usually it is ‘X’.
- **ALG** if the calculator is in **ALGEBRAIC** mode, **RPN** if it is in **RPN** mode.
- **{HOME}** or **{HOME ESSAI}** to show the name of the current directory (for example, the main directory HOME or the ESSAI subdirectory).

1.b The calculation history:
General principle: on the screen, the input expression (prefixed by :) is displayed left-justified, and the result is right-justified.

1.c The menu:
Menu commands are accessed using the following keys: F1 F2 F3 F4 F5 F6.
When the menu has more than 6 commands, press the **NXT** key to display the next portion of the menu. The menu can also contain subdirectories (which, in turn, contain a set of commands): they can be recognized because their menu item has a small bar across the upper-left corner. To execute a menu command, simply press the corresponding Fi key.

2. The keyboard:
You should find:

- the **ON** key, to turn on the calculator, and to interrupt a calculation while it is in progress. To turn off the calculator, press **red-shift** and then **ON**.
- two “shift” keys, one blue and one red; they allow a single key to have more than one function.
- the **ALPHA** key, to enter alphabetical characters (uppercase by default). To keep alphabetic mode active for more than one subsequent keystroke, it is necessary to press **ALPHA** twice. To exit this mode, press **ALPHA** again. To toggle between uppercase and lowercase alphabetic entry modes, press **blue-shift ALPHA** while the calculator is in alphabetic entry mode.
- the **ENTER** key; it either enters or confirms a command.
- four arrow keys (left, right, up, down); they move the cursor when you are either in the editor or in a command list.
1.2 Calculator modes

The calculator can work in several modes.
You can choose: -algebraic or reverse poland notation mode (\texttt{ALG} or \texttt{RPN})
-real or complex mode (R or C)
-exact or approximate mode (= or \texttt{\sim})
-immediate or step-by-step mode...

\textbf{WARNING:} this book assumes that the calculator is in algebraic, real, exact, immediate mode (R = ALG).

Type: \texttt{CASCFG} (Computer Algebra System ConFiG) to set the calculator in real, exact, immediate mode. While working, you can type \texttt{CASCFG} to restore this configuration (the calculator automatically changes mode -asking you for a permission- when it is appropriate!).

\textbf{CHECK}

Check now that your calculator is indeed in algebraic, real, exact mode. In order to do this:
press \texttt{MODE} to check that the operating mode really is algebraic; if it is not, select algebraic either with the \texttt{choos} menu item, or pressing the $+/-$ key.

While you are in the \texttt{MODE} screen, activate the \texttt{cas} menu item and check that neither numeric, nor approx, nor complex are enabled. If one ore more of these modes are enabled, disable them all using the \texttt{chk} menu item.

Notice: for pedagogic applications, it is often interesting to enable the step/step mode, to make the calculator execute its calculations step by step.

Now select the \texttt{ok} menu item to confirm all changes made in \texttt{cas}, and then \texttt{ok} again to confirm the choices you made in \texttt{MODE}.

Now you are in algebraic, real, exact mode.

\textbf{WARNING:}
\texttt{THIS BOOK ASSUMES THAT THE CALCULATOR IS IN THIS MODE.}

You are now in the \texttt{HOME} directory.

You can simply type in the calculation you want executed, for example: $1 + 1$, and then press \texttt{ENTER}

The result is displayed (right justified), and the input expression $1 + 1$, preceded by :, goes up in the history area (left justified).

It is possible to copy this expression in the command line by pressing the \texttt{HIST} key (the up arrow allows you to select the expression to copy, and the \texttt{echo} menu item copies and simplifies it).

It is also possible to reuse the last result (denoted \texttt{ANS(1)}) using the \texttt{ANS} (red-shift \texttt{ENTER}) key, as well as previous results (denoted \texttt{ANS(2)}) and so on.

You can do both exact calculations and approximate ones; for example:
$\sqrt{2}$ followed by \texttt{ENTER} does not evaluate $\sqrt{2}$ and keeps the result exact, but the same expression followed by red-shift \texttt{ENTER} \texttt{(→NUM)} returns the approximate value of $\sqrt{2}$ with 12 significant digits, keeping the calculator in exact mode.

Of course, if you want to do only numeric calculations, you can enable approx mode (\texttt{MODE} key, and then \texttt{cas} menu item); in this mode, the \texttt{ENTER} key does the calculation numerically, evaluating both constants and variables.
1.3 Notation

In this book, the four arrow keys are represented by the following four triangles:

\[ \triangle \leftarrow \rightarrow \uparrow \downarrow \]

The delete arrow (deletion of the character to the left of the cursor) is represented by:

\[ \leftarrow \]

The red arrow over the 0 (zero) key is represented by:

\[ \rightarrow \]

The \texttt{STO} key is represented, in a program, by:

\[ \texttt{STO} \leftarrow \texttt{or} \rightarrow \]

The carriage return (in red, over the decimal point key), is represented by:

\[ \leftarrow \rightarrow \]

1.4 Flags

The vary majority of commands takes system flags into account. Each flag has its own unique number, and has a default value. If you want to change the value of a flag, you can do it by pressing the \texttt{MODE} key, then F1 to select the \texttt{flags} menu item and enter the flag management screen.

When you toggle the flag you want changed, its new function appears on the screen.

If you know in advance the flag number, you can also change its value with the \texttt{SF} and \texttt{CF} commands.

For example, to change flag number \(-117\) (that is, the flag controlling the display of menus), you type:

\[ \texttt{SF}(-117) \]

(most menus are now displayed across the bottom of the screen, instead of using pop-up command lists). After this command:

\[ \texttt{FS?}(-117) \text{ returns } 1. \text{ and } \texttt{FC?}(-117) \text{ is } 0. \]

To have most menus displayed using pop-up command lists again, you type:

\[ \texttt{CF}(-117) \] (\texttt{FS?}(-117) is now 0. and \texttt{FC?}(-117) is 1.).

2 Important keys

2.1 The APPS key

This key, when pressed, displays a list of all calculator’s application.

2.1.1 Plot functions

This command list has the following items:

\textbf{Equation entry}. This item acts the same as the key sequence \texttt{blue-shift F1 (Y=3D)}.

\textbf{Plot window}. This item acts the same as the key sequence \texttt{blue-shift F2 (WIN)}.
Graph display. This item acts the same as the key sequence blue-shift F3 (GRAPH).

Plot setup. This item acts the same as the key sequence blue-shift F4 (2D/3D).

Table setup. This item acts the same as the key sequence blue-shift F5 (TBLSET).

Table display. This item acts the same as the key sequence blue-shift F6 (TABLE).

For more information, refer to chapter 3.

2.1.2 I/O functions

This list contains the commands that allow your calculator to interface with a computer. For example, the fifth item is: Transfer.

If you press 5, and then ok, the calculator opens the Transfer window and displays:

Port : Wire
Type : Kermit (or XModem)

This window allows you to transfer a file interactively. You can do the same thing from the command line, too; for example, these are the steps you should follow to use the Linux kermit program:

- Connect both the calculator and the computer to the serial link cable.
- On the computer, type:
  
  kermit
  and then serv
- On the HP49G type:
  
  SEND('NOM')
  to copy the NOM variable from your HP49G to your computer.
- Or else,
  
  On the HP49G type:
  
  KGET('NOM')
  to copy the NOM variable from your computer to your HP49G.

2.1.3 Constants library

This item displays a list of 40 physical constants. These constants are denoted by their symbol and either their name or their value (if the value menu item is selected). They are followed by their measurement units, if the unit menu item is selected. They can also be copied on the command line, by pressing the ->stk menu key.

2.1.4 Numeric solver

This item acts the same as the sequence of keys: red-shift 7 (NUM.SLV).

2.1.5 Time & date

This item acts the same as the sequence of key: red-shift 9 (TIME).

2.1.6 Equation writer

This item acts the same as the EQW key. See section 2.1 for more details.
2.1.7 File manager
This item acts the same as the sequence of keys: blue-shift APPS (FILES).
See section 2.5 for more details.

2.1.8 Matrix writer
This item acts the same as the sequence of keys: blue-shift EQW (MTRW).
See section 2.2 for more details.

2.1.9 Text editor
This item opens the command line: notice that it is possible to write on more than one line (by pressing red − shift • (←) to open a new line).

2.1.10 Math menu
This item acts the same as the sequence of keys: blue-shift SYMB (MTH).

2.1.11 CAS menu
This item opens a command list with the following entries:
1. ARITHMETIC corresponding to the blue-shift 1 (ARIT) menu
2. ALGEBRA corresponding to the red-shift 4 (ALG) menu
3. COMPLEX corresponding to the red-shift 1 (CMPLX) menu
4. CALCULUS corresponding to the blue-shift 4 (CALC) menu
5. EXP&LN corresponding to the blue-shift 8 (EXP&LN) menu
6. SYMBOLIC SOLVER corresponding to the blue-shift 7 (S.SLV) menu
7. MATRICES corresponding to the blue-shift 5 (MATRICES) menu
8. CONVERT corresponding to the blue-shift 6 (CONVERT) menu
9. TRIGONOMETRIC corresponding to the red-shift 8 (TRIG) menu
Refer to chapter 4 for more information.

2.2 The MODE key
This key allows you to tune the operating mode of your calculator: Algebraic or RPN mode, to examine and change the calculator’s flags (F1 key), to tune the cas (F3 key), and to change the way your calculator displays data on the screen with disp (F4 key).

For example (see also page 5) the flag -117 can be either:
choose boxes to have the calculator display its menus using popup windows
or
soft menu to have the calculator display its menus across the bottom of the screen.

2.3 The TOOL key
This key displays a menu containing:
edit, to edit the first line of the history (or the highlighted line).
view, to view the first line of the history (or the highlighted line).
rcl, the same as the key sequence blue − shift STO ▷ (RCL) (see page 15).
sto ▷ the same as the key STO ▷.
purge, the same as the command PURGE (see page 15). clear, to delete the current command line, leaving the cursor at the beginning of the line (this isn’t the same as CANCEL that kills the current command line!). Beware, when the command line is not active, clear deletes the whole history; in this case, it is the same as red−shift ← (CLEAR).

2.4 The UNDO key (red-shift HIST)
This key is very useful, because it undoes the last command executed.

2.5 The VAR key
This key displays, across the bottom of the screen, a menu containing the names of all variables in the current directory (press NXT to view them all!). See section 2.4 for more information.

2.6 The EQW key
This key invokes the equation editor. It can always be used, even in the matrix editor. Also, from the equation editor it is possible to access the history (see 1.11). For more information, see section 2.1.

2.7 The MTRW key (blue-shift EQW)
This key invokes the matrix editor, to enter a matrix. If you want to enter a vector instead, make sure that the vect menu option is selected.

To enter a matrix: you enter the first line, then you move the cursor to the beginning of the next line; when you finish entering the following lines, the cursor automatically wraps around.

See section 2.2 for more details.

2.8 The SYMB key
This key opens a menu containing the most basic symbolic functions, divided by category. The sub-menus contain the cas functions an high-school student usually needs. These functions, and many others, can be found in the corresponding cas menus, too.

Example: The SYMBOLIC ARITH MENU is a portion of the INTEGER sub-menu of ARITH (blue-shift 1).

2.9 The MTH (blue-shift SYMB) key
This key opens the mathematics functions menu. There are: The hyperbolic functions (sub-menu 4), like:

SINH ASINH COSH ACOSH TANH ATANH

The functions:
and some functions used on real numbers (sub-menu 5), like:

**FLOOR(X)**, returning the largest integral value not grater than \( X \).

**CEIL(X)**, returning the smallest integral value not less than \( X \).

**RND(X,n)**, that rounds \( X \) to \( n \) decimal digits.

**TRNC(X,n)**, that truncates \( X \) to \( n \) decimal digits.

### 2.10 The UNITS (red-shift 6) key

The **UNITS** menu has 127 measurement units, divided by category.

To use a measurement unit, you must write the unit preceded by \_ (red-shift \_).

You can convert from one measurement unit to another using the **CONVERT** function (you can find it into the **Tools** sub-menu of the **UNITS** menu).

**Example:**

Enter:

\[
\text{CONVERT(12\_cm,1\_m)}
\]

The result is:

\[0.12\_m\]

### 2.11 The HIST key

This key allows you to access the history while you are typing a command. The same key allows also to access the history from inside either the equation editor, or the matrix editor.

It is important to know that the object copied from the history is both copied AND evaluated.

If you want to use a result again without reevaluating it, you must use:

**ANS(1)** or **ANS(2)** (red-shift ENTER (ANS(1)).

If you want to reuse a command, you can also press **blue-shift HIST** (CMD); this key gives you a list containing the last commands you used.

### 3 Data entry

#### 3.1 The equation editor

##### 3.1.1 Entering the equation writer

The **EQW** key (**EQuationWriter**) allows you to enter the equation editor at any time, from the command line. It is a very efficient editor useful to write, simplify and work on mathematical expressions.

While you are in the equation editor you can type expressions in, knowing that any operator you type always operates either on the expression next to the cursor, on on the selected expression.

You must not be worried about entering parentheses, you simply select!

You must imagine that mathematical expressions are like a tree, (not necessarily a binary one), and understand that the four arrow keys allow you to visit the tree in a natural way (the right and left keys allow you to go from a sub-tree to another, the up and down keys allow you to go up and down in the tree hierarchy, the “shifted” right and left keys allow you to accomplish various selections; see the second example on page [10]).
3.1.2 How to select?

You can enter selection mode in two ways:

- The \( \triangle \) key enters selection mode and selects the expression element next to the cursor. Then, you can enlarge the selection to the sub-tree located immediately to the right of your present selection, by pressing \( \triangleright \).

- The \( \triangleright \) key enters selection mode and selects the sub-tree next to the cursor.

- Warning: if you are entering a function with more than one argument, (like, for example, \( \sum \), \( \int \) or \( \text{AND} \)), the \( \triangleright \) arrow allows you go from one argument to another. Therefore, you must always use the \( \triangle \) key to start selection mode in this case (see 2.1.4).

Equation writer examples:

- Example 1
  Type:
  
  \[
  2 + X \cdot 3 - X
  \]
  You obtain:
  
  \[
  2 + X \cdot 3 - X
  \]
  ENTER ENTER gives the following result:
  
  \[
  2 + 2 \cdot X
  \]
  Type:
  
  \[
  2 + X \triangleright \cdot 3 - X
  \]
  You obtain:
  
  \[
  (2 + X) \cdot 3 - X
  \]
  ENTER ENTER gives the following result:
  
  \[
  6 + 2 \cdot X
  \]
  Type:
  
  \[
  2 + X \triangleright \cdot 3 \triangle - X
  \]
  You obtain:
  
  \[
  (2 + X) \cdot (3 - X)
  \]
  ENTER ENTER gives the following result:
  
  \[
  -(X^2 - X - 6)
  \]

- Example 2
  If you want to enter:
  
  \[
  X^2 - 3 \cdot X + 1
  \]
  You type:
  
  \[
  X y^2 2 \triangleright - 3 X + 1
  \]
• Example 3
You want to enter: \[ \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \]

Here, the root of the tree is a +, and there are four sub-trees; each sub-tree has a ÷ as root, and has two leaves.

First of all, you must press \texttt{EQW}, and then you enter the first sub-tree:

\[ \frac{1}{2} \]

Then, you select this sub-tree with \( \triangleright \)

press \(+\)

and enter the second sub-tree:

\[ \frac{1}{3} \]

Then, you select this sub-tree with \( \triangleright \)

press \(+\)

and enter the third sub-tree:

\[ \frac{1}{4} \]

Then, you select this sub-tree with \( \triangleright \)

press \(+\)

and enter the fourth sub-tree:

\[ \frac{1}{5} \]

Last, press \( \triangleright \) again to select the last sub-tree you entered.

Now, the expression you want:

\[ \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \]

has been entered into the equation writer, and \( \frac{1}{6} \) is selected.

Visit the tree to select:

\[ \frac{1}{3} + \frac{1}{4} \]

You must press \( \triangleleft \)
to select \( \frac{1}{6} \); next, \( \text{red – shift} \triangleleft \)
allows you to extend the selection to two contiguous sub-trees, in this example:

\[
\frac{1}{3} + \frac{1}{4}
\]

Notice that:
You can evaluate the selected portion of the expression with:

\text{red} - \text{shift SYMB} (\text{EVAL})

You obtain:

\[
\frac{1}{2} + \frac{7}{12} + \frac{1}{5}
\]

Now, if you want to evaluate

\[
\frac{1}{2} + \frac{1}{5}
\]

first of all you must do a permutation in order to make \( \frac{1}{2} \) and \( \frac{1}{5} \) adjacent, pressing

\text{blue} - \text{shift}<

thus exchanging the selected element with his left neighbor.

You obtain:

\[
\frac{7}{12} + \frac{1}{2} + \frac{1}{5}
\]

and \( \frac{7}{12} \) is selected. Then,

\text{red} - \text{shift}>

selects

\[
\frac{1}{2} + \frac{1}{5}
\]

You can now do EVAL again.

\section*{3.1.3 How to modify an expression}

To replace the selection with another expression, you can directly type the new expression.

To remove a selection without deleting the selected expression, press:

\[
\Leftarrow
\]

To delete the selected expression, press:

\text{red} - \text{shift} \Leftarrow (\text{CLEAR})

To delete the unary operator at the root of the selected sub-tree, press:

\text{blue} - \text{shift} \Leftarrow (\text{DEL})

For example, to replace \( \sin(\text{expr}) \) with \( \cos(\text{expr}) \), you delete \( \sin(\text{expr}) \) (selecting \( \sin(\text{expr}) \) and pressing \text{blue} - \text{shift} \Leftarrow), then you enter: \text{cos}.

To delete a binary operator, you must use the \text{edit} menu option, make the correction in the command-line editor, and return to the equation writer with \text{ENTER}.

The \text{HIST} key (when used inside the equation editor) allows you to enter the history and to copy a history element into the equation writer with the \text{echo} menu option.
3.1.4 How to enter AND, ∫ and ∑

To enter AND, you type it in alpha mode and you press ▶.

To enter the ∫ symbol, you press:

\[ \text{red – shift TAN (∫)} \]

To enter the ∑ symbol, you press:

\[ \text{red – shift SIN (∑)} \]

The cursor automatically moves where input is required, and you can move it using ▶.

The expressions you enter follow the selection rules explained above, but you must use △ to
enter selection mode.

Warning: do not use the index variable i in summations, because i denotes the complex
number that solves the equation \( x^2 + 1 = 0 \).

You must also understand that ∑ is able to calculate symbolically the summations of rational fractions, and the hypergeometric series admitting a discrete primitive (starting from ROM
version 1.11).

In numeric mode, ∑ performs approximate calculations (for example, \( \sum_{k=0}^{4} \frac{1}{k!} = 2.70833333334 \),
instead of \( 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} = \frac{65}{24} \) (the ! symbol can be typed in pressing alpha red – shift 2).

3.1.5 Cursor mode

Cursor mode allows you to select a big expression fast:
press red-shift EQW (’ ) to enter cursor mode (or press on the curs menu option). Next, use
the arrows to enclose your selection in a box and press ENTER to select the box contents, or
CANCEL to cancel the operation.

3.1.6 To view all

Pressing on the big menu option, you make the font used to display the expression either bigger
or smaller: sometimes, making the font smaller allows you to view a big expression as a whole
on the screen.

If this is not yet enough, select the view option of the TOOL menu.

3.2 The matrix writer

To invoke the matrix writer, press: blue-shift EQW (MTRW).

You can then enter the elements belonging to the first line of the matrix by pressing ENTER
after each entry (you can use the equation editor to write them, too!). Next, you move the cursor
to the beginning of the second line with the arrow keys (the cursor automatically wraps around
when you finish entering the following lines).
To enter a negative number, for example $-2$, enter $+/- 2$.

If you want to enter a vector, make sure that the vect menu option is selected.

Notice that in Algebraic mode you must enter the matrix elements one at a time (pressing ENTER after each element), but in RPN mode you can write more than one element separating them with spaces; pressing ENTER then enters them all.

3.3 The text editor

This is the line that opens under the history to type a command in. It is a full-fledged text editor, where you can: select an expression (with BEGIN END), either cut it (CUT) or copy it (COPY) into a buffer, and then paste it at the current cursor position (PASTE).

Notice also that these commands work in EQW and MTRW, too.

3.3.1 BEGIN END

Move the cursor on the first character of the text you want to select, then press: red-shift APPS (BEGIN).

Then, move the cursor on the character that follows the last character to select, and press: red-shift MODE (END).

Your selection will be highlighted.

3.3.2 COPY

red-shift VAR (COPY) copies the selection into a buffer.

3.3.3 CUT

red-shift STO (CUT) copies the selection into a buffer, and deletes it from the command line.

3.3.4 PASTE

red-shift NXT (PASTE) pastes the contents of the buffer at the current cursor position (you must have previously done either COPY, or CUT, to put something into the buffer).

3.4 Variables

You can store objects into variables, and reference them using the variable name.

Be sure to notice the difference between $A$ et ‘$A$’: $A$ is evaluated (it denotes the execution of variable contents), while ‘$A$’ is not (it denotes the variable name).

For example:

STO(B, ‘$A$’): stores the contents of B into A.
STO(‘$B$’, ‘$A$’): means that B and A will have the same contents from now on.
VAR displays a menu listing all variables you have created in the current directory, as well as all
its subdirectories (you can distinguish variables from subdirectories because subdirectories have a small bar across the upper-left corner of their menu item).

The blue-shift APPS (FILES) application displays the whole variable tree starting from HOME, as well as the archive memory, and greatly simplifies variable management.

3.4.1 STO

STO allows you to create a variable and to store an object into it.

**WARNING**: STO is prefixed if you type it in alpha mode, and is infixed if you use the STO key. (from now on, this key will be denoted by either STO> or >).

Examples:

Type:

STO(1,’A’)

or

use the STO> key, displayed on screen with > :

to enter:

1 STO > A (1 > A).

Notice that, in the latter case, you don’t put ’ ’ around A.

The variable A is created, and it contains 1.

Enter:

≪12≫ STO> P

P is a variable containing the program ≪12≫, that displays 12.

3.4.2 RCL

RCL takes a variable name, surrounded by ’ , as argument, and displays the variable contents.

To recall the contents of a variable, entering the variable name is enough, UNLESS the variable contains a program (because, in the latter case, the program is executed).

In the last examples:

A displays 1 and P displays 12

but:

RCL(’A’) displays 1 and RCL(’P’) displays ≪12≫.

3.4.3 PURGE

PURGE allows you to delete a variable and its contents.

You can find PURGE in the TOOL menu.

Example:

PURGE(’A’)

15
3.4.4 Predefined variables

The name of the current symbolic variable is stored in VX (and it will usually be X), therefore you should either not use X as an ordinary variable, or purge X before doing symbolic calculations.

EPS holds the value of epsilon used by the EPSX0 command (see 4.20.1).

EQ holds the last equation you plotted.

MATRIX holds the last matrix used as argument to either JORDAN, EGV or EGVL.

MODULO holds the value of p when you do symbolic calculations in the Z/p.Z ring.

PERIOD holds the period of the function of which you want to calculate the Fourier coefficients (see 4.7.16).

PRIMIT holds the antiderivative of the last function you asked the calculator to integrate.

REALASSUME holds the names of the symbolic variables you want the calculator assume to be reals (by default, X, t and all auxiliary integration variables used).

SYSTEM holds the last system of equations used as argument to either rref or RREF, if the system has at least a parameter.

3.5 Directories

At the beginning, you only have the HOME directory; it is the ancestor of any other directory you will create in the future.

3.5.1 Creating a directory

Press blue-shift APPS (FILES) to display the tree structure of your directories.

Select the directory you want to be the parent directory (for example HOME) and press ok.

A menu containing edit copy move... is displayed; press NXT and select new (new variable or directory) with F3.

Do not fill the Object field, but fill Name instead (to do this, simply type the name you chose, and then press ok of the menu.

Then, select Directory with F3 (chk), and press ok of the menu.

Last, press CANCEL to return in HOME.

Check, by pressing VAR, that your directory has actually been created.

You can also create a directory with the CRDIR command.
You make the parent directory current, and then type:

```
CRDIR(‘NOMREP’)
```

to create a subdirectory named NOMREP.

### 3.5.2 Working in a directory

Working in a directory is simple: simply press VAR to display the subdirectory names in the menu area, and then open the subdirectory you want by pressing on the F1 corresponding to its name, followed by ENTER.

To climb up in the directory tree, press:

```
blue-shift VAR (UPDIR)
```

### 3.5.3 Deleting, renaming, moving a directory

Press blue-shift APPS (FILES) to display your directory tree.

Select the directory you want to delete, rename, move, and press the ok menu key.

A menu containing `edit copy move...purge rename...` is displayed.

- `purge` deletes the directory, if it is empty.
- `rename` gives the directory a new name.
- `copy` copies the directory (the arrow keys are used to indicate the destination, and ok confirms).
- `move` moves the directory (the arrow keys are used to indicate the destination, and ok confirms).

### 4 Plotting graphs

#### 4.1 Plot windows

##### 4.1.1 Equation entry

This window is activated with the following sequence of keys:

```
blue-shift F1 (Y=)
```

It allows you to define the equation to be plotted.

##### 4.1.2 Plot window

This window is activated with the following sequence of keys:

```
blue-shift F2 (WIN)
```

It allows you to define the plot window and to enter the lower and upper boundaries of the independent plot variable.

If boundaries are set to Default, they are assumed to be equal to the horizontal size of the plot window.

To reset a parameter to Default, you must press NXT, and then press the reset menu key.

##### 4.1.3 Graph display

This window is activated with the following sequence of keys:

```
blue-shift F3 (GRAPH)
```

It allows you to draw the plot when you have set all its parameters.
4.1.4 Plot setup
This window is activated with the following sequence of keys:
blue-shift F4 (2D/3D).
It allows you to choose the plot type, the equation to be plotted and the plot variables.

4.1.5 Table setup
This window is activated with the following sequence of keys:
blue-shift F5 (TBLSET).
It allows you to initialize a table.

4.1.6 Table display
This window is activated with the following sequence of keys:
blue-shift F6 (TABLE).
It displays the table you initialized with TBLSET.

4.2 Plot setup
4.2.1 Plot type
You can select the plot type using the choos menu key of the PLOT SETUP (blue-shift F4 (2D/3D)) window.

Here, the most common plot types will be described, such as:

Function to plot a function in cartesian coordinates.

Polar to plot a function in polar coordinates.

Parametric to plot a parametric function.

Truth to plot the solutions of an equation (the pixel at \((x,y)\) is turned on iff \(EQ\) is true).

Diff Eq to plot the solutions of the differential equation \(y' = f(x,y)\).
You can plot the solution satisfying \(y(x_0) = y_0\) on the interval \([a,b]\).
To do this, put in H-View the values of \(a\) and \(b\), then \(x_0\) into Init and \(y_0\) into Init-Soln.
The solution is plotted in two steps: first, set Final to \(b\) to plot the solution on \([x_0, b]\), draw
the plot, then set Final to \(a\) to plot the solution on \([a, x_0]\) and draw the plot again.

Slopefield to draw the slope field of the differential equation \(y' = f(x,y)\).

Fast3D to plot a surface defined by \(z = f(x,y)\).
The plot can be rotated using NXT, TOOL and the arrow \(\triangle < \triangleright \triangleright \\downarrow\) keys, to have a good view of
the surface.

4.2.2 The equation
You can enter the equation in many ways:
-you can store it into the EQ variable.
-you can enter it in the window opened by blue-shift F1 (Y=).
-you can enter it in the EQ field of the PLOT SETUP window, opened by blue-shift F4 (2D/3D).
-you can also use the \texttt{cas} function \texttt{PLOT}. It has a functions as argument, stores it into \texttt{EQ} and opens the \texttt{PLOT SETUP} window.

Notice that \texttt{EQ} can be a list of equations; in this case, all of them will be plotted on the same graphic.

You can also add \texttt{en} equation to the list of equations stored in \texttt{EQ}, with the aid of the \texttt{cas} function \texttt{PLOTADD}.

\subsection*{4.2.3 Independent variable and equation types}

The equation type depends on the plot type you have selected and on the independent variable you chose.

