Triple products of Coleman’s families and their periods
(a joint work with S. Boecherer)

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Abstract

For a prime number $p \geq 5$, we consider three classical cusp eigenforms

$$f_j(z) = \sum_{n=1}^{\infty} a_{n,j} e(nz) \in S_{k_j}(N_j, \psi_j), \ (j = 1, 2, 3)$$

of weights $k_1, k_2, k_3$, of conductors $N_1, N_2, N_3$, and of nebentypus characters $\psi_j \text{ mod } N_j$.

According to H.Hida [Hi86] and R.Coleman [CoPB], one can include each $f_j \ (j = 1, 2, 3)$ (under suitable assumptions on $p$ and on $f_j$) into a $p$-adic analytic family

$$k'_j \mapsto \{f_{j,k'_j} = \sum_{n=1}^{\infty} a_n(f_{j,k'_j}) q^n\}$$

of cusp eigenforms $f_{j,k'_j}$ of weights $k'_j$ in such a way that $f_{j,k_j} = f_j$, and that all their Fourier coefficients $a_n(f_{j,k'_j})$ are given by certain $p$-adic analytic functions $k'_j \mapsto a_{n,j}(k'_j)$.

The purpose of this talk is to describe a four variable $p$-adic
$L$ function attached to Garrett’s triple product of three Coleman’s families

$$k_j' \mapsto \left\{ \left. f_{j,k_j'} = \sum_{n=1}^{\infty} a_{n,j}(k_j')q^n \right\} \right.$$ 

of cusp eigenforms of three fixed slopes $\sigma_j = v_p(\alpha_{p,j}^{(1)}(k_j')) \geq 0$

where $\alpha_{p,j}^{(1)} = \alpha_{p,j}(k_j')$ is an eigenvalue (which depends on $k_j'$) of Atkin’s operator $U = U_p$ acting on Fourier expansions by

$$U(\sum_{n \geq 0} a_n q^n) = \sum_{n \geq 0} a_{np} q^n.$$ 

Let us consider the product of three eigenvalues:

$$\lambda = \lambda(k_1', k_2', k_3') = \alpha_{p,1}^{(1)}(k_1')\alpha_{p,2}^{(1)}(k_2')\alpha_{p,3}^{(1)}(k_3')$$

and assume that the slope of this product

$$\sigma = v_p(\lambda(k_1', k_2', k_3')) = \sigma(k_1', k_2', k_3') = \sigma_1 + \sigma_2 + \sigma_3$$

is constant and positive for all triplets $(k_1', k_2', k_3')$ in an appropriate $p$-adic neighbourhood of the fixed triplet of weights $(k_1, k_2, k_3)$. The each value $\sigma_j$ is fixed.
We consider the $p$-adic weight space $X$ containing all $(k'_j, \psi_j)$. Our $p$-adic $L$-functions are Mellin transforms of certain measures with values in $\mathcal{A}$, where $\mathcal{A} = \mathcal{A}(\mathcal{B})$ denotes an affinoid algebra associated with an affinoid space $\mathcal{B}$ as in [CoPB], where $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3$, is an affinoid neighbourhood around $(k_1, k_2, k_3) \in X^3$ (with a given integers $k_j$ and fixed Dirichlet characters $\psi_j \mod N$).

We construct such a measure from higher twists of classical Siegel-Eisenstein series, which produce distributions with values in certain Banach $\mathcal{A}$-modules $\mathcal{M} = \mathcal{M}(N; \mathcal{A})$ of triple modular forms with coefficients in the algebra $\mathcal{A}$.
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0 Introduction and Main results

Let $p$ be a prime. The purpose of this talk is to describe a four variable $p$-adic $L$ function attached to Garrett’s triple product of three Coleman’s families

$$k_j' \mapsto \left\{ f_{j,k_j'} = \sum_{n=1}^{\infty} a_{n,j}(k_j')q^n \right\}$$

of cusp eigenforms of three fixed slopes $\sigma_j = v_p(\alpha_{p,j}(k_j')) \geq 0$ where $\alpha_{p,j} = \alpha_{p,j}(k_j')$ is an eigenvalue (which depends on $k_j'$) of Atkin’s operator $U = U_p$ acting on Fourier expansions by

$$U(\sum_{n \geq 0} a_nq^n) = \sum_{n \geq 0} a_{np}q^n.$$ 

For this purpose we use the theory of $p$-adic integration with values in spaces of nearly holomorphic modular forms (in the sense of Shimura, see [ShiAr]).
A family of slope $\sigma > 0$ of cusp eigenforms $f_{k'}$ of weight $k' \geq 2$ containing $f$

\[ k' \mapsto f_{k'} = \sum_{n=1}^{\infty} a_n(k')q^n \]
\[ \in \overline{\mathbb{Q}}[q] \subset \mathbb{C}_p[q] \]

A model example of a $p$-adic family (not cusp and $\sigma = 0$):

Eisenstein series
\[ a_n = \sum_{d|n} d^{k'-1}, f_{k'} = E_{k'} \]

1) the Fourier coefficients $a_n(k')$ of $f_{k'}$
and one of the Satake $p$-parameters $\alpha_p(k')$
are given by certain $p$-adic analytic functions $k' \mapsto a_n(k')$ for $(n, p) = 1$

2) the slope is constant and positive:
\[ \text{ord}(\alpha_p(k')) = \sigma > 0 \]
1 Generalities on triple products

Consider three primitive cusp eigenforms

\[ f_j(z) = \sum_{n=1}^{\infty} a_{n,j} q^n \in S_{k_j}(N_j, \psi_j), \quad (j = 1, 2, 3) \quad (1.1) \]

of weights \( k_1, k_2, k_3 \), of conductors \( N_1, N_2, N_3 \), and of nebentypus characters \( \psi_j \mod N_j \), and let \( \chi \) denote a varying Dirichlet character.

