p-adic zeta functions and quasimodular forms (Fonctions zêta p-adiques et formes quasimodulaires)

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Main object: automorphic L-functions and their p-adic avatars

For an algebraic group G over a number field K these L functions (or simply zeta-functions in Shimura's terminology) are defined as certain Euler products.

Example ($G = \mathrm{GL}(2), K = \mathbb{Q},\ L_f(s) = \sum_{n \geq 1} a_n n^{-s},\ s \in \mathbb{C}$)

Here $f(z) = \sum_{n \geq 0} a_n q^n$ is a modular form on the upper-half plane

$$H = \{z \in \mathbb{C} | \mathrm{Im}\,(z) > 0\} = \mathrm{SL}(2)/\mathrm{SO}(2), q = e^{2\pi i z}.$$

An Euler product has the form

$$L_f(s) = \prod_{p \ \ primes} (1 - a_p p^{-s} + \psi_f(p) p^{k-1-2s})^{-1}$$

where k is the weight and ψ_f the Dirichlet character of f. It is defined iff the representation π_f attached to f is irreducible. Recall that π_f is generated by the lift $\mathbf{f}$ of f to the group $G(\mathbb{A})$, where $\mathbb{A}$ is the ring of adeles $\mathbb{A} = \{x = (x_\infty, x_p) | x_\infty \in \mathbb{R}, x_p \in \mathbb{Q}_p$ such that $x_p \in \mathbb{Z}_p$ for all but a finite number of $p\}$.

A p-adic avatar of $L_f(s)$ (Manin-Mazur)

is a p-adic analytic function $L_{f,p}(s,\chi)$ of p-adic arguments $s \in \mathbb{Z}_p, \chi$ mod p^r which interpolates algebraic numbers

$$L_f^*(s,\chi)/\omega^{\pm} \in \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p = \hat{\bar{\mathbb{Q}}}_p \quad \text{(the Tate field)}$$

for $1 \le s \le k-1$, $\omega^{\pm}$ are periods of f where the complex analytic L function of f is defined for all $s \in \mathbb{C}$ so that in the absolutely convergent case $\mathrm{Re}(s) > (k+1)/2$

$$L_f^*(s,\chi) = (2\pi)^{-s}\Gamma(s)\sum_{n\ge 1}\chi(n)a_n n^{-s}$$

which extends to holomorphic function with a functional equation. According to Manin and Shimura, this number is algebraic if the period is chosen according to the parity $\chi(-1)(-1)^s = \pm 1$.

Constructions of p-adic avatars

In general case of an irreducible automorphic representation of the adelic group $G(\mathbb{A}_K)$ there is an L-function

$$L(s, \pi, r, \chi) = \prod_{\mathfrak{p} \text{ primes} \in K} \prod_{j=1}^{m} (1 - \beta_{j,\mathfrak{p}} N\mathfrak{p}^{-s})^{-1}$$

where

$$\prod_{j=1}^{m} (1 - \beta_{j,\mathfrak{p}} X) = \det(1_m - r(\mathrm{diag}(\alpha_{i,\mathfrak{p}})_i X))$$

$\alpha_{i,\mathfrak{p}}$ are the Satake parameters of $\pi = \bigotimes_v \pi_v$, $v \in \Sigma_K$ (places in K),$\mathfrak{p} = \mathfrak{p}_v$. Here $h_v = \mathrm{diag}(\alpha_{i,\mathfrak{p}_v})_i$ live in the Langlands group ${}^L G(\mathbb{C})$, $r : {}^L G(\mathbb{C}) \to \mathrm{GL}_m(\mathbb{C})$ denotes its representation, $\chi : \mathbb{A}_K^* / K^* \to \mathbb{C}^*$ is a character of finite order.

Constructions extend to rather general automorphic representations on Shimura varieties via

- Petersson products with a fixed automorphic form, or
- linear forms coming from the Fourier coefficients (or Whittaker functions), or throught the
- CM-values (special points on Shimura varieties),

Accessible cases: symplectic and unitary groups

$G = \mathrm{GL}_1$ over $\mathbb{Q}$ (Kubota-Leopoldt-Mazur) for the Dirichlet L-function $L(s, \chi)$.

$G = \mathrm{GL}_1$ over a totally real field F (Deligne-Ribet, using algebraicity result by Klingen.

$G = \mathrm{GL}_1$ over a CM-field K,i.e. a totally imaginary extension of a totally real field F (N.Katz, Manin-Vishik).

$G = \mathrm{GSp}_n$ (the Siegel modular case) or the unitary groups over a CM-field K

$$G = G(\varphi) = \{\alpha \in \mathrm{GL}_m(K) | \alpha \varphi^t \alpha^\rho = \nu(\alpha)\varphi\}, \nu(\alpha) \in F^*,$$

where $\varphi = \eta_n = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$ or $\varphi = \begin{pmatrix} 1_n & 0 \\ 0 & -1_m \end{pmatrix}$, see C. Skinner and E. Urban, [MC] and Shimura G., [Sh00].

A new method of constructing p-adic zeta-functios

The combinatorial structure of the Fourier coefficients of the holomorphic forms used in these constructions is quite complicated. We present a method of simplification using nearly-holomorphic and general quasimodular forms, related to algebraic automorphic forms.

It gives a new method of constructing p-adic zeta-functios using general quasi-modular forms and their Fourier coefficients.

The symmetric space

$$\mathcal{H} = G(\mathbb{R})/(\text{maximal-compact subgroup } \mathcal{K} \times \text{Center})$$

parametrizes certain families of abelian varieties $A_z (z \in \mathcal{H})$ so that $F \subset \mathrm{End}(A_z) \otimes \mathbb{Q}$.