Depending on this, you enter an equation of type:

\begin{itemize}
  \item $f(x)$ to plot \( y = f(x) \) in cartesian coordinates, if \( x \) is the independent variable and the plot type is \texttt{Function}.
  \item $f(t)$ to plot \( r = f(t) \) in polar coordinates, if \( t \) is the independent variable and the plot type is \texttt{Polar}.
  \item $x(t) + i \cdot y(t)$ to plot \((x = x(t), y = y(t))\) in parametric coordinates, if \( t \) is the independent variable and the plot type is \texttt{Parametric}.
  \item $f(x, y) > 0$ to highlight the corresponding portion of the \( x, y \) plane, if \( x \) and \( y \) are the independent variables and the plot type is \texttt{Truth}.
  \item $f(t, y)$ to plot the solutions of the differential equation \( y' = f(t, y) \) if \( t \) is the independent variable, \( y \) is the solution variable and the plot type is \texttt{Diff Eq}.
  \item $f(t, y)$ to plot the slope field of the differential equation \( y' = f(t, y) \), if \( t \) is the independent variable, \( y \) is the solution variable and the plot type is \texttt{Slopefield}.
  \item $f(x, y)$ to plot the surface defined by \( z = f(x, y) \) if \( x \) and \( y \) are the independent variables and the plot type is \texttt{Fast3D}.
\end{itemize}

Sometimes, the name of the second independent variable can be changed; by default its name is \( y \). This name is always tagged by \texttt{Depend}, even if it does correspond to an independent variable! Do not take the word \texttt{Depend} into account in this case.

\subsection*{4.3 Drawing the plot}

Before drawing a plot, you must set up many parameters. When you have set all parameters up, to draw a plot press on: \texttt{erase draw} (if you want to erase the last plot you made) or \texttt{draw} (if you want yo keep the last plot you made).

Using the menu of one of the following windows:

\begin{itemize}
  \item \texttt{PLOT SETUP} (\texttt{blue-shift F4} \texttt{(2D/3D)})
  \item \texttt{PLOT} (\texttt{blue-shift F1} \texttt{(Y=3D)})
\end{itemize}
PLOT WINDOW (blue-shift F2 (WIN)).

You can also press:
blue-shift F3 (GRAPH) to draw the new plot without erasing the previous one.

You can review the last plot you made by pressing on <.

5 Symbolic calculations

5.1 Integers (and Gauss integers)

In this chapter, all integers can be freely replaced by Gauss integers, as an argument for all functions described here.

5.1.1 Infinite-precision integers

The calculator can handle infinite-precision integers, for example:

100!

The symbol ! can be obtained either pressing alpha red – shift 2, or using red-shift CAT (CHARS).

In the latter case, you select ! in CHAR (with the arrow keys), and then copy it into the command line using the echo1 menu key.

Since the decimal representation of 100! is very long, you can view the result using the TOOL key, followed by the view menu key.

The HIST and the up arrow keys allow you to climb up through the history, and the view menu key allows you to review previous results.

5.1.2 DEFINE

Consider the following exercise:

Calculate the first six Fermat numbers $F_k = 2^{2^k} + 1$ for $k = 1..6$, and check whether they are prime.

Type the expression:

\[ 2^{2^2} + 1 \]

to find 17, then invoke the ISPRIME?() command with ANS(1) as argument.

You can find this command in the ARITH (blue-shift 1) menu, sub-menu 1 INTEGER (or you can type it in α mode).

The answer is 1., meaning true.

With the aid of the history, (HIST) you can copy the expression $2^{2^2} + 1$ into the command line, and modify it to read as:

\[ 2^{2^3} + 1 \]

Otherwise, you can type the expression $2^{2^k} + 1$ STO FK, then do 3 STO k, and so on...

Otherwise, and it is the better choice, you can define the function $F(K)$ with the aid of DEF (blue-shift 2), entering:

```
DEFINE(F(K) = 2^{2^k} + 1)
```
The result is NOVAL and F is added to the variables (press on VAR to check this).

For $K = 5$ you enter: $F(5)$ obtaining: $4294967297$

You can factorize $F_5$ with FACTOR; you find it in the ALG (red-shift 4) menu.

Type: $\text{FACTOR}(F(5))$

You obtain: $641 \cdot 6700417$

For $F(6)$ you find: $18446744073709551617$

Factorizing the result with FACTOR, the result is: $274177 \cdot 67280421310721$

Notice the difference in notation between:

\[
2.5 = \frac{5}{2}
\]

and

\[
2 \cdot 5 = 10
\]

### 5.1.3 GCD

GCD denotes the greatest common divisor of two integers (or of two lists of integers with the same size).

Enter: $\text{GCD}(18, 15)$

You obtain: 3

Enter: $\text{GCD}([18, 28], [15, 21])$

You obtain: \{3, 7\}

because $\text{GCD}(18, 15) = 3$ and $\text{GCD}(28, 21) = 7$.

### 5.1.4 LGCD

LGCD denotes the greatest common divisor of a list of integers.

Enter: $\text{LGCD}([18, 15, 21, 36])$

You obtain: 3
5.1.5 SIMP2

SIMP2 has two integers as arguments (or two lists of integers). These integers are assumed to represent a fraction: the first element of the list is the fraction’s numerator, the second is the denominator. SIMP2 returns a list of two integers representing, under the same assumptions, the input fraction simplified.

Enter: \[ \text{SIMP2}(18, 15) \]
You obtain: \{6, 5\}

Enter: \[ \text{SIMP2}([18, 28], [15, 21]) \]
You obtain: \{6, 5, 4, 3\}

5.1.6 LCM

LCM denotes the least common multiple of two integers (or of two lists of integers).

Enter: \[ \text{LCM}(18, 15) \]
You obtain: 90

5.1.7 FACTOR

FACTOR factorizes its argument into a product of prime factors.

Enter: \[ \text{FACTOR}(90) \]
You obtain: \(2.3^2.5\)

5.1.8 FACTORS

FACTORS does the same, but the result is given as a list, containing the prime factors and their exponents.

Type: \[ \text{FACTORS}(90) \]
You obtain: \{2, 1, 3, 2, 5, 1\}.

5.1.9 DIVIS

DIVIS returns a list containing all divisors of a given integer.

Type: \[ \text{DIVIS}(36) \]
You obtain: \{1, 3, 9, 2, 6, 18, 4, 12, 36\}

22
5.1.10 IQUOT

IQUOT returns the integer quotient of the euclidean division of two integers.

Type:

\[ \text{IQUOT}(148, 5) \]

You obtain:

29

5.1.11 IREMINDER MOD

IREMAINDER returns the integer remainder of the euclidean division of two integers.

You type:

\[ \text{IREMAINDER}(148, 5) \]

or

\[ 148 \text{ MOD } 5 \]

You obtain:

3

The difference between IREMINDER and MOD is that the former works with both integers and Gauss integers.

Try:

\[ \text{IREMAINDER}(148!, 5! + 2) \]

(you obtain ! with alpha red-shift 2).

5.1.12 IDIV2

IDIV2 returns a list containing the quotient and the remainder of the euclidean division between two integers, in that order.

Type:

\[ \text{IDIV2}(148, 5) \]

You obtain:

\{29, 3\}

In step-by-step mode, the calculator shows the division process like it is taught at school.

5.1.13 ISPRIME?

ISPRIME?(N) returns 1. (true) if N is pseudo-prime, and returns 0. (false) if N is not prime.

Definition: For all integers less than $10^{14}$ pseudo-primality and primality are the same! ...beyond $10^{14}$ a pseudo-prime integer has a very high probability to be prime (see the Rabin algorithm in section 7.6).

Type:

\[ \text{ISPRIME?}(13) \]

You obtain:

1.

Type:

\[ \text{ISPRIME?}(14) \]

You obtain:

0.
5.1.14 NEXTPRIME

NEXTPRIME(N) returns the smallest pseudo-prime number greater than N.

Type: 

NEXTPRIME(75)

You obtain: 

79

5.1.15 PREVPRIME

PREVPRIME(N) returns the largest pseudo-prime number less than N.

Type: 

PREVPRIME(75)

You obtain: 

73

5.1.16 IEGCD

IEGCD(A,B) returns the extended GCD (Bézout identity) of two integers, that is, IEGCD(A,B) returns {D,U,V} so that AU+BV=D and D=GCD(A,B).

Type: 

IEGCD(48,30)

You obtain: 

{6,2,−3}

In fact: 

2·48+(−3)·30=6

5.1.17 IABCUV

IABCUV(A,B,C) returns {U,V} so that AU+BV=C.

C must be a multiple of GCD(A,B) for a solution to exist.

Type: 

IABCUV(48,30,18)

You obtain: 

{6,−9}

5.1.18 ICHINREM

ICHINREM([A,P],[B,Q]) returns a vector [X, N] so that:

X=A (mod P) and X=B (mod Q).

The solution X exists if P and Q are mutually prime, and all solutions are congruent modulo N = P · Q

Example:

Solve: 

\[
\begin{align*}
X & = 3 \pmod{5} \\
X & = 9 \pmod{13}
\end{align*}
\]

Type: 

ICHINREM([3,5],[9,13])
You obtain: \([-147, 65]\)
that is, \(X = -147 \pmod{65}\)

5.1.19 \textbf{PA2B2}

\textbf{PA2B2} decomposes a prime integer \(p\), congruent to 1 modulo 4, as follows: \(p = a^2 + b^2\). The calculator returns the result as \(a + bi\).

Type: \(\text{PA2B2}(17)\)

You obtain: \(4 + i\)
that is, \(17 = 4^2 + 1^2\)

5.1.20 \textbf{EULER}

\textbf{EULER} denotes the Euler’s totient function of an integer. \(\text{EULER}(n)\) is the number of integers less than \(n\) and prime with \(n\).

Type: \(\text{EULER}(21)\)

you obtain: \(12\)

In fact, the set:
\(E = \{2, 4, 5, 7, 8, 10, 11, 13, 15, 16, 17, 19\}\) is the set of integers less than 21 and prime with 21, and it has 12 elements.

5.2 \textbf{Rationals}

Type: \(\frac{123}{12} + \frac{57}{21}\)
and then \text{ENTER}; the answer is: \(\frac{363}{28}\)
with \text{red-shift} \text{ ENTER} (\rightarrow \text{NUM}) the answer is:
\(12.9642857143\)

If you mix both representations, for example:
\(\frac{1}{2} + 0.5\)
the calculator demands to enable \text{approx} mode to carry out the calculation; you should answer \text{yes} to obtain:
\(1\).

Now, return to exact mode (\text{MODE cas} menu keys, and so on...).
5.2.1 PROPFRAC

PROPFRAC(A/B) rewrites the fraction \( \frac{A}{B} \) as:

\[
Q + \frac{R}{B} \quad \text{with} \quad 0 \leq R < B
\]

Type:

\[
\text{PROPFRAC}(43 \div 12)
\]

You obtain:

\[3 + \frac{7}{12}\]

5.2.2 FXND

FXND has a fraction as argument, and returns a list containing the fraction’s numerator and denominator simplified.

Type:

\[
\text{FXND}(42 \div 12)
\]

You obtain:

\[\{7, 2\}\]

5.2.3 SIMP2

SIMP2 (see 4.1.5) has a list of two integers representing a fraction as argument and, like FXND, returns a list containing the fraction’s numerator and denominator simplified.

Type:

\[
\text{SIMP2}(\{42, 12\})
\]

You obtain:

\[\{7, 2\}\]

5.3 Reals

Type:

\[
\text{EXP}(\pi \times \sqrt{20})
\]

followed by ENTER; the answer is:

\[
\text{EXP}(2 \times \sqrt{5} \times \pi)
\]

with red-shift ENTER (→NUM) the answer is:

1263794.7537

5.4 Complex numbers

Type:

\[
(1 + 2 \cdot i)^2
\]

follower by ENTER.

If you aren’t in complex mode, the calculator asks for a mode change: you should answer yes to obtain the answer:

\[-(3 - 4 \cdot i)\]
Notice that this expression is not simplified beforehand (all results always put in evidence complex numbers with a positive real part in exact mode).

In the red-shift 1 (CMPLX) menu you will find the following functions, having a complex-valued expression as argument:
- \textsc{arg} returns the argument of its input.
- \textsc{abs} returns the modulo of its argument.
- \textsc{conj} returns the conjugate of its argument.
- \textsc{re} returns the real part of its argument.
- \textsc{im} returns the imaginary part of its argument.
- \textsc{neg} returns the opposite of its argument.
- \textsc{sign} returns the quotient between its argument and its argument’s modulo.

For example:
Type:
\[ \textsc{arg}(3 + 4 \cdot i) \]
you obtain:
\[ \text{atan}(\frac{4}{3}) \]

5.5 Algebraic expressions

5.5.1 \texttt{factor}

\texttt{factor} factorizes the expression given as argument.

Example:
Factorize \( x^4 + 1 \)
Type:
\[ \text{factor}(x^4 + 1) \]
You can find \texttt{factor} in the \texttt{alg} (red-shift 4) menu (or you can type it in \textalpha\ mode).

In real mode, the answer is:
\[ (x^2 + \sqrt{2} \cdot x + 1) \cdot (x^2 - \sqrt{2} \cdot x + 1) \]

In complex mode (to enable this mode, press \texttt{mode}, then the \texttt{cas} menu key, then check the \texttt{complex} field with \texttt{chk} and press \texttt{ok ok}) the answer is:
\[ (2 \cdot x + (1 + i) \cdot \sqrt{2}) \cdot (2 \cdot x - (1 - i) \cdot \sqrt{2}) \cdot (2 \cdot x + (1 - i) \cdot \sqrt{2}) \cdot (2 \cdot x - (1 + i) \cdot \sqrt{2}) \]

5.5.2 \texttt{expand eval}

\texttt{expand} and \texttt{eval} have an expression as argument; they expand and simplify their input.

Example:
Doing \texttt{expand(ans(1))}, you obtain again
\[ x^4 + 1. \]
5.5.3 SUBST

SUBST has two arguments: an expression depending on a parameter, and an equality (parameter=substitution value).
SUBST does the commanded substitution in the input expression, and returns the result.
Type:

\[
\text{SUBST}(A^2 + 1, A = 2)
\]
You obtain:

\[2^2 + 1\]

5.5.4 PREVAL

PREVAL has three arguments: an expression \(F(VX)\) depending on the variable stored in \(VX\) and two expressions: \(A\) and \(B\).
PREVAL evaluates \(F(B) - F(A)\).
PREVAL is useful to compute a definite integral given an antiderivative: you evaluate the antiderivative between the upper and lower limits of the integral.
Type:

\[
\text{PREVAL}(X^2 + X, 2, 3)
\]
You obtain:

\[12 - 6\]

5.6 Functions

5.6.1 DERVX

Let \(f(x)\) be:

\[
f(x) = \frac{x}{x^2 - 1} + \ln\left(\frac{x + 1}{x - 1}\right)
\]
Determine the derivative of \(f\).
You can find DERVX in the menu:
CALC (blue-shift 4), sub-menu 1.DERIV. & INT..., third position (or you can type it in mode \(\alpha\)).

Type either:

\[
\text{DERVX}\left(\frac{x}{x^2 - 1} + \ln\left(\frac{x + 1}{x - 1}\right)\right)
\]
or, if you have previously stored \(f(x)\) into \(F\):

\[
\text{DERVX}(F)
\]
or, if you have defined \(F(X)\) using DEFINE: (DEFINE\((F(X) = \frac{x}{x^2 - 1} + \ln\left(\frac{x + 1}{x - 1}\right))\))

\[
\text{DERVX}(F(X))
\]
The result is an involved expression that you can simplify by copying it into the command line (△ ENTER ENTER)).
You obtain:

\[-\frac{3 \cdot X^2 - 1}{X^4 - 2 \cdot X^2 + 1}\]

5.6.2 DERIV

DERIV has two arguments: an expression (or a functions), and a variable (or a vector containing more than one variable name) (see multivariate functions, paragraph 4.16.1).
DERIV returns the derivative of the expression (or function) with respect to the given variable(s) (useful to calculate partial derivatives!).
Example:
Suppose you should calculate:

\[\frac{\partial(x \cdot y^2 \cdot z^3 + x \cdot y)}{\partial z}\]

Type:

DERIV(X \cdot Y^2 \cdot Z^3 + X \cdot Y, Z)

You obtain:

\[3 \cdot X \cdot Y^2 \cdot Z^2\]

5.6.3 INTVX

Let \( f(x) \) be:

\[f(x) = \frac{x}{x^2 - 1} + \ln\left(\frac{x + 1}{x - 1}\right)\]

Determine an antiderivative of \( f \).
You can find INTVX in the CALC (blue-shift 4) menu, 1. DERIV. & INT... sub-menu, eighth position (or you can type it in α mode).
You type either:

INTVX\left(\frac{X}{X^2 - 1} + \ln\left(\frac{X + 1}{X - 1}\right)\right)

or, if you have previously stored \( f(x) \) into \( F \):

INTVX(F)

or, if you have defined \( F(X) \) using DEFINE (DEFINE(F(X) = \( \frac{X}{X^2 - 1} + \ln(\frac{X + 1}{X - 1})\))):

INTVX(F(X))

You obtain:

\[X \cdot \ln(\frac{X + 1}{X - 1}) + \frac{3}{2} \cdot \ln(|X - 1|) + \frac{3}{2} \cdot \ln(|X + 1|)\]

Exercise 1
Calculate:

\[\int \frac{2}{x^6 + 2 \cdot x^4 + x^2} \, dx\]

Type:

INTVX\left(\frac{2}{X^6 + 2 \cdot X^4 + X^2}\right)
You obtain:
\[-3 \cdot \text{ATAN}(X) - \frac{2}{X} = \frac{X}{X^2 + 1}\]

Notice:
You can also enter the expression using the equation writer (EQW key):
\[
\int_1^X \frac{2}{X^6 + 2 \cdot X^4 + X^2} \, dX
\]
This gives the same result, barring an integration constant equal to
\[
\frac{3 \cdot \pi + 10}{4}
\]

Exercise 2
Calculate:
\[
\int \sin(x) + \sin(2 \cdot x) \, dx
\]
Type:
\[
\text{INTVX}(\frac{1}{\sin(X) + \sin(2 \cdot X)})
\]
You find:
\[-\frac{1}{6} \cdot \ln(|\cos(X) - 1|) + \frac{1}{2} \cdot \ln(|\cos(X) + 1|) + \frac{-2}{3} \cdot \ln(|2 \cdot \cos(X) + 1|) - \ln(2)\]

5.6.4 LIMIT
Find, for \(n > 2\), the limit when \(x\) approaches 0 of:
\[
\frac{n \times \tan(x) - \tan(n \times x)}{\sin(n \times x) - n \times \sin(x)}
\]
You can use the LIMIT command, found in the menu:
\text{CALC (blue-shift 4), sub-menu 2 LIMIT & SERIES} (or you can type it in \(\alpha\) mode).
Type:
\[
\text{LIMIT}\left(\frac{N \cdot \text{TAN}(X) - \text{TAN}(N \cdot X)}{\text{SIN}(N \cdot X) - N \cdot \text{SIN}(X)}, 0\right)
\]
You obtain:
\[2\]
Find the limit, when \(x\) approaches \(+\infty\), of:
\[
\sqrt{x} + \sqrt{x + \sqrt{x}} - \sqrt{x}
\]
Type:
\[
\text{LIMIT}\left(\sqrt{X} + \sqrt{X + \sqrt{X}} - \sqrt{X}, +\infty\right)
\]
After a moment, you obtain:
\[\frac{1}{2}\]
Notice that you obtain \(+\infty\) pressing:
\[\text{+/-} \quad \text{+/-} \quad \text{+/-} \quad \text{+/-} \quad \infty \quad \text{(blue - shift 0)}\]
5.6.5 LIMIT and ∫

Find the limit, when \( a \) approaches \(+\infty\), of:

\[
\int_{2}^{a} \left( \frac{x}{x^2 - 1} + \ln\left(\frac{x + 1}{x - 1}\right) \right) \, dx
\]

In the equation writer, type:

\[
\int_{2}^{+\infty} \left( \frac{X}{X^2 - 1} + \text{LN}\left(\frac{X + 1}{X - 1}\right) \right) \, dX
\]

Notice that you obtain \(+\infty\) pressing:

\(+/-\) \(+/-\) \(\infty\) (blue – shift 0)

You obtain:

\(+\infty - \frac{7 \cdot \text{LN}(3)}{2}\)

and, after simplification:

\(+\infty\)

5.6.6 IBP

IBP has two arguments: an expression you can write as \(u(x) \cdot v'(x)\) and \(v(x)\).

IBP returns a list containing \(u(x) \cdot v(x)\) and \(-v(x) \cdot u'(x)\), that is, the two terms you must calculate when doing an integration by parts.

You must then integrate the second term and add the result to the first term to obtain an antiderivative of \(u(x) \cdot v'(x)\) (this is handy in RPN mode!).

Type:

\(\text{IBP}(\text{LN}(X), X)\)

You obtain:

\({X \cdot \text{LN}(X), -1}\)

Notice: If the first argument of IBP is a list of two elements, IBP only operates on the last element of the list, and adds the integration result to the first element (so that it is easy to invoke IBP multiple times in algebraic mode).

5.6.7 RISCH

RISCH has two arguments: an expression and the name of a variable.

RISCH returns an antiderivative of the first argument with respect to the variable given as the second argument.

Type:

\(\text{RISCH}\left( (2 \cdot X^2 + 1) \cdot \text{EXP}(X^2 + 1), X \right)\)

You obtain:

\(X \cdot \text{EXP}(X^2 + 1)\)
5.7 Trigonometric expressions

5.7.1 TEXPAND

TEXPAND has a trigonometric expression as argument.
TEXPAND expands this expression with respect to \( \sin(x) \) and \( \cos(x) \).

Example 1:
Type:

\[
\text{TEXPAND}(\cos(x + y))
\]
You obtain:
\[
\cos(y) \cdot \cos(x) - \sin(y) \cdot \sin(x)
\]

Example 2:
Type:

\[
\text{TEXPAND}(\cos(3 \cdot x))
\]
You obtain:
\[
4 \cdot \cos(x)^3 - 3 \cdot \cos(x)
\]

Example 3:
Type:

\[
\text{TEXPAND}\left(\frac{\sin(3 \cdot x) + \sin(7 \cdot x)}{\sin(5 \cdot x)}\right)
\]
You obtain, after one simplification step (\(\triangle\) ENTER):
\[
4 \cdot \cos(x)^2 - 2
\]

5.7.2 TLIN

TLIN has a trigonometric expression as argument.
TLIN linearizes this expression in function of \( \sin(n \cdot x) \) and \( \cos(n \cdot x) \).

Example:
Type:

\[
\text{TLIN}(\cos(x) \cdot \cos(y))
\]
You obtain:
\[
\frac{1}{2} \cdot \cos(x - y) + \frac{1}{2} \cdot \cos(x + y)
\]

Example 2:
Type:

\[
\text{TLIN}(\cos(x)^3)
\]
You obtain:
\[
\frac{1}{4} \cdot \cos(3 \cdot x) + \frac{3}{4} \cdot \cos(x)
\]
Example 3:
Type:

\[ TLIN(4 \cdot \cos(x)^2 - 2) \]

You obtain:

\[ 2 \cdot \cos(2 \cdot x) \]

5.7.3 TCOLLECT

TCOLLECT has a trigonometric expression as argument. TCOLLECT linearizes this expression in function of \( \sin(n \cdot x) \) and \( \cos(n \cdot x) \), then collects in real mode sines and cosines of the same angle.

Type:

\[ \text{TCOLLECT} (\sin(x) + \cos(x)) \]

You obtain:

\[ \sqrt{2} \cdot \cos(x - \frac{\pi}{4}) \]

5.7.4 ACOS2S

ACOS2S has a trigonometric expression as argument. ACOS2S rewrites this expression replacing \( \arccos(x) \) with \( \frac{\pi}{2} - \arcsin(x) \).

Type:

\[ \text{ACOS2S} (\arccos(x) + \arcsin(x)) \]

You obtain:

\[ \frac{\pi}{2} \]

5.7.5 ASIN2C

ASIN2C has a trigonometric expression as argument. ASIN2C rewrites this expression replacing \( \arcsin(x) \) with \( \frac{\pi}{2} - \arccos(x) \).

Type:

\[ \text{ASIN2C} (\arccos(x) + \arcsin(x)) \]

You obtain:

\[ \frac{\pi}{2} \]
5.7.6  ASIN2T

ASIN2T has a trigonometric expression as argument.
ASIN2T rewrites this expression replacing \( \arcsin(x) \) with \( \arctan \left( \frac{x}{\sqrt{1-x^2}} \right) \).
Type:

\[
\text{ASIN2T} \left( \arcsin(X) \right)
\]

You obtain:

\[
\arctan \left( \frac{x}{\sqrt{1-x^2}} \right)
\]

5.7.7  ATAN2S

ATAN2S has a trigonometric expression as argument.
ATAN2S rewrites this expression replacing \( \arctan(x) \) with \( \arcsin \left( \frac{x}{\sqrt{1+x^2}} \right) \).
Type:

\[
\text{ATAN2S} \left( \arctan(X) \right)
\]

You obtain:

\[
\arcsin \left( \frac{x}{\sqrt{x^2+1}} \right)
\]

5.7.8  SINCOS

SINCOS accepts as argument an expression containing complex exponentials.
SINCOS rewrites this expression in function of \( \sin(x) \) and \( \cos(x) \).
Type:

\[
\text{SINCOS} \left( \exp(i \cdot X) \right)
\]

You obtain:

\[
\cos(X) + i \cdot \sin(X)
\]

5.7.9  TAN2SC

TAN2SC has a trigonometric expression as argument.
TAN2SC rewrites this expression replacing \( \tan(x) \) with \( \frac{\sin(x)}{\cos(x)} \).
Type:

\[
\text{TAN2SC} \left( \tan(X) \right)
\]

You obtain:

\[
\frac{\sin(X)}{\cos(X)}
\]
5.7.10 TAN2SC2

TAN2SC2 has a trigonometric expression as argument. TAN2SC2 rewrites this expression replacing \( \tan(x) \) with \( \frac{\sin(2 \cdot x)}{1 + \cos(2 \cdot x)} \) (or with \( \frac{1 - \cos(2 \cdot x)}{\sin(2 \cdot x)} \) if you prefer sines, that is, when flag -116 is set to Prefer sin(); see [1.4] for more details).

Type:

\[ \text{TAN2SC2}(\tan(x)) \]

You obtain:

\[ \frac{\sin(2 \cdot x)}{1 + \cos(2 \cdot x)} \]

5.7.11 HALFTAN

HALFTAN has a trigonometric expression as argument. HALFTAN rewrites \( \sin(x) \), \( \cos(x) \) and \( \tan(x) \) terms of the expression in function of \( \tan(\frac{x}{2}) \).

Type:

\[ \text{HALFTAN}\left(\frac{\sin(2 \cdot x)}{1 + \cos(2 \cdot x)}\right) \]

You obtain, after simplification:

\[ \tan(x) \]

Type:

\[ \text{HALFTAN}(\sin(x)^2 + \cos(x)^2) \]

You obtain (\( \text{SQ}(x) = x^2 \)):

\[ \left( \frac{2 \cdot \tan(\frac{x}{2})}{\text{SQ}(\tan(\frac{x}{2}))+1} \right)^2 + \left( \frac{1 - \text{SQ}(\tan(\frac{x}{2}))}{\text{SQ}(\tan(\frac{x}{2}))+1} \right)^2 \]

You obtain, after simplification:

\[ 1 \]

5.7.12 TRIG

TRIG has a trigonometric expression as argument. TRIG simplifies this expression using the identity: \( \sin(x)^2 + \cos(x)^2 = 1 \).