The triple product with a Dirichlet character \( \chi \) is defined as the following complex \( L \)-function (an Euler product of degree eight):

\[
L(f_1 \otimes f_2 \otimes f_3, s, \chi) = \prod_{p \nmid N_1 N_2 N_3} L((f_1 \otimes f_2 \otimes f_3)_p, \chi(p)p^{-s}), \quad (1.2)
\]
where \( L((f_1 \otimes f_2 \otimes f_3)_p, X)^{-1} = \) \( \det \left( 1_8 - X \begin{pmatrix} \alpha_{p,1}^{(1)} & 0 \\ 0 & \alpha_{p,1}^{(2)} \end{pmatrix} \otimes \begin{pmatrix} \alpha_{p,2}^{(1)} & 0 \\ 0 & \alpha_{p,2}^{(2)} \end{pmatrix} \otimes \begin{pmatrix} \alpha_{p,3}^{(1)} & 0 \\ 0 & \alpha_{p,3}^{(2)} \end{pmatrix} \right) \)

\[ = \prod_{\eta} (1 - \alpha_{p,1}^{(\eta(1))} \alpha_{p,2}^{(\eta(2))} \alpha_{p,3}^{(\eta(3))} X) \]

\[ = (1 - \alpha_{p,1}^{(1)} \alpha_{p,2}^{(1)} \alpha_{p,3}^{(1)} X)(1 - \alpha_{p,1}^{(1)} \alpha_{p,2}^{(1)} \alpha_{p,3}^{(2)} X) \ldots (1 - \alpha_{p,1}^{(2)} \alpha_{p,2}^{(2)} \alpha_{p,3}^{(2)} X) \]

product taken over all \( \eta : \{1, 2, 3\} \to \{1, 2\} \), and

\[ 1 - a_{p,j} X - \psi_j(p) p^{k_j - 1} X^2 = (1 - \alpha_{p,j}^{(1)}(p) X)(1 - \alpha_{p,j}^{(2)}(p) X), \]

are three Hecke \( p \)-polynomials of forms \( f_j \). We always assume that

\[ k_1 \geq k_2 \geq k_3, \text{ and } k_1 \leq k_2 + k_3 - 2 \text{ ("balanced" weights)} \]
We use the corresponding normalized $L$ function (see [De79], [Co], [Co-PeRi]), which has the form:

\[
\Lambda(f_1 \otimes f_2 \otimes f_3, s, \chi) = \Gamma_C(s)\Gamma_C(s - k_3 + 1)\Gamma_C(s - k_2 + 1)\Gamma_C(s - k_1 + 1)L(f_1 \otimes f_2 \otimes f_3, s, \chi),
\]

where $\Gamma_C(s) = 2(2\pi)^{-s}\Gamma(s)$. The Gamma-factor determines the critical values $s = k_1, \cdots, k_2 + k_3 - 2$ of $\Lambda(s)$, which we explicitly evaluate (like $\zeta(2) = \frac{\pi^2}{6}$).

A functional equation of $\Lambda(s)$ has the form:

\[s \mapsto k_1 + k_2 + k_3 - 2 - s.\]
According to H.Hida [Hi86] and R.Coleman [CoPB], one can include each $f_j$ ($j = 1, 2, 3$) (under suitable assumptions on $p$ and on $f_j$) into a $p$-adic analytic family

$$f_j : k'_j \mapsto \{ f_{j, k'_j} = \sum_{n=1}^{\infty} a_n(f_{j, k'_j}) q^n \}$$

of cusp eigenforms $f_{j, k'_j}$ of weights $k'_j$ in such a way that $f_{j, k_j} = f_j$, and that all their Fourier coefficients $a_n(f_{j, k'_j})$ are given by certain $p$-adic analytic functions $k'_j \mapsto a_{n,j}(k'_j)$.

**Our previous result:** a two variable $p$-adic $L$-function constructed interpolating on all $k'$ a function

$$(k', s) \mapsto L^*(f_{k'}, s, \chi)(s = 1, \cdots, k' - 1)$$

2 Statement of the problem

Given three $p$-adic analytic families $f_j$ of positive slope $\sigma_j > 0$, to construct a four-variable $p$-adic $L$-function attached to Garrett’s triple product of these families

(we may view such function as interpolating the special values $\Lambda(f_1, k'_1 \otimes f_2, k'_2 \otimes f_3, k'_3, s, \chi)$ at critical points $s = k'_1, \cdots, k'_2 + k'_3 - 2$; we prove that these values are algebraic numbers after dividing by certain “periods”). However the construction uses directly modular forms, and not the $L$-values in question, and a comparison of special values of two functions is done after the construction.

We consider the $p$-adic weight space $X = \text{Hom}_{\text{cont}}(Y, \mathbb{C}_p^*)$. This is an analytic space over $\mathbb{C}_p$, which consists of all continuous characters of the profinite group $Y = (\mathbb{Z}/N\mathbb{Z})^* \times \mathbb{Z}_p^*$, containing all classical weights $(k'_j, \psi_j), j = 1, 2, 3$. 
Consider the product of the Satake parameters
\[ \lambda_p = \alpha_{p,1}^{(1)} \alpha_{p,2}^{(1)} \alpha_{p,3}^{(1)} = \lambda_p(k_1', k_2', k_3') \]

We assume that \( |\alpha_{p,j}^{(1)}|_p \leq |\alpha_{p,j}^{(2)}|_p \), and that the slope \( \sigma = \text{ord}_p(\lambda_p(k_1', k_2', k_3')) \) is constant and positive for all triplets \((k_1', k_2', k_3')\) in a \( p \)-adic neighbourhood \( \mathcal{B} \subset X^3 \) of the fixed triplet of weights \((k_1, k_2, k_3)\).