The CM-points z correspond to a maximal multiplication ring $\mathrm{End}(A_z)$.

For $GL(2)$, N.Katz used arithmetical elements (real-analytic and p-adic)

instead of holomorphic forms in these representation spaces. These elements correspond also to quasimodular forms coming from derivatives which can be defined in general using Shimura's arithmeticity and the Maass-Shimura operators.

A relation real-analytic $\leftrightarrow$ p-adic modular forms comes from the notion of p-adic modular forms invented by J.-P.Serre [Se73] as p-adic limits of q-expansions of modular forms with rational coefficients for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$.

The present method of constructing p-adic automorphic L-functions uses general quasimodular forms, and their link to algebraic p-adic modular forms.

Real-analytic and p-adic modular forms

In Serre's case for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, the ring $\mathcal{M}_p$ of p-adic modular forms contains $\mathcal{M} = \oplus_{k\geq 0}\mathcal{M}_k(\Gamma, \mathbb{Z}) = \mathbb{Z}[E_4, E_6]$, and it contains $E_2 = 1 - 24\sum_{n\geq 1}\sigma_1(n)q^n$. On the other hand,

$$\tilde{E}_2 = -\frac{3}{\pi y} + E_2 = -12S + E_2, \text{ where } S = \frac{1}{4\pi y},$$

is a nearly holomorphic modular form (its coefficients are polynomials of S over $\mathbb{Q}$). Let $\mathcal{N}$ be the ring of such forms. Then

$$\tilde{E}_2|_{S=0} = E_2,$$

and it was proved by J.-P.Serre that E_2 is a p-adic modular form. Elements of the ring $\mathcal{M}^\sharp = \mathcal{N}|_{S=0}$ are quasimodular forms. These phenomena are quite general and can be used in computations and proofs.

In June 2014 in a talk in Grenoble, S.Boecherer extended these results to the Siegel modular case.

Given a quasimodular form f, how to find a nearly holomorphic form $\tilde{f}$ such that $\tilde{f}|_{S=0} = f$?

The first method: use the structure theorem as the polynomial ring [MaRo5],

$$\mathcal{N} = \mathbb{Q}[\tilde{E}_2, E_4, E_6], \quad \mathcal{M}^\sharp = \mathbb{Q}[E_2, E_4, E_6],$$

the answer is $\tilde{f} = P(\tilde{E}_2, E_4, E_6)$ for any $f = P(E_2, E_4, E_6)$.
The second method: $\tilde{f}$ can be recovered from the transformation law of f (see [MaRo5]): if for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$,

$$(cz+d)^{-k} f(\gamma z) = \sum_{t=0} \left(\frac{c}{cz+d}\right)^t f_t(z) = P_{z,f}(X) \text{ then}$$

$$\tilde{f} = P_{z,f}\left(\frac{1}{2iy}\right) = P_{z,f}(-2i\pi S), \text{ where } S = \frac{1}{4\pi y}, y = \frac{1}{4\pi S}.$$

Remark: the derivative $D = \frac{1}{2\pi i}\frac{d}{dz} = q\frac{d}{dq}$, acts on $\mathcal{M}^\sharp$.
It corresponds to Shimura's differential operator δ on $\mathcal{N}$.

Examples : (a) Serre's p-adic modular form E_2 is quasi-modular in the following sense

The transformation law of $f = E_2$ is: for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$,

$$(cz+d)^{-2}E_2(\gamma(z)) = E_2(z) + \frac{6}{i\pi}\left(\frac{c}{cz+d}\right) = P_{z,E_2}\left(\frac{c}{cz+d}\right),$$

where $P_{z,E_2}(X) = E_2(z) + \frac{6}{i\pi}X$ is a polynomial with holomorphic coefficients for $X = \dfrac{c}{cz+d}$.

The same polynomial $P_{z,E_2}(X)$ gives the corresponding nearly holomorphic function $\tilde{E}_2$ as its value at $X = -2i\pi S$:

$$\tilde{E}_2 = -\frac{3}{\pi y} + E_2 = -12S + E_2 = P_{z,E_2}(\frac{1}{2iy}) = P_{z,E_2}(-2i\pi S).$$

This function has rational coefficients of q and S, so it is "algebraic".

Examples : (b) Derivative of a modular form is not a modular form, but it is quasi-modular (Don Zagier, 1994)

If $f = D^r g$ where $g \in \mathcal{M}_l(\Gamma)$ is a holomorphic modular form of weight ℓ, then the transformation law of weight $k = \ell + 2r$ is

$$(cz+d)^{-k} D^r g(\gamma z) = \sum_{t=0}^{r} \binom{r}{t} \frac{\Gamma(r+\ell)}{\Gamma(t+\ell)} \left(\frac{1}{2\pi i} \cdot \frac{c}{cz+d} \right)^{r-t} D^t g(z).$$

In this way we get SHIMURA's differential operator

$$\delta_\ell^r g = \widetilde{D^r g} = \sum_{t=0}^{r} \binom{r}{t} \frac{\Gamma(r+\ell)}{\Gamma(r-t+\ell)} (2\pi i)^{-t} (-2\pi i S)^t D^{r-t} g(z)$$

$$= \sum_{t=0}^{r} \binom{r}{t} \frac{\Gamma(r+\ell)}{\Gamma(r-t+\ell)} (-S)^t D^{r-t} g(z), \text{ where } S = \frac{1}{4\pi y},$$

which preserves the rationality of the coefficients of S and q. It comes from the above transformation law of $D^r g$ with $X = \dfrac{c}{cz+d}$ replaced by $-2\pi i S$:

$$P_{z,D^r g}(X) = \sum_{t=0} \binom{r}{t} \frac{\Gamma(r+\ell)}{\Gamma(t+\ell)} (X/2\pi i)^{r-t} D^t g(z).$$