Type:

\[ \text{TRIG}(\sin(x)^2 + \cos(x)^2 + 1) \]

You obtain:

\[ 2 \]
5.7.13 TRIGSIN

TRIGSIN has a trigonometric expression as argument. TRIGSIN simplifies this expression using the identity: \( \sin(x)^2 + \cos(x)^2 = 1 \), privileging and preserving \( \sin(x) \) terms.
Type:

\[
\text{TRIGSIN}(\sin(x)^4 + \cos(x)^2 + 1)
\]

You obtain:

\[
\sin(x)^4 - \sin(x)^2 + 2
\]

5.7.14 TRIGCOS

TRIGCOS has a trigonometric expression as argument. TRIGCOS simplifies this expression using the identity: \( \sin(x)^2 + \cos(x)^2 = 1 \), privileging and preserving \( \cos(x) \) terms.
Type:

\[
\text{TRIGCOS}(\sin(x)^4 + \cos(x)^2 + 1)
\]

You obtain:

\[
\cos(x)^4 - \cos(x)^2 + 2
\]

5.7.15 TRIGTAN

TRIGTAN has a trigonometric expression as argument. TRIGTAN simplifies this expression using the identity: \( \sin(x)^2 + \cos(x)^2 = 1 \), privileging and preserving \( \tan(x) \) terms.
Type:

\[
\text{TRIGTAN}(\sin(x)^4 + \cos(x)^2 + 1)
\]

You obtain:

\[
\frac{2 \cdot \tan(x)^4 + 3 \cdot \tan(x)^2 + 2}{\tan(x)^4 + 2 \cdot \tan(x)^2 + 1}
\]

5.7.16 FOURIER

FOURIER has two arguments: an expression \( f(x) \) and an integer \( n \). FOURIER returns the Fourier coefficient \( c_n \) of \( f(x) \). \( f(x) \) is assumed to be a periodic function defined on the interval \([0, T]\), with period \( T \). (\( T \) is the current value of the PERIOD variable).
If \( f \) is piecewise continuous:

\[
f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{i n \pi x / T}
\]

Example: Find the Fourier coefficients of the function \( f \); the period of \( f \) is \( 2\pi \), and \( f \) is defined on \([0, 2\pi]\) as \( f(x) = x^2 \).
Type:

\[
2 \cdot \pi \ \text{STO} \triangleright \text{PERIOD}
\]
You obtain after simplification:

\[ \frac{2 \cdot i \cdot N \cdot \pi + 2}{N^2} \]

So, if \( n \neq 0 \):

\[ c_n = \frac{2 \cdot i \cdot N \cdot \pi + 2}{N^2} \]

Then, type:

\[ \text{FOURIER}(X^2, 0) \]

You obtain:

\[ \frac{4 \cdot \pi^2}{3} \]

So, if \( n = 0 \):

\[ c_0 = \frac{4 \cdot \pi^2}{3} \]

\[ \text{5.8 Exponentials and Logarithms} \]

\[ \text{5.8.1 EXPLN} \]

EXPLN has a trigonometric expression as argument.
EXPLN rewrites the trigonometric expression in terms of exponentials and logarithms WITHOUT linearization.
EXPLN demands to put the calculator in complex mode.
Type:

\[ \text{EXPLN} (\sin(X)) \]

You obtain:

\[ \exp(i \cdot X) - \frac{1}{2 \cdot i} \frac{\exp(i \cdot X)}{\exp(-i \cdot X)} \]

\[ \text{5.8.2 LIN} \]

LIN has an expression containing exponentials and trigonometric functions as argument.
LIN linearizes the expression (that is, it rewrites the expression in terms of \( \exp(n \cdot x) \)).
LIN demands to put the calculator in complex mode when the input expression contains trigonometric functions.

Example 1:
Type:

\[ \text{LIN} (\sin(X)) \]

You obtain:

\[ -(\frac{1}{2} \cdot \exp(1 \cdot X)) + \frac{1}{2} \cdot \exp(-i \cdot X)) \]
Example 2:
Type:
\[ \text{LIN}((\cos(X)^2) \]
You obtain:
\[-(\frac{1}{4} \cdot \exp(2 \cdot i \cdot X)) + \frac{1}{2} + \frac{1}{4} \cdot \exp(-2 \cdot i \cdot X) \]

Example 3:
Type:
\[ \text{LIN}((\exp(X) + 1)^3) \]
You obtain:
\[ 3 \cdot \exp(X) + 1 + 3 \cdot \exp(2 \cdot X) + \exp(3 \cdot X) \]

5.8.3 LNCOLLECT
LNCOLLECT has an expression containing logarithms as argument.
LNCOLLECT collects the logarithmic terms. Therefore, it is better to use it on a factorized expression (using FACTOR beforehand).
Type:
\[ \text{LNCOLLECT}(\ln(X + 1) + \ln(X - 1)) \]
You obtain:
\[ \ln((X + 1)(X - 1)) \]

5.8.4 TSIMP
TSIMP has an expression as argument; it simplifies the expression rewriting it in function of complex exponentials (enabling complex mode in the process), and then reducing the number of variables as returned by LVAR (see section 4.20.2).
Use TSIMP only as a last resort.
Type:
\[ \text{TSIMP}(\frac{\sin(3 \cdot X) + \sin(7 \cdot X)}{\sin(5 \cdot X)}) \]
You obtain after simplification (that is, after copying the result 2 times):
\[ \frac{\exp(i \cdot X)^4 + 1}{\exp(i \cdot X)^2} \]

5.9 Polynomials
5.9.1 GCD
GCD returns the gcd (greatest common divisor) of two polynomials (or of two lists of polynomials with the same length).
Type:
\[ \text{GCD}(X^2 + 2 \cdot X + 1, X^2 - 1) \]
You obtain: \[ X + 1 \]

Type: \[ \text{GCD}\{\{X^2 + 2 \cdot X + 1, X^3 - 1\}, \{X^2 - 1, X^2 + X - 2\}\} \]

You obtain: \[ \{X + 1, X - 1\} \]

5.9.2 LGCD

LGCD denotes the gcd (greatest common divisor) of a list of polynomials.

LGCD returns a list containing the given list of polynomials and the GCD of all polynomials of the list.

Type: \[ \text{LGCD}\{(X^2 + 2 \cdot X + 1, X^3 + 1, X^2 - 1, X^2 + X)\} \]

You obtain: \[ \{\{X^2 + 2 \cdot X + 1, X^3 + 1, X^2 - 1, X^2 + X\}, X + 1\} \]

5.9.3 SIMP2

SIMP2 has two polynomials (or two lists of polynomials with the same length) as arguments. These two polynomials are considered as representing a rational fraction. SIMP2 returns the simplified rational fraction, represented as a list of two polynomials.

Type: \[ \text{SIMP2}(X^3 - 1, X^2 - 1) \]

You obtain: \[ \{X^2 + X + 1, X + 1\} \]

5.9.4 LCM

LCM returns the lcm (least common multiple) of two polynomials (or of two lists of polynomials with the same length).

Type: \[ \text{LCM}(X^2 + 2 \cdot X + 1, X^2 - 1) \]

You obtain: \[ (X^2 + 2 \cdot X + 1) \cdot (X - 1) \]

5.9.5 FACTOR

FACTOR has either a polynomial or a list of polynomials as argument. FACTOR factors its input.

Type: \[ \text{FACTOR}(X^2 + 2 \cdot X + 1) \]

You obtain: \[ (X + 1)^2 \]

Type: \[ \text{FACTOR}(X^4 - 2 \cdot X^2 + 1) \]
You obtain:
\((X - 1)^2 \cdot (X + 1)^2\)

Type:
```
FACTOR({X^3 - 2 \cdot X^2 + 1, X^2 - X})
```

You obtain:
\[
\left\{ \frac{(X - 1) \cdot (2 \cdot X + -1 + \sqrt{5}) \cdot (2 \cdot X - (1 + \sqrt{5}))}{4}, X \cdot (X - 1) \right\}
\]

**5.9.6 FACTORS**

FACTORS has either a polynomial or a list of polynomials as argument.

FACTORS returns a list containing the factors of the polynomial and their exponents.

Type:
```
FACTORS(X^2 + 2 \cdot X + 1)
```

you obtain:
\[
\{X + 1, 2\}
\]

Type:
```
FACTORS(X^4 - 2 \cdot X^2 + 1)
```

You obtain:
\[
\{X - 1, 2, X + 1\}
\]

Type:
```
FACTORS({X^3 - 2 \cdot X^2 + 1, X^2 - X})
```

You obtain:
\[
\left\{ \{X - 1, 2 \cdot X + -1 + \sqrt{5}, 1, 2 \cdot X - (1 + \sqrt{5}), 1, 4, -1\}, \{X, 1, X - 1, 1\} \right\}
\]

**5.9.7 DIVIS**

DIVIS has either a polynomial or a list of polynomials as argument, and returns the list of its divisors.

Type:
```
DIVIS(X^4 - 1)
```

You obtain:
\[
\{1, X^2 + 1, X - 1, X^3 - X^2 + X - 1, X + 1, X^3 + X^2 + X + 1, X^2 - 1, X^4 - 1\}
\]

**5.9.8 QUOT**

QUOT returns the quotient of the division between two polynomials.

Type:
```
QUOT(X^2 + 2 \cdot X + 1, X)
```

You obtain:
\[
X + 2
\]
5.9.9 REMAINDER

REMAINDER returns the remainder of the division between two polynomials.
Type: \[\text{REMAINDER}(X^3 - 1, X^2 - 1)\]
you obtain: \[X - 1\]

5.9.10 DIV2

Returns a list containing both the quotient and the remainder of the division between two polynomials.
Type: \[\text{DIV2}(X^2 + 2 \cdot X + 1, X)\]
You obtain: \[\{X + 2, 1\}\]
The step-by-step mode can be of interest here, because it displays the intermediate steps of the division process.

5.9.11 EGCD

This command applies the Bézout identity (Extended Greatest Common Divisor). \[\text{EGCD}(A[X], B[X])\] returns \[\{D[X], U[X], V[X]\}\], where \(D, U, V\) satisfy the following relation:
\[D[X] = U[X] \cdot A[X] + V[X] \cdot B[X]\]
Type: \[\text{EGCD}(X^2 + 2 \cdot X + 1, X^2 - 1)\]
You obtain: \[\{2 \cdot X + 2, 1, -1\}\]

5.9.12 ABCUV

This command applies the Bézout identity like \(\text{EGCD}\) but, now, the arguments are three polynomials, \(A, B, C\) (\(C\) must be a multiple of \(\text{GCD}(A,B)\)):
\[\text{ABCUV}(A[X], B[X], C[X])\] returns \[\{U[X], V[X]\}\], where \(U, V\) satisfy the following:
\[C[X] = U[X] \cdot A[X] + V[X] \cdot B[X]\]
Type: \[\text{ABCUV}(X^2 + 2 \cdot X + 1, X^2 - 1, X + 1)\]
You obtain: \[\left\{\frac{1}{2}, -\frac{1}{2}\right\}\]
Type: \[\text{ABCUV}(X^2 + 2 \cdot X + 1, X^2 - 1, X^3 + 1)\]
You obtain: \[\left\{\frac{X^2 - X + 1}{2}, \frac{X^2 - X + 1}{2}\right\}\]
5.9.13 HORNER

HORNER has two arguments: a polynomial $P[X]$ and a number $a$; it returns a list containing $Q[X]$ (quotient of $P[X]$ divided by $X - a$), $a$, and $P[a]$.

Type:

```
HORNER(X^4 + 2 \cdot X^3 - 3 \cdot X^2 + X - 2, 1)
```

You obtain:

```
{X^3 + 3 \cdot X^2 + 1, 1, -1}
```

5.9.14 PTAYL

Rewrites a polynomial $P[X]$ in function of the powers of $X - a$.

PTAYL has two arguments: a polynomial $P$ and a number $a$.

Type:

```
PTAYL(X^2 + 2 \cdot X + 1, 2)
```

you obtain the polynomial $Q[X]$:

```
X^2 + 6 \cdot X + 9
```

Warning, notice that:

```
P(X) = Q(X - 2)
```

5.9.15 ZEROS

ZEROS has two arguments: a polynomial $P$ and a variable name.

ZEROS returns a list containing the zeros of $P$ with respect to the given variable, WITHOUT their multiplicity.

Type:

```
ZEROS(X^4 - 1, X)
```

You obtain:

- in real mode

```
{-1, 1}
```

- in complex mode

```
{-1, 1, -i, i}
```

5.9.16 PROOT

PROOT is the numeric command of the HP48.

PROOT has a vector containing the coefficients of a monovariate polynomial (ordered by decreasing powers of the polynomial’s variable) as argument.

PROOT returns a vector whose elements are the roots of the polynomial.

To find the roots of $P[x] = x^5 - 2 \cdot x^4 + x^3$, type:

```
PROOT([1, -2, 1, 0, 0, 0])
```

You obtain:

```
[0., 0., 0., 1., 1.]
```

The result means that 0 is a triple root, and 1 is a double root of $P[x]$. 

42
5.9.17  \textbf{FROOTS}

\textbf{FROOTS} has a rational function $F[x]$ as argument.
\textbf{FROOTS} returns a vector whose components are the roots and the poles of $F[x]$, followed by their multiplicity.

Type:
\[
\text{FROOTS}\left(\frac{x^5 - 2 \cdot x^4 + x^3}{x - 2}\right)
\]

You obtain:
\[\{2, -1, 0, 3, 1, 2\}\]

The result means that: 2 is a pole of order 1, 0 is a triple root, and 1 is a double root of $F[x] = \frac{x^5 - 2 \cdot x^4 + x^3}{x - 2}$.

5.9.18  \textbf{PCOEF}

\textbf{PCOEF} is the numeric command of the HP48.

\textbf{PCOEF} has a vector containing the roots of a polynomial $P[x]$ as argument.
\textbf{PCOEF} returns a vector whose components are the coefficients of the polynomial $P[x]$ (ordered by decreasing powers of the polynomial’s variable).

Type:
\[
\text{PCOEF}([1, 2, 0, 0, 3])
\]

You obtain:
\[\{1, -6, 11, -6, 0, 0\}\]

This means that $P[x] = (x - 1) \cdot (x - 2) \cdot x \cdot x \cdot (x - 3)$ is equal to:
\[x^5 - 6 \cdot x^4 + 11 \cdot x^3 - 6 \cdot x^2\]

5.9.19  \textbf{FCOEF}

\textbf{FCOEF} has as argument a vector whose components are the roots and poles of a rational function $F[x]$, followed by their multiplicity.
\textbf{FCOEF} returns the rational function $F[x]$.

Type:
\[
\text{FCOEF}([1, 2, 0, 3, 2, -1])
\]

You obtain:
\[
\frac{x^5 - 2 \cdot x^4 + x^3}{x - 2}
\]

since $(x - 1)^2 \cdot x^3 = x^5 - 2 \cdot x^4 + x^3$

5.9.20  \textbf{CHINREM}

\textbf{CHINREM} has two vectors as arguments; each vector has two polynomials as components.
\textbf{CHINREM} returns a vector with two polynomials as components.
\textbf{CHINREM}([A[X], R[X]], [B[X], Q[X]]) finds the polynomials $P[X]$ and $S[X]$ satisfying the following relations:


There always is a solution $P[X]$ if $R[X]$ and $Q[X]$ are mutually primes, and all solutions are
congruent modulo \( S[X] = R[X] \cdot Q[X] \).

Find the solutions \( P[X] \) of:

\[
\begin{align*}
P[X] &= X \pmod{X^2 + 1} \\
P[X] &= X - 1 \pmod{X^2 - 1}
\end{align*}
\]

Type: \( \text{CHINREM}([X, X^2 + 1], [X - 1, X^2 - 1]) \)

You obtain:

\[
[\frac{-X^2 - 2 \cdot X + 1}{2}, \frac{-X^4 - 1}{2}]
\]

that is, \( P[X] = \frac{-X^2 - 2 \cdot X + 1}{2} \pmod{\frac{-X^4 - 1}{2}} \)

5.9.21 \ TRUNC

\( \text{TRUNC} \) truncates a polynomial to a given order.

\( \text{TRUNC} \) has two arguments: a polynomial and \( X^n \).

\( \text{TRUNC} \) returns the polynomial truncated to order \( n - 1 \) (no terms of order \( \geq X^n \)).

Type: \( \text{TRUNC}((1 + X + \frac{1}{2} \cdot X^2)^3, X^4) \)

You obtain:

\[
4 \cdot X^3 + \frac{9}{2} \cdot X^2 + 3 \cdot X + 1
\]

5.9.22 \ LAGRANGE

\( \text{LAGRANGE} \) has as argument a matrix with two rows and \( n \) columns:

the first row corresponds to the abscissa values \( x_i \), and the second row corresponds to ordinate values \( y_i \) (\( i = 1..n \)).

\( \text{LAGRANGE} \) returns the polynomial \( P \) of degree \( n - 1 \), so that \( P(x_i) = y_i \).

Type: \( \text{LAGRANGE}([[1, 3], [0, 1]]) \)

You obtain:

\[
\frac{X - 1}{2}
\]

in fact \( \frac{X - 1}{2} = 0 \) for \( x = 1 \) and \( \frac{X - 1}{2} = 1 \) for \( x = 3 \)

5.9.23 \ LEGENDRE

\( \text{LEGENDRE} \) has as argument an integer value \( n \).

\( \text{LEGENDRE} \) returns the non trivial polynomial solution of the differential equation:

\[
(x^2 - 1) \cdot y'' - 2 \cdot x \cdot y' - n(n + 1) \cdot y = 0
\]

Type: \( \text{LEGENDRE}(4) \)

You obtain:

\[
\frac{35 \cdot X^4 - 30 \cdot X^2 + 3}{8}
\]
5.9.24 HERMITE

HERMITE has as argument an integer value \( n \).
HERMITE returns the Hermite polynomial of degree \( n \).
Type: \( \text{HERMITE}(n) \)
You obtain:
\[
64 \cdot X^6 - 480 \cdot X^4 + 720 \cdot X^2 - 120
\]

5.9.25 TCHEBYCHEFF

TCHEBYCHEFF has as argument an integer value \( n \).
If \( n > 0 \), TCHEBYCHEFF returns the polynomial \( T_n \):
\[
T_n[x] = \cos(n \cdot \arccos(x))
\]
If \( n < 0 \) TCHEBYCHEFF returns the the polynomial \( T_n \):
\[
T_n[x] = \frac{\sin(n \cdot \arccos(x))}{\sin(\arccos(x))}
\]
Type: \( \text{TCHEBYCHEFF}(n) \)
You obtain:
\[
8 \cdot X^4 - 8 \cdot X^2 + 1
\]
in fact:
\[
\cos(4 \cdot x) = R\text{e}((\cos(x) + i \cdot \sin(x))^4)
\]
\[
\cos(4 \cdot x) = \cos(x)^4 - 6 \cdot \cos(x)^2 \cdot (1 - \cos(x)^2) + ((1 - \cos(x)^2)^2.
\]
\[
\cos(4 \cdot x) = T_4(\cos(x)).
\]
Type: \( \text{TCHEBYCHEFF}(-4) \)
You obtain:
\[
8 \cdot X^3 - 4 \cdot X
\]
in fact:
\[
\sin(4 \cdot x) = \sin(x) \cdot (8 \cdot \cos(x)^3 - 4 \cdot \cos(x)).
\]

5.9.26 REORDER

REORDER has two arguments: an expression and a vector containing an ordered list of variables.
REORDER reorders the input expression following the order of variables given by its second argument.
Type: \( \text{REORDER}(x^2 + 2 \cdot X \cdot A + A^2 + Z^2 - X \cdot Z, [A, X, Z]) \)
You obtain:
\[
A^2 + 2 \cdot X \cdot A + X^2 - Z \cdot X + Z^2
\]
5.10 Rational fractions

5.10.1 FXND

FXND has a rational fraction as argument, and returns a list containing the simplified numerator and denominator of this fraction.

Type:

\[
\text{FXND}(\frac{X^2 - 1}{X - 1})
\]

You obtain:

\[
\{X + 1, 1\}
\]

5.10.2 SIMP2

SIMP2 has two polynomials (or two lists of polynomials with the same length) as arguments. These two polynomials are considered as representing a rational fraction. SIMP2 simplifies the rational fraction and returns the result as a list of two polynomials.

Type:

\[
\text{SIMP2}(x^3 - 1, x^2 - 1)
\]

You obtain:

\[
\{x^2 + x + 1, x + 1\}
\]

5.10.3 PROPFRAC

PROPFRAC has a rational fraction as argument.

PROPFRAC rewrites the rational fraction to put its integer part in evidence and returns the result. In other words, \(\text{PROPFRAC}(A(X)/B(X))\) rewrites the rational fraction \(\frac{A(X)}{B(X)}\) as:

\[
Q(X) + \frac{R(X)}{B(X)}
\]

where \(R(X) = 0\) or \(0 \leq \text{deg}(R(X)) < \text{deg}(B(X))\).

Type:

\[
\text{PROPFRAC}\left(\frac{5 \cdot X + 3}{X + 2}\right)
\]

You obtain:

\[
5 \cdot X - 12 + \frac{21}{X + 2}
\]

5.10.4 PARTFRAC

To decompose into partial fractions a rational fraction like

\[
\frac{x^5 - 2 \cdot x^3 + 1}{x^4 - 2 \cdot x^3 + 2 \cdot x^2 - 2 \cdot x + 1}
\]

you can use the PARTFRAC command.

You can find this command in the ARITH (blue-shift 1) menu, sub-menu 2.POLYNOMIAL..., position 14 (or you can type it in \(\alpha\) mode).

Type:

\[
\text{PARTFRAC}\left(\frac{x^5 - 2 \cdot x^3 + 1}{x^4 - 2 \cdot x^3 + 2 \cdot x^2 - 2 \cdot x + 1}\right)
\]
In real mode, you obtain:

\[ X + 2 + \frac{-1}{X - 1} + \frac{x - 3}{x^2 + 1} \]

In complex mode, you obtain instead:

\[ X + 2 + \frac{1 - 3 i}{X + 1} + \frac{-1}{X - 1} + \frac{1 + 3 i}{X - 1} \]

**5.11 Modular calculations**

You can do calculations “modulo \( p \)”, that is, in either \( \mathbb{Z}/p\mathbb{Z} \) or \( \mathbb{Z}/p\mathbb{Z}[X] \).

Warning: for some commands, \( p \) must be a prime number.

The calculator uses the symmetrical representation of elements (-1 instead of 6 modulo 7).

The value of \( p \) must be stored into the \( \text{MODULO} \) variable in the HOME directory.

**All the following examples assume that** \( p=13 \).

### 5.11.1 MODSTO

To store into \( \text{MODULO} \) the value of \( p \) (for example \( p=13 \)) you can use either:

- `MODE cas MODULO ...`
- `13 STO ⊿ MODULO` (if the current directory is HOME), or `MODSTO(13)`.

`MODSTO` allows you to change the value of the \( \text{MODULO} \) variable in the HOME directory.

For example, you will type: `MODSTO(5)` or `5 STO ⊿ MODULO` to do your calculations modulo 5.

**All the following examples assume that** \( p=13 \).

### 5.11.2 ADDTMOD

`ADDTMOD` performs an addition in \( \mathbb{Z}/p\mathbb{Z}[X] \).

Type:

\[ \text{ADDTMOD}(11X + 5, 8X + 6) \]

You obtain:

\[ 6X - 2 \]

### 5.11.3 SUBTMOD

`SUBTMOD` performs a subtraction in \( \mathbb{Z}/p\mathbb{Z}[X] \).

Type:

\[ \text{SUBTMOD}(11X + 5, 8X + 6) \]

You obtain:

\[ 3X - 1 \]

### 5.11.4 MULTMOD

`MULTMOD` performs a multiplication in \( \mathbb{Z}/p\mathbb{Z}[X] \).

Type:

\[ \text{MULTMOD}(11X + 5, 8X + 6) \]

You obtain:

\[ -(3X^2 - 2X - 4) \]
The arguments of DIV2MOD are two polynomials $A[X]$ and $B[X]$. The result is a list containing both the quotient and the remainder of the euclidean division of $A[X]$ by $B[X]$ in $\mathbb{Z}/p\mathbb{Z}[X]$.

\[ {\text{DIV2MOD}(X^3 + X^2 + 1, 2X^2 + 4)} \]

Since:
\[ X^3 + X^2 + 1 = (2X^2 + 4) \cdot \left(\frac{X + 1}{2}\right) + \frac{5X - 4}{4} \]

you obtain:
\[ \left\{ \frac{X + 1}{2}, \frac{5X - 4}{4} \right\} \]

Then, with
\[ \text{EXPANDMOD}\left(\left\{ \frac{X + 1}{2}, \frac{5X - 4}{4} \right\} \right) \]

you obtain:
\[ \left\{ -(6X + 6), -(2X + 1) \right\} \]

The arguments of DIVMOD are two polynomials $A[X]$ and $B[X]$. The result is the rational fraction $\frac{A[X]}{B[X]}$ simplified in $\mathbb{Z}/p\mathbb{Z}[X]$.

\[ {\text{DIVMOD}(2X^2 + 5, 5X^2 + 2X - 3)} \]

You obtain:
\[ \frac{5X + 3}{6X + 6} \]

POWMOD returns $X$ raised to $N$ in $\mathbb{Z}/p\mathbb{Z}[X]$.

The current value $p$ of MODULO must be a prime number less than 100.

\[ {\text{POWMOD}(2X + 1, 5)} \]

You obtain:
\[ 6 \cdot X^5 + 2 \cdot X^4 + 2 \cdot X^3 + X^2 - 3 \cdot X + 1 \]

because:
\[ 10 = -3 \pmod{13} \quad 40 = 1 \pmod{13} \quad 80 = 2 \pmod{13} \quad 32 = 6 \pmod{13} \]

INVMOD has an integer as argument.

INVMOD returns the reciprocal of this integer in $\mathbb{Z}/p\mathbb{Z}$.

\[ {\text{INVMOD}(5)} \]

You obtain (since $5 \times -5 = -25 = 1 \pmod{13}$):
\[ -5 \]
5.11.9 GCDMOD

GCDMOD has two polynomials as arguments. GCDMOD returns the GCD of the polynomials in \( \mathbb{Z}/p\mathbb{Z}[X] \).

Type:

\[
\text{GCDMOD}(2X^2 + 5X^2 + 2X - 3)
\]

You obtain:

\[
-(4X - 5)
\]

5.11.10 EXPANDMOD

EXPANDMOD has a polynomial expression as argument. EXPANDMOD expands this expression in \( \mathbb{Z}/p\mathbb{Z}[X] \).

Type:

\[
\text{EXPANDMOD}((2X^2 + 12)(5X - 4))
\]

You obtain:

\[
-(3X^3 - 5X^2 + 5X - 4)
\]

5.11.11 FACTORMOD

FACTORMOD has a polynomial as argument. FACTORMOD factorizes the polynomial in \( \mathbb{Z}/p\mathbb{Z}[X] \) if \( p \leq 97 \) and \( p \) is prime.

Type:

\[
\text{FACTORMOD}(-(3X^3 - 5X^2 + 5X - 4))
\]

You obtain:

\[
-((3X - 5)(X^2 + 6))
\]

5.11.12 RREFMOD

RREFMOD solves a system of linear equations \( AX = B \) in \( \mathbb{Z}/p\mathbb{Z} \).

The argument is the matrix \( A \) augmented with the vector \( B \) as its rightmost column. The result is a matrix composed of \( A_1 \) and \( B_1 \), where \( A_1 \) has zeros both above and under its principal diagonal, and the system \( A_1X = B_1 \) is equivalent to \( AX = B \).

Type:

\[
\text{RREFMOD}([[1, 2, 9],[3, 10, 0]])
\]

to solve

\[
\begin{align*}
  x + 2 \cdot y & = 9 \\
  3 \cdot x + 10 \cdot y & = 0
\end{align*}
\]

You obtain:

\[
\begin{bmatrix}
  2 & 0 & 6 \\
  0 & 4 & -1
\end{bmatrix}
\]

that is, \( 2X = 6 \) and \( 4Y = -1 \) or, which is the same, \( X = 3 \) \( Y = 3 \) (since \( -4 + 3 = 1 \) (mod13)).
5.12 Limited and asymptotic expansions

5.12.1 DIVPC

DIVPC has three arguments: two polynomials \( A(X) \), \( B(X) \) (with \( B(0) \neq 0 \)) and an integer \( n \).
DIVPC returns the quotient \( Q(X) \) of the division of \( A(X) \) by \( B(X) \), with \( \text{deg}(Q) \leq n \) or \( Q = 0 \).
That is, \( Q[X] \) is the series expansion, limited to order \( n \), of \( \frac{A(X)}{B(X)} \) about \( X = 0 \).
Type:
\[
\text{DIVPC}(1 + X^2 + X^3, 1 + X^2, 5)
\]
You obtain:
\[
1 + X^3 - X^5
\]
Warning: the calculator demands to enable the “increasing powers” mode; answer yes to proceed.

5.12.2 TAYLOR0

TAYLOR0 has only one argument: a function of \( x \) to be expanded, and returns its Taylor expansion, limited to relative order 4, about \( x = 0 \) (where \( x \) is the current CAS variable).
Type:
\[
\text{TAYLOR0}\left(\frac{\tan(P \cdot X) - \sin(P \cdot X)}{\tan(Q \cdot X) - \sin(Q \cdot X)}\right)
\]
You obtain:
\[
\frac{P^5 - Q^2 \cdot P^3}{4 \cdot Q^3} \cdot X^2 + \frac{P^3}{Q^3}
\]
Warning: “relative order 4” means to expand up to relative order 4 both numerator and denominator (here, absolute order 5 for both numerator and denominator); this gives an expansion of order 2 (5-3), because \( X^3 \) can be factored out in both the expanded numerator and denominator.