The existence of families of slope \( \sigma > 0 \) was established by R. Coleman in [CoPB]

He gave an example with 
\[ p = 7, \ f = \Delta, \ k = 12 \]
\[ a_7 = \tau(7) = -7 \cdot 2392, \ \sigma = 1. \]

A program in PARI for computing such families is contained in [CST98] (see also the Web-page of W. Stein, http://modular.fas.harvard.edu/ )
Our method

uses a version of Garrett’s integral representation for the triple \( L \)-functions of the form: for \( r = 0, \cdots, k'_2 + k'_3 - k'_1 - 2 \),
\[
\Lambda(f_{1,k'_1} \otimes f_{2,k'_2} \otimes f_{3,k'_3}, k'_2 + k'_3 - r, \chi) = \int \int \int \frac{\tilde{f}_{1,k'_1}(z_1)\tilde{f}_{2,k'_2}(z_2)\tilde{f}_{3,k'_3}(z_3)E(z_1, z_2, z_3; -r, \chi)}{(\Gamma_0(N^2p^{2v})/\mathbb{H})^3} \prod_j \left( \frac{dx_jdy_j}{y_j^2} \right)
\]
where \( E \) is the triple modular form of weight \((k'_1, k'_2, k'_3)\), and of fixed character \((\psi_1, \psi_2, \psi_3)\), obtained from a nearly holomorphic Siegel-Eisenstein series by applying Boecherer’s higher twist and Ibukiyama’s differential operator;

the theory of \( p \)-adic integration with values in \( \mathcal{A} \)-modules \( \mathcal{M}_T(\mathcal{A}) \) of triple nearly holomorphic modular forms over \( \mathcal{A} \)-adic Banach algebras \( \mathcal{A} \), which allows to view \( E \) as an element of \( \mathcal{M}_T(\mathcal{A}) \);

the spectral theory of Atkin’s \( \mathcal{U} \)-operator allows to evaluate the integral using a projection of \( \mathcal{M}_T(\mathcal{A}) \) to the \( \lambda \)-part \( \mathcal{M}_T(\mathcal{A})^\lambda \).
Here $\mathcal{A} = \mathcal{A}(\mathcal{B})$ is a certain $p$-adic Banach algebra of functions on an open analytic subspace $\mathcal{B} \subset X^3$ in the product of three copies of the weight space $X = \text{Hom}_{\text{cont}}(Y, \mathbb{C}_p^*)$. This is an analytic space over $\mathbb{C}_p$, which consists of all continuous characters of the profinite group $Y = (\mathbb{Z}/N\mathbb{Z})^* \times \mathbb{Z}_p^*$.

The weights $k'$ correspond to certain points in the weight space $X$. Let $\mathcal{B}_j \subset X$ denote an open analytic subspace containing $k_j \in X$, and let $\mathcal{A}_j = \mathcal{A}_j(\mathcal{B}_j)$ be the $p$-adic Banach algebra attached to $\mathcal{B}_j$. Any series $f_j = \sum_{n \geq 1} a_n q^n \in \mathcal{A}_j[[q]]$ produces a family of $q$-expansions

$$\{ f_{j,k'_j} = \text{ev}_{k'_j}(f_j) = \sum_{n \geq 1} \text{ev}_{k'_j}(a_n) q^n \in \mathbb{C}_p[[q]] \},$$

which happen to be classical modular forms in $\overline{\mathbb{Q}}[[q]]$ under a fixed embedding $i_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ and the specialization maps $\text{ev}_{k'_j} : \mathcal{A}_j \to \mathbb{C}_p$. 

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• We may assume that \( \mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3 \), and consider the specialization maps \( ev_{k'} : \mathcal{A} \to \mathbb{C}_p \), where \( k' = (k'_1, k'_2, k'_3) \in \mathcal{B} \).

• We construct an analytic function \( \mathcal{L}_\mu : X \to \mathcal{A} = \mathcal{A}(\mathcal{B}) \) as the \( p \)-adic Mellin transform

\[
\mathcal{L}_\mu(x) = \int_Y x(y) \, d\mu(y) \quad \text{(where } x \in X = \text{Hom}_{\text{cont}}(Y, \mathbb{C}_p^*), x = x(y)),
\]

of a certain measure \( \mu \) with values in \( \mathcal{A} \) on the profinite group \( Y \).

We obtain the function in question \( \mathcal{L}_\mu(x, s) \) by evaluation at \( s = ((s_1, \psi_1), (s_2, \psi_2), (s_3, \psi_3)) \in \mathcal{B} \): this is a \( p \)-adic analytic function in four variables \( (x, s) \in X \times \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3 \subset X \times X \times X \times X \):

\[
\mathcal{L}_\mu(x, s) := ev_s(\mathcal{L}_\mu(x)) \quad (x \in X, \ s \in \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3, \ \mathcal{L}_\mu(x) \in \mathcal{A}).
\]
• We check an equality relating the values of the constructed analytic function $L_\mu(x, s)$ at the arithmetical characters $x = y_p^r \chi \in X$, and at triple weights $s = (k'_1, k'_2, k'_3) \in \mathcal{B}$, with the normalized critical special values

$$L^*(f_{1, k'_1} \otimes f_{2, k'_2} \otimes f_{3, k'_3}, \ k'_2 + k'_3 - 2 - r, \chi) \ (r = 0, \cdots, k'_2 + k'_3 - k'_1 - 2),$$

for certain Dirichlet characters $\chi \mod Np^v, v \geq 1$. These are algebraic numbers, embedded into $\mathbb{C}_p = \hat{\mathbb{Q}}_p$ (the Tate field of $p$-adic numbers). The normalisation of $L^*$ includes at the same time Gauss sums, Petersson scalar products, powers of $\pi$, $\lambda_p(k'_1, k'_2, k'_3)$, and a certain finite Euler product.
3 Arithmetical nearly holomorphic modular forms

Let $\mathcal{A}$ be a commutative ring (a subring of $\mathbb{C}$ or a normed $\mathcal{O}$-algebra $\mathcal{A}$ where $\mathcal{O}$ is the ring of integers in a finite extension $K$ of $\mathbb{Q}_p$).