Using algebraic and p-adic modular forms

There are several methods to compute various L-values starting from the constant term of the Eisenstein series in [Se73],

$$G_k(z) = \frac{\zeta(1-k)}{2} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n = \frac{\Gamma(k)}{(2\pi i)^k} \sum_{(c,d)}{}' (cz+d)^{-k},$$

and using Petersson products of nearly-holomorphic Siegel modular forms and arithmetical automorphic forms as in [Sh00]:
the Rankin-Selberg method,
the doubling method (pull-back method).
A known example is the standard zeta function $D(s,f,\chi)$ of a Siegel cusp eigenform $f \in \mathcal{S}_k^n(\Gamma)$ of genus n (with local factors of degree $2n+1$) and χ a Dirichlet character.
Theorem (the case of even genus n (Courtieu-A.P.), via the Rankin-Selberg method) gives a p-adic interpolation of the normailzed critical values $D^*(s,f,\chi)$ using Andrianov-Kalinin integral representation of these values $1+n-k \le s \le k-n$ through the Petersson product $\langle f, \theta_{T_0}\delta^r E\rangle$ where δ^r is a certain composition of Maass-Shimura differential operators, θ_{T_0} a theta-series of weight $n/2$, attached to a fixed $n \times n$ matrix T_0.
Theorem (the case of odd genus (Boecherer-Schmidt) and Anh-Tuan Do (non-ordinary case, PhD Thesis of March 2014)), via the doubling method) uses Boecherer–Garrett–Shimura identity (a pull-back formula):

A pull-back formula

allows to compute the critical values through certain double Petersson product by integrating over $z \in \mathbb{H}_n$ the identity:

$$\Lambda(l+2s,\chi)D(l+2s-n,f,\chi)f = \langle f(w), E^{2n}_{l,\nu,\chi,s}(\mathrm{diag}[z,w])\rangle_w.$$

Here $k = l+\nu$, $\nu \geq 0$, $\Lambda(l+2s,\chi)$ is a product of special values of Dirichlet L-functions and Γ-functions, $E^{2n}_{l,\nu,\chi,s}$ a higher twist of a Siegel-Eisenstein series on $(z,w) \in \mathbb{H}_n \times \mathbb{H}_n$ (see [Boe85], [Boe-Schm]).

A p-adic construction uses congruences for the L-values, expressed through the Fourier coefficients of the Siegel modular forms and nearly-modular forms.

In the present approach of computing the Petersson products and L-values, an injection of algebraic nearly holomorphic modular forms into p-adic modular forms is used.

Applications to families of Siegel modular forms are constructed.

As a consequence, explicit two-parameter families are constructed.

A recent discovery by Takashi Ichikawa (Saga University), [Ich12], J. reine angew. Math., [Ich13]

allows to inject nearly-holomorphic arithmetical (vector valued) Siegel modular forms into *p-adic modular forms*.

Via the Fourier expansions, the image of this injection is represented by certain *quasimodular holomorphic forms* like $E_2 = 1 - 24 \sum_{n \geq 1} \sigma_1(n) q^n$, with algebraic Fourier expansions.

This description provides many advantages, both computational and theoretical, in the study of algebraic parts of Petersson products and L-values, which we would like to develop here.

This work is related to a recent preprint [BoeNa13] by S. Boecherer and Shoyu Nagaoka where it is shown that Siegel modular forms of level $\Gamma_0(p^m)$ are p-adic modular forms. Moreover they show that derivatives of such Siegel modular forms are p-adic. Parts of these results are also valid for vector-valued modular forms.

Arithmetical nearly-holomorphic Siegel modular forms

Nearly-holomorphic Siegel modular forms over a subfield k of $\mathbb{C}$ are certain $\mathbb{C}^d$-valued smooth functions f of $Z = X + \sqrt{-1}Y \in \mathbb{H}_n$ given by the following expression

$$f(Z) = \sum_T P_T(S) q^T,$$

where T runs through the set B_n of all half-integral semi-positive matricies, $S = (4\pi Y)^{-1}$ a symmetric matrix,
$q^T = \exp(2\pi\sqrt{-1}\mathrm{tr}(TZ))$, $P_T(S)$ are vectors of degree d whose entries are polynomials over k of the entries of S.

Review of the algebraic theory

Following [Ha81], consider the columns $Z_1, Z_2, \dots, Z_n$ of Z and the $\mathbb{Z}$-lattice L_Z in $\mathbb{C}^n$ generated by $\{E_1, \dots, E_n, Z_1, \dots, Z_n\}$, where $E_1, \dots, E_n$ are the columns of the identity matrix E. The torus $\mathcal{A}_Z = \mathbb{C}^n / L_Z$ is an abelian variety, and there is an analytic family $\mathcal{A} \longrightarrow \mathbb{H}_n$ whose fiber over the point Z is $\mathcal{A}_Z$.