5.12.3 TAYLR

Determine a limited Taylor expansion of order 2 near \( x = 0 \) of:
\[
\frac{3 \cdot \tan(x) - \tan(3 \cdot x)}{x \cdot (1 - \cos(3 \cdot x))}
\]
Since the denominator has order 3, to obtain a limited expansion of order 2 about \( x = 0 \), it is necessary to expand the numerator near \( x = 0 \) up to order 5. To do this, use the TAYLR command; you can find it in the CALC (blue-shift 4) menu, sub-menu 2. LIMITS & SER... at position 5 (or you can type it in α mode).
TAYLR is compatible with the HP48.
Type:
\[
\text{TAYLR}\left(\frac{3 \cdot \tan(X) - \tan(3 \cdot X)}{X \cdot (1 - \cos(3 \cdot X))}, X, 5\right)
\]
You obtain:
\[
-\frac{16}{9} \cdot (1 + \frac{19}{4} \cdot X^2)
\]
5.12.4 SERIES

- expansion about \( x=a \)
  Example:
  Determine a series expansion limited to order 4 about \( x = \frac{\pi}{6} \) of \( \cos(2x)^2 \).
  You use the SERIES command; you can find it in the CALC (blue-shift 4) menu, position 8 (or you can type it in \( \alpha \) mode).
  Type:
  
  \[
  \text{SERIES}(\cos(2 \cdot x)^2, x = \frac{\pi}{6}, 4)
  \]
  You obtain:
  
  \[
  \begin{align*}
  \text{Limit} & : \frac{1}{4} \quad \text{Equiv} : \frac{1}{4} \\
  \text{Expans} & : \left(-\frac{8}{3}h^4 + \frac{8\sqrt{3}}{3}h^3 + 2h^2 - \sqrt{3}h + \frac{1}{4}\right) \\
  \text{Remain} & : \frac{h^5}{4} \ h = X - \frac{\pi}{6}
  \end{align*}
  \]

- expansion about \( x=+\infty \) or \( x=-\infty \)
  Example 1:
  Determine an expansion of \( \arctan(x) \), of order 5, about \( x=+\infty \), assuming \( h = \frac{1}{x} \).
  Type:
  
  \[
  \text{SERIES}(\text{ATAN}(X), X = +\infty, 5)
  \]
  You obtain:
  
  \[
  \begin{align*}
  \text{Limit} & : \frac{\pi}{2} \quad \text{Equiv} : \frac{\pi}{2} \\
  \text{Expans} & : \left(\frac{\pi}{2} - h + \frac{h^3}{3} - \frac{h^5}{5}\right) \text{ Remain} : \frac{\pi h^6}{2} \ h = \frac{1}{X}
  \end{align*}
  \]

Example 2:
Determine an expansion of \( (2x-1)e^{\frac{1}{x-1}} \), of order 2, about \( x=+\infty \), assuming \( h = \frac{1}{x} \).
Type:

\[
\text{SERIES}((2X - 1) \cdot \text{EXP}\left(\frac{1}{X - 1}\right), X = +\infty, 3)
\]
You obtain:

\[
\begin{align*}
\text{Limit} & : +\infty \quad \text{Equiv} : \frac{2}{h} \\
\text{Expans} & : \left(\frac{2 + h + 2h^2 + \frac{17h^3}{6}}{h}\right) \text{ Remain} : 2h^3 \ h = \frac{1}{X}
\end{align*}
\]

Example 3:
Determine an expansion of \( (2x-1)e^{\frac{1}{x-1}} \), of order 2, about \( x=-\infty \), assuming \( h = -\frac{1}{x} \).
Type:

\[
\text{SERIES}((2X - 1) \cdot \text{EXP}\left(\frac{1}{X - 1}\right), X = -\infty, 3)
\]
You obtain:

\[
\begin{align*}
\text{Limit} & : -\infty \quad \text{Equiv} : -\frac{2}{h} \\
\text{Expans} & : \left(-2 + h - 2h^2 + \frac{17h^3}{6}\right) \text{ Remain} : -2h^3 \ h = -\frac{1}{X}
\end{align*}
\]
unidirectional expansions

You must give a positive real number (for example 4.) as the order to do an unidirectional expansion about the point \( x = a \) with \( x > a \), and a negative real number (for example -4.) to do an unidirectional expansion about \( x = a \) with \( x < a \).

Example 1:
Determine an expansion of \( \frac{(1+X)^{1/3}}{X^3} \), of order 2, about \( X = 0^+ \).
Type:

\[
\text{SERIES}\left(\frac{(1+X)^{1/3}}{X^3}, X, 2.\right)
\]

You obtain:

\[
\{\text{Limit} : +\infty \text{ Equiv} : \frac{e}{h^3} \text{ Expans} : \left(-\frac{e \cdot h - 2 \cdot e}{2 \cdot h^3}\right) \text{ Remain} : \frac{e}{h} \ h = X\}
\]

Example 2:
Determine an expansion of \( \frac{(1+X)^{1/3}}{X^3} \), of order 2, about \( X = 0^- \).
Type:

\[
\text{SERIES}\left(\frac{(1+X)^{1/3}}{X^3}, X, -2.\right)
\]

You obtain:

\[
\{\text{Limit} : -\infty \text{ Equiv} : \frac{e}{h^3} \text{ Expans} : \left(-\frac{e \cdot h - 2 \cdot e}{2 \cdot h^3}\right) \text{ Remain} : \frac{e}{h} \ h = X\}
\]

Example 3:
Determine an expansion of \( \frac{(1+X)^{1/3}}{X^3} \), of order 2, about \( X = 0 \).
Type:

\[
\text{SERIES}\left(\frac{(1+X)^{1/3}}{X^3}, X, 2.\right)
\]

You obtain:

\[
\{\text{Limit} : \infty \text{ Equiv} : \frac{e}{h^3} \text{ Expans} : \left(-\frac{e \cdot h - 2 \cdot e}{2 \cdot h^3}\right) \text{ Remain} : \frac{e}{h} \ h = X\}
\]

5.12.5 LIMIT

LIMIT has as arguments an expression depending on a variable, and an equality (the variable equated to the value at which you want to calculate the limit of the expression).

Often, it is better to surround the expression with quotes, to avoid a preliminary transformation of the expression into normal form (thus, avoiding a rational simplification) before the limit computation actually takes place.
For example, type:

\[
\text{LIMIT}'(2X - 1) \cdot \text{EXP}\left(\frac{1}{X - 1}\right)' , X = +\infty
\]

You obtain:

\[+\infty\]
5.13 Matrices

5.13.1 TRAN

TRAN has a matrix A as argument.
TRAN returns the input matrix A transposed.
Type:

\[
\text{TRAN}\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right)
\]
You obtain:
\[
\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}
\]

5.13.2 TRN

TRN has a matrix A as argument.
TRN returns the adjoint of A (transpose of the conjugate) of A (this is the HP48 command).
Type:

\[
\text{TRN}\left(\begin{bmatrix} i & 1+i \\ 1 & 1-i \end{bmatrix}\right)
\]
After simplification, you obtain:
\[
\begin{bmatrix} -i & 1 \\ 1-i & 1+i \end{bmatrix}
\]

5.13.3 MAD

MAD has a square matrix A, of order n, as argument.
MAD returns a list containing the determinant of A, the inverse of A, a list containing the matrix coefficients of a polynomial \(Q\), and of the characteristic polynomial \(P\) of A.
We have:

\[
P(x) = (-1)^n \cdot \det(A - x \cdot I)
\]
The polynomial with matrix coefficients \(P(A) - P(x) \cdot I\) is divisible by \(A - x \cdot I\) (since its value is zero for \(x = A\)). Let \(Q(x)\) be their quotient.
Since \(P(A) = 0\), we have \(P(A) - P(x) \cdot I = -P(x) \cdot I = (A - x \cdot I) \cdot Q(x)\).
Therefore, \(Q(x)\) is also the co-matrix of \(A - x \cdot I\) and the following holds:
\[
Q(x) = I \cdot x^{n-1} + \ldots + B_0, \text{ where } B_0 \text{ is the co-matrix of } A \text{ (with the sign exchanged if } n \text{ is even!)}
\]
Type:

\[
\text{MAD}\left(\begin{bmatrix} 4 & 1 & -2 \\ 1 & 2 & -1 \\ 2 & 1 & 0 \end{bmatrix}\right)
\]
You obtain:
\[
\{8, \begin{bmatrix} \frac{1}{8} & -\frac{1}{2} & \frac{3}{8} \\ \frac{1}{8} & \frac{1}{2} & \frac{3}{8} \end{bmatrix}\},
\{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},
\begin{bmatrix} -2 & 1 & -2 \\ 1 & -4 & -1 \\ 2 & 1 & -6 \end{bmatrix}\},
\begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & 2 \\ -3 & -2 & 7 \end{bmatrix}\}
\]
\[
X^3 - 6 \cdot X^2 + 12 \cdot X - 8
\]
5.13.4 HADAMARD

HADAMARD has two matrices $A$ and $B$, with the same size, as arguments.
HADAMARD returns the element-by-element product between $A$ and $B$.

Type:

\[
\text{HADAMARD} \left( \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \right)
\]

You obtain:

\[
\begin{bmatrix} 5 & 12 \\ 21 & 32 \end{bmatrix}
\]

5.13.5 AXM

If given a symbolic matrix as argument, AXM returns an equivalent (but approximate) numeric matrix, and vice versa.

Type:

\[
\text{AXM}([[[1/2, 2], [3, 4]])
\]

You obtain:

\[
[[0.5, 2], [3, 4]]
\]

5.13.6 AXL

When AXL is given a matrix as argument, it returns the same matrix rewritten as a list of lists. Vice versa, AXL transforms a list of lists into a matrix.

Type:

\[
\text{AXL}([[[1, 2], [3, 4]])
\]

You obtain:

\[
\{\{1, 2\}\{3, 4\}\}
\]

Type:

\[
\text{AXL}([\{\{1, 2\}\{3, 4\}\}])
\]

You obtain:

\[
[[1, 2], [3, 4]]
\]

5.13.7 EGV

EGVL has a matrix $A$, of order $n$, as argument.
EGVL returns a vector containing the $n$ eigenvalues of $A$.

\textbf{NOTICE:} If $A$ is a symbolic matrix, you will obtain only the eigenvalues that the CAS is able to determine (because it is necessary to symbolically factorize the characteristic polynomial!)

Type:

\[
\text{EGV} \left( \begin{bmatrix} 4 & 1 & -2 \\ 1 & 2 & -1 \\ 2 & 1 & 0 \end{bmatrix} \right)
\]

You obtain:

\[
[2, 2, 2]
\]
5.13.8 EGV

EGV has a matrix $A$, of order $n$, as argument.
EGV returns a list containing the matrix of the $n$ column eigenvectors of $A$ and the vector of the $n$ eigenvalues of $A$ (the same notice given for EGVL is valid here, too).
Type:

$$\text{EGV} \begin{bmatrix} 4 & 1 & -2 \\ 1 & 2 & -1 \\ 2 & 1 & 0 \end{bmatrix}$$

You obtain:

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}, [2,2,2]$$

5.13.9 PCAR

PCAR has a matrix $A$, of order $n$, as argument.
PCAR returns the characteristic polynomial $P$ of $A$ ($P[x] = (-1)^n \cdot \det(A - x \cdot I)$) Type:

$$\text{PCAR} \begin{bmatrix} 4 & 1 & -2 \\ 1 & 2 & -1 \\ 2 & 1 & 0 \end{bmatrix}$$

You obtain:

$$X^3 - 6 \cdot X^2 + 12 \cdot X - 8$$

5.13.10 JORDAN

JORDAN has a matrix $A$, of order $n$, as argument.
JORDAN returns a list composed by the minimal polynomial $M$ of $A$, the characteristic polynomial $P$ of $A$, the list of the eigenvectors and characteristic vectors (each vector is preceded by its characteristic value), and the vector of the $n$ eigenvalues of $A$.
Type:

$$\text{JORDAN} \begin{bmatrix} 4 & 1 & -2 \\ 1 & 2 & -1 \\ 2 & 1 & 0 \end{bmatrix}$$

You obtain:

$$\{X^3 - 6X^2 + 12X - 8, X^3 - 6X^2 + 12X - 8, \}$$

$$\{\text{Char} : 2 : [1,0,0], \text{Char} : 2 : [2,1,2], \text{Eigen} : 2 : [1,0,1], [2,2,2]\}$$

5.13.11 HILBERT

HILBERT has an integer $n$ as argument.
HILBERT returns the square Hilbert matrix of order $n$ whose elements are given by:

$$a_{i,j} = \frac{1}{i+j-1}$$

Type:

$$\text{HILBERT}(4)$$
You obtain:

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
\end{bmatrix}
\]

5.13.12 VANDERMONDE

VANDERMONDE has as argument a vector whose components are denoted by \(x_i\).

VANDERMONDE returns the corresponding Vandermonde matrix (the \(k\)-th row of the matrix is the vector whose components are \(x_i^{k-1}\)).

Type:

\[
\text{VANDERMONDE([A, B, C])}
\]

You obtain:

\[
\begin{bmatrix}
1 & 1 & 1 \\
A & B & C \\
A^2 & B^2 & C^2 \\
\end{bmatrix}
\]

5.13.13 LCXM

LCXM has as arguments two integers, \(n\) and \(p\), and a program accepting as arguments \(i\) (a row number) and \(j\) (a column number) and yielding the value of \(a_{i,j}\).

LCXM returns a \(n \cdot p\) matrix having coefficients \(a_{i,j}\). Type:

\[
\text{LCXM(2, 3, \ll\ll \ll I \ll J \ll I + J \gg \gg )}
\]

You obtain:

\[
\begin{bmatrix}
2 & 3 & 4 \\
3 & 4 & 5 \\
\end{bmatrix}
\]

5.14 Vectors

In the blue-shift SYMB (MTH) menu, you can find the functions to compute:
- the absolute value of a vector: ABS
- the dot product of two vectors: DOT
- the cross product of two vectors: CROSS

5.15 Quadratic forms

5.15.1 QXA

QXA has two arguments: a quadratic form \(q\) and a vector whose components are the form’s variables.

QXA returns a list of two elements: the matrix \(A\) associated with \(q\) and the vector denoting the variables of the quadratic form.

Type:

\[
\text{QXA(2 \cdot X \cdot Y , [X, Y])}
\]

You obtain:

\[
\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} , [X, Y] \}
\]
5.15.2 AXQ

AXQ has two arguments: a symmetric matrix \( A \) representing a quadratic form \( q \) and a vector whose components are the quadratic form’s variables. AXQ returns a list of two elements: the quadratic form \( q \) and the vector denoting the form’s variables.

Type:

\[ AXQ([[0,1],[1,0]], [X,Y]) \]

You obtain:

\[ \{2 \cdot X \cdot Y, [X,Y]\} \]

5.15.3 GAUSS

GAUSS has two arguments: a quadratic form \( q \) and a vector whose components are the quadratic form’s variables. GAUSS returns a list of 4 elements: the diagonal elements of a diagonal matrix \( B \) (obtained expressing \( q \) as a sum of squares), the matrix of change of base \( Q \), \( q \) expressed as a sum of squares, and the vector denoting the form’s variables.

We have (if we denote as \( A \) the matrix associated with \( q \)):

\[ ^tQ \cdot B \cdot Q = A \]

Type:

\[ GAUSS(2 \cdot X \cdot Y , [X,Y]) \]

You obtain:

\[ \{[\frac{1}{2}, -2], \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}, \frac{1}{2}(\frac{Y-X}{2})^2 + \frac{1}{2}(Y+X)^2, [X,Y]\} \]

5.15.4 SYLVESTER

SYLVESTER has one argument: a symmetric matrix representing a quadratic form \( q \). SYLVESTER returns a list of two elements: the diagonal elements of the diagonal matrix \( B \) (obtained expressing \( q \) as a sum of squares) and the matrix of change of base \( Q \).

We have:

\[ ^tQ \cdot B \cdot Q = A \]

Type:

\[ SYLVESTER([[0,1],[1,0]]) \]

You obtain:

\[ \{[\frac{1}{2}, -2], \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\} \]

5.16 Functions of multiple variables

5.16.1 DERIV

DERIV has two arguments: an application \( F \) from \( R^n \) in \( R \) and a vector of \( R^n \) denoting the variable names. DERIV returns the gradient of \( F \) \( ([\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}, \frac{\partial F}{\partial Z}] \) if \( n = 3 \).

Type:

\[ DERIV(2 \cdot X^2 \cdot Y - X \cdot Z^2, [X,Y,Z]) \]
After simplification, you obtain:

\[ [4 \cdot Y \cdot X - Z^3, 2 \cdot X^2, -(3 \cdot Z^2 \cdot X)] \]

### 5.16.2 LAPL

LAPL has two arguments: an application \( F \) from \( R^n \) in \( R \) and a vector of \( R^n \) denoting the variable names.

LAPL returns the laplacian of \( F \) (\( \frac{\partial^2 F}{\partial X^2} + \frac{\partial^2 F}{\partial Y^2} + \frac{\partial^2 F}{\partial Z^2} \) if \( n = 3 \)).

**Type:**

\[ \text{LAPL}(2.0 \cdot Y - X \cdot Z^3, [X, Y, Z]) \]

You obtain:

\[ 4 \cdot Y - 6 \cdot X \cdot Z \]

### 5.16.3 HESS

HESS has two arguments: an application \( F \) from \( R^n \) in \( R \) and a vector of \( R^n \) denoting the variable names.

HESS returns a list containing the hessian of \( F \), the gradient of \( F \) and the vector of the variable names.

**Type:**

\[ \text{HESS}(2.0 \cdot X^2 \cdot Y - X \cdot Z^3, [X, Y, Z]) \]

You obtain:

\[
\begin{bmatrix}
4.0 & 4.0 & -1 \\
4.0 & 0 & 0 \\
-1 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
4.0 \cdot Y - Z, 2.0 \cdot X^2, -X \end{bmatrix}, [X, Y, Z]
\]

Now, to obtain the critical points of \( F \), in RPN mode you can type:

\[ \text{SOLVE} \]

directly, because on the stack you have: \([4.0 \cdot Y - Z, 2.0 \cdot X^2, -X]\) and \([X, Y, Z]\)

In ALGEBRAIC mode, you must enter instead:

\[ \text{SOLVE}((4.0 \cdot Y - Z, 2.0 \cdot X^2, -X), [X, Y, Z]) \]

### 5.16.4 DIV

DIV has two arguments: a vectorial function \( F \) (application from \( R^n \) in \( R^n \)) and a vector of \( R^n \) denoting the variable names.

DIV returns the divergence of \( F \):

\[ \text{DIV}([A, B, C], [X, Y, Z]) = \frac{\partial A}{\partial X} + \frac{\partial B}{\partial Y} + \frac{\partial C}{\partial Z} \text{ (here } n = 3) \]

**Type:**

\[ \text{DIV}([X \cdot Z, -Y^2, 2 \cdot X^2], [X, Y, Z]) \]

You obtain:

\[ Z - 2 \cdot Y \]
5.16.5  **CURL**

Here \( n = 3 \).
CURL has two arguments: a vectorial function \( F \) (application from \( \mathbb{R}^3 \) in \( \mathbb{R}^3 \)) and a vector of \( \mathbb{R}^3 \) denoting the variable names.
CURL returns the rotor of \( F \).

\[
\text{CURL}([A, B, C], [X, Y, Z]) = \left[ \frac{\partial C}{\partial Y} - \frac{\partial B}{\partial Z}, \frac{\partial A}{\partial Z} - \frac{\partial C}{\partial X}, \frac{\partial B}{\partial X} - \frac{\partial A}{\partial Y} \right]
\]

Type:

\[
\text{CURL}([X.Z, -Y^2, 2.X^2], [X, Y, Z])
\]

You obtain:

\[
[2.X^2, X - 2.Y.2X, 0]
\]

5.17  **Equations**

5.17.1  **EXLR**

EXLR has an equation as argument.
EXLR returns a list containing the left and right hand sides of the equation.

Type:

\[
\text{EXLR}(A = B)
\]

You obtain:

\[
\{A, B\}
\]

5.17.2  **SOLVEVX**

SOLVEVX has as argument either an equation between two expressions of the variable stored in VX, or an expression \((=0\) is assumed then).
SOLVEVX solves the equation.
Example 1:

Type:

\[
\text{SOLVEVX}(X^4 - 1 = 3)
\]

In real mode, you obtain:

\[
\{X = -\sqrt{2}, X = \sqrt{2}\}
\]

In complex mode, you obtain:

\[
\{X = -\sqrt{2}, X = \sqrt{2}, X = -(i \cdot \sqrt{2}), X = i \cdot \sqrt{2}\}
\]

Example 2:

Type:

\[
\text{SOLVEVX}((X - 2).\text{SIN}(X))
\]

In real mode, you obtain:

\[
\{X = -(2 \cdot \pi \cdot n_1), X = 2 \cdot \pi \cdot n_1, X = 2\}
\]
5.17.3 SOLVE

SOLVE has as argument either an equation between two expressions or an expression (=0 is assumed then), and the name of a variable.

SOLVE can also solve a system of equations: to do this, put the equations into a vector and the variable names into another vector.

SOLVE solves either the equation or the system of equations.

Example 1:

Type:

\[ \text{SOLVE}(X^4 - 1 = 3, X) \]

In real mode, you obtain:

\[ \{ X = -\sqrt{2}, X = \sqrt{2} \} \]

In complex mode, you obtain:

\[ \{ X = -\sqrt{2}, X = \sqrt{2}, X = -(i \cdot \sqrt{2}), X = i \cdot \sqrt{2} \} \]

Example 2:

Type:

\[ \text{SOLVE}([X + Y = 1, X - Y], [X, Y]) \]

You obtain:

\[ \{ [X = \frac{1}{2}, Y = \frac{1}{2}] \} \]

5.17.4 ISOL

ISOL isolates a variable in an expression or equation (the variable must appear only once). This command is the same as the HP48 one.

ISOL has two arguments: either an expression or an equation, and the name of the variable to isolate.

Type:

\[ \text{ISOL}(X^4 - 1 = 3, X) \]

In real mode, you obtain:

\[ \{ X = -\sqrt{2}, X = \sqrt{2} \} \]

In complex mode, you obtain:

\[ \{ X = -\sqrt{2}, X = \sqrt{2}, X = -(i \cdot \sqrt{2}), X = i \cdot \sqrt{2} \} \]

Warning: if flag 01 (Principal value) is set, ISOL always returns only one solution.

5.18 Linear systems

In this paragraph, we call “augmented matrix” of the system \( A \cdot X = B \) (or matrix “representing” the system \( A \cdot X = B \)), the matrix obtained augmenting the matrix \( A \) to the right with the column vector \( B \).
5.18.1 \textbf{REF}

\textbf{REF} solves a linear system of equations written in matrix form:

\[ A \cdot X = B \]

The \textbf{REF} command is in the MATRICES (\texttt{blue-shift 5}) menu, 5 LINEAR SYST... sub-menu. The argument of \textbf{REF} is the augmented matrix of the system (the matrix obtained augmenting matrix \( A \) to the right with the column vector \( B \)). The result is a matrix \([A_1,B_1] : A_1 \) has zeros under its principal diagonal, and the solutions of:

\[ A_1 \cdot X = B_1 \]

are the same as:

\[ A \cdot X = B \]

For example, to solve the system:

\[
\begin{align*}
3 \cdot x + y &= -2 \\
3 \cdot x + 2 \cdot y &= 2 
\end{align*}
\]

Type (using \texttt{blue-shift EQW (MTRW)} to enter the matrix):

\[
\text{REF}([[3,1,-2],[3,2,2]])
\]

You obtain:

\[
\begin{bmatrix}
1 & \frac{1}{3} & \frac{-2}{3} \\
0 & 1 & \frac{4}{3}
\end{bmatrix}
\]

5.18.2 \textbf{rref}

\textbf{rref} solves a linear system of equations written in matrix form:

\[ A \cdot X = B \]

The \textbf{rref} command is in the MATRICES (\texttt{blue-shift 5}) menu, 5 LINEAR SYST... sub-menu. The argument of \textbf{rref} is the augmented matrix of the system (the matrix obtained augmenting matrix \( A \) to the right with the column vector \( B \)). The result is a list containing the list of pivot elements used by the command and a matrix \([A_1,B_1] : A_1 \) has zeros both above and under its principal diagonal, and the solutions of:

\[ A_1 \cdot X = B_1 \]

are the same as:

\[ A \cdot X = B \]

It is interesting to use \textbf{rref} in step-by-step mode, setting the Step/Step (MODE cas chk) flag. For example, to solve the system:

\[
\begin{align*}
3 \cdot x + y &= -2 \\
3 \cdot x + 2 \cdot y &= 2 
\end{align*}
\]

Type (using \texttt{blue-shift EQW (MTRW)} to enter the matrix):

\[
\text{rref}([[3,1,-2],[3,2,2]])
\]

You obtain:

\[
\{\text{Pivots : \{1 1\} } \begin{bmatrix}
3 & 0 & -6 \\
0 & 1 & 4
\end{bmatrix}\}
\]
5.18.3 RREF

RREF is the same as \texttt{rref}, but it does not return the pivots.

RREF is the HP48 command; this is its result in step-by-step mode, on the HP49, to solve the system:

\[
\begin{align*}
3 \cdot x + y &= -2 \\
3 \cdot x + 2 \cdot y &= 2
\end{align*}
\]

Type (using \texttt{blue-shift EQW (MTRW)} to enter the matrix):

\[
\text{RREF}([[3, 1, -2], [3, 2, 2]])
\]

You obtain:

\[
\begin{align*}
L_2 &= L_2 - L_1 \\
\begin{bmatrix}
3 & 1 & -2 \\
3 & 2 & 2
\end{bmatrix}
\end{align*}
\]

after \texttt{ok}:

\[
\begin{align*}
L_1 &= L_1 - L_2 \\
\begin{bmatrix}
3 & 1 & -2 \\
0 & 1 & 4
\end{bmatrix}
\end{align*}
\]

after \texttt{ok}:

Reduction result:

\[
\begin{align*}
\begin{bmatrix}
3 & 0 & -6 \\
0 & 1 & 4
\end{bmatrix}
\end{align*}
\]

after \texttt{ok} the result (with all 1 on the diagonal of A1) is stored in the history:

\[
\begin{bmatrix}
1 & 0 & -2 \\
0 & 1 & 4
\end{bmatrix}
\]

5.18.4 LINSOLVE

LINSOLVE solves a linear system of equations.

The \texttt{LINSOLVE} command is in the \texttt{MATRICES (blue-shift 5), 5.LINEAR SYST...} sub-menu, position 1.

Type:

\[
\text{LINSOLVE()}
\]

Then, enter \texttt{MATRIXWRITER} by pressing \texttt{blue-shift EQW (MTRW)} (with the cursor positioned between the two parentheses of \texttt{LINSOLVE}).

Set the \texttt{vect} menu option (if it isn’t already set), and enter the equations (possibly with the help of \texttt{EQW}).

\[
\begin{align*}
2 \cdot X + Y + Z &= 1 \text{ ENTER} \\
X + Y + 2 \cdot Z &= 1 \text{ ENTER} \\
X + 2 \cdot Y + Z &= 4 \text{ ENTER}
\end{align*}
\]
Then, enter the unknown variables: \[X \ Y \ Z\]

and ENTER

If step-by-step mode is enabled (MODE cas Step/Step), you obtain:

\[
\begin{bmatrix}
2 & 1 & 1 \\
1 & 2 & 1 \\
1 & 2 & 1
\end{bmatrix}
\]

after ok:

\[
\begin{bmatrix}
2 & 1 & 1 \\
0 & 1 & 3 \\
1 & 2 & 1
\end{bmatrix}
\]

and so on... Last, you obtain: Result

\[
\begin{bmatrix}
8 & 0 & 0 \\
0 & 8 & 0 \\
0 & 0 & 8
\end{bmatrix}
\]

after ok,

\[
\begin{bmatrix}
X = -\frac{1}{2} \\
Y = \frac{5}{2} \\
Z = -\frac{1}{2}
\end{bmatrix}
\]

is stored into the history.

5.19 Differential equations

5.19.1 LDEC

LDEC directly solves linear differential equations; in order to do this, LAP and ILAP (see 4.19.3) are used internally.

For second order linear equations, the arguments are the second member of the equation and the characteristic equation.

For first order systems of linear equations, the arguments are the second member of the equation (a vector) and the matrix of the system.

Example 1:

Solve:

\[
y'' - 6 \cdot y' + 9 \cdot y = x \cdot e^{3 \cdot x}
\]

Type:

\[\text{LDEC}(X \cdot \text{EXP}(3 \cdot X), X^2 - 6 \cdot X + 9)\]

You obtain:

\[
\left(\frac{x^3}{6} - (3 \cdot C0 - C1) \cdot X + C0\right) \cdot \text{EXP}(3 \cdot X)
\]

C0 and C1 are arbitrary integration constants \(y(0) = C0, \ y'(0) = C1\).

Example 2:
Solve:

\[
Z' = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix} Z + \begin{bmatrix} 0 \\ X \cdot \text{EXP}(3 \cdot X) \end{bmatrix}
\]

This is the same example as before, with \(Z = [y, y']\).