Arithmetical nearly holomorphic modular forms (in the sense of Shimura, [ShiAr]) are certain formal series

$$g = \sum_{n=0}^{\infty} a(n; R) q^n \in \mathcal{A}[q][R],$$

with the property that for $\mathcal{A} = \mathbb{C}$, $z = x + iy \in \mathbb{H}$, $R = (4\pi y)^{-1}$, the series converges to a $\mathbb{C}^\infty$-modular form on $\mathbb{H}$ of a given weight $k$ and Dirichlet character $\psi$. The coefficients $a(n; R)$ are polynomials in $\mathcal{A}[R]$. If $\deg_R a(n; R) \leq r$ for all $n$, we call $g$ nearly holomorphic of type $r$ (it is annihilated by $(\frac{\partial}{\partial z})^{r+1}$, see [ShiAr]).
We use the notation $\mathcal{M}_{k,r}(N, \psi, A)$ or $\tilde{\mathcal{M}}(N, \psi, A)$ for $A$-modules of such forms. (In our constructions the weight $k$ varies).

A known example (see the introduction to [ShiAr]) is given by the series

$$- 12R + E_2 := -12R + 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n)q^n$$

$$= \frac{3}{\pi^2} \lim_{s \to 0} y^s \sum_{m_1, m_2 \in \mathbb{Z}} (m_1 + m_2z)^{-2}|m_1 + m_2z|^{-2s}, (R = (4\pi y)^{-1})$$

where $\sigma_1(n) = \sum_{d|n} d$.

There is the action of the Shimura differential operator

$$\delta_k : \mathcal{M}_{k,r}(N, \psi, A) \to \mathcal{M}_{k+2,r+1}(N, \psi, A),$$

given over $\mathbb{C}$ by $\delta_k(f) = (\frac{1}{2\pi i} \frac{\partial}{\partial z} - \frac{k}{4\pi y})f$. 

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This operator is a correction of the Ramanujan operator

\[
\theta(\sum_{n=0}^{\infty} a_n q^n) = \sum_{n=1}^{\infty} n a_n q^n = \frac{1}{2\pi i} \frac{\partial}{\partial z} \left(\sum_{n=0}^{\infty} a_n q^n\right) = q \frac{\partial}{\partial q} \left(\sum_{n=0}^{\infty} a_n q^n\right),
\]

which does not preserve the modularity. For example \(\theta \Delta = E_2 \Delta\), where \(E_2\) is a quasimodular form (in the sense of Kaneko and Zagier, see [Ka-Za]).

Notice that \(\delta_k f = (\theta - kR)f\), and that Serre’s operator \(f \mapsto \theta f - \frac{k}{12} E_2 f\) takes \(M_k\) to \(M_{k+2}\).

Note that that the arithmetical twist operator

\[
\theta\chi(\sum_{n=0}^{\infty} a_n q^n) = \sum_{n=1}^{\infty} \chi(n) a_n q^n
\]

is a natural analog of the Ramanujan operator.
Let $\mathcal{A}$ be a commutative ring. The tensor product over $\mathcal{A}$

$$M_{k,r,T}(N, \psi, \mathcal{A}) = M_{k_1,r}(N, \psi_1, \mathcal{A}) \otimes M_{k_2,r}(N, \psi_2, \mathcal{A}) \otimes M_{k_3,r}(N, \psi_3, \mathcal{A})$$

consists of **triple arithmetical modular forms** as certain formal series of the form

$$g = \sum_{n_1,n_2,n_3=0}^{\infty} a(n_1, n_2, n_3; R_1, R_2, R_3) q_1^{n_1} q_2^{n_2} q_3^{n_3} \in \mathcal{A}[q_1, q_2, q_3][R_1, R_2, R_3],$$

where $z_j = x_j + iy_j \in \mathbb{H}$, $R_j = (4\pi y_j)^{-1}$, with the property that for $\mathcal{A} = \mathbb{C}$, the series converges to a

$\mathbb{C}^\infty$-modular form on $\mathbb{H}^3$ of a given weight $(k_1, k_2, k_3)$ and character $(\psi_1, \psi_2, \psi_3)$, $j = 1, 2, 3$. The coefficients $a(n_1, n_2, n_3; R_1, R_2, R_3)$ are polynomials in $\mathcal{A}[R_1, R_2, R_3]$. Examples of such modular forms come from the restriction to the diagonal of Siegel modular forms of degree 3.
4 Siegel-Eisenstein series

Recall some definitions concerning Siegel modular forms.

Let $J_{2m} = \begin{pmatrix} 0_m & -1_m \\ 1_m & 0_m \end{pmatrix}$. The symplectic group

$$\text{Sp}_m(\mathbb{R}) = \{ g \in \text{GL}_{2m}(\mathbb{R}) \mid t \cdot J_{2m} \cdot g = J_{2m} \},$$

acts on the Siegel upper half plane

$$\mathbb{H}_m = \{ z = t \cdot z \in M_m(\mathbb{C}) \mid \text{Im}z > 0 \}$$

by $g(z) = (az + b)(cz + d)^{-1}$, where we use the bloc notation

$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Sp}_{2m}(\mathbb{R})$, and the congruence subgroup