Let us consider the quotient space $\mathbb{H}_n/\Gamma(N)$ of the Siegel upper half space $\mathbb{H}_n$ of degree n by the integral symplectic group

$$\Gamma(N) = \left\{ \gamma = \begin{pmatrix} A_\gamma & B_\gamma \\ C_\gamma & D_\gamma \end{pmatrix} \;\middle|\; \begin{matrix} A_\gamma \equiv D_\gamma \equiv 1_n \\ B_\gamma \equiv C_\gamma \equiv 0_n \end{matrix} \right\}$$

If $N > 3$, $\Gamma(N)$ acts without fixed points on $\mathcal{A} = \mathcal{A}_n$ and the quotient is a smooth algebraic family $\mathcal{A}_{n,N}$ of abelian varieties with level N structure over the quasi-projective variety $\mathcal{H}_{n,N}(\mathbb{C}) = \mathbb{H}_n/\Gamma(N)$ defined over $\mathbb{Q}(\zeta_N)$, where ζ_N is a primitive N-th root of 1.

For positive integers n and N, $\mathcal{H}_{n,N}$ is the moduli space classifying principally polarized abelian schemes of relative dimension n with symplectic level N structure.

De Rham and Hodge vector bundles

The fiber varieties $\mathcal{A}$ and $\mathcal{A}_{n,N}$ give rise to a series of vector bundles over $\mathbb{H}_n$ and $\mathcal{H}_{n,N}(\mathbb{C})$.

Notations

- $\mathcal{H}^1_{DR}(\mathcal{A}/\mathbb{H}_n)$ and $\mathcal{H}^1_{DR}(\mathcal{A}_{n,N}/\mathcal{H}_{n,N})$
  the relative algebraic De Rham cohomology bundles of dimension $2n$ over $\mathbb{H}_n$ and $\mathcal{H}_{n,N}$) respectively. Their fibers at $Z \in \mathbb{H}_n$ are $H^1 := \mathrm{Hom}_{\mathbb{C}}(L_Z \otimes \mathbb{C}, \mathbb{C})$ generated by α_j, β_j:

  $$\alpha_i(\sum_j a_j E_j + b_j Z_j) = a_i, \quad \beta_i(\sum_j a_j E_j + b_j Z_j) = b_i \quad (i = 1, \cdots, n).$$

- $\mathcal{H}^1_\infty$ the C^∞ vector bundle associated to $\mathcal{H}^1_{DR}$ (over $\mathbb{H}_n$ and $\mathcal{H}_{n,N}$). It splits as a direct sum $\mathcal{H}^1_\infty = \mathcal{H}^{1,0}_\infty \otimes \mathcal{H}^{0,1}_\infty$ and induces the Hodge decomposition on the De Rham cohomology of each fiber.
- The summand $\omega = \mathcal{H}^{1,0}_\infty$ is the bundle of relative 1-forms for either $\mathcal{A}/\mathbb{H}_n$ or $\mathcal{A}_{n,N}/\mathcal{H}_{n,N}$. Let us denote by $\pi : \mathcal{A}_{n,N} \to \mathcal{H}_{n,N}$ the universal abelian scheme with 0-section s, and by the Hodge bundle of rank n defined as

  $$\mathbb{E} = \pi_*(\Omega^1_{\mathcal{A}_{n,N}/\mathcal{H}_{n,N}}) = s^*(\Omega^1_{\mathcal{A}_{n,N}/\mathcal{H}_{n,N}})$$

- The bundle of holomorphic 1-forms on the base $\mathbb{H}_n$ or on $\mathcal{H}_{n,N}$, is denoted Ω.

Algebraic Siegel modular forms

are defined as global sections of $\mathbb{E}_\rho$, the locally free sheaf on $\mathcal{H}_{n,N} \otimes R$ obtained from twisting the Hodge bundle $\mathbb{E}$ by ρ.

Definition. Let R be a $\mathbb{Z}[1/N, \zeta_N]$-algebra. For an algebra homomorphism $\rho : \mathrm{GL}_n \to \mathrm{GL}_d$ over R, define algebraic Siegel modular forms over R as elements of $\mathcal{M}_\rho(R) = H_0(\mathcal{H}_{n,N} \otimes R, \mathbb{E}_\rho)$, called of weight ρ, degree n, level N.

If $\rho = \det^{\otimes k} : \mathrm{GL}_n \to \mathbb{G}_m$, then elements of $\mathcal{M}_k(R) = \mathcal{M}_{\det^{\otimes k}}(R)$ are called of weight k. For $R = \mathbb{C}$, each $Z \in \mathbb{H}_n$, let $\mathcal{A}_Z = \mathbb{C}^n/(\mathbb{Z}^n + \mathbb{Z}^n \cdot Z)$ be the corresponding abelian variety over $\mathbb{C}$, and $(u_1, ..., u_n)$ be the natural coordinates on the universal cover $\mathbb{C}^n$ of $\mathcal{A}_Z$. Then $\mathbb{E}$ is trivialized over $\mathbb{H}_n$ by $du_1, ..., du_n$, and $f \in \mathcal{M}\rho(\mathbb{C})$ is a complex analytic section of $\mathbb{E}_\rho$ on $\mathcal{H}_{n,N}(\mathbb{C}) = \mathbb{H}_n/\Gamma(N)$. Hence, an element $f \in \mathcal{M}_\rho(\mathbb{C})$ is a $\mathbb{C}^d$-valued holomorphic function on $\mathbb{H}_n$ satisfying the ρ-automorphic condition:

$$f(Z) = \rho(C_\gamma Z + D_\gamma)^{-1} \cdot f(\gamma(Z)) \left(Z \in \mathbb{H}_n, \gamma = \begin{pmatrix} A_\gamma & B_\gamma \\ C_\gamma & D_\gamma \end{pmatrix} \right),$$

because $\mathcal{A}_Z \xrightarrow{\sim} \mathcal{A}_{\gamma(Z)}; {}^t(u_1, ..., u_n) \mapsto (CZ + D)^{-1} \cdot {}^t(u_1, ..., u_n)$, and γ acts equivariantly on the trivialization of $\mathbb{E}$ over $\mathbb{H}_n$ as the left multiplication by $(CZ + D)^{-1}$.