Type:

\[\text{LDEC}([0, X \cdot \text{EXP}(3 \cdot X)], [[0, 1][[-9, 6]])\]

You obtain:

\[
\left(\frac{X^3}{6} - (3 \cdot V1 - V2) \cdot X + V1) \cdot \text{EXP}(3 \cdot X),
\right.
\]

\[
\left(\frac{X^3}{2} + \frac{X^2}{2} - (9 \cdot V1 - 3 \cdot V2) \cdot X + V2) \cdot \text{EXP}(3 \cdot X)\right]
\]

\(V1\) and \(V2\) are arbitrary integration constants \((Z(0) = [V1, V2])\).

### 5.19.2 DESOLVE and SUBST

The \texttt{DESOLVE} command is in the \texttt{S.SLV} (blue-shift 7) menu, at position 1, or in the \texttt{CALC} (blue-shift 4) menu, DIFFERENTIAL EQNS... sub-menu.

\texttt{DESOLVE} solves other types of differential equations.

Its arguments are: the differential equation (here \(y'\) is written as \(d1Y(X)\) and \(y\) is written as \(Y(X)\)).

**Example 1:**

Solve:

\[y'' + y = \cos(x)\]

\(y(0) = c_0\)

\(y'(0) = c_1\)

Type:

\[\text{DESOLVE}(d1d1Y(X) + Y(X) = \cos(X), Y(X))\]

You obtain:

\[
Y(X) = c_0 \cdot \cos(X) + \frac{X + 2 \cdot c_1}{2} \cdot \sin(X)
\]

Now, you can assign a value to the integration constants, using the \texttt{SUBST} command; you can find it in the \texttt{ALG} (red-shift 4) menu, at position 6. If you want the solutions that satisfy \(y(0) = 1\), you type:

\[\text{SUBST}(Y(X) = c_0 \cdot \cos(X) + \frac{X + 2 \cdot c_1}{2} \cdot \sin(X), c_0 = 1)\]

You obtain:

\[
Y(X) = \frac{2 \cdot \cos(X) + (X + 2 \cdot c_1) \cdot \sin(X)}{2}
\]

**Example 2:**

Solve:

\[y'' + y = \cos(x)\]

\(y(0) = 1\)

\(y'(0) = c_1\)

To have the solutions satisfying \(y(0) = 1\) you can also directly type:

\[\text{DESOLVE}([d1d1Y(X) + Y(X) = \cos(X), Y(0) = 1], Y(X))\]

In this case, you obtain:

\[
Y(X) = \cos(X) + \frac{X + 2 \cdot c_1}{2} \cdot \sin(X)
\]

64
5.19.3 LAP ILAP

You can find these commands in the CALC (blue-shift 4) menu, 3.DIFFERENTIAL... submenu, at positions 2 and 3. Laplace transform (LAP) and inverse Laplace transform (ILAP) are useful to solve linear differential equations with constant coefficients, for example:

\[ y'' + p \cdot y' + q \cdot y = f(x) \quad y(0) = a \quad y'(0) = b \]

The following relations hold:

\[ \text{LAP}(Y)(P) = \int_{0}^{\infty} e^{-P \cdot X} Y(X) dX \]
\[ \text{ILAP}(F)(T) = \frac{1}{2 \cdot \pi} \int_{C} e^{2 \cdot \pi \cdot T} dZ \]

where \( C \) is a closed contour enclosing the poles of \( F \)

The following property is used:

\[ \text{LAP}(Y')(P) = -Y(0) + P \cdot \text{LAP}(Y)(P) \]

The solution is then:

\[ \text{ILAP}\left(\frac{\text{LAP}(F(X)) + (X + P) \cdot A + B}{X^2 + P \cdot X + Q}\right) \]

Example:

Solve:

\[ y'' - 6 \cdot y' + 9 \cdot y = x \cdot e^{3 \cdot x} \]
\[ y(0) = a \]
\[ y'(0) = b \]

Type:

\[ \text{LAP}(X \cdot \text{EXP}(3 \cdot X)) \quad \text{ENTER} \]

You obtain:

\[ \frac{1}{X^2 - 6 \cdot X + 9} \]

Type:

\[ \text{ILAP}\left(\frac{\text{ANS}(1) + (X - 6) \cdot A + B}{X^2 - 6 \cdot X + 9}\right) \]

You obtain the solution \( y \):

\[ \left(\frac{X^3}{6} - (3 \cdot A - B) \cdot X + A\right) \cdot \text{EXP}(3 \cdot X) \]

5.20 Other functions

5.20.1 EPSX0

EPSX0 has as argument an expression; in the expression, it replaces all numeric values whose magnitude is smaller than EPS with zero and returns the result.

Type:

\[ \text{EPSX0}(0.001 + X) \]

You obtain (when EPS=0.01):

\[ 0 + X \]

You obtain (when EPS=0.0001):

\[ .001 + X \]
5.20.2 LVAR
LVAR has an expression as argument, and returns a list containing the expression and a vector whose components are the independent variables of the expression.
Type:
\[
\text{LVAR}(X.Y.SIN(X))
\]
You obtain:
\[
\{X.Y.SIN(X), [\text{SIN}(X), X, Y]\}
\]

5.20.3 LNAME
LNAME has an expression as argument, and returns a vector whose components are the symbolic variable names the expression contains.
Type:
\[
\text{LNAME}(X.Y.SIN(X))
\]
You obtain:
\[
[X, Y]
\]

5.20.4 XNUM
XNUM has either an expression or an array as argument.
XNUM enables approximate mode and returns the numeric approximation of its argument.
Type:
\[
\text{XNUM}(\sqrt{2})
\]
You obtain:
\[
1.41421356237
\]

5.20.5 XQ
XQ has a real numeric expression as argument.
XQ enables exact mode and returns either a rational or a real approximation of the expression.
Type:
\[
\text{XQ}(1.41422)
\]
You obtain:
\[
\frac{66441}{46981}
\]
Type:
\[
\text{XQ}(1.414213562)
\]
You obtain:
\[
\sqrt{2}
\]

5.21 New commands
5.21.1 qr
qr has a square matrix as argument.
qr factors the matrix as \(Q^\ast R\), where \(Q\) is an orthogonal matrix and \(R\) is a triangular matrix. For example, enter:
\[
\text{qr}\left(\begin{bmatrix} 3 & 5 \\ 4 & 5 \end{bmatrix}\right)
\]
You obtain:
\[
\begin{bmatrix}
  \begin{array}{cc}
    3 & 4 \\
    4 & 5 \\
  \end{array}
\end{bmatrix},
\begin{bmatrix}
  5 & 7 \\
  0 & 1 \\
\end{bmatrix}
\]

5.21.2 GRAMSCHMIDT

GRAMSCHMIDT has two arguments: a vector representing a base of a vectorial space and the definition of a scalar product in that space, expressed as a function.

GRAMSCHMIDT returns an orthonormal base of the vectorial space with respect to the given scalar product.

Example:

In the vectorial space of polynomials of degree less than 2, we consider the scalar product defined by:

\[
P \cdot Q = \int_{-1}^{1} P(x) \cdot Q(x) dx
\]

We first express the scalar product as a function named PS :

\[
\langle \rightarrow P \ Q \ 
\langle \rightarrow \text{PREVAL}(\text{INTVX}(P \ast Q), -1, 1) \ 
\rangle \rangle \ 
\text{STO} \ 
\langle \rightarrow \text{PS} \rangle
\]

Next, we type:

\[
\text{GRAMSCHMIDT}([1, 1 + X], \text{RCL}'(PS'))
\]

We obtain:

\[
\begin{bmatrix}
  1/\sqrt{2} & \sqrt{6} \cdot X/2
\end{bmatrix}
\]

5.21.3 SYST2MAT

SYST2MAT has two arguments: a vector containing a system of linear equations, and a vector whose elements are the system’s variables.

SYST2MAT rewrites the system in matrix notation, and returns the matrix.

For example, type:

\[
\text{SYST2MAT}([X + Y, X - Y = 2], [X, Y])
\]

You obtain:

\[
\begin{bmatrix}
  1 & 1 & 0 \\
  1 & -1 & -2
\end{bmatrix}
\]

Warning: you must purge the system’s variables (X and Y in this example) beforehand.

5.21.4 CHOLESKY

CHOLESKY has as argument a square matrix \( M \), positive by definition.

CHOLESKY returns an upper triangular matrix \( P \) so that:

\[
^tP \ast P = M
\]

For example, type:

\[
\text{CHOLESKY}\left(\begin{bmatrix}
  1 & 1 \\
  1 & 5
\end{bmatrix}\right)
\]

You obtain:

\[
\begin{bmatrix}
  1 & 1 \\
  0 & 2
\end{bmatrix}
\]

67
5.21.5 DIAGMAP

DIAGMAP has two arguments: a diagonalizable matrix and a function defining an holomorphic operator. DIAGMAP applies the operator to the matrix, and returns the result. Warning! The matrix must be diagonalizable.

For example, we define the function PH as follows:
\[
\ll M \ll \exp(M) \gg \sto \phi
\]
Next, we type:
\[
\text{DIAGMAP}\left(\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \ll M \ll \exp(M) \gg \gg\right)
\]
or:
\[
\text{DIAGMAP}\left(\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \text{RCL('PH')}\right)
\]
We obtain:
\[
\begin{bmatrix} \exp(1) & -\exp(1) + \exp(2) \\ 0 & \exp(2) \end{bmatrix}
\]

5.21.6 ISOM

ISOM has as argument a matrix representing a two or three-dimensional linear isometry. ISOM returns the list of the isometry’s characteristics elements and either +1 (for direct isometries) or -1 (for indirect isometries).

For example, type:
\[
\text{ISOM}\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\right)
\]
You obtain:
\[
\{[1 \ 0 \ -1] \ -1\}
\]
This result means that the isometry is a simmetry with respect to the plane \( x - z = 0 \).

If you type:
\[
\text{ISOM}(\sqrt{2}/2 \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix})
\]
You obtain:
\[
\{\pi/4, 1\}
\]
therefore, this isometry is a rotation of \( \pi/4 \) radians.

5.21.7 MKISOM

In a three-dimensional space, MKISOM has the list of the characteristics elements of an isometry, and either +1 (denoting a direct isometry) or -1 (denoting an indirect isometry) as arguments.

In a two-dimensional space, MKISOM has the characteristic element of an isometry (either an angle or a vector) and either +1 (denoting a direct isometry) or -1 (denoting an indirect isometry) as arguments.

MKISOM returns the matrix representing the given isometry.

For example, type:
\[
\text{MKISOM}([-1, 2, -1], \pi, 1)
\]
You obtain the matrix of a rotation with axis $[-1, 2, -1]$ and angle $\pi$:

$$\frac{1}{3} \begin{bmatrix}
-2 & -2 & 1 \\
-2 & 1 & -2 \\
1 & -2 & -2 \\
\end{bmatrix}$$

For example, type:

$$\text{MKISOM}(\{\pi\}, -1)$$

You obtain the matrix of a symmetry with respect to the origin:

$$\begin{bmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1 \\
\end{bmatrix}$$

Type:

$$\text{MKISOM}(\{(1, 1, 1) \cdot \frac{\pi}{3}\}, -1)$$

You obtain the matrix of a rotation with axis $[1, 1, 1]$ and angle $\frac{\pi}{3}$ combined with a symmetry with respect to the plane $x + y + z = 0$:

$$\begin{bmatrix}
0 & -1 & 0 \\
0 & 0 & -1 \\
-1 & 0 & 0 \\
\end{bmatrix}$$

Type:

$$\text{MKISOM}(\frac{\pi}{2}, 1)$$

You obtain the matrix of a rotation of $\frac{\pi}{2}$ radians in two dimensions:

$$\begin{bmatrix}
0 & -1 \\
1 & 0 \\
\end{bmatrix}$$

Type:

$$\text{MKISOM}([1, 1], -1)$$

You obtain the matrix of a symmetry with respect to $y = x$ in two dimensions:

$$\begin{bmatrix}
0 & -1 \\
-1 & 0 \\
\end{bmatrix}$$

5.21.8 KER

KER has as argument a matrix representing a linear application $f$ in terms of the standard basis. KER returns a list of vectors; they are a basis of the kernel of $f$.

For example, type:

$$\text{KER} \left( \begin{bmatrix}
1 & 1 & 2 \\
2 & 1 & 3 \\
3 & 1 & 4 \\
\end{bmatrix} \right)$$

You obtain:

$$\{[1, 1, -1]\}$$
5.21.9 IMAGE

IMAGE has as argument a matrix representing a linear application $f$ in terms of the standard basis.
IMAGE returns a list of vectors; they are a basis of the image of $f$.
For example, type:

```
IMAGE( [1 1 2
        2 1 3
        3 1 4 ]
```

You obtain:

`[[1, 0, -1], [1, 1, 2]]`

5.21.10 IBASIS

IBASIS has as arguments two lists of vectors, defining two vectorial spaces.
IBASIS returns a list containing the vectors of a basis of the intersection between these two vectorial spaces.
For example, type:

```
IBASIS([ [1, 2] ], [ [2, 4] ])
```

You obtain:

`[[1, 2]]`

5.21.11 BASIS

BASIS has as argument a list of vectors, defining a vectorial subspace of $\mathbb{R}^n$.
BASIS returns a list containing the vectors of a basis of the vectorial subspace.
For example, type:

```
BASIS([ [1, 2, 3], [1, 1, 1], [2, 3, 4] ])
```

You obtain:

`[[1, 0, -1], [0, 1, 2]]`

5.21.12 AUGMENT

AUGMENT has as arguments two vectors, or two lists, or a list and an element.
AUGMENT concatenates its arguments.
For example, type:

```
AUGMENT([1, 2], 3)
```

You obtain:

`{1 2 3}`

5.21.13 CYCLOTOMIC

CYCLOTOMIC has an integer $n$ as argument.
CYCLOTOMIC returns the cyclotomic polynomial of order $n$. This is a polynomial having the $n$-th primitive roots of the unity as zeros.
For example, when $n = 4$ the fourth roots of the unity are: $\{1, i, -1, -i\}$;
among them, the primitive roots are: $\{i, -i\}$.
Therefore, the cyclotomic polynomial of order 4 is $(X - i)(X + i) = X^2 + 1$.
Another example; if you type:

```
CYCLOTOMIC(20)
```

70
You obtain:

\[ x^8 - x^6 + x^4 - x^2 + 1 \]

**5.21.14 STURM**

STURM has a polynomial \( P \) as argument. STURM returns a list containing the Sturm’s sequences of \( P \) and their multiplicities. The Sturm sequence \( R_1, R_2, \ldots \) can be obtained from a square-free factor \( F \) of \( P \) as follows:

- \( R_1 \) is the opposite of the remainder of the euclidean division of \( F \) by \( F' \);
- then, \( R_2 \) is the opposite of the remainder of the euclidean division of \( F' \) by \( R_1 \), and so on, until \( R_k = 0 \).

For example, type:

\[ \text{STURM}(x^3 + 1) \]

You obtain:

\[ \{ [1], -1, [x^3 + 1, -3x^2 - 1], 1. \} \]

The first element of the list denotes that the denominator of \( P \) (that is, the element with \(-1 \) power), is 1.

**5.21.15 STURMAB**

STURMAB has three arguments: a polynomial \( P \) and two numbers: \( a \) and \( b \). STURMAB returns a list containing an element with the same sign as \( P(a) \) and the number of zeros of \( P \) in \([a, b]\). For example, type:

\[ \text{STURMAB}(x^2 \cdot (x^3 + 2), -2, 0) \]

You obtain:

\[ \{-6, 1\} \]

**5.21.16 P2C**

Permutations are defined with the image list \( \{P(1), P(2), \ldots, P(n)\} \).

For example, defining the permutation \( P \) as \( P = \{3, 2, 1\} \) means that:

- \( P(1) = 3 \), \( P(2) = 2 \), \( P(3) = 1 \).

A cycle is denoted with a list containing the images of an element through the cycle; for example, defining a cycle \( C \) as \( C = \{3, 2, 1\} \) means that:

- \( C(3) = 2 \), \( C(2) = 1 \), \( C(1) = 3 \).

Accordingly, a decomposition into cycles is denoted with a list of lists. P2C has a permutation as argument. P2C returns its decomposition into cycles and its signature. For example, type:

\[ \text{P2C}\{3, 4, 5, 2, 1\} \]

You obtain:

\[ \{ \{1, 3, 5\}, \{2, 4\}, -1 \} \]
5.21.17  C2P
C2P has a list of cycles as argument.
C2P returns the permutation having the input list as decomposition into cycles (see also P2C).
For example, type:
\[
\text{C2P}([\{1, 3, 5\}, \{2, 4\}])
\]
You obtain:
\[
\{3, 4, 5, 2, 1\}
\]

5.21.18  CIRC
CIRC has two permutations as arguments.
CIRC returns the permutation obtained by composition: \((1\text{st argument} \circ 2\text{nd argument})\).
For example, type:
\[
\text{CIRC}([\{3, 4, 5, 2, 1\}, \{2, 1, 4, 3, 5\}])
\]
You obtain:
\[
\{4, 3, 2, 5, 1\}
\]

5.21.19  FDISTRIB
FDISTRIB performs a full distribution of multiplication with respect to addition in a single step.
For example, type:
\[
\text{FDISTRIB}((X + 1) \cdot (X + 2) \cdot (X + 3))
\]
You obtain:
\[
X^3 + 6X^2 + 11X + 6
\]

5.21.20  DISTRIB
DISTRIB applies the distributive property of multiplication with respect to addition once.
DISTRIB, when invoked multiple times, allows you to perform a full distribution of multiplication
with respect to addition step-by-step.
For example, type:
\[
\text{DISTRIB}((X + 1) \cdot (X + 2) \cdot (X + 3))
\]
You obtain:
\[
X \cdot (X + 2) \cdot (X + 3) + 1 \cdot (X + 2) \cdot (X + 3)
\]

5.21.21  POWEXPAND
POWEXPAND rewrites a power as a product.
For example, type:
\[
\text{POWEXPAND}((X + 1)^3)
\]
You obtain:
\[
(X + 1) \cdot (X + 1) \cdot (X + 1)
\]
You can also expand \((x+1)^3\) step-by-step, by invoking DISTRIB many times on the result obtained above.
5.21.22 SIMPLIFY

SIMPLIFY attempts to simplify the expression given as argument automatically. Like all other automatic simplification tools, you should not demand miracles to this command. For example, type:

\[
\text{SIMPLIFY}\left(\frac{\sin(3 \cdot X) + \sin(7 \cdot X)}{\sin(5 \cdot X)}\right)
\]

After simplification, you obtain:

\[4 \cdot \cos(X)^2 - 2\]

5.21.23 EXP2POW

EXP2POW rewrites an expression like

\[\exp(n \times \ln(x))\]

as a power of \(x\). For example, type:

\[
\text{EXP2POW}(\exp(N \times \ln(X)))
\]

You obtain:

\[X^N\]

Notice the difference with respect to LNCOLLECT:

We have:

\[
\text{LNCOLLECT}(\exp(N \times \ln(X))) = \exp(N \times \ln(X))
\]

\[
\text{LNCOLLECT}(\exp(\ln(X)/3)) = \exp(\ln(X)/3)
\]

But:

\[
\text{EXP2POW}(\exp(\ln(X)/3)) = X^{1/3}
\]

5.21.24 MSLV

MSLV solves numerically a system of non-polynomial equations. MSLV has three vectors as arguments: a vector containing the equations, a vector containing the system’s variables, and a vector containing an initial guess for the solution. MSLV returns a vector containing an approximate solution of the given system of equations. While the command is running, the first display line shows the last estimate \(\mathbf{V}\), and the second line shows the modulo of \(\Delta \mathbf{V}\). For example, type:

\[
\text{MSLV}([\sin(X) + Y', X + \sin(Y) = 1'], [X, Y], [0, 0])
\]

You obtain:

\[1.82384112611, -0.968154636174]\]

5.21.25 PMINI

PMINI has a matrix \(A\) as argument. PMINI returns another matrix, whose first “non-zero row” is the minimal polynomial of \(A\). For example, type:

\[
\text{PMINI}([[1, 0], [0, 1]])
\]

73
In step-by-step mode, you obtain:

$L_2 = L_2 - L_1$
\[
\begin{bmatrix}
1 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & X \\
1 & 0 & 0 & 1 & X^2 \\
\end{bmatrix}
\]

$L_3 = L_3 - L_1$
\[
\begin{bmatrix}
1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & X - 1 \\
1 & 0 & 0 & 1 & X^2 \\
\end{bmatrix}
\]

Reduction result
\[
\begin{bmatrix}
1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & X - 1 \\
0 & 0 & 0 & 0 & X^2 - 1 \\
\end{bmatrix}
\]

So, the minimal polynomial of $A$: \[
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix}
\]
is:

\[X - 1\]

5.21.26 IBERNOULLI

IEEE 754 has an integer $n$ as argument.

IEEE 754 returns the $n^{th}$ Bernoulli’s number $B(n)$.

The following relation holds:

\[
\frac{t}{e^t - 1} = \sum_{n=0}^{+\infty} \frac{B(n)}{n!} t^n
\]

Remember that the Bernoulli’s polynomials $B_k$ are defined as:

$B_0 = 1$

$B_k'(x) = kB_k(x)$

\[
\int_0^1 B_k(x) dx = 0
\]

Bernoulli’s numbers are defined as:

$B(n) = B_n(0)$

For example, type:

IEEE 754(6)

You obtain:

\[
\begin{bmatrix}
1 \\
42 \\
\end{bmatrix}
\]

5.21.27 GAMMA

Returns the value of the $\Gamma$ function at the given point.

The $\Gamma$ function is defined as:

\[
\Gamma(x) = \int_0^{+\infty} e^{-t} t^{x-1} dt
\]

We have:

\[
\Gamma(1) = 1
\]
\[ \Gamma(x + 1) = x \cdot \Gamma(x) \]

For example, type: \texttt{GAMMA(5)}

You obtain: 24

Type: \texttt{GAMMA(1/2)}

You obtain: \[ \sqrt{\pi} \]

5.21.28 PSI

PSI has two numbers \( a \) and \( n \) as arguments.

PSI returns the value of the \( n \)-th derivative of the Digamma function at \( a \). The Digamma function is defined as the derivative of \( \ln(\Gamma(x)) \).

For example, type: \texttt{PSI(3,1)}

You obtain: \[ -\frac{5}{4} + \frac{1}{6} \cdot \pi^2 \]

5.21.29 Psi

Psi has as argument a number, \( a \).

Psi returns the value of the Digamma function at \( a \).

The Digamma function is defined as the derivative of \( \ln(\Gamma(x)) \), so we have: \( \text{PSI}(a,0) = \text{Psi}(a) \).

For example, type: \texttt{Psi(3)}

You obtain: \(.922784335098\)

5.21.30 RESULTANT

RESULTANT has two polynomials as arguments.

RESULTANT returns the resultant of the two polynomials. The resultant of two polynomials is the last non-null remainder of the Euclidean algorithm, and is also the determinant of their Sylvester matrix \( S \).

The Sylvester matrix \( S \) of two polynomials \( A[X] = \sum_{i=0}^{n} a_i X^i \) and \( B[X] = \sum_{i=0}^{m} a_i X^i \) is a square matrix with \( m + n \) rows and columns; its first \( m \) rows are made from the coefficients of \( A[X] \):

\[
\begin{pmatrix}
  s_{11} = a_n & s_{12} = a_{n-1} & \cdots & s_{1(n+1)} = a_0 & 0 & \cdots & 0 \\
  s_{21} = 0 & s_{22} = a_n & \cdots & s_{2(n+1)} = a_1 & s_{2(n+2)} = a_0 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  s_{m1} = 0 & s_{m2} = 0 & \cdots & s_{m(n+1)} = a_{m-1} & s_{m(n+2)} = a_{m-2} & \cdots & a_0 \\
\end{pmatrix}
\]
and the following $n$ rows are made in the same way from the coefficients of $B[X]$:

$$
\begin{pmatrix}
  s_{(m+1)1} &= b_m & s_{(m+1)2} &= b_{m-1} & \cdots & s_{(m+1)(m+1)} &= b_0 & 0 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \ddots & \vdots
\end{pmatrix}
$$

For example, type:

RESULTANT($x^3 - p \cdot x + q, 3x^2 - p$)

You obtain:

$-4p^3 + 27q^2$

5.21.31 SEVAL

SEVAL has an expression as argument.
SEVAL simplifies the expression, operating on all but the top-level operator of the expression.
For example, type:

SEVAL(SIN(3 \cdot X - X) + SIN(X + X))

You obtain:

$\text{SIN}(2 \cdot X) + \text{SIN}(2 \cdot X)$

5.21.32 SIGMA

SIGMA has two arguments: the first argument is a function $f(x)$ of a variable $x$ given as the second argument. SIGMA returns the discrete antiderivative of the input function, that is, the function $G$ that satisfies the relation: $G(x+1) - G(x) = f(x)$.
For example, type:

SIGMA(X \cdot X!, X)

You obtain:

$X!$

5.21.33 SIGMAVX

SIGMAVX has as argument a function $f$ of the current variable VX.
SIGMAVX returns the discrete antiderivative of the input function, that is a function $G$ that satisfies the relation: $G(x+1) - G(x) = f(x)$.
For example, type:

SIGMAVX($x^2$)

You obtain:

$$
\frac{2x^3 - 3x^2 + x}{6}
$$

5.21.34 VER

VER returns the version number of your CAS.
Type:

VER
You obtain:

4.20000124

This result means that you have a version 4 CAS, dated 24 January 2000.
Instead, VERSION returns the version number of the calculator’s ROM as a whole.
Type:
VERSION
You obtain:

```
{ "Version HP49 - B & Revision #1.17 - 4" "Copyright HP 1999"
}
```

5.21.35  **TABVAR**

**TABVAR** has as argument an expression whose derivative is rational.
**TABVAR** returns the variation table of the expression with respect to the current variable.
Type:

```
TABVAR(LN(X) + X)
```

In step-by-step mode you obtain:
\[ F =: (LN(X) + X) \]
\[ F' =: (\frac{1}{X} + 1) \]
\[ \rightarrow: \frac{X + 1}{X} \]
Variation table:

```
-\infty \ ? \ 0 \ + \ +\infty \ X
\ ? \ ? \ -\infty \ \uparrow \ +\infty \ F
```

5.21.36  **SIGNTAB**

**SIGNTAB** has a rational expression as argument.
**SIGNTAB** returns the sign table of the expression with respect to the current variable.
Type:

```
SIGNTAB(X^2 + X)
```

You obtain:

\{ -\infty + -1 \ - 0 + +\infty \}

5.21.37  **TABVAL**

**TABVAL** has an expression and a list of numbers as arguments.
**TABVAL** stores the input expression into the EQ variable and returns a list. This list holds the expression itself and a list of values; the latter list contains the value the expression assumes when the current CAS variable is replaced by the list of numbers given as input.
For example, type:
```
TABVAL(X^2 + X, \{1,2,3\})
```

You obtain:

\{X^2 + X, \{\{1,2,3\}, \{2,6,12\}\}\}

5.21.38  **PLOT**

**PLOT** has an expression as argument.
**PLOT** stores the input expression into the EQ variable and opens the PLOT SETUP window.
For example, type:
```
PLOT(X^2 + X)
```

77
Now, by pressing ERASE DRAW, you obtain the plot of the expression stored in EQ, and
\[ x^2 + x \]
is stored into the history.

5.21.39  PLOTADD

PLOTADD has an expression as argument.
PLOTADD appends this expression to the list of equations currently stored in EQ and opens the
PLOT SETUP window.
Type:
\[ \text{PLOTADD}(x^2 - x) \]
Now, by pressing ERASE DRAW, you draw the plots of all expressions stored in EQ one over another,
and
\[ x^2 - x \]
is stored into the history.

5.21.40  SCROLL

SCROLL has a graphics object as argument.
SCROLL displays the graphics object, without touching PICT.
For example, suppose you have just plotted the function \( F(X) \). The plot is automatically stored
into the reserved variable PICT.
Therefore, PICT contains the current graphics variable. You can save its contents, typing:
\[ \text{RCL}(\text{PICT}) \text{ STO} \rightarrow \text{GRF} \]
Next, if you type:
\[ \text{SCROLL}(\text{GRF}) \]
you display the plot of \( F(X) \) again.

5.21.41  GROBADD

GROBADD has two graphics objects as arguments.
GROBADD concatenates its arguments and returns the result.
For example, if you previously saved the plot of \( F(X) \) into GRF, and the plot of \( H(X) \) into GRH, you
can type:
\[ \text{GROBADD}(\text{GRF}, \text{GRH}) \text{ STO} \rightarrow \text{GRFH} \]
This command stores into GRFH a graphics object containing the plots of both \( F(X) \) and \( H(X) \), one
below the other.