$\Gamma_0^m(N) = \{ \gamma \in \text{Sp}_m(\mathbb{Z}) \mid \gamma \equiv (0 \ 0) \}$ $\subset \text{Sp}_m(\mathbb{Z})$. 
A Siegel modular form \( f \in \mathcal{M}_k(\Gamma_0^m(N), \chi) \) of degree \( m > 1 \), weight \( k \) and a Dirichlet character \( \chi \mod N \) is a holomorphic function \( f : \mathbb{H}_m \to \mathbb{C} \) such that for every \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0^m(N) \) one has

\[
f(\gamma(z)) = \chi(\det d) \det(cz + d)^k f(z).
\]

The Fourier expansion of \( f \) uses the semi-group \( B_m = \{ \mathcal{J} = t \mathcal{J} \geq 0 | \mathcal{J} \text{ half-integral} \} : \)

\[
f(z) = \sum_{\mathcal{J} \in B_m} a(\mathcal{J}) q^\mathcal{J} \in \mathbb{C}[q^{B_m}](\text{a formal } q\text{-expansion } \in \mathbb{C}[q^{B_m}]),
\]

where \( q^\mathcal{J} = \exp(2\pi i \text{tr}(\mathcal{J}z)) \)

\[
= \prod_{i=1}^{m} q_{ii}^{\mathcal{J}_{ii}} \prod_{i<j} q_{ij}^{2\mathcal{J}_{ij}} \subset \mathbb{C}[q_{11}, \ldots, q_{mm}][q_{ij}, q_{ij}^{-1}]_{i,j=1,\ldots,m}
\]

and \( q_{ij} = \exp(2\pi(\sqrt{-1}z_{i,j})) \).
Example 4.1 (Siegel-Eisenstein series) (see in [Nag2], p.408):

\[ E_4^{(2)}(z) = 1 + 240q_{11} + 240q_{22} + 2160q_{11}^2 + (240q_{12}^{-2} + 13440q_{12}^{-1} \]
\[ \quad \quad 30240 + 13440q_{12} + 240q_{12}^2)q_{11}q_{22} + 2160q_{22}^2 + \ldots \]

\[ E_6^{(2)}(z) = 1 - 504q_{11} - 504q_{22} - 16632q_{11}^2 + (-540q_{12}^{-2} + 44352q_{12}^{-1} \]
\[ \quad \quad 166320 + 44352q_{12} - 504q_{12}^2)q_{11}q_{22} - 16632q_{22}^2 + \ldots \]
Arithmetical nearly holomorphic Siegel modular forms

Consider a commutative ring $\mathcal{A}$, the formal variables $q = (q_{i,j})_{i,j=1,...,m}$, $R = (R_{i,j})_{i,j=1,...,m}$, and the ring of formal Fourier series

$$\mathcal{A}[q^{B_m}][R_{i,j}] = \left\{ f = \sum_{\mathcal{J} \in B_m} a(\mathcal{J}, R)q^{\mathcal{J}} \mid a(\mathcal{J}, R) \in \mathcal{A}[R_{i,j}] \right\} \quad (4.1)$$

(over the complex numbers this notation corresponds to $q^{\mathcal{J}} = \exp(2\pi i \text{tr} (\mathcal{J}z))$, $R = (4\pi \text{Im}(z))^{-1}$).

The formal Fourier expansion of a nearly holomorphic Siegel modular form $f$ with coefficients in $\mathcal{A}$ is a certain element of $\mathcal{A}[q^{B_m}][R_{i,j}]$. We call $f$ arithmetical in the sense of Shimura [ShiAr], if $\mathcal{A} = \overline{\mathbb{Q}}$. 
4.1 Algebraic differential operators of Maass and Shimura

Let us consider the Maass differential operator (see \cite{Maa}) $\Delta_m$ of degree $m$, acting on complex $C^\infty$-functions on $\mathbb{H}_m$ by:

$$\Delta_m = \det(\tilde{\partial}_{ij}), \quad \tilde{\partial}_{ij} = 2^{-1}(1 + \delta_{ij})\partial/\partial_{ij},$$

(4.2)

its algebraic version is the Ramanujan operator of degree $m$:

$$\Theta_m := \det\left(\frac{1}{2\pi i} \tilde{\partial}_{ij}\right) = \det(\theta_{ij}) = \frac{1}{(2\pi i)^m} \Delta_m,$$

(4.3)

where $\theta_{ij} = 2^{-1}(1 + \delta_{ij})q_{ij}\frac{\partial}{\partial q_{ij}}, \Theta_m(q^\mathcal{J}) = \det(\mathcal{J})q^\mathcal{J}$. 


The Shimura differential operator (see [Shi76, ShiAr]):

$$\delta_k f(z) = \det(R)^{k+1-z} \Theta_m \left[ \det(R)^{z-1-k} f \right], \quad \text{where } R = (4\pi y)^{-1},$$

acts on arithmetic nearly holomorphic Siegel modular forms, and the composition is defined

$$\delta^{(r)}_k = \delta_{k+2r-2} \circ \cdots \circ \delta_k : \tilde{M}^m_k(N, \psi; \overline{\mathbb{Q}}) \to \tilde{M}^m_{k+2r}(N, \psi; \overline{\mathbb{Q}}), \quad (4.4)$$

where

$$\delta_k f(z) = \left(\frac{-1}{4\pi}\right)^m \det(y)^{-1} \det(z - \bar{z})^{z-k} \Delta_m \left[ \det(z - \bar{z})^{k-z+1} f(z) \right].$$
Let \( f = \sum_{\mathcal{I} \in B_m} c(\mathcal{I}) q^{\mathcal{I}} \in \mathcal{M}_k^m(N, \psi) \) be a formal holomorphic Fourier expansion. One shows that \( \delta_k^{(r)} f \) is given by

\[
\delta_k^{(r)} f = \sum_{\mathcal{I} \in B_m} Q(R, \mathcal{I}; k, r) c(\mathcal{I}) q^{\mathcal{I}}.
\]