Algebraic Fourier expansion

can be defined algebraically using an algebraic test object over the ring $\mathcal{R}_n = \mathbb{Z}[\![q_{11}, \ldots, q_{nn}]\!][q_{ij}, \ q_{ij}^{-1}]_{i,j=1,\cdots,n}$, where $q_{i,j} (1 \leq i, j \leq n)$ are variables with symetry $q_{i,j} = q_{j,i}$.
Mumford constructs in [Mu72] an object represented over $\mathcal{R}_n$ as

$$(\mathbb{G}_m)^n / \langle (q_{i,j})_{i=1,\cdots,n} \big| 1 \leq j \leq n \rangle, (\mathbb{G}_m)^n = \mathrm{Spec}(\mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]).$$

For the level N, at each 0-dimensional cusp c on $\mathcal{H}_{n,N}^*$ (Satake's minimal compactification of $\mathcal{H}_{n,N}$), this construction gives an abelian variety over the formal power series ring

$$\mathcal{R}_{n,N} = \mathbb{Z}[1/N, \zeta_N][\![q_{11}^{1/N}, \ldots, q_{nn}^{1/N}]\!][q_{ij}^{\pm 1/N}]_{i,j=1,\cdots,n}.$$

with a symplectic level N structure, and $\omega_i = dx_i/x_i$ $(1 \leq i \leq n)$ form a basis of regular 1-forms.
We may view algebraically Siegel modular forms as certain sections of vector bundles over $\mathcal{H}_{n,N}$. Using the morphism $\mathrm{Spec}(\mathcal{R}_{n,N}) \to \mathcal{H}_{n,N}$, $\mathbb{E}$ becomes $(\mathcal{R}_{n,N} \otimes R)^n$ in the basis $\omega_i = dx_i/x_i$ $(1 \leq i \leq n)$ of regular 1-forms.

Fourier expansion map and q-expansion principle

For an algebraic representation $\rho : \mathrm{GL}_n \to \mathrm{GL}_d$, $\mathbb{E}_\rho$ becomes in the above basis ω_i

$$\mathbb{E}_\rho \times_{\mathcal{H}_{n,N}\otimes R} \mathrm{Spec}(\mathcal{R}_{n,N} \otimes R) = (\mathcal{R}_{n,N} \otimes R)^d.$$

For an R-module M, the space of Siegel modular forms with coefficients in M of weight ρ is defined as $\mathcal{M}_\rho(M) = H^0(\mathcal{H}_{n,N} \otimes R, \mathbb{E}_\rho \otimes_R M)$. Then the evaluation on Mumford's abelian scheme gives a homomorphism

$$F_c : \mathcal{M}_\rho(M) \to (\mathcal{R}_{n,N} \otimes_{\mathbb{Z}[1/N,\zeta_N]} M)^d$$

which is called the Fourier expansion map associated with c. According to [Ich13], Theorem 2, F_c satisfies the following

q-expansion principle:

If M' is a sub R-module of M and $f \in \mathcal{M}_\rho(M)$ satisfies that $F_c(f) \in (\mathcal{R}_{n,N} \otimes_{\mathbb{Z}[1/N,\zeta_N]} M')^d$, then $f \in \mathcal{M}_\rho(M')$.

Algebraic nearly holomorphic forms as formal Fourier expansions over a commutative ring A

Algebraically we use the notation

$$q^T = \prod_{i=1}^{n} q_{ii}^{T_{ii}} \prod_{i<j} q_{ij}^{2T_{ij}} \in A[\![q_{11}, \dots, q_{nn}]\!][q_{ij},\ q_{ij}^{-1}]_{i,j=1,\cdots,n}$$

(with $q^T = \exp(2\pi i \mathrm{tr}(TZ))$, $q_{ij} = \exp(2\pi(\sqrt{-1}Z_{i,j}))$ for $A = \mathbb{C}$).
The elements q^T form a multiplicative semi-group so that $q^{T_1} \cdot q^{T_2} = q^{T_1+T_2}$, and one may consider f as a formal q-expansion over an arbitrary ring A via elements of the semi - group algebra $A[\![q^{B_n}]\!]$.
Algebraic definition of arithmetical nearly holomorphic forms, see [Sh00] $f \in S_e(Sym^2(A^n), A[\![q^{B_n}]\!]^d)$, where S_e denotes the A-polynomial mappings of degree e on symmetric matricies $S \in Sym^2(A^n)$ of order n with vector values in $A[\![q^{B_n}]\!]^d$.
Notation: $f = \sum_T a_T(S)q^T \in \mathcal{N}(A)$
General quasimodular forms. For all $f = \sum_T a_T(S)q^T \in \mathcal{N}(A)$ define general quasimodular forms as elements of the form $\kappa(f) = \sum_T a_T(0)q^T = f|_{S=0}$.
Notation: $\kappa(f) \in \mathcal{M}^\sharp(A)$

Computing the Petersson products

The Petersson product of a given modular form $f(Z) = \sum_T a_T q^T \in \mathcal{M} \subset \mathcal{M}_\rho(\bar{\mathbb{Q}})$ by another modular form $h(Z) = \sum_T b_T q^T \in \mathcal{M} \subset \mathcal{M}_{\rho^*}(\bar{\mathbb{Q}})$ produces a linear form

$$\ell_f : h \mapsto \frac{\langle f, h \rangle}{\langle f, f \rangle}$$

defined over a subfield $k \subset \bar{\mathbb{Q}}$. Thus ℓ_f can be expressed through the Fourier coefficients of h in the case when there is a finite basis of the dual space consisting of certain Fourier coefficients:

$$\ell_{T_i} : h \mapsto b_{T_i} \quad (i = 1,\ n).$$

It follows that $\ell_f(h) = \sum_i \gamma_i b_{T_i},$ where $\gamma_i \in k$.