6  Bac 99 and HP49G

6.1 Introduction

First of all, enter CASCFG (Computer Algebra System ConFiG) to put the calculator in algebraic
mode and to initialize it.
The commands you will use can be found in the menu displayed by the SYMB key, and in the following sub-menus:

ALGEBRA (FACTOR LIN SUBST)
ARITHMETIC (IEGCD ISPRIME? PROPFRAC)
CALCULUS (DERIVX DERIV INTVX INT LIMIT)
GRAPH (SIGNTAB TABVAR)
TRIGONOMETR (TEXPAND)
and in the red-shift 1 (CMPLX) menu:

RE IM

Remember that after entering each command, it is necessary to press ENTER to execute it; this notice will often be omitted in the following.

Here, you find the mathematics test of the “Bac” 1999 test (S series) solved. Effort has been made to leverage the HP49G capabilities as much as possible, but you will notice that it is still the student’s duty to explain and justify his/her calculations, and to make a bit of reasoning.

6.2 Exercise 1

The goal of this exercise is to plot the curve Γ represented by \( M : \frac{1}{2} \cdot z^2 - z \) when \( m \), represented by \( z \), is a circle \( C \) of center \( O \) and radius 1. Let \( t \) be a real in \([−\pi, \pi]\) and \( m \) the point of \( C \) corresponding to \( z = e^{i \cdot t} \).

1. Calculation of the coordinates of \( M \):
   
   With the help of EQW, we enter the expression \( \frac{1}{2} \cdot z^2 - z \).
   
   We type:
   
   \[
   \text{EQW} \quad \alpha \cdot Z \cdot y \cdot 2 \div 2 \div \alpha \cdot Z \quad \text{ENTER}
   \]
   
   The expression is now in the command line, and we store it into the variable \( M \):
   
   \[\text{STO} \downarrow \alpha \cdot M\]
   
   Since \( z = e^{i \cdot t} \) we type:
   
   \[\text{SUBST}(M, Z = \exp(i \times t))\]
   
   the answer is:
   
   \[\frac{\exp(i \cdot t)^2 - 2 \cdot \exp(i \cdot t)}{2}\]
   
   Now, we linearize the expression, using the history to copy the previous expression down:
   
   \[\text{LIN(HIST ENTER) ENTER}\]
   
   the answer is:
   
   \[\frac{\frac{1}{2} \cdot \exp(2 \cdot i \cdot t) + \exp(i \cdot t)}{2}\]
   
   Notice that by copying the expression down, the calculator simplifies it, too:
   
   \[\Delta \text{ENTER ENTER}\]
   
   gives:
   
   \[\frac{\exp(2 \cdot i \cdot t) - 2 \cdot \exp(i \cdot t)}{2}\]

\(^2\text{Translator’s note: In France, the “Bac” (short form of “Baccalauréat”) certificate is awarded to students that successfully complete the upper secondary school course, at about age 18.}\)
• Now we want to look at the real part of this expression:

\[
\text{RE( HIST ENTER) ENTER}
\]

the answer is:
\[
\frac{\cos(t \cdot 2) - 2 \cdot \cos(t)}{2}
\]

At this point, we can define the function \( x(t) \), by typing:

\[
\text{DEFINE (X(t) = HIST ENTER) ENTER}
\]

• Then, we calculate the imaginary part (we must climb up in the history to retrieve the original expression \( \frac{\exp(2 \cdot i \cdot t) - 2 \cdot \exp(i \cdot t)}{2} \)), typing:

\[
\text{IM(HIST ENTER) ENTER}
\]

the answer is:
\[
\frac{\sin(t \cdot 2) - 2 \cdot \sin(t)}{2}
\]

To define the function \( y(t) \), we must now type:

\[
\text{DEFINE(Y(t) = HIST ENTER) ENTER}
\]

2. To determine an axis of symmetry of \( \Gamma \), we want to calculate \( x(-t) \) and \( y(-t) \); for this, we type:

\[
X(-t) \text{ ENTER}
\]

the answer is:
\[
\frac{\cos(t \cdot 2) - 2 \cdot \cos(t)}{2}
\]

Therefore: \( x(-t) = x(t) \)

After this, we type:

\[
Y(-t) \text{ ENTER}
\]

the answer is:
\[
\frac{-\sin(t \cdot 2) + 2 \cdot \sin(t)}{2}
\]

Therefore: \( y(-t) = -y(t) \)

If \( M_1(x(t), y(t)) \) is on \( \Gamma \), \( M_2(x(-t), y(-t)) \) is on \( \Gamma \), too.

We have just showed that \( M_1 \) and \( M_2 \) are symmetric with respect to \( Ox \); from this we conclude that the \( Ox \) axis is an axis of symmetry of \( \Gamma \).

3. Calculation of \( x'(t) \):

We type:

\[
\text{DERIV(X(t), t)}
\]

the answer is:
\[
\frac{2 \cdot (-2 \cdot \sin(t \cdot 2)) - 2 \cdot (-\sin(t))}{4}
\]

that is, after simplification (\( \triangle \ \text{ENTER ENTER} \)):

\[
-(\sin(t \cdot 2) - \sin(t))
\]
We now expand the expression (transformation of \( \sin(2 \cdot t) \)); we type:

\[
\text{TEXPAND( HIST ENTER) ENTER}
\]

the answer is:

\[-(\sin(t) \cdot 2 \cdot \cos(t) - \sin(t))\]

Now we factorize:

\[
\text{FACTOR( HIST ENTER) ENTER}
\]

the answer is:

\[-\sin(t) \cdot (2 \cdot \cos(t) - 1)\]

At last, we can define \( x'(t) \) by typing:

\[
\text{DEFINE( X1(t) = HIST ENTER) ENTER}
\]

4. Calculation of \( y'(t) \):

We type:

\[
\text{DERIV(Y(t), t)}
\]

the answer is:

\[
\frac{2 \cdot (2 \cdot \cos(t \cdot 2)) - 2 \cdot \cos(t)}{4}
\]

that is, after simplification (\( \triangle \) ENTER ENTER):

\[
\cos(t \cdot 2) - \cos(t)
\]

We now expand the expression (transformation of \( \cos(2 \cdot t) \)); we type:

\[
\text{TEXPAND( HIST ENTER) ENTER}
\]

the answer is:

\[2 \cdot \cos(t)^2 - 1 - \cos(t)\]

Now we factorize:

\[
\text{FACTOR( HIST ENTER) ENTER}
\]

the answer is:

\[(\cos(t) - 1) \cdot (2 \cdot \cos(t) + 1)\]

At last, we can define \( y'(t) \) by typing:

\[
\text{DEFINE( Y1(t) = HIST ENTER) ENTER}
\]

5. Variations of \( x(t) \) and \( y(t) \)

To do this we draw on the same plot both \( x(t) \) and \( y(t) \); we type: blue-shift F4 (2D/3D): the PLOT SETUP window opens.

We choose function as plot type, with the help of the choos menu key.

Then, we enter

\[\{X(t), Y(t)\}\]

into the equation (EQ) field, and \( t \) as independent variable, followed by ENTER.

Afterwards, we type blue-shift F2 (WIN), to set the plot window parameters.
6. Trace of \( \Gamma \):

- **Values of \( x(t) \) and \( y(t) \)**
  
  We calculate the values of \( x(t) \) and \( y(t) \) for \( t = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi \), by typing:

  \[
  X(0) \text{ ENTER} \\
  \text{answer: } -\frac{1}{2} \\
  X(\pi \div 3) \text{ ENTER} \\
  \text{answer: } -\frac{3}{4} \\
  X(2 \times \pi \div 3) \text{ ENTER} \\
  \text{answer: } \frac{1}{4} \\
  X(\pi) \text{ ENTER} \\
  \text{answer: } \frac{3}{2} \\
  Y(0) \text{ ENTER} \\
  \text{answer: } 0 \\
  Y(\pi \div 3) \text{ ENTER} \\
  \text{answer: } -\frac{\sqrt{3}}{4} \\
  Y(2 \times \pi \div 3) \text{ ENTER} \\
  \text{answer: } -\frac{3}{4} \cdot \sqrt{3} \\
  Y(\pi) \text{ ENTER} \\
  \text{answer: } 0
  \]

- **Slope of tangents \( m = \frac{y'(t)}{x'(t)} \)**
  
  We calculate the values of \( \frac{y'(t)}{x'(t)} \) for \( t = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi \), by typing:

  \[
  \text{LIMIT( } Y1(t)/X1(t), t = 0 ) \text{ ENTER} \\
  \text{answer: } 0 \\
  \text{LIMIT( } Y1(t)/X1(t), t = \pi \div 3 ) \text{ ENTER} \\
  \text{answer: } \infty \\
  \text{LIMIT( } Y1(t)/X1(t), t = 2 \times \pi \div 3 ) \text{ ENTER} \\
  \text{answer: } 0 \\
  \text{LIMIT( } Y1(t)/X1(t), t = \pi ) \text{ ENTER} \\
  \text{answer: } \infty
  \]

These are the variations of \( x(t) \) and \( y(t) \):

| \( t \) | \( 0 \) | \( \frac{\pi}{3} \) | \( \frac{2\pi}{3} \) | \( \pi \) |
|-----------|----------|----------------|-----------------|
| \( x'(t) \) | \( - \) | \( 0 \) | \( + \) | \( ? \) | \( + \) | \( 0 \) |
| \( y'(t) \) | \( 0 \) | \( - \) | \( ? \) | \( - \) | \( 0 \) | \( + \) | \( ? \) |
| \( m \) | \( 0 \) | \( \infty \) | \( 0 \) | \( \infty \) |
• Plotting $\Gamma$:
  We now plot the parametric curve.
  We type: 
  blue-shift F4 (2D/3D) and the PLOT SETUP opens up.
  We choose the parametric plot type, with the help of the choos menu key. Then, we enter
  \[ X(t) + i \times Y(t) \]
  into the equation (EQ) field and $t$ as independent variable, followed by ENTER.
  We then press blue-shift F2 (WIN) to set the plot window parameters.

6.3 Exercise 2 (specialized)

Let be, for a natural $n$:

\[ a_n = 4 \times 10^n - 1, \quad b_n = 2 \times 10^n - 1 \quad \text{and} \quad c_n = 2 \times 10^n + 1 \]

We type:

\[
\begin{align*}
\text{DEFINE}(A(N)) &= 4 \cdot 10^N - 1 \\
\text{DEFINE}(B(N)) &= 2 \cdot 10^N - 1 \\
\text{DEFINE}(C(N)) &= 2 \cdot 10^N + 1
\end{align*}
\]

1. • a) Calculate $a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3$:

   It suffices to type:

   \[
   \begin{align*}
   A(1) & \quad \text{answer 39} \\
   B(1) & \quad \text{answer 19} \\
   C(1) & \quad \text{answer 21} \\
   A(2) & \quad \text{answer 399} \\
   B(2) & \quad \text{answer 199} \\
   C(2) & \quad \text{answer 201} \\
   A(3) & \quad \text{answer 3999} \\
   B(3) & \quad \text{answer 1999} \\
   C(3) & \quad \text{answer 2001}
   \end{align*}
   \]
• b) number of digits and divisibility
Here, the calculator is useful only to do some evaluation trials for different values of

\[ n \]

We know that integers \( n \) satisfying:

\[
10^n \leq n < 10^{n+1}
\]

have \( (n + 1) \) digits when they are written using the decimal representation.

We have:

\[
3 \cdot 10^n < a_n < 4 \cdot 10^n \\
10^n < b_n < 2 \cdot 10^n \\
2 \cdot 10^n < c_n < 3 \cdot 10^n
\]

so the decimal representation of \( a_n, b_n, c_n \) has \( (n + 1) \) digits.

Moreover, \( d_n = 10^n - 1 \) is divisible by 9, since all digits of its decimal representation

are 9.

We have

\[ a_n = 3 \cdot 10^n + d_n \]

and

\[ c_n = 3 \cdot 10^n - d_n \]

so \( a_n \) and \( c_n \) are divisible by 3.

• c) \( b_3 \) is prime
We type:

\[ \text{ISPRIME?}(B(3)) \]

the answer is:

1.

that means true

To show that \( b_3 = 1999 \) actually is prime, we only have to test if 1999 is divisible by

all prime numbers less than or equal to \( \sqrt{1999} \).

Since \( 1999 < 2025 = 45^2 \), we check the divisibility of 1999 with \( n = 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41 \).

Since 1999 is not divisible by any of these numbers, we conclude that 1999 actually is

prime.

• d) \( a_n = b_n \times c_n \)
We type:

\[ B(N) \cdot C(N) \]

The answer is:

\[ 4 \cdot (10^n)^2 - 1 \]

that is the same value as \( a_n \)

Factorization of \( a_6 \) into prime factors
We type:

\[ \text{FACTOR}(A(6)) \]

The answer is:

\[ 3 \cdot 23 \cdot 29 \cdot 1999 \]
• e) $b_n$ et $c_n$ are mutually prime.
Here, the calculator is useful only to do some evaluation trials for different values of $n$...
To show that $c_n$ and $b_n$ are mutually prime, we observe that:

$$c_n = b_n + 2$$

So, the common divisors of $c_n$ and $b_n$ are the common divisors of $b_n$ and 2, and are the common divisors of $c_n$ and 2, too. $b_n$ and 2 are mutually prime, because $b_n$ is a prime number not equal to 2. Therefore,

$$GCD(c_n, b_n) = GCD(c_n, 2) = GCD(b_n, 2) = 1$$

2. Consider the equation:

$$b_3 \cdot x + c_3 \cdot y = 1$$

• a) There is at least a solution, because we can apply the Bézout identity. In fact, the Bézout theorem states that:
If $a$ and $b$ are mutually prime, two values $x$ and $y$ do exist so that:

$$a \cdot x + b \cdot y = 1$$

Therefore, the equation:

$$b_3 \cdot x + c_3 \cdot y = 1$$

has at least a solution.

• b) We type:

$$\text{IEGCD}(B(3), C(3))$$

the answer is:

$$\{1, 1000, -999\}$$

this means that:

$$1 = b_3 \times 1000 + c_3 \times (-999)$$

so, we have just found a (particular) solution:

$x = 1000$, $y = -999$.

By hand, we write:

c_3 = b_3 + 2 and $b_3 = 999 \times 2 + 1$
so, $b_3 = 999 \times (c_3 - b_3) + 1$ and:

$$b_3 \times 1000 + c_3 \times (-999) = 1$$

• c) Here, the calculator cannot find the general solution.
We have:

$$b_3 \cdot x + c_3 \cdot y = 1$$

and

$$b_3 \times 1000 + c_3 \times (-999) = 1$$

subtracting the second equation from the first, we have:

$$b_3 \cdot (x - 1000) + c_3 \cdot (y + 999) = 0$$

85
and then:

\[ b_3 \cdot (x - 1000) = -c_3 \cdot (y + 999) \]

Applying the Gauss theorem: \( c_3 \) is prime with \( b_3 \), so \( c_3 \) divides \((x - 1000)\). Therefore, it exists \( k \in \mathbb{Z} \) so that:

\[ (x - 1000) = k \times c_3 \]

and

\[ -(y + 999) = k \times b_3 \]

Reciprocally, let be:

\[ x = 1000 + k \times c_3 \]

and

\[ y = -999 - k \times b_3 \text{ for } k \in \mathbb{Z} \]

We have:

\[ b_3 \cdot x + c_3 \cdot y = b_3 \times 1000 + c_3 \times (-999) = 1 \]

The general solution therefore is: for any \( k \in \mathbb{Z} \):

\[ x = 1000 + k \times c_3 \]

\[ y = -999 - k \times b_3 \]

6.4 Exercise 2 (not specialized)

Let us consider the succession

\[ u_n = \int_0^2 \frac{2x + 3}{x + 2} e^x \, dx \]

1. a) Variations of \( g(x) = \frac{2x + 3}{x + 2} \) for \( x \in [0, 2] \)

We type:

\[ \text{DEFINE}(G(X) = \frac{2X + 3}{X + 2}) \]

then, we type:

\[ \text{TABVAR}(G(X)) \]

We obtain:

\[ -\infty \uparrow -2 \uparrow +\infty \quad X \quad \uparrow \quad \infty \uparrow 2 \quad F \]

The first line gives the sign of \( f'(x) \) depending on \( x \), and the second one gives the variations of \( f(x) \).

From this result, we conclude that \( g(x) \) increases in the interval \([0, 2]\).

Notice that if the calculator is in step-by-step mode (to enable this mode, you must press on MODE, then on the cas menu key, and set Step/Step using the chk menu key followed by ok ok), we obtain (the input function is denoted by \( F \)):

\[ F := \frac{2 \cdot X + 3}{X + 2} \]

86
After pressing the ok menu key, we obtain:

$$F' := \frac{2 \cdot (X + 2) - (2 \cdot X + 3)}{SQ(X + 2)}$$

and, using the \( \nabla \) arrow key to scroll the screen:

$$\rightarrow \frac{1}{(X + 2)^2}$$

At last, pressing the ok menu key again displays the variation table.

If the step-by-step mode is disabled, it is possible to calculate the derivative with:

\[ \text{DERVX}(G(X)) \]

We want to calculate \( g(0) \) and \( g(2) \); to do this we type:

\[ G(0) \]

answer \( \frac{3}{4} \)

\[ G(2) \]

answer \( \frac{7}{4} \)

\[ \frac{3}{2} \leq g(x) \leq \frac{7}{4} \quad \text{for} \quad x \in [0, 2] \]

- b) Here the calculator is not useful at all... it is enough to notice that:

\[ e^{x/2} \geq 0 \quad \text{for} \quad x \in [0, 2] \]

to show that, for \( x \in [0, 2] \), we have:

\[ \frac{3}{2}e^{x/2} \leq g(x)e^{x/2} \leq \frac{7}{4}e^{x/2} \]

- c) We integrate the inequality written above, typing:

\[ \int_0^2 e^{x/2} \, dX \]

We obtain:

\[ N \cdot e^{x/2} - N \]

From this, we conclude that:

\[ \frac{3}{2}(ne^{x/2} - n) \leq u_n \leq \frac{7}{4}(ne^{x/2} - n) \]

To justify the previous calculation, we must also say that a primitive of \( e^{x/2} \) is \( n \cdot e^{x/2} \).

If we don’t know this, we can always type:

\[ \text{INTVX}(\text{EXP}(X \div N)) \]

and the answer is: \( N \cdot e^{x/2} \)
• d) We look for the limit of \((ne^\frac{2}{n} - n)\) when \(n \to +\infty\):

\[
\text{LIMIT}(N \cdot \text{EXP}(2 \div N) - N, \ N = +\infty)
\]

We obtain:

\[2\]

To explain this result, we must say that:

\[
\lim_{x \to 0} \frac{e^x - 1}{x} = 1
\]

and, as a consequence:

\[
\lim_{n \to +\infty} \frac{e^{\frac{2}{n}} - 1}{\frac{2}{n}} = 1
\]

or, also:

\[
\lim_{n \to +\infty} (e^{\frac{2}{n}} - 1) \cdot n = 2
\]

If \(L\) does exist, letting \(n\) go towards \(+\infty\) in the inequalities of point 1b), we obtain:

\[
\frac{3}{2} \cdot 2 \leq L \leq \frac{7}{4} \cdot 2
\]

2. • a) \(g(x) = 2 - \frac{1}{x+2}\) and calculation of \(I = \int_0^2 g(x)dx\)

We type:

\[
\text{PROPFRAC}(G(X))
\]

We obtain:

\[2 - \frac{1}{X + 2}\]

To calculate \(I\), we type in the equation editor (EQW key):

\[
\int_0^2 g(X)dX
\]

We obtain:

\[-(\ln(2) - 4)\]

Doing it by hand, we have \(2x + 3 = 2(x + 2) - 1\), so

\[g(x) = 2 - \frac{1}{x + 2}\]

We now integrate term by term between 0 et 2, obtaining:

\[
\int_0^2 g(x)dx = [2x - \ln(x + 2)]|_{x=0}^{x=2}
\]

that is, since \(\ln 4 = 2 \ln 2\):

\[
\int_0^2 g(x)dx = 4 - \ln 2
\]
b) Here, the calculator is not useful... it is enough to notice that $e^{x}$ increases for $x \in [0, 2]$, to obtain the inequality:

$$1 \leq e^{x} \leq e^{2}$$

then, since $g(x)$ is positive on $[0, 2]$, we have:

$$g(x) \leq g(x)e^{x} \leq g(x)e^{2}$$

integrating we have:

$$I \leq u_{n} \leq e^{2}I$$

c) Convergence of $u_{n}$

We look for the limit of $e^{x}$ when $n \to +\infty$:

$$\text{LIMIT}(\exp(2 \div N), N = +\infty)$$

We obtain:

$$1$$

Actually, $\frac{2}{n}$ goes towards 0 when $n$ goes towards $+\infty$, so $e^{\frac{2}{n}}$ goes towards $e^{0} = 1$ when $n$ goes towards $+\infty$.

When $n$ goes towards $+\infty$, $u_{n}$ stays between $I$ and a quantity that goes towards $I$ (see the inequalities in point 2b)).

Therefore, $u_{n}$ converges and its limit is $I$.

We have shown that:

$$L = I = 4 - \ln 2$$

6.5 Problem

Part A

Let a function $f$ be defined on $[0, +\infty[$ as

$$f(x) = \left(1 - \frac{1}{x}\right)(\ln x - 2)$$

We type (using the Equation Writer):

DEFINE(F(X) = (1 - 1 \div X) \times (LN(X) - 2))

This is, in detail, the sequence of keys we must press ($<$ represents the cursor):

DEFINE(F(X) = <$)

To enter the expression that follows, using the equation editor, we press the EQW key.

We now are in the equation editor; we type:

$$1 - 1 \div X \uparrow \uparrow \uparrow \times LN(X) \uparrow \uparrow \uparrow \downarrow - 2 \text{ ENTER}$$

In the command line, we have:

DEFINE(F(X) = (1 - \frac{1}{X}) \cdot (LN(X) - 2))

At this point, we press ENTER to execute it.

$F$ is added to the variable's menu and NOVAL is displayed in the screen.

To check our work, we type $F(X)$; we should obtain:

$$\frac{(X - 1) \cdot LN(X) - (2 \cdot X - 2)}{X}$$
1. Limit of \( f \) in \(+\infty\) and 0.

We type:

\[
\text{LIMIT}(F(X), +\infty)
\]

answer \(+\infty\)

then,

\[
\text{LIMIT}(F(X), 0)
\]

answer \(+\infty\)

2. Calculation of \( f'(x) \).

We type:

\[
\text{DERVX}(F(X))
\]

We obtain:

\[
\frac{\ln(x) + x - 3}{x^2}
\]

3. \( u(x) = \ln x + x - 3 \)

Now, we type:

\[
\text{DEFINE}(U(X) = \ln(x) + x - 3)
\]

• a) Variations of \( u \).

We type:

\[
\text{TABVAR}(U(X))
\]

The calculator asks to enable complex mode: answer YES.

We obtain:

\[
\begin{array}{cccc}
-\infty & + & -1 & - 0 & +\infty \\
-\infty & \uparrow & i\pi - 4 & \downarrow & -\infty & \uparrow & +\infty & F
\end{array}
\]

Warning!!!!

Only the portion of the table in which \( x > 0 \) must be taken into account (when \( x < 0 \) the calculator assumes \( \ln(x) \) exists, and has a complex value).

• b) \( u(x) = 0 \) has an unique solution \( \alpha \) in \([2, 3]\).

From a) \( u \) increases in \([0, +\infty[\).

We type:

\[
U(2) \text{ red – shift ENTER}
\]

answer \(-0.306...\)

then,

\[
U(3) \text{ red – shift ENTER}
\]

answer \(1.098...\)

Thereafter, we calculate:

\[
U(2.20) \text{ red – shift ENTER}
\]

answer \(-0.306...\)

and

\[
U(2.21) \text{ red – shift ENTER}
\]

answer \(-0.306...\)

From the theorem of intermediate values (\( u \) is both increasing and continuous in \([2, 3]\),
therefore $u$ zeroes itself exactly once between 2 et 3 (since $u(2) < 0$ and $u(3) > 0$).
So, if we call $\alpha$ the unique zero of $u$ in $[2, 3]$, we have:

$$2.20 < \alpha < 2.21$$

since $u(2.20) < 0$ and $u(2.21) > 0$.

- c) Sign of $u(x)$ in $]0, +\infty[$
  The sign of $u(x)$ can be deducted from the variation table of $u$; we have:
  $$\begin{align*}
  u(x) &< 0 \quad \text{for } x < \alpha \\
  u(x) &= 0 \quad \text{for } x = \alpha \\
  u(x) &> 0 \quad \text{for } x > \alpha
  \end{align*}$$

4. 
- a) Variations of $f$.
  We must do the variation table by hand, because the derivative of $f$ is not a rational function... and the calculator is not yet able to handle this case!
Since the sign of $f'(x)$ is the same as of $u(x)$, we have:

<table>
<thead>
<tr>
<th>$f'(x)$</th>
<th>$0$</th>
<th>$-\frac{1}{A}(\ln(A) - 2)$</th>
<th>$+\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>$+\infty$</td>
<td>$\frac{A^2 - 2 \cdot A + 1}{A}$</td>
<td>$+\infty$</td>
</tr>
</tbody>
</table>

- b) $f(\alpha) = -\frac{(\alpha-1)^2}{\alpha}$
We have:
$u(\alpha) = 0$, so $\ln(\alpha) = 3 - \alpha$
We type in the equation writer:

$$(1 - \frac{1}{A})(\ln(A) - 2)$$

then we highlight LN(A),
open the ALG (red-shift 4) menu,
press the (SUBST) key
to complete the command \text{SUBST}(\text{LN}(A), \text{LN}(A) = 3 - A),
and press \text{ENTER} \text{ENTER}.
We obtain:

$$- \frac{A^2 - 2 \cdot A + 1}{A}$$

Now,

$$\text{FACTOR}( - \frac{A^2 - 2 \cdot A + 1}{A} )$$
returns

$$\frac{(A - 1)^2}{A}$$
We type:

$$\text{DERIV}( - \frac{(A - 1)^2}{A}, A )$$
We obtain:

$$- \frac{(A^2 - 1)}{A^2}$$
So, the function $v(x) = -\frac{(x-1)^2}{x}$ decreases for $x > 1$.
We obtain a bounded approximation of $f(\alpha)$ calculating:
We type:
\[-(1.21)^2 \quad \text{red - shift ENTER}\]
answer \(-0.662488\)

\[-(1.2)^2 \quad \text{red - shift ENTER}\]
answer \(-0.65454\)...
so, we have:
\[-0.663 < f(\alpha) < -0.654\]
This is an approximation with tolerance \(9 \cdot 10^{-3}\) (since \(0.663 - 0.654 = 9 \cdot 10^{-3}\))
or, again:
\[-0.67 < f(\alpha) < -0.65\]
that is an approximation with tolerance \(2 \cdot 10^{-2}\) (since \(0.67 - 0.65 = 2 \cdot 10^{-2}\))

5. \(\bullet\) a) Sign of \(f\)
We notice that \(f(1) = 0\) and that \(f(e^2) = 0\)
We type:
\[F(1)\]  
answer : 0
\[F(\text{EXP}(2))\]  
answer : 0
These are the variations of \(f\) and the sign of \(f(x)\):

\[
\begin{array}{c|cccccc}
 f'(x) & 0 & -1 & -\alpha & +e^2 & +\infty \\
 f(x) & +\infty & \downarrow & 0 & \downarrow & \frac{e^2-1}{\alpha}(\ln(\alpha)-2) & \uparrow & 0 & \uparrow & +\infty \\
 f(x) & +\infty & +0 & - & \approx -0.66 & - & 0 & + & +\infty \\
\end{array}
\]

\(\bullet\) b) Plot \(C\) of \(f\)
We open the PLOT SETUP (blue_shift F4) menu, we choose function and F(X) for EQ, then we set the window parameters in WIN (blue_shift F2)

Part B
Let \(H\) be the antiderivative of \(f\) in \([0, +\infty[\) and \(\Gamma\) its plot.
Be careful with the notations; they are not the same as those of the textbook!

1. \(\bullet\) a) Variations of \(H\)
Since \(H'(x) = f(x)\) we have the following variation table:

\[
\begin{array}{c|cccc}
 f(x) & 0 & +1 & -e^2 & +\infty \\
\end{array}
\]

\(\bullet\) b) Tangent lines for \(x = 1\) and \(x = e^2\)
It is \(f(1) = 0\) and \(f(e^2) = 0\). The tangents of \(\Gamma\) for the abscissas 1 and \(e^2\) have a zero slope, and are therefore horizontal.