Here we use a universal polynomial \((4.5)\) which can be defined for all \( k \in \mathbb{C} \), and it expresses the action of the Shimura operator on the exponential (of degree \( m \)):

\[
\delta_k^{(r)} (q^{\mathcal{I}}) = Q(R, \mathcal{I}; k, r) q^{\mathcal{I}}.
\]

If \( m = 1 \), \( r \) arbitrary (see \([\text{Shi76}]\)),

\[
\delta_k^{(r)} = \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} \frac{\Gamma(k + r)}{\Gamma(k + j)} R^{r-j} \theta^j.
\]
If \( r = 1 \), \( m \) arbitrary, one has (see \[\text{Maa}\]):

\[
\delta_k f(z) = \sum_{\mathcal{J} \in B_m} c(\mathcal{J}) \sum_{l=0}^{m} (-1)^{m-l} c_{m-l}(k+1-\kappa) \text{tr} \left( t \rho_{m-l}(R) \cdot \rho_l^*(\mathcal{J}) \right) q^{\mathcal{J}}
\]

where \( R = (4\pi y)^{-1} = (R_{i,j}) \in M_m(\mathbb{R}) \), \( c_m(\alpha) = \frac{\Gamma_m(\alpha + \kappa)}{\Gamma_m(\alpha + \kappa - 1)} \),

\[
\Gamma_m(s) = \pi^{m(m-1)/4} \prod_{j=0}^{m-1} \Gamma(s - (j/2)).
\]

Here we use the natural representation \( \rho_r : \text{GL}_m(\mathbb{C}) \rightarrow \text{GL}(\wedge^r \mathbb{C}^m) \) (\( 0 \leq r \leq m \)) of the group \( \text{GL}_m(\mathbb{C}) \) on the vector space \( \Lambda^r \mathbb{C}^m \). Thus \( \rho_r(z) \) is a matrix of size \( \binom{m}{r} \times \binom{m}{r} \) composed of the subdeterminants of \( z \) of degree \( r \). Put \( \rho^*_r(z) = \det(z) \rho_{m-r}(tz)^{-1} \).

Then the representations \( \rho_r \) and \( \rho^*_r \) turn out to be polynomial representations so that for each \( z \in M_m(\mathbb{C}) \) the linear operators \( \rho_r(z), \rho^*_r(z) \) are well defined.
In general (see [CourPa], Theorem 3.14) one has:

\[ Q(R, \mathcal{T}) = Q(R; k, r) \]

\[ = \sum_{t=0}^{r} \left( \begin{array}{c} r \\ t \end{array} \right) \det(\mathcal{T})^{r-t} \sum_{|L| \leq mt-t} R_L(\kappa - k - r)Q_L(R, \mathcal{T}), \]

\[ Q_L(R, \mathcal{T}) = \text{tr} \left( t^{\rho_{m-l_1}(R)} \rho_{l_1}^*(\mathcal{T}) \right) \cdots \text{tr} \left( t^{\rho_{m-l_t}(R)} \rho_{l_t}^*(\mathcal{T}) \right). \]

In (4.5), \( L \) goes over all the multi-indices \( 0 \leq l_1 \leq \cdots \leq l_t \leq m \), such that \( |L| = l_1 + \cdots + l_t \leq mt - t \), and \( R_L(\beta) \in \mathbb{Z}[1/2][\beta] \) in (4.5) are polynomials in \( \beta \) of degree \( (mt - |L|) \) (used with \( \beta = \kappa - k - r \)).

Note the differentiation rule of degree \( m \) (see [Sh83], p.466):

\[ \Delta(fg) = \sum_{r=0}^{m} \text{tr} \left( t^{\rho_r}(\bar{\partial}/\partial z)f \cdot \rho_{m-r}^*(\bar{\partial}/\partial z)g \right). \]
Example 4.2 (Siegel-Eisenstein series of higher level)

\[ G^*(z, s; k, \chi, N) \]

\[ = \det(y)^s \sum_{c,d} \chi(\det c) \det(cz + d)^{-k} |\det(cz + d)|^{-2s} \cdot \]

\[ \cdot \tilde{\Gamma}(k, s)L_N(k + 2s, \chi) \left( \prod_{i=1}^{[m/2]} L_N(2k + 4s - 2i, \chi^2) \right), \text{ where} \]

\((c, d)\) runs over all “non-associated coprime symmetric pairs” with \(\det(c)\) coprime to \(N\), \(\kappa = (m + 1)/2\), and for \(m\) odd the \(\Gamma\)-factor has the form: \((2.134)\)

\[ \tilde{\Gamma}(k, s) = i^{mk}2^{-m(k+1)}\pi^{-m(s+k)}\Gamma_m(k + s). \]

We use this series with \(m = 3\), \(\kappa = \frac{m + 1}{2} = 2\), \([m/2] = 1\).
Theorem 4.3 (Siegel, Shimura [Sh83], P. Feit [Fei86]) Let $m$ be an odd integer such that $2k > m$, and $N > 1$ be an integer, then:

For an integer $s$ such that $s = -r$, $0 \leq r \leq k - \kappa$, there is the following Fourier expansion

$$G^*(z, -r) = G^*(z, -r; k, \chi, N) = \sum_{A_m \ni \mathcal{T} \geq 0} a(\mathcal{T}, R) q^{\mathcal{T}}, \quad (4.7)$$

where for $s > (m + 2 - 2k)/4$ in (4.7) the only non-zero terms occur for positive definite $\mathcal{T} > 0$, polynomials $Q(R, \mathcal{T}; k - 2r, r)$ are given by (4.5), and for all $\mathcal{T} > 0$, $\mathcal{T} \in A_m$, where

$$a(\mathcal{T}, R) = M(\mathcal{T}, \chi, k - 2r) \cdot \det(\mathcal{T})^{k - 2r - \kappa} Q(R, \mathcal{T}; k - 2r, r), \quad (4.8)$$

$$M(\mathcal{T}, k - 2r, \chi) = \prod_{\ell \mid \det(2\mathcal{T})} M_{\ell}(\mathcal{T}, \chi(\ell)\ell^{-k+2r}) \quad (4.9)$$

is a finite Euler product, in which $M_{\ell}(\mathcal{T}, x) \in \mathbb{Z}[x]$. □
5  Distributions and admissible measures