Applications to constructions of p-adic L-functions

There exist two kinds of L-functions

- Complex L-functions (Euler products) on $\mathbb{C} = \mathrm{Hom}(\mathbb{R}_+^*, \mathbb{C}^*)$.
- p-adic L-functions on the $\mathbb{C}_p$-analytic group $\mathrm{Hom}_{cont}(\mathbb{Z}_p^*, \mathbb{C}_p^*)$ (Mellin transforms L_μ of p-adic measures μ on $\mathbb{Z}_p^*$).

Both are used in order to obtain a number (L-value) from an automorphic form. Such a number can be algebraic (after normalization) via the embeddings,

$$\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}, \quad \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p = \widehat{\overline{\mathbb{Q}}}_p.$$

and we may compare the complex and p-adic L-values at many points How to define and to compute p-adic L-functions?
The Mellin transform of a p-adic distribution μ on $\mathbb{Z}_p^*$ gives an analytic function on the group of p-adic characters

$$x \mapsto L_\mu(x) = \int_{\mathbb{Z}_p^*} x(y) d\mu, \quad x \in X_{\mathbb{Z}_p^*} = \mathrm{Hom}_{cont}(\mathbb{Z}_p^*, \mathbb{C}_p^*).$$

A general idea is to construct p-adic measures directly from Fourier coefficients of modular forms proving Kummer-type congruences for L-values. Here we present a new method to construct p-adic L-functions via quasimodulat forms:

How to prove Kummer-type congruences using the Fourier coefficients?

Suppose that we are given some L-function $L_f^*(s,\chi)$ attached to a Siegel modular form f and assume that for infinitely many "critical pairs" (s_j,χ_j) one has an integral representation

$\boxed{L_f^*(s_j,\chi_j) = \langle f, h_j\rangle}$ with all $h_j = \sum_T b_{j,T} q^T \in \mathcal{M}$ in a certain finite-dimensional space $\mathcal{M}$ containing f and defined over $\bar{\mathbb{Q}}$.

We want to prove the following Kummer-type congruences:

$$\forall x \in \mathbb{Z}_p^* \sum_j \beta_j \chi_j x^{k_j} \equiv 0 \mod p^N \Longrightarrow \sum_j \beta_j \frac{L_f^*(s_j,\chi_j)}{\langle f, f\rangle} \equiv 0 \mod p^N.$$

for any choice of $\beta_j \in \bar{\mathbb{Q}}$, $k_j = \begin{cases} s_j - s_0 & \text{if } s_0 = \min_j s_j \text{ or} \\ k_j = s_0 - s_j & \text{if } s_0 = \max_j s_j. \end{cases}$

Using the above expression for $\ell_f(h_j) = \sum_i \gamma_{i,j} b_{j,T_i}$, the above congruences reduce to

$$\sum_{i,j} \gamma_{i,j}\beta_j b_{j,T_i} \equiv 0 \mod p^N.$$

Reduction to a finite dimensional case

In order to prove the congruences

$$\sum_{i,j} \gamma_{i,j} \beta_j b_{j,T_i} \equiv 0 \mod p^N.$$

in general we use the functions h_j which belong only to a certain infinite dimensional $\overline{\mathbb{Q}}$-vector space $\mathcal{M} = \mathcal{M}(\overline{\mathbb{Q}})$

$$\mathcal{M}(\overline{\mathbb{Q}}) := \bigcup_{m \geq 0} \mathcal{M}_{\rho^*}(Np^m, \overline{\mathbb{Q}}).$$

Starting from the functions h_j, we use their caracteristic projection $\pi = \pi^\alpha$ on the characteristic subspace $\mathcal{M}^\alpha$ (of generalized eigenvectors) associated to a non-zero eigenvalue α Atkin's U-operator on f which turns out to be of fixed finite dimension so that for all j, $\pi^\alpha(h_j) \in \mathcal{M}^\alpha$.

From holomorphic to nearly holomorphic and p-adic modular forms

Next we explain, how to treat the functions h_j which belong to a certain infinite dimensional $\overline{\mathbb{Q}}$-vector space $\mathcal{N} \subset \mathcal{N}_\rho(\overline{\mathbb{Q}})$ (of *nearly holomorphic modular forms*).

Usually, h_j can be expressed through the functions $\delta^{k_j}(\varphi_0(\chi_j))$ for a certain non-negative power k_j of the Maass-Shimura-type differential operator applied to a holomorphic form $\varphi_0(\chi_j)$.

Then the idea is to proceed in two steps:

1) to pass from the infinite dimensional $\overline{\mathbb{Q}}$-vector space $\mathcal{N} = \mathcal{N}(\overline{\mathbb{Q}})$ of nearly holomorphic modular forms,

$$\mathcal{N}(\overline{\mathbb{Q}}) := \bigcup_{m\geq 0} \mathcal{N}_{k,r}(Np^m, \overline{\mathbb{Q}}) \text{ (of the depth } r).$$

to a fixed finite dimensional characteristic subspace $\mathcal{N}^\alpha \subset \mathcal{N}(Np)$ of U_p in the same way as for the holomorphic forms.