2. Calculation of \(H(x)\)
• a) Calculation of $\int_1^x \ln t \, dt$
In the equation editor, we type:
$$\int_1^x \ln(T) dT$$
We obtain:
$$x \cdot \ln(x) - (x - 1)$$
We can also ask for the antiderivative of $\ln x$, we type:
$$\text{INTVX}(\ln(x))$$
We obtain:
$$x \cdot \ln(x) - x$$
By hand, we set $u = \ln(t)$ and $dv = dt$, so $du = \frac{dt}{t}$ and $v = t$ we have:
$$\int_1^x \ln t \, dt = [t \cdot \ln t]_1^x - \int_1^x \frac{dt}{t} = x \cdot \ln x - \int_1^x dt = x \cdot \ln x - (x - 1)$$

• b) Expanding the expression of $f(x)$ we obtain:
$$f(x) = \ln x - \frac{\ln x}{x} + \frac{2}{x} - 2$$

• c) Expression of $H(x)$
We type (using the equation writer):
$$\text{DEFINE}(H(X) = \int_1^x F(T) dT)$$
then,
$$H(X) \ \text{ENTER}$$
We obtain:
$$-\frac{\ln(x)^2 - (2 \cdot x + 4) \cdot \ln(x) + 6 \cdot x - 6}{2}$$
We do the calculation by hand, integrating term by term the expression of $f(x)$ found in 2b); we have:
$$\int_1^x \ln t \, dt = x \ln x - x + 1$$
$$-\int_1^x \frac{\ln t}{t} \, dt = -\frac{(\ln x)^2}{2}$$
$$\int_1^x \frac{2}{t} \, dt = 2 \ln x$$
that’s why:
$$H(x) = x \ln x - x + 1 - \frac{(\ln x)^2}{2} + 2 \ln x - 2x + 2$$
$$H(x) = -\frac{(\ln x)^2}{2} + (x + 2) \ln x - 3x + 3$$
3. a) We type

\[
\text{LIMIT}(\ln(x)/x, x = 0)
\]

answer 0
From the course, we know that “\(x\) dominates \(\ln(x)\)”; this explains the result!
We type:

\[
\text{LIMIT}(H(x), x = 0)
\]

answer \(-\infty\)
We have:

\[
H(x) = x \ln x + \ln x \left(\frac{4 - \ln x}{2}\right) - 3x + 3
\]

When \(x\) goes towards 0, the first term of \(H(x)\) goes towards 0, its second term goes towards \(-\infty\) (because \(\ln x\) goes towards \(-\infty\) when \(x\) goes towards 0), and the following terms go towards 3.
Therefore,

\[
\lim_{x \to 0} H(x) = -\infty
\]

b) We type:

\[
\text{LIMIT}(H(x), x = +\infty)
\]

answer \(+\infty\)
We put in evidence \(x \ln x\) at the beginning of \(H(x)\), obtaining:

\[
H(x) = x \ln x \left(1 - \frac{\ln x}{2x} + \frac{2}{x} - \frac{3}{\ln x}\right) + 3
\]

\[
\lim_{x \to +\infty} x \ln x = +\infty
\]
and the term within parentheses goes towards 1 when \(x\) goes towards \(+\infty\), so

\[
\lim_{x \to +\infty} H(x) = +\infty
\]

c) Variations of \(H(x)\)
We calculate \(H(e^2)\)
We type:

\[
H(\exp(2))
\]

The result is the expression obtained replacing, in \(H(x)\), \(X\) with \(\exp(2)\).
We copy this expression to simplify it: (\(\triangle\) ENTER ENTER), obtaining:

\[
-(\exp(2) - 5)
\]

then,

\(\triangle\) ENTER red – shift ENTER

answer -2.38905...
We have also an approximate value of \(H(e^2)\).
We recall the table already worked out in 1a)

\begin{align*}
\begin{array}{c|ccc}
\hline
f(x) & 0 & + & 1 - e^2 & + & +\infty \\
\hline
H(x) & -\infty & | & 0 & \downarrow & \simeq -2.39 & | & +\infty \\
\hline
\end{array}
\end{align*}
• d) Plot \( C \) of \( f \) and plot \( \Gamma \) of \( H \)

We open the PLOT SETUP (blue_shift F4) window, choose function and put \( \{ F(X), H(X) \} \) in the EQ field, then we set the window parameters using WIN (blue_shift F2).

4. Area calculation
We have calculated:

\[
\int_{1}^{e^2} f(x) \, dx
\]

obtaining:

\[-(\text{EXP}(2) - 5)\]

this integral is negative because the function is below the \( x \) axis between 1 and \( e^2 \). Since the measurement unit is 2cm, the area expressed in \( \text{cm}^2 \) is therefore equal to \(-4H(e^2)\text{cm}^2\), that is:

\[4 \cdot (\text{EXP}(2) - 5) \text{ cm}^2\]

that is,

\[A = (4e^2 - 20) \text{ cm}^2\]

or

\[9.55 \text{ cm}^2 < A < 9.56 \text{ cm}^2\]

6.6 Conclusion
We have showed that a good leverage of the HP49G calculator’s capabilities enabled us to solve a large portion of the problems...

However, we must also notice that for arithmetic problems, more reasoning is needed: then, the calculator is useful to do checks and verifications...

7 Programming

7.1 Programming in algebraic mode

7.1.1 Entering a program
You write a program in the command line, between delimiters \( \ll \) and \( \gg \)

7.1.2 Saving a program
It is enough to append \( \text{STO} \gg \text{NOMDUPROGRAMME} \)

after the last \( \gg \) delimiter.

7.1.3 Editing a program
When entering a program, if the program syntax is bad, the calculator automatically places the command line cursor where the compiler has detected the error, so you can immediately fix it!!!

Instead, if the error is detected during program execution, you must enter:

\( \text{VISIT(’NOMDUPROGRAMME’)} \) to edit the program stored in the \text{NOMDUPROGRAMME} variable.

You can then edit it; \( \text{ENTER} \) saves the edited program.
7.1.4 Executing a program

If the program has no parameters, you type its name in the command line; if it has one or more parameters, the program name must be followed by its arguments, enclosed in brackets and delimited by commas.

Example: PGCD(45, 75)

7.1.5 Modifying a program and saving it with another name

You type:
RCL(‘NOMDUPROGRAMME’) and press the edit menu key.

Then, you modify the program as you wish, and append

STO⟩ NOUVEAUNOM

after the last ⟩ delimiter.

7.2 Program comments

It is a good habit to put comments in programs.

In algorithmic language a comment starts with // and ends at the end of the line.

On the HP49G, a comment starts with @ and ends at the end of the line or when another @ is encountered.

You obtain the @ character by pressing red-shift ENTER.

Warning!!! The compiler deletes all comments so, to preserve your comments, you should store the program as a text string and then compile it with STR →; this makes things a little more difficult, though...

7.3 Variables

7.3.1 Variable names

Variables are places where you can store values, numbers, expressions and any other object.

7.3.2 Local variables

The HP49G has local variables. Local variables are both declared and initialized (the initialization is mandatory!) with → (red − shift 0)

In RPN mode, you can define and initialize more than one variable at once, for example:

≪ 1 2 → A B ≫ subprogrambody ≫

In Algebraic mode, each definition must be followed by a subprogram (with delimiters ≫). The arrow must be surrounded by spaces (these spaces are automatically inserted when the calculator is not in α mode).

Example :

≪ 3.14 → PI ≫ 2 * PI * R ≫ STO⟩ PER

In this example, we have written a program named PER.
PI is a local variable defined and initialized by the code fragment 3.14 → PI. The scope of this
variable is limited to the subprogram that follows its definition (here, \(2 \pi R\)).

On the other hand, \(R\) is a global variable (it must exist before the program \(\text{PER}\) is executed).

If, during the execution of a program, you want to store a value into a variable (either local or global), you must use the \(\text{STO} >\) operator.

### 7.3.3 Parameters

When you define a function, it is possible to use parameters.

For example, if you want \(R\) to be a parameter of function \(\text{PER}\), you should write:

\[
\langle \rightarrow R \langle 3.14 \rightarrow \pi \langle 2 \pi R \rangle \rangle \rangle \text{STO} > \text{PER}
\]

Parameter \(R\) behaves the same as local variables; the only difference is that it is initialized when the function is invoked.

The invocation is done when, for example, you evaluate: \(\text{PER}(5)\).

To define a function with two arguments, the correct syntax, in both \(\text{RPN}\) and \(\text{Algebraic}\) modes is:

\[
\langle \rightarrow A B \langle ... \rangle \rangle
\]

### 7.4 Data input

#### 7.4.1 Translation in algorithmic language

To denote that the user enters a value into variable \(A\) during the execution of a program, in algorithmic language you write: \text{input A}.

To input values into \(A\) and \(B\) you write: \text{input A,B}.

#### 7.4.2 Translation for the HP49G in RPN mode

`'A' PROMPTSTO`

or

"A" "" INPUT STR-> EVAL 'A' STO

#### 7.4.3 Translation for the HP49G in Algebraic mode

To input a value into a local variable \(A\), you type:

\[
\ldots 0 \rightarrow A \langle \text{PROMPTSTO}(\text{'A')} \ldots \rangle \text{ or }
\ldots 0 \rightarrow A \langle \text{PROMPTSTO}('A'); 0 \rightarrow B \langle \text{PROMPTSTO}(\text{'B')} \ldots \rangle\rangle
\]

If you enter only:

\[
\ldots \text{PROMPTSTO}('A'), A \text{ is then a global variable, or }
\ldots \text{PROMPTSTO}('A') \text{; PROMPTSTO}(\text{'B'}), A \text{ and } B \text{ are then global variables.}
\]

You can use \text{INPUT}, too. For example, if a local variable \(A\) must contain a character string, you can enter:

\[
\text{INPUT}('A =\ldots') \rightarrow A
\]

or, if \(A\) must contain a number:

\[
\ldots \text{INPUT}('A =\ldots') \rightarrow A
\]

\[
\langle \text{OBJ} \rightarrow (A) \rightarrow A
\]

\[
\langle \ldots \rangle
\]

\[
\rangle
\]

97
7.5 Data output

7.5.1 Translation in algorithmic language

In algorithmic language, you write:

```
print "A=",A
```

7.5.2 Translation for the HP49G in RPN mode

Usually it is enough to push results into the stack, so that you can reuse them; if this is the case, you simply write:

```
A B
```

You can also tag the result, as follows:

```
A "A=" ->TAG
```

HALT halts the program.

7.5.3 Translation for the HP49G in Algebraic mode

Only the last result is written into the history.

To display intermediate results, you can use either:

```
DISP("A = " + A,3)
```

here, 3 is the line number on which the result is displayed, or:

```
MSGBOX("A = + → STR(A))
```

CLEAR clears the screen.

FREEZE(7) stops the program and freezes the screen so that you can see the results you have previously written on it.

7.6 Sequence of instructions, or “block”

A “block” is a sequence of one or more instructions.

7.6.1 Translation in algorithmic language

In algorithmic language, you can use either the space or a newline to terminate an instruction.

7.6.2 Translation for the HP49G in RPN mode

Like the algorithmic language, spaces and newlines act as instruction separators.

7.6.3 Translation for the HP49G in algebraic mode

When the HP49G works in algebraic mode, ; acts as instruction separator.

You can type ; pressing red-shift SPC simultaneously.

7.7 Store instruction

The store instruction is used to store a value or an expression into a variable.
7.7.1 Translation in Algorithmic language
In algorithmic language, the store operation is denoted, for example, by:
2*A→B to store 2*A into B.

7.7.2 Translation for the HP49G in RPN mode
When the HP49G works in RPN mode, you must use the postfix notation and the ST0 command:
2 A * 'B' ST0

7.7.3 Translation for the HP49G in Algebraic mode
When the HP49G works in algebraic mode, you use the STO key, that is displayed on the calculator screen as: △ (and will be denoted here by STO△).

7.8 Conditional branch instructions
7.8.1 Translation in algorithmic language
if condition then action end if
if condition then action1 else action2 end if
Example:
if A = 10 or A < B then B-A→B else A-B→A end if

7.8.2 Translation for the HP49G in RPN mode
IF condition THEN action END
IF condition THEN action1 action2 END
Warning: you must use the postfix notation and == to denote the equality test operator.
The above mentioned example can be written as:
IF A 10 == A B < OR THEN B A - 'B' STO ELSE A B - 'A' STO END
you can also write:
IF 'A==10' 'A < B' OR THEN ...

7.8.3 Translation for the HP49G in Algebraic mode
IF condition THEN action END
IF condition THEN action1 ELSE action2 END
Example: Pay attention to always use == to denote the equality test!
IF A == 10 OR A < B THEN B A STO △ B ELSE A B STO △ A END

7.9 "For" loops
7.9.1 Translation in algorithmic language
for I from A to B do action end for
for I from A to B (step P) do action end for

7.9.2 Translation for the HP49G in RPN mode
A B FOR I action NEXT
A B FOR I action P STEP

99
7.9.3 Translation for the HP49G in Algebraic mode

FOR (I , A , B) action NEXT
FOR (I , A , B) action STEP P

It is not necessary to declare the loop variable I in advance. I is automatically declared and initialized by: FOR (I,...)...

7.10 “While” loops

7.10.1 Translation in algorithmic language

while condition do action end while

7.10.2 Translation for the HP49G in RPN mode

WHILE condition REPEAT action END

7.10.3 Translation for the HP49G in Algebraic mode

WHILE condition REPEAT action END

7.11 Boolean (or conditional) expressions

A boolean (or conditional) expression is a function having a boolean value, that is, either true or false.

7.11.1 Translation in algorithmic language

To express a simple condition you use the following operators: = < > ≤ ≥ ≠

7.11.2 Translation for the HP49G in RPN mode

Warning, you must use the postfix notation and == to denote the equality test operator.

7.11.3 Translation for the HP49G in Algebraic mode

Warning, for the HP49G, the equality test operator is denoted by: ==

7.12 Logical operators

7.12.1 Translation in algorithmic language

To express complex conditions, you can use the following logical operators: or, and, and not.

7.12.2 Translation for the HP49G in RPN mode

Warning, you must use the postfix notation.
or, and, and not are denoted, respectively, by OR, AND, and NOT.

7.12.3 Translation for the HP49G in Algebraic mode

or, and, and not are denoted on the HP49G by OR, AND, and NOT.
7.13 Functions

In a function, you do not perform data entry explicitly: instead, you use function parameters, that are automatically initialized when the function is called.

In a function, you must be able to reuse the result at will: in algorithmic language, you use the return command instead of print.

7.13.1 Translation in algorithmic language

You can write, for example:

```
function addition(A,B)
  return A+B
end function
```

This means that:

- The result of the function will be displayed when the function is executed.
- The function can be used in an expression.

7.13.2 Translation for the HP49G in RPN mode

When the HP49G works in RPN mode, it is assumed that function arguments are on the stack when the function is invoked.

Inside the function, the arguments become local variables initialized with the topmost elements of the stack. The result of the function is pushed on the stack.

The example above can be written as:

```
≪→ A B
  ≪ A B + ≫
≫
'ADDITION' STO
```

To invoke the function you must push the intended values of A and B on stack levels 2 and 1; after the execution of ADDITION their sum will be on stack level 1.

7.13.3 Translation for the HP49G in Algebraic mode

The example above can be written as:

```
≪→ A B
  ≪ A + B ≫
≫
STO> ADDITION
```

To invoke the function, you enter:

```
ADDITION(4,5)
```
7.14 Lists

7.14.1 Translation in algorithmic language

In the algorithmic language, \{ and \} are used as list delimiters. For example, \{\} denotes the empty list, and \{1, 2, 3\} is a list of 3 elements. The + operator can be used to concatenate two lists together or a list with an element:

\{1, 2, 3\}→TAB

TAB + 4 →TAB (now TAB contains \{1, 2, 3, 4\})

TAB[2] denotes the second element of TAB, 2 in this example.

7.14.2 Translation for the HP49G in RPN mode

It is not necessary to declare the maximum length of a list in advance. The list delimiters are \{ and \}. For example \{1 2 3\} is a list of 3 elements, and \{\} is an empty list.

You can retrieve the P-th element of list L and put it on the stack using:

L P GET or 'L' P GET

To update the P-th element of L (for example, set it to 0) you must enter:

'L' P 0 PUT or L P 0 PUT 'L' STO

Actually, L P 0 PUT leaves the updated list on the stack, and:

'L' P 0 PUT updates list L in-place instead.

The + operator can be used to concatenate two lists together or a list with an element.

7.14.3 Translation for the HP49G in algebraic mode

It is not necessary to declare the maximum length of a list in advance. The list delimiters are \{ and \}. For example \{1 2 3\} is a list of 3 elements, and \{\} is an empty list.

You can retrieve the P-th element of list L using:

L[P] or GET (L, P )

To update the P-th element of L (for example, set it to 0) you must enter:

PUT(L, P, 0) STO→ L

or

PUT('L', P, 0)

Actually, PUT(L, P, 0) returns the updated list, and:

PUT ('L', P, 0) updates list L in-place instead.

The + operator can be used to concatenate two lists together or a list with an element. The SEQ command allows you to create a list; for example, if you type:

SEQ('X * X', 'X', 4, 10, 1)

you obtain:

\{16, 25, 36, 49, 64, 81, 100\}

7.15 Example: The sieve of Erastothenes

7.15.1 Description

To find the prime numbers less than or equal to \(N\):

1. Put the numbers from 1 to \(N\) into a list.
2. Mark 1 and store 2 into the variable $P$.
   If $P \cdot P \leq N$ we must process all elements from $P$ to $N$.

3. Mark all multiples of $P$ starting from $P \cdot P$.

4. Increment $P$ by 1
   If $P \cdot P$ is less than or equal to $N$, we must process the non-marked elements from $P$ to $N$.

5. Store into $P$ the smallest non-marked elements of the list.

6. Repeat steps 3, 4 and 5 while $P \cdot P$ is less than or equal to $N$.

7.15.2 Algorithmic language

```plaintext
function crible(N)
    local TAB, PREM, I, P
    // TAB and PREM are lists
    {} -> TAB
    {} -> PREM
    for I from 2 to N do
        TAB + I -> TAB
    end for
    0 + TAB -> TAB
    2 -> P
    // Done steps 1 and 2
    // marking has been implemented replacing the element to be marked with 0
    // TAB is the list 0 2 3 4 ... N
    while $P \cdot P \leq N$ do
        for I from P to int(N/P) do
            0 -> TAB[I*P]
        end for
        // We marked all multiples of P starting from $P \cdot P$
        P + 1 -> P
        // We search the smallest number <= N, not marked, between P and N
        while ($P \cdot P \leq N$) and (TAB[P]=0) do
            P + 1 -> P
        end while
    end while
    // We copy the result into PREM
    for I from 2 to N do
        if TAB[I] != 0 then
            PREM + I -> PREM
        end if
    end for
    return PREM
```

103
7.15.3 HP49G, RPN mode

This is the program CRIBLE:

N is the parameter whose actual value must be on the stack.
The local variables are:
P and I (integers),
TA and PREM (lists).

%%HP: T(3)A(R)F(.); << {} {} 2 1 → N TA PREM P I
≪ 0 'X' 'X' 2 N 1 SEQ + 'TA' STO
  WHILE P P * N ≤ REPEAT
      P N P / FLOOR FOR I
          TA I P * 0 PUT 'TA' STO
      NEXT
      1 'P' STO+
      WHILE P P * N ≤ TA P GET 0 == AND REPEAT
          1 'P' STO+
  END
END
2 N FOR I
    IF TA I GET 0 ≠ THEN
        I 'PREM' STO+
    END
NEXT
PREM
≫

7.15.4 HP49G, Algebraic mode

This is the program CRIBLE; the user must type, for example:
CRIBLE(100), to execute it.

≪ → N
  ≪ 0+SEQ('I','I',1,N,1) → TA
  ≪ 2 → P
      ≪ WHILE P * P ≤ N REPEAT
          FOR (I , P , FLOOR(N/P))
              PUT('TA',I*P,0)
          NEXT;
          P + 1 STO P;
      WHILE P*P ≤ N AND GET(TA,P) == 0 REPEAT
          P + 1 STO P
      END
END;
{2} → PREM
  ≪ FOR (I,3,N)
      IF TA(I) ≠ 0 THEN
          PREM + I STO PREM;
      END
  NEXT;

104
8 Arithmetic programs

8.1 Calculating the GCD using the Euclide’s algorithm

This algorithm is rooted on the recursive definition of GCD:
\[
\begin{align*}
GCD(A, 0) &= A \\
GCD(A, B) &= GCD(B, A \mod B) \text{ if } B \neq 0
\end{align*}
\]

The algorithm can be described as follows:
we carry out this sequence of euclidean divisions:
\[
\begin{align*}
A &= B \times Q_1 + R_1 \quad 0 \leq R_1 < B \\
B &= R_1 \times Q_2 + R_2 \quad 0 \leq R_2 < R_1 \\
R_1 &= R_2 \times Q_3 + R_3 \quad 0 \leq R_3 < R_2 \\
&\ldots
\end{align*}
\]

After a finite number of steps, it exists an integer \( n \) so that: \( R_n = 0 \).
We obtain then:
\[
\begin{align*}
GCD(A, B) &= GCD(B, R_1) = \ldots \\
GCD(R_{n-1}, R_n) &= GCD(R_{n-1}, 0) = R_{n-1}
\end{align*}
\]

8.1.1 Algorithmic language

- Iterative implementation
  If \( B \neq 0 \) we calculate \( R = A \mod B \) then, using \( B \) instead of \( A \) (storing \( B \) into \( A \)) and \( R \) instead of \( B \) (storing \( R \) into \( B \)) we repeat the process until \( B = 0 \); when this happens, the GCD is \( A \).

```
function GCD(A,B)
local R
while B \neq 0 do
    A mod B->R
    B->A
    R->B
end while
return A
end function
```

- Recursive implementation
  We simply write out the recursive definition given above.
function GCD(A,B)
if B ≠ 0 then
    return GCD(B,A mod B)
else
    return A
end if
end function

8.1.2 HP49G, RPN mode

- Iterative implementation
  ≪ 0 → A B R
  ≪ WHILE B 0 ≠ REPEAT
      A B MOD 'R' STO
      B 'A' STO
      R 'B' STO
    END
    A
  ≫

- Recursive implementation
  ≪ 0 → A B
  ≪ IF B 0 ≠ THEN
      B A B MOD PGCDR
  ELSE
      A
  END
  ≫

The program must be stored into variable PGCDR.

8.1.3 HP49G, Algebraic mode

- Iterative implementation
  ≪ → A, B
  ≪ 0 → R
  ≪ WHILE B ≠ 0 REPEAT
      A MOD B STO → R;
      B STO → A;
      R STO → B
    END;
    A
  ≫
  ≫ STO → PGCD
To execute the program you enter, for example, PGCD(45,75).
• Recursive implementation

            \[ \begin{align*}
              & A, B \\
              & \text{IF } B \neq 0 \text{ THEN} \\
              & \text{PGCD}(B, A \mod B) \\
              & \text{ELSE} \\
              & A \\
              & \text{END}
            \end{align*} \]

To execute the program you enter, for example, PGCDR(45, 75).

Notice: If you use the symbolic function IREMAINDER instead of MOD in the programs above, PGCD (or PGCDR) can then have Gauss integers as arguments, too.

8.2 Bézout identity using the Euclidean algorithm

In this section, the function Bezout(A, B) returns the list \{U, V, GCD(A, B)\}, where U and V satisfy: \( A \times U + B \times V = GCD(A, B) \).

8.2.1 Iterative implementation without lists

The Euclidean's algorithm allows us to find a pair U and V satisfying:

\( A \times U + B \times V = GCD(A, B) \)

Actually, if we denote \( A_0 \) and \( B_0 \) the starting values of \( A \) and \( B \), we have:

\[
\begin{align*}
  A &= A_0 \times U + B_0 \times V \quad \text{with } U = 1 \text{ and } V = 0 \\
  B &= A_0 \times W + B_0 \times X \quad \text{with } W = 0 \text{ and } X = 1
\end{align*}
\]

Next, we make \( A, B, U, V, W, X \) evolve so that the above relations continue to be satisfied.

If:

\( A = B \times Q + R \quad 0 \leq R < B \quad (R = A \mod B \text{ and } Q = E(A/B)) \)

We can write:

\[
\begin{align*}
  R &= A - B \times Q = A_0 \times (U - W \times Q) + B_0 \times (V - X \times Q) = \\
  A_0 \times S + B_0 \times T \quad \text{with } S = U - W \times Q \text{ and } T = V - X \times Q
\end{align*}
\]

We must now repeat the process with B in place of A (B->A W->U X->V) and R in place of B (R->B S->W T->X).

We can write the whole algorithm as follows:

```plaintext
function Bezout(A,B)
local U,V,W,X,S,T,Q,R
1->U 0->V 0->W 1->X
while B \neq 0 do
  A mod B->R
  E(A/B)->Q
  //R=A-B*Q
  U-W*Q->S
  V-X*Q->T
  B->A W->U X->V
```

107
R->B S->W T->X
end while
return {U, V, A}
end function

8.2.2 Iterative implementation with lists
The algorithm above can be simplified using less variables: in order to do this, we introduce the lists LA, LB, and LR to store the terms \{U, V, A\}, \{W, X, B\} and \{S, T, R\}.

```
function Bezout (A,B)
local LA LB LR
{1, 0, A}->LA
{0, 1, B}->LB
while LB[3] \neq 0 do
LA-LB*E(LA[3]/LB[3])->LR
LB->LA
LR->LB
end while
return LA
end function
```

8.2.3 Recursive version with lists
The Bezout function can be defined recursively by: \textit{Bezout}(A,0) = \{1, 0, A\} If \(B \neq 0\) we must define \textit{Bezout}(A, B) in function of \textit{Bezout}(B, R), where \(R = A - B \times Q\) and \(Q = E(A/B)\).

We have:

\[
\text{Bezout}(B, R) = LT = \{W, X, \gcd(B, R)\}
\]

with \(W \times B + X \times R = \gcd(B, R)\)

Therefore:

\[
W \times B + X \times (A - B \times Q) = \gcd(B, R) \quad \text{or, again,}
X \times A + (W - X \times Q) \times B = \gcd(A, B).
\]

So, if \(B \neq 0\) and \textit{Bezout}(B, R) = LT we have:
\[
\]

```
function Bezout (A,B)
local LT Q R
if B \neq 0 do
E(A/B)->Q
A-B*Q->R
Bezout(B,R)->LT
return {LT[2], LT[1]-LT[2]*Q, LT[3]}
else return {1, 0, A}
end if
end function
```
8.2.4 HP49G, RPN mode

- Iterative implementation with lists.
  At the very beginning, A et B contain the two numbers for which we are seeking the Bézout identity, later they denote the lists LA and LB mentioned in the algorithm.
  
  ```plaintext
  %%HP: T(3)A(R)F(.); \{ \} \rightarrow A B R
  \{ 1 0 \} 'A' STO+
  \{ 0 1 \} 'B' STO+
  \WHILE B 3 GET 0 \neq \REPEAT
  A B A 3 GET B 3 GET / FLOOR * - 'R' STO
  B 'A' STO
  R 'B' STO
  \END
  A
  \rightarrow
  \rightarrow
  \rightarrow
  ```

- Recursive implementation with lists
  
  ```plaintext
  %%HP: T(3)A(R)F(.); \{ \} \rightarrow A B T
  \IF B 0 \neq \THEN
  B A B MOD BEZOUR 'T' STO
  T 2 GET DUP A B / FLOOR *
  T 1 GET SWAP -
  T 3 GET + + +
  \ELSE
  \{ 1 0 \} A +
  \END
  \rightarrow
  \rightarrow
  ```

This program must be stored into the variable BEZOUR.

8.2.5 HP49G, Algebraic mode

- Iterative implementation with lists.
  At the very beginning, A et B contain the two numbers for which we are seeking the Bézout identity, later they denote the lists LA and LB mentioned in the algorithm.
  
  ```plaintext
  \{ \} \rightarrow A,B
  \{ 1.0.A \} STO> A;
  \{ 0.1.B \} STO> B;
  \WHILE B[3] \neq 0 \REPEAT
  A \rightarrow B * FLOOR(A[3]/B[3]) STO> R;
  B STO> A;
  R STO> B
  \END
  A
  \rightarrow
  \rightarrow
  STO> BEZOUT
  ```
To execute the program you type, for example, \texttt{BEZOUT(45,75)}.

- Recursive implementation with lists

\[
\begin{align*}
\text{STO} & \to \text{A, B} \\
\text{STO} & \to \{\} \to T \\
\text{STO} & \to \{\text{IF B} \neq 0 \text{ THEN} \\
& \quad \text{BEZOUR(B, A \text{ MOD} B) STO} \to T; \\
& \quad \{T[2], T[1] = T[2] \times \text{FLOOR}(A/B), T[3]\} \\
\text{ELSE} & \quad \{1, 0, A\} \\
\text{END} & \} \\
\end{align*}
\]

To execute the program you type, for example, \texttt{BEZOUR(45,75)}.

Notice:
If you use the symbolic function \texttt{IREMAINDER} instead of \texttt{MOD} in the programs above, \texttt{PGCD} (or \texttt{PGCDR}) can then have Gauss integers as arguments, too.

### 8.3 Factorization

#### 8.3.1 Algorithms and their translations

- First algorithm
  We check, for all integers D from 2 to N, whether N is divided by D.
  If D divides N, we search the factors of N/D ... and so on.
  All factors are stored into the list FACT.

\[
\begin{align*}
\text{function} & \ \text{facprem(N)} \\
\text{local} & \ \text{D FACT} \\
2 & \to D \\
\{\} & \to \text{FACT} \\
\text{while} & \ N \neq 1 \ \text{do} \\
\text{if} & \ N \mod D = 0 \ \text{then} \\
\text{FACT} & + D \to \text{FACT} \\
N/D & \to N \\
\text{else} \\
D+1 & \to D \\
\text{end if} \\
\text{end while} \\
\text{return} & \ \text{FACT} \\
\text{end function}
\end{align*}
\]

- First improvement
  We only check potential factors between 2 and \(E(\sqrt{N})\).

\[
\begin{align*}
\text{function} & \ \text{facprem(N)} \\
\text{local} & \ \text{D FACT} \\
\end{align*}
\]
2 -> D
{} -> FACT

while D*D ≤ N do

    if N mod D = 0 then
        FACT + D -> FACT
        N/D -> N
    else
        D+1 -> D
    end if

end while

FACT + N -> FACT
return FACT
end function

• Second improvement
We check if 2 divides N, then we check only odd potential factors D between 3 and E(√N).
In the FACT list, each factor is now followed by its exponent:
facprem(12)={2,2,3,1}.

function facprem(N)
local K D FACT
{}->FACT
0 -> K

while N mod 2 = 0 do
    K+1 -> K
    N/2 -> N
end while

if K ≠0 then

    FACT + {2 K} -> FACT

end if

3 ->D

while D*D ≤ N do

    0 -> K
    while N mod D = 0 do
        K+1 -> K
        N/D -> N
    end while

    if K ≠0 then

        FACT + {D K} -> FACT

    end if

    D+2 -> D

end while
if \( N \neq 1 \) then

\[
\text{FACT + \{N 1\} } \rightarrow \text{FACT}
\]
end if

return FACT
end function

8.3.2 HP49G, RPN mode

%%HP: T(1)A(R)F(.); This is the translation of the second improvement:

\[
\begin{align*}
&\text{≪} \; 0 \; 3 \; \{\} \; \rightarrow \; N \; K \; D \; \text{FACT} \\
&\text{≪} \; \text{WHILE} \; N \; \text{MOD} \; 0 \; == \; \text{REPEAT} \\
&\text{1} \; \text{'K'} \; \text{STO}+ \\
&\text{1} \; \text{'N'} \; 2 \; \text{STO}/ \\
&\text{END} \\
&\text{IF} \; K \; 0 \; \neq \; \text{THEN} \\
&\{2 \; K\} \; \text{FACT'} \; \text{STO} \\
&\text{END} \\
&\text{WHILE} \; N \; \text{D} \; \text{D} \; * \; \geq \; \text{REPEAT} \\
&\text{0} \; \text{'K'} \; \text{STO} \\
&\text{WHILE} \; N \; \text{D} \; \text{MOD} \; 0 \; == \; \text{REPEAT} \\
&\text{1} \; \text{'K'} \; \text{STO}+ \\
&\text{1} \; \text{'N'} \; \text{D} \; \text{STO}/ \\
&\text{END} \\
&\text{IF} \; K \; 0 \; \neq \; \text{THEN} \\
&\{D \; K\} \; \text{FACT'} \; \text{STO}+ \\
&\text{END} \\
&2 \; \text{'D'} \; \text{STO}+ \\
&\text{END} \\
&\text{IF} \; N \; 1 \; \neq \; \text{THEN} \\
&\{N \; 1\} \; \text{FACT'} \; \text{STO}+ \\
&\text{END} \\
&\rightarrow
\end{align*}
\]

8.3.3 HP49G, algebraic mode

\[
\begin{align*}
&\rightarrow \; N \\
&\rightarrow \; 0 \; \rightarrow \; K \\
&\rightarrow \; \text{WHILE} \; N \; \text{MOD} \; 2 \; == \; 0 \; \text{REPEAT} \\
&\text{K} + 1 \; \text{STO} \; \triangleright \; K; \\
&N/2 \; \text{STO} \; \triangleright \; N \\
&\text{END;} \\
&\{\} \; \rightarrow \; \text{FACTO} \\
&\rightarrow \; \text{IF} \; K \; \neq \; 0 \; \text{THEN} \\
&\text{FACTO} + \{2, \; K\} \; \text{STO} \; \triangleright \; \text{FACTO} \\
&\text{END;} \\
&3 \; \rightarrow \; D \\
&\rightarrow \; \text{WHILE} \; D \; \ast \; D \; \leq \; N \; \text{REPEAT} \\
&0 \; \text{STO} \; \triangleright \; K;
\end{align*}
\]
WHILE N MOD D == 0 REPEAT
  K + 1 STO▷ K;
  N/D STO▷ N;
END;
IF K ≠ 0 THEN
  FACTO + {D,K} STO▷ FACTO
END;
D + 2 STO▷ D
END;
IF N ≠ 1 THEN
  FACTO + {N,1} STO▷ FACTO
END;
FACTO;

FACTEUR

To execute the program you type, for example, FACTEUR(45).

8.4 Calculation of \(A^P \mod N\)

8.4.1 Algorithmic language

- First algorithm
  We use local variables PUIS et I.
  We make an iterative program so that at each step, PUIS represents \(A^I \mod N\).

  function puismod (A, P, N)
  local PUIS, I
  1->PUIS
  for I from 1 to P do
    A*PUIS mod N ->PUIS
  end for
  return PUIS
end function

- Second algorithm
  We use only one local variable, PUI, but we update P so that at each iteration step we always have:
  \(result = PUI * A^P \mod N\)

  function puismod (A, P, N)
  local PUI
  1->PUI
  while P>0 do
    A*PUI mod N ->PUI
    P-1->P
  end while
  return PUI
Third algorithm

We can improve the previous program by noticing that:
\[ A^{2^*P} = (A * A)^P. \]
So, when \( P \) is even, the following is true:
\[ PUI * A^P = PUI * (A * A)^{P/2} \mod N \]
and when \( P \) is odd, the following is true:
\[ PUI * A^P = PUI * A * A^{P-1} \mod N. \]
The result is a fast algorithm to compute \( A^P \mod N \).

```plaintext
function puismod(A, P, N)
    local PUI
    1->PUI
    while P>0 do
        if P mod 2 =0 then
            P/2->P
            A*A mod N->A
        else
            A*PUI mod N ->PUI
            P-1->P
        end if
    end while
    return PUI
end function
```

We can also notice that if \( P \) is odd, then \( P-1 \) is even.
We can write:

```plaintext
function puismod(A, P, N)
    local PUI
    1->PUI
    while P>0 do
        if P mod 2 =1 then
            A*PUI mod N ->PUI
            P-1->P
        end if
        P/2->P
        A*A mod N->A
    end while
    return PUI
end function
```

Recursive program

We can leverage the following recurrence relation:
\[ A^0 = 1, \quad A^{P+1} \mod N = (A^P \mod N) * A \mod N \]

```plaintext
function puismod(A, P, N)
    if P>0 then
        return puismod(A, P-1, N)*A mod N
    end if
end function
```
else
return 1
end if
end function

• Fast, recursive program

function puimod(A, P, N)
if P>0 then
if P mod 2 =0 then
return puimod(A*A, P/2, N)
else
return puimod(A, P-1, N)*A mod N
end if
else
return 1
end if
end function

8.4.2 HP49G, RPN mode

The user must push on the stack:
A, P, N to obtain $A^P \mod N$.

This is the translation of the fast, iterative algorithm:

```
1 → A P N PUI
WHILE P 0 > REPEAT
    IF P MOD 2 == 1 THEN
        A PUI * N MOD 'PUI' STO
        'P' STO-
    END
    P 2 / 'P' STO
    A A * N MOD 'A' STO
END
PUI
```

```
We can store the program into PUIMOD (using 'PUIMOD' STO).
```

8.4.3 HP49G, Algebraic mode

This is the translation of the fast, iterative algorithm:

```
1 → A P N
WHILE P > 0 REPEAT
    IF P MOD 2 == 1 THEN
        A PUI MOD N STO PUI
        P - 1 STO P;
    END;
    P/2 STO P;
```

A * A MOD N STO ⊳ A;
END;
PUI

≫
≫ STO ⊳ PUIMOD
We can type, for example, PUIMOD(45,32,13) to execute it.

8.5 The function “isprime”
8.5.1 Algorithmic language

• First algorithm

We are about to write a boolean function of N, returning TRUE if N is prime, and FALSE
if it is not.

In order to do this, we check whether N has a factor ≠ 1 and ≤ $E(\sqrt{N})$ (integer part of
the square root of N).

The special case $N = 1$ is handled separately!

We use a boolean variable PREM, initially set to TRUE and changed to be FALSE when
we find a factor of N.

function isprime(N)
local PREM, I, J

$E(\sqrt{N})$ ->J

if N = 1 then
    FALSE->PREM
else
    TRUE->PREM
end if
2->I

while PREM and I ≤ J do

    if N mod I = 0 then
        FALSE->PREM
    else
        I+I->I
    end if
end while
return PREM
end function

• First improvement

We notice that first of all, we can check if N is even and, if it is not, only check whether it
has odd factors.
function iswprime(N)
    local PREM, I, J
    E(√N) -> J
    if (N = 1) or (N mod 2 = 0) and N≠2 then
        FALSE->PREM
    else
        TRUE->PREM
    end if
    3->I
    while PREM and I ≤ J do
        if N mod I = 0 then
            FALSE->PREM
        else
            I+2->I
        end if
    end while
    return PREM
end function

• Second improvement
We check if N can be divided by 2 or by 3, else we check whether N has a factor that can be expressed as either 6 × k − 1 or 6 × k + 1.

function isprime(N)
    local PREM, I, J
    E(√N) -> J
    if (N = 1) or (N mod 2 = 0) or ( N mod 3 = 0) then
        FALSE->PREM
    else
        TRUE->PREM
    end if
    if N=2 or N=3 then
        TRUE->PREM
    end if
    5->I
    while PREM and I ≤ J do
        if (N mod I = 0) or (N mod I+2 =0) then
            FALSE->PREM
        else
            I+6->I
        end if
    end while
    return PREM
end function
8.5.2 HP49G, RPN mode

We translate the last algorithm listed above: the result is either 0 (false) or 1 (true).

\[
\begin{align*}
\text{≪} & \text{ DUP } \sqrt{ } \text{ FLOOR 0 5 } \rightarrow \text{ N J PREM I} \\
\text{≪} & \text{ IF N 1 == N 2 MOD 0 == 3D = 3D OR N 3 MOD 0 == OR THEN} \\
& \text{ 0 'PREM' STO} \\
\text{ELSE} & \text{ 1 'PREM' STO} \\
\text{END} \\
\text{IF N 2 == N 3 == OR THEN} & \text{ 1 'PREM' STO} \\
\text{END} \\
\text{WHILE PREM I J ≤ AND REPEAT} & \\
& \text{ IF N I MOD 0 == N I 2 + MOD 0 == OR THEN} \\
& \text{ 0 'PREM' STO} \\
\text{ELSE} & \text{ I 6 + 'I' STO} \\
\text{END} \\
\text{END} \\
\text{PREM} & \text{≫} \\
\text{≫} \\
\text{≫} \\
\text{≫} \\
\text{≫} \\
\text{≫} \\
\text{STO≫ PREM?}
\end{align*}
\]

8.5.3 HP49G. Algebraic mode

\[
\begin{align*}
\text{≪} & \rightarrow \text{ N} \\
\text{≪} & \text{ 0 } \rightarrow \text{ P} \\
\text{≪} & \text{ IF N MOD 2 == 0 OR N MOD 3 == 0 OR N == 1 THEN} \\
& \text{ 0 STO≫ P} \\
\text{ELSE;} & \\
& \text{ 1 STO≫ P;} \\
\text{END;} \\
\text{IF N == 2 OR N == 3 THEN} & \\
& \text{ 1 STO≫ P;} \\
\text{END;} \\
\text{FLOOR(}\sqrt{\text{N}}\text{)} & \rightarrow \text{ J} \\
\text{≪} & \text{ 5 } \rightarrow \text{ I} \\
\text{≪} & \text{ WHILE I ≤ J AND P REPEAT} \\
& \text{ IF N MOD I == 0 OR N MOD (I + 2) == 0 THEN} \\
& \text{ 0 STO≫ P;} \\
\text{ELSE;} & \\
& \text{ I + 6 STO≫ I;} \\
\text{END;} \\
\text{END;} \\
\text{P} & \text{≫} \\
\text{≫} \\
\text{≫} \\
\text{≫} \\
\text{≫} \\
\text{STO≫ PREM?}
\end{align*}
\]
to execute this program you type, for example, \texttt{PREM? (45789)}.

### 8.6 Rabin’s probabilistic method

If \( N \) is prime, all integers \( K \) less that \( N \) are prime with \( N \); therefore, applying the Fermat theorem we can state that:

\[
K^{N-1} \equiv 1 \pmod{N}
\]

for all integers \( K \) less than \( N \). Instead, if \( N \) is not prime, integer values \( K \) satisfying:

\[
K^{N-1} \equiv 1 \pmod{N}
\]

are rare.

To be more precise, it can be shown that if \( N > 4 \), the probability to randomly find such an integer \( K \) is less than 0.25.

An integer \( N \) satisfying \( K^{N-1} = 1 \pmod{N} \) for 20 random trials of \( K \) is a pseudo-prime integer.

The Rabin’s probabilistic method consists of randomly generate an integer \( K \) (1 \( < \) \( K \) \(< \) \( N \)) and calculate:

\[
K^{N-1} \pmod{N}
\]

If \( K^{N-1} = 1 \pmod{N} \) we repeat the process with a different value of \( K \); if, instead, \( K^{N-1} \neq 1 \pmod{N} \) we are sure that \( N \) is not prime.

If \( K^{N-1} = 1 \pmod{N} \) for 20 random trials of \( K \) we can state that \( N \) is prime with a small probability of error, less than \( 0.25^{20} \), that is about \( 10^{-12} \).

Of course, this method is very fast, and is widely used to check whether very big integers are pseudo-prime.

#### 8.6.1 Algorithmic language

We assume that: Hasard(\( N \)) returns a random integer between 0 and \( N-1 \).

We calculate:

\[
K^{N-1} \mod{N}
\]

using the fast power algorithm described above (see page 113).

We denote as:

\texttt{puismod}(K, P, N) the function calculating and returning \( K^P \mod N \).

function \texttt{isprime}(N)
local K, I, P
1->I
1->P
while P = 1 and I < 20 do
    hasard(N-2)+2->K
    puismod(K, N-1, N)->P
    I+1->I
end while
if P = 1 then
    return TRUE
else
    return FALSE
end if
end function

119
8.6.2 HP49G, RPN mode

%%HP: T(3)A(R)F(.); We assume that the function PUIMOD, taking from the stack three arguments A, K, N, and returning \( A^K \mod N \) is already available.

\begin{verbatim}
1 0 1 → N I K P
0 RDZ
WHILE P 1 == 20 I > AND REPEAT
   1 'I' STO+
   RAND N 2 - * FLOOR 2 + 'K' STO
   K N 1 - N PUIMOD 'P' STO
END
IF P 1 == THEN
   1
ELSE
   0
END

≫
\end{verbatim}

8.6.3 HP49G, Algebraic mode

\begin{verbatim}
→ N
1 → I
0 → K
1 → P
RDZ(0);
WHILE P == 0 AND I < 20 REPEAT
   1 + I STO> I;
   FLOOR((N - 2) * RAND) + 2 STO> K;
   PUIMOD(K, N - 1, N) STO> P;
END;
IF P == 1 THEN
   1
ELSE
   0
END;
P
≫
≫
≫ RABIN
\end{verbatim}

To execute this program you can type, for example, \texttt{RABIN(45313)}.

Notice:
We can also use the built-in command \texttt{POWMOD} and write:

- In RPN mode:
  \begin{verbatim}
  N MODSTO
  K N 1 - POWMOD 'P' STO
  \end{verbatim}
instead of:
\[ K \cdot N \cdot 1 - N \cdot \text{PUIMOD} \; 'P' \; \text{STO} \]

- In Algebraic mode:
  MODSTO(N);
  POWMOD(K,N-1) \; \text{STO} \;> \; P

instead of:
  PUIMOD(K,N-1,N) \; \text{STO} \;> \; P
Index

\[\triangle < \triangleright \triangledown, 5\]
\[\leftarrow, 5\]
\[\leftarrow, 9\]
\[\rightarrow, 5\]
\[\triangleright \text{STOP}, 5\]
\[= \sim, 4\]

ABCUV, 41
ABS, 27, 56
ACOS2S, 33
ADDTMOD, 47
ARG, 27
ASIN2C, 33
ASIN2T, 34
ATAN2S, 34
AUGMENT, 70
AXL, 54
AXM, 54
AXQ, 57

BASIS, 70

C2P, 72
CASCFG, 4
CF, 5
CHINREM, 43
CHOLESKY, 67
CIRC, 72
CLEAR, 98
CONJ, 27
COPY, 14
CROSS, 55
CURL, 59
CUT, 14

cyclotomic, 70

DEFINE, 20
DERIV, 29, 57
DERVX, 28
DESOVLVE, 94, 129
DIAGMPL, 68
DISP, 98
DISTRIB, 72
DIV, 58
DIV2, 41
DIV2MOD, 48
DIVIS, 22, 40
DIVMOD, 48

DIVPC, 50
DOT, 56
EGCD, 41
EGV, 55
EGVL, 54
EPSX0, 65
EULER, 25
EVAL, 27, 127
EXLR, 59
EXP2POW, 73
EXPAND, 27, 127
EXPANDMOD, 49
EXPLN, 37

FACTOR, 22, 27, 39, 127
FACTORMOD, 49
FACTORS, 22, 49, 127
FC?, 5
FCOEF, 43
FDISTRIB, 72
FOURIER, 36
FREEZE, 98
FROOTS, 43
FS?, 5
FXND, 26, 46

GAMMA, 74
GAUSS, 57
GCD, 21, 39
GCDMOD, 49
GET, 102
GRAMSCHMIDT, 67
GROBADD, 78

HADAMARD, 54
HALFTAN, 35
HERMITE, 45
HESS, 58
HILBERT, 55
HORNER, 42

IABCUV, 24
ISBASIS, 70
IBERNOULLI, 74
IBP, 31
ICHINREM, 74

IDIV2, 23

122
TABVAL, 77
TABVAR, 77
TAN2SC, 34
TAN2SC2, 35
TAYLOR0, 50
TAYLRO, 50
TAYLR, 50
TCHEBYCHEFF, 25
TCOLLECT, 33
TEXPAND, 32
TLIN, 32
TRAN, 53
TRIG, 35
TRIGCOS, 36
TRIGSIN, 36
TRIGTAN, 36
TRN, 53
TRUNC, 44
TSIMP, 38
VANDERMONDE, 56
VER, 76
VISIT, 95
XNUM, 66
XQ, 66
ZEROS, 42
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.1.1</td>
<td>Entering a program</td>
<td>95</td>
</tr>
<tr>
<td>6.1.2</td>
<td>Saving a program</td>
<td>95</td>
</tr>
<tr>
<td>6.1.3</td>
<td>Editing a program</td>
<td>95</td>
</tr>
<tr>
<td>6.1.4</td>
<td>Executing a program</td>
<td>96</td>
</tr>
<tr>
<td>6.1.5</td>
<td>Modifying a program and saving it with another name</td>
<td>96</td>
</tr>
<tr>
<td>6.2</td>
<td>Program comments</td>
<td>96</td>
</tr>
<tr>
<td>6.3.1</td>
<td>Variable names</td>
<td>96</td>
</tr>
<tr>
<td>6.3.2</td>
<td>Local variables</td>
<td>96</td>
</tr>
<tr>
<td>6.3.3</td>
<td>Parameters</td>
<td>97</td>
</tr>
<tr>
<td>6.4.1</td>
<td>Translation in algorithmic language</td>
<td>97</td>
</tr>
<tr>
<td>6.4.2</td>
<td>Translation for the HP49G in RPN mode</td>
<td>97</td>
</tr>
<tr>
<td>6.4.3</td>
<td>Translation for the HP49G in Algebraic mode</td>
<td>97</td>
</tr>
<tr>
<td>6.5</td>
<td>Data output</td>
<td>98</td>
</tr>
<tr>
<td>6.6.1</td>
<td>Translation in algorithmic language</td>
<td>98</td>
</tr>
<tr>
<td>6.6.2</td>
<td>Translation for the HP49G in RPN mode</td>
<td>98</td>
</tr>
<tr>
<td>6.6.3</td>
<td>Translation for the HP49G in Algebraic mode</td>
<td>98</td>
</tr>
<tr>
<td>6.7</td>
<td>Store instruction</td>
<td>98</td>
</tr>
<tr>
<td>6.7.1</td>
<td>Translation in Algorithmic language</td>
<td>98</td>
</tr>
<tr>
<td>6.7.2</td>
<td>Translation for the HP49G in RPN mode</td>
<td>98</td>
</tr>
<tr>
<td>6.7.3</td>
<td>Translation for the HP49G in Algebraic mode</td>
<td>98</td>
</tr>
<tr>
<td>6.8</td>
<td>Conditional branch instructions</td>
<td>99</td>
</tr>
<tr>
<td>6.8.1</td>
<td>Translation in algorithmic language</td>
<td>99</td>
</tr>
<tr>
<td>6.8.2</td>
<td>Translation for the HP49G in RPN mode</td>
<td>99</td>
</tr>
<tr>
<td>6.8.3</td>
<td>Translation for the HP49G in Algebraic mode</td>
<td>99</td>
</tr>
<tr>
<td>6.9</td>
<td>“For” loops</td>
<td>99</td>
</tr>
<tr>
<td>6.9.1</td>
<td>Translation in algorithmic language</td>
<td>99</td>
</tr>
<tr>
<td>6.9.2</td>
<td>Translation for the HP49G in RPN mode</td>
<td>99</td>
</tr>
<tr>
<td>6.9.3</td>
<td>Translation for the HP49G in Algebraic mode</td>
<td>100</td>
</tr>
<tr>
<td>6.10</td>
<td>“While” loops</td>
<td>100</td>
</tr>
<tr>
<td>6.10.1</td>
<td>Translation in algorithmic language</td>
<td>100</td>
</tr>
<tr>
<td>6.10.2</td>
<td>Translation for the HP49G in RPN mode</td>
<td>100</td>
</tr>
<tr>
<td>6.10.3</td>
<td>Translation for the HP49G in Algebraic mode</td>
<td>100</td>
</tr>
<tr>
<td>6.11</td>
<td>Boolean (or conditional) expressions</td>
<td>100</td>
</tr>
<tr>
<td>6.11.1</td>
<td>Translation in algorithmic language</td>
<td>100</td>
</tr>
<tr>
<td>6.11.2</td>
<td>Translation for the HP49G in RPN mode</td>
<td>100</td>
</tr>
<tr>
<td>6.11.3</td>
<td>Translation for the HP49G in Algebraic mode</td>
<td>100</td>
</tr>
<tr>
<td>6.12</td>
<td>Logical operators</td>
<td>100</td>
</tr>
<tr>
<td>6.12.1</td>
<td>Translation in algorithmic language</td>
<td>100</td>
</tr>
<tr>
<td>6.12.2</td>
<td>Translation for the HP49G in RPN mode</td>
<td>100</td>
</tr>
<tr>
<td>6.12.3</td>
<td>Translation for the HP49G in Algebraic mode</td>
<td>100</td>
</tr>
<tr>
<td>6.13</td>
<td>Functions</td>
<td>101</td>
</tr>
<tr>
<td>6.13.1</td>
<td>Translation in algorithmic language</td>
<td>101</td>
</tr>
<tr>
<td>6.13.2</td>
<td>Translation for the HP49G in RPN mode</td>
<td>101</td>
</tr>
<tr>
<td>6.13.3</td>
<td>Translation for the HP49G in Algebraic mode</td>
<td>101</td>
</tr>
<tr>
<td>6.14</td>
<td>Lists</td>
<td>102</td>
</tr>
<tr>
<td>6.14.1</td>
<td>Translation in algorithmic language</td>
<td>102</td>
</tr>
<tr>
<td>6.14.2</td>
<td>Translation for the HP49G in RPN mode</td>
<td>102</td>
</tr>
<tr>
<td>6.14.3</td>
<td>Translation for the HP49G in Algebraic mode</td>
<td>102</td>
</tr>
<tr>
<td>6.15</td>
<td>Example: The sieve of Eratosthenes</td>
<td>102</td>
</tr>
<tr>
<td>6.15.1</td>
<td>Description</td>
<td>102</td>
</tr>
<tr>
<td>6.15.2</td>
<td>Algorithmic language</td>
<td>103</td>
</tr>
<tr>
<td>6.15.3</td>
<td>HP49G, RPN mode</td>
<td>104</td>
</tr>
<tr>
<td>6.15.4</td>
<td>HP49G, Algebraic mode</td>
<td>104</td>
</tr>
</tbody>
</table>
### 7 Arithmetic programs

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.1</td>
<td>Calculating the GCD using the Euclidean algorithm</td>
<td>105</td>
</tr>
<tr>
<td>7.1.1</td>
<td>Algorithmic language</td>
<td>105</td>
</tr>
<tr>
<td>7.1.2</td>
<td>HP49G, RPN mode</td>
<td>106</td>
</tr>
<tr>
<td>7.1.3</td>
<td>HP49G, Algebraic mode</td>
<td>106</td>
</tr>
<tr>
<td>7.2</td>
<td>Bézout identity using the Euclidean’s algorithm</td>
<td>107</td>
</tr>
<tr>
<td>7.2.1</td>
<td>Iterative implementation without lists</td>
<td>107</td>
</tr>
<tr>
<td>7.2.2</td>
<td>Iterative implementation with lists</td>
<td>108</td>
</tr>
<tr>
<td>7.2.3</td>
<td>Recursive version with lists</td>
<td>108</td>
</tr>
<tr>
<td>7.2.4</td>
<td>HP49G, RPN mode</td>
<td>109</td>
</tr>
<tr>
<td>7.2.5</td>
<td>HP49G, Algebraic mode</td>
<td>109</td>
</tr>
<tr>
<td>7.3</td>
<td>Factorization</td>
<td>110</td>
</tr>
<tr>
<td>7.3.1</td>
<td>Algorithms and their translations</td>
<td>110</td>
</tr>
<tr>
<td>7.3.2</td>
<td>HP49G, RPN mode</td>
<td>112</td>
</tr>
<tr>
<td>7.3.3</td>
<td>HP49G, algebraic mode</td>
<td>112</td>
</tr>
<tr>
<td>7.4</td>
<td>Calculation of ( A^p \text{ mod } N )</td>
<td>113</td>
</tr>
<tr>
<td>7.4.1</td>
<td>Algorithmic language</td>
<td>113</td>
</tr>
<tr>
<td>7.4.2</td>
<td>HP49G, RPN mode</td>
<td>115</td>
</tr>
<tr>
<td>7.4.3</td>
<td>HP49G, Algebraic mode</td>
<td>115</td>
</tr>
<tr>
<td>7.5</td>
<td>The function “isprime”</td>
<td>116</td>
</tr>
<tr>
<td>7.5.1</td>
<td>Algorithmic language</td>
<td>116</td>
</tr>
<tr>
<td>7.5.2</td>
<td>HP49G, RPN mode</td>
<td>118</td>
</tr>
<tr>
<td>7.5.3</td>
<td>HP49G, Algebraic mode</td>
<td>118</td>
</tr>
<tr>
<td>7.6</td>
<td>Rabin’s probabilistic method</td>
<td>119</td>
</tr>
<tr>
<td>7.6.1</td>
<td>Algorithmic language</td>
<td>119</td>
</tr>
<tr>
<td>7.6.2</td>
<td>HP49G, RPN mode</td>
<td>120</td>
</tr>
<tr>
<td>7.6.3</td>
<td>HP49G, Algebraic mode</td>
<td>120</td>
</tr>
</tbody>
</table>