This step controls Petersson products using conjugate f^0 of an eigenfunction f_0 of $U(p)$:

$$\langle f^0, h\rangle = \alpha^{-m}\langle f^0, |U(p)^m h\rangle = \langle f^0, \pi^\alpha(h)\rangle.$$

From holomorphic to nearly holomorphic and p-adic modular forms (continued)

2) To apply Ichikawa's mapping $\iota_p : \mathcal{N}(Np) \to \mathcal{M}_p(Np)$ to a certain space $\mathcal{M}_p(Np)$ of p-adic Siegel modular forms. Assume algebraically,

$$h_j = \sum_T b_{j,T}(S) q^T \mapsto \kappa(h_j) = \sum_T b_{j,T}(0) q^T,$$

which is also a certain Siegel quasi-modular form. Under this mapping, computation become much easier, as the action of δ^{k_j} becomes simply a k_j-power of the Ramanujan Θ-operator

$$\Theta : \sum_T b_T q^T \mapsto \sum_T \det(T) b_T q^T,$$

in the scalar-valued case. In the vector-valued case such operators were studied in [BoeNa13].
After this step, proving the Kummer-type congruences reduces to those for the Fourier coefficients the quasimodular forms $\kappa(h_j(\chi_j))$ which can be explicitely evaluated using the Θ-operator.

Computing with Siegel modular forms over a ring A

There are several types of Siegel modular forms (vector-valued, nearly-holomorphic, quasi-modular, p-adic). Consider modular forms over a ring $A = \mathbb{C}, \mathbb{C}_p, \Lambda = \mathbb{Z}_p[\![T]\!], \cdots$ as certain formal Fourier expansions over A.

Let us fix the congruence subgroup Γ of a nearly holomorphic modular form $f \in \mathcal{N}_\rho$ and its depth r as the maximal S-degree of the poynomial Fourier Fourier coefficients $a_T(S)$ of a nearly holomorphic form

$$f = \sum_T a_T(S) q^T \in \mathcal{N}_\rho(A),$$

over R, and denote by $\mathcal{N}_{\rho,r}(\Gamma, A)$ the A-module of all such forms. This module is often locally-free of finite rank, that is, it becomes a finite-dimensional F-vector space over the fraction field $F = Frac(A)$.

Types of modular forms

- $\mathcal{M}_\rho$ (holomorphic vector-valued Siegel modular forms attached to an algebraic representation $\rho : \mathrm{GL}_n \to \mathrm{GL}_d$)
- $\mathcal{N}_\rho$ (nearly holomorphic vector-valued Siegel modular forms attached to ρ over a number field $k \subset \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$))
- $\mathcal{M}_\rho^\sharp$ (quasi-modular vector-valued forms attached to ρ)
- $\mathcal{M}_\rho^\flat$ (algebraic p-adic vector-valued forms attached to ρ over a number field $k \subset \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$)

Definitions and interrelations:

- $\mathcal{M}_{\rho,r}^\sharp = \kappa(\mathcal{N}_\rho) \subset \mathcal{R}_{n,\infty}^d$, where $\kappa : f \mapsto f|_{S=0} = \sum_T P_T(0) q^T$, where $\mathcal{R}_{n,\infty} = \mathbb{C}[\![q_{11}, \ldots, q_{nn}]\!][q_{ij},\ q_{ij}^{-1}]_{i,j=1,\cdots,n}$.
- $\mathcal{M}_{\rho,r}^\flat(R,\Gamma) = F_c(\iota_p(\mathcal{N}_{\rho,r}(R,\Gamma))) \subset \mathcal{R}_{n,p}^d$, where $\mathcal{R}_{n,p} = \mathbb{C}_p[\![q_{11}, \ldots, q_{nn}]\!][q_{ij},\ q_{ij}^{-1}]_{i,j=1,\cdots,n}$.

Let us fix the level Γ, the depth r, and a subring R of $\bar{\mathbb{Q}}$, then all the R-modules $\mathcal{M}_\rho(R,\Gamma)$, $\mathcal{N}_{\rho,r}(R,\Gamma)$, $\mathcal{M}_{\rho,r}^\sharp(R,\Gamma)$, $\mathcal{M}_{\rho,r}^\flat(R,\Gamma)$ are then locally free of finite rank.
In interesting cases, there is an inclusion $\mathcal{M}_{\rho,r}^\sharp(R,\Gamma) \hookrightarrow \mathcal{M}_{\rho,r}^\flat(R,\Gamma)$.
If $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, $k = 2$, $P = E_2$ is a p-adic modular form, see [Se73], p.211.
Question: Prove it in general! (after discussions with S.Boecherer and T.Ichikawa)

Computing with families of Siegel modular forms

Let $\Lambda = \mathbb{Z}_p[\![T]\!]$ be the Iwasawa algebra, and consider Serre's ring

$$\mathcal{R}_{n,\Lambda} = \Lambda[\![q_{11}, \ldots, q_{nn}]\!][q_{ij}^{\pm 1}]_{i,j=1,\cdots,n}.$$

For any pair (k, χ) as above consider the homomorphisms:

$$\kappa_{k,\chi} : \Lambda \to \mathbb{C}_p, \mathcal{R}^d_{n,\Lambda} \mapsto \mathcal{R}^d_{n,\mathbb{C}_p}, \text{ where } T \mapsto \chi(1+p)(1+p)^k - 1.$$

Definition (families of Siegel modular forms)

Let $\mathbf{f} \in \mathcal{R}^d_{n,\Lambda}$ such that for infinitely many pairs (k, χ) as above,

$$\kappa_{k,\chi}(\mathbf{f}) \in \mathcal{M}_{\rho_k}((i_p(\bar{\mathbb{Q}}))) \overset{F_c}{\hookrightarrow} \mathcal{R}^d_{n,\mathbb{C}_p}$$

is the Fourier expansion at c of a Siegel modular form over $\bar{\mathbb{Q}}$.
All such $\mathbf{f}$ generate the Λ-submodule $\mathcal{M}_{\rho_k}(\Lambda) \subset \mathcal{R}^d_{n,\Lambda}$ of Λ-adic Siegel modular forms of weight ρ.
In the same way, the Λ-submodule $\mathcal{M}^\sharp_{\rho_k}(\Lambda) \subset \mathcal{R}_{n,\Lambda}$ of Λ-adic Siegel quasi-modular forms is defined.

Examples of families of Siegel modular forms

can be constructed via differential operators of Maass $\Delta = \det(\frac{1+\delta_{ij}}{2}\frac{\partial}{\partial z_{ij}})$, so that $\Delta q^T = \det(T)q^T$. Shimura's operator $\delta_k f(Z) = (-4\pi)^{-n}\det(Z-\bar{Z})^{\frac{1+n}{2}-k}\Delta(\det(Z-\bar{Z})^{k-\frac{1+n}{2}+1}f(Z)$ acts on q^T using $\rho_r : \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}(\wedge^r\mathbb{C}^n)$ and its adjoint ρ_r^*:

$$\delta_k(q^T) = \sum_{l=0}^{n}(-1)^{n-l}c_{n-l}(k+1-\frac{1+n}{2})\mathrm{tr}({}^t\rho_{n-l}(S)\rho_l^*(T))q^T,$$

where $c_{n-l}(s) = s(s-\frac{1}{2})\cdots(s-\frac{n-l-1}{2})$, $S = (2\pi i(\bar{z}-z))^{-1}$.

- Nearly holomorphic Λ-adic Siegel-Eisenstein series as in [PaSE] can be produced from the pairs $(-s,\chi)$: if s is a nonpositive integer such that $k+2s > n+1$,

$$E_k(Z,s,\chi) = \prod_{i=0}^{-s-1} c_n(k+2s+2i)^{-1}\delta_{k+2s}^{(-s)}(E_{k+2s}(Z,0,\chi)).$$

Examples of families of Siegel modular forms (continued)

- Ichikawa's construction: quasi-holomorphic (and p-adic) Siegel - Eisenstein series obtained in [Ich13] using the injection ι_p

$$\iota_p(\pi^{ns}E_k(Z,s,\chi)) = \prod_{i=0}^{-s-1} c_n(k+2s+2i)^{-1} \sum_T \det(T)^{-s} b_{k+2s}(T)q^T,$$

  where
  $E_{k+2s}(Z,0,\chi) = \sum_T b_{k+2s}(T)q^T, k+2s > n+1, s \in \mathbb{Z}$.
- A two-variable family is for the parameters $(k+2s, s), k+2s > n+1, s \in \mathbb{Z}$ will be now constructed.

Normalized Siegel-Eisenstein series of two variables

Let us start with an explicit family described in [Ike01], [PaSE], [Pa91] as follows

$$\mathcal{E}_k^n = E_k^n(z) 2^{n/2} \zeta(1-k) \prod_{i=1}^{[n/2]} \zeta(1-2k+2i) = \sum_T a_T(\mathcal{E}_k^n) q^T,$$

where for any non-degenerate matrice T of quadratic character ψ_T:

$$a_T(\mathcal{E}_k^n)$$
$$= 2^{-\frac{n}{2}} \det T^{k-\frac{n+1}{2}} M_T(k) \times \begin{cases} L(1-k+\frac{n}{2}, \psi_T) C_T^{\frac{n}{2}-k+(1/2)}, & n \text{ even}, \\ 1, & n \text{ odd}, \end{cases}$$

($C_T =$ cond(ψ_T), $M_T(k)$ a finite Euler product over $\ell | \det(2T)$.
Starting from the holomorphic series of weight $k > n+1$ and $s = 0$, let us move to all points $(k+2s, s), k+2s > n+1, s \in \mathbb{Z}, s \leq 0$. Then Ichikawa's construction is applicable and it provides a two-variable family.

Examples of families of Siegel modular forms (continued)

- Ikeda-type families of cusp forms of even genus [PaIsr11] (reported in Luminy, May 2011). Start from a p-adic family

$$\varphi = \{\varphi_{2k}\} : 2k \mapsto \varphi_{2k} = \sum_{n=1}^{\infty} a_n(2k) q^n \in \overline{\mathbb{Q}}[\![q]\!] \subset \mathbb{C}_p[\![q]\!],$$

  where the Fourier coefficients $a_n(2k)$ of the normalized cusp Hecke eigenform φ_{2k} and one of the Satake p-parameters $\alpha(2k) := \alpha_p(2k)$ are given by certain p-adic analytic functions $k \mapsto a_n(2k)$ for $(n, p) = 1$. The Fourier expansions of the modular forms $F = F_{2n}(\varphi_{2k})$ can be explicitly evaluated where $L(F_{2n}(\varphi), St, s) = \zeta(s) \prod_{i=1}^{2n} L(\varphi, s + k + n - i)$. This sequence provide an example of a p-adic family of Siegel modular forms.
- Ikeda-Myawaki-type families of cusp forms of $n = 3$, [PaIsr11] (reported in Luminy, May 2011).
- Families of Klingen-Eisenstein series extended in [JA13] from $n = 2$ to a general case (reported in Journées Arithmétiques, Grenoble, July 2013).

Thank you!

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