Notation

\( \mathcal{A} \) \hspace{1cm} (a \( p \)-adic Banach algebra)
\( V \) \hspace{1cm} (an \( \mathcal{A} \)-module)
\( \mathcal{C}(Y, \mathcal{A}) \) \hspace{1cm} (the \( \mathcal{A} \)-Banach algebra of continuous functions on \( Y \))
\( \mathcal{C}^{loc-\text{const}}(Y, \mathcal{A}) \) \hspace{1cm} (the \( \mathcal{A} \)-algebra of locally constant functions on \( Y \))
Definition 5.1  

a) A distribution $\mathcal{D}$ on $Y$ with values in $V$ is an $A$-linear form

$$\mathcal{D} : C^{loc-const}(Y, A) \rightarrow V, \quad \varphi \mapsto \mathcal{D}(\varphi) = \int_Y \varphi d\mathcal{D}.$$ 

b) A measure $\mu$ on $Y$ with values in $V$ is a continuous $A$-linear form

$$\mu : C(Y, A) \rightarrow V, \quad \varphi \mapsto \int_Y \varphi d\mu.$$ 

The integral $\int_Y \varphi d\mu$ can be defined for any continuous function $\varphi$, and any bounded distribution $\mu$, using the Riemann sums.
Admissible measures of Amice-Vélu

Let $h$ be a positive integer. A more delicate notion of an $h$-admissible measure was introduced in [Am-V] by Y. Amice, J. Vélu (see also [MTT], [V]):

**Definition 5.2**

a) For $h \in \mathbb{N}, h \geq 1$ let $\mathcal{P}^h(Y, \mathcal{A})$ denote the $\mathcal{A}$-module of locally polynomial functions of degree $< h$ of the variable $y_p : Y \to \mathbb{Z}_p^\times \hookrightarrow \mathcal{A}_p^\times$; in particular, $\mathcal{P}^1(Y, \mathcal{A}) = \mathcal{C}^{loc-const}(Y, \mathcal{A})$

(the $\mathcal{A}$-submodule of locally constant functions). Let also denote
$\mathcal{C}^{loc-an}(Y, \mathcal{A})$ the $\mathcal{A}$-module of locally analytic functions, so that $\mathcal{P}^1(Y, \mathcal{A}) \subset \mathcal{P}^h(Y, \mathcal{A}) \subset \mathcal{C}^{loc-an}(Y, \mathcal{A}) \subset \mathcal{C}(Y, \mathcal{A})$.

b) Let $V$ be a normed $\mathcal{A}$-module with the norm $| \cdot |_{p,V}$. For a given
positive integer $h$ an $h$-admissible measure on $Y$ with values in $V$ is an $A$-module homomorphism
\[
\tilde{\Phi} : \mathcal{P}^h(Y, A) \to V
\]
such that for fixed $a \in Y$ and for $v \to \infty$ the following growth condition is satisfied:
\[
\left| \int_{a + (Np^v)} (y_p - a_p)^{h'} d\tilde{\Phi} \right|_{p, V} = o(p^{-v(h'-h)}) \quad (5.10)
\]
for all $h' = 0, 1, \ldots, h - 1$, $a_p := y_p(a)$

The condition (5.10) allows to integrate only the locally-analytic functions: there exists a unique extension of $\tilde{\Phi}$ to $\mathcal{C}^{loc-an}(Y, A) \to V$ (via the embedding $\mathcal{P}^h(Y, A) \subset \mathcal{C}^{loc-an}(Y, A)$).

This condition allows to integrate locally-analytic functions on $Y$ along $\tilde{\Phi}$ using Taylor’s expansions!
5.1 $U_p$–Operator and the method of canonical projection

We construct an $h$-admissible measure $\widetilde{\Phi}^\lambda : \mathcal{P}^h(Y, \mathcal{A}) \to \mathcal{M}(\mathcal{A})$ out of a sequence of distributions $\Phi_r : \mathcal{P}^1(Y, \mathcal{A}) \to \mathcal{M}(\mathcal{A})$ on local monomials $y^r_p$ of each degree $r$ by

$$\int_{(a)_v} y^r_p d\widetilde{\Phi}^\lambda = \pi_{\lambda}(\Phi_r((a)_v)), \text{ where } \Phi_r((a)_v) \in M = \mathcal{M}(\mathcal{A}).$$

Here $\Phi_r$ take values in an $\mathcal{A}$-module

$$M = \mathcal{M}(\mathcal{A}) \subset \mathcal{A}[q_1, q_2, q_3][R_1, R_2, R_3]$$

of nearly holomorphic triple modular forms over $\mathcal{A}$ (for $0 \leq r \leq h - 1$, $h = \lceil 2\text{ord}_p \lambda_p \rceil + 1$)
Here $\mathcal{A}$ is an $\mathbb{C}_p$-algebra, and $\lambda \in \mathcal{A}^\times$ is a fixed non-zero eigenvalue of triple Atkin’s operator $U_T = U_{T,p}$, acting on $\mathcal{M}(\mathcal{A})$,

$$\pi_\lambda : \mathcal{M}(\mathcal{A}) \to \mathcal{M}(\mathcal{A})^\lambda$$

is the canonical projection operator onto the maximal $\mathcal{A}$-submodule $\mathcal{M}(\mathcal{A})^\lambda$ over which the operator $U_T - \lambda I$ is nilpotent (we call $\mathcal{M}(\mathcal{A})^\lambda$ the $\lambda$-characteristic submodule of $\mathcal{M}(\mathcal{A})$). The projector $\pi_\lambda$ is defined by its kernel:

$$\text{Ker} \pi_\lambda := \bigcap_{n \geq 1} \text{Im} (U_T - \lambda I)^n.$$
Triple modular forms are certain formal series

\[ g = \sum_{n_1, n_2, n_3 = 0}^{\infty} a(n_1, n_2, n_3; R_1, R_2, R_3) q_1^{n_1} q_2^{n_2} q_3^{n_3} \]

\[ \in \mathcal{A}[q_1, q_2, q_3][R_1, R_2, R_3], \text{ where } z_j = x_j + iy_j \in \mathbb{H}, \ R_j = (4\pi y_j)^{-1}, \]

with the property that for \( \mathcal{A} = \mathbb{C} \), the series converges to a \( \mathbb{C}\infty \)-modular form on \( \mathbb{H}^3 \) of a given weight \((k_1, k_2, k_3)\) and character \((\psi_1, \psi_2, \psi_3), \ j = 1, 2, 3\). The coefficients \( a(n_1, n_2, n_3; R_1, R_2, R_3) \) are polynomials in \( \mathcal{A}[R_1, R_2, R_3] \), and the triple Atkin’s operator is given by

\[ U_T(g) = \sum_{n_1, n_2, n_3 = 0}^{\infty} a(pn_1, pn_2, pn_3; pR_1, pR_2, pR_3) q_1^{n_1} q_2^{n_2} q_3^{n_3}. \]
Eigenfunctions of $U_p$ and of $U_p^*$.

Recall that for a primitive cusp eigenform $f_j = \sum_{n=1}^{\infty} a_n(f)q^n$, there is an eigenfunction $f_{j,0} = \sum_{n=1}^{\infty} a_n(f_{j,0})q^n \in \mathbb{Q}[q]$ of $U = U_p$ with the eigenvalue $\alpha = \alpha_{p,j}^{(1)} \in \mathbb{Q}$ ($U(f_0) = \alpha f_0$) given by

$$f_{j,0} = f_j - \alpha_{p,j}^{(2)} f_j|V_p = f_j - \alpha_{p,j}^{(2)} p^{-k/2} f_j \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$$  \hspace{1cm} (5.11)$$

$$\sum_{n=1}^{\infty} a_n(f_{j,0}) n^{-s} = \sum_{n=1}^{\infty} a_n(f_j) n^{-s} (1 - \alpha_{p,j}^{(1)} p^{-s} \n q^n. \hspace{1cm} (5.12)$$

Moreover, there is an eigenfunction $f_{j}^0$ of $U_p^*$ given by

$$f_{j}^0 = f_{j,0}^\rho \left|_k \begin{pmatrix} 0 & -1 \\ Np & 0 \end{pmatrix} \right., \text{ where } f_{j,0}^\rho = \sum_{n=1}^{\infty} a(n, f_0) q^n. \hspace{1cm} (5.12)$$

Therefore, $U_T(f_{1,0} \otimes f_{2,0} \otimes f_{3,0}) = \lambda(f_{1,0} \otimes f_{2,0} \otimes f_{3,0})$. 

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Let us describe in more detail critical values of the $L$ function $L(f_1 \otimes f_2 \otimes f_3, s, \chi)$.

For an arbitrary Dirichlet character $\chi \mod Np^v$ consider the following Dirichlet characters:

$$\chi_1 \mod Np^v = \chi, \quad \chi_2 \mod Np^v = \psi_2 \bar{\psi}_3 \chi,$$

$$\chi_3 \mod Np^v = \psi_1 \bar{\psi}_3 \chi, \quad \psi = \chi^2 \psi_1 \psi_2 \bar{\psi}_3;$$

later on we impose the condition that the conductors of the corresponding primitive characters $\chi_{0,1}, \chi_{0,2}, \chi_{0,3}$ are $Np$-completes (i.e. have the same prime divisors as resp. those of $Np$).
Theorem A (algebraic properties of the triple product)
Assume that $k_2 + k_3 - k_1 \geq 2$, then for all pairs $(\chi, r)$ such that the corresponding Dirichlet characters $\chi_j$ have $Np$-complete conductors, and $0 \leq r \leq k_2 + k_3 - k_1 - 2$, we have that

$$\Lambda(f^\rho_1 \otimes f^\rho_2 \otimes f^\rho_3, k_2 + k_3 - 2 - r, \psi_1 \psi_2 \chi)$$

$$\frac{\langle f^\rho_1 \otimes f^\rho_2 \otimes f^\rho_3, f^\rho_1 \otimes f^\rho_2 \otimes f^\rho_3 \rangle_T}{\langle f^\rho_1 \otimes f^\rho_2 \otimes f^\rho_3, f^\rho_1 \otimes f^\rho_2 \otimes f^\rho_3 \rangle_T} \in \overline{\mathbb{Q}}$$

where

$$\langle f^\rho_1 \otimes f^\rho_2 \otimes f^\rho_3, f^\rho_1 \otimes f^\rho_2 \otimes f^\rho_3 \rangle_T := \langle f^\rho_1, f^\rho_1 \rangle_N \langle f^\rho_2, f^\rho_2 \rangle_N f^\rho_3, f^\rho_3 \rangle_N$$

$$= \langle f_1, f_1 \rangle_N \langle f_2, f_2 \rangle_N f_3, f_3 \rangle_N.$$
Fix a positive integer $N$, a Dirichlet character $\psi \mod N$ and consider the commutative profinite group

$$Y = Y_{N,p} = \lim_{\leftarrow v} (\mathbb{Z}/Np^v\mathbb{Z})^*$$

and its group $X_{N,p} = \text{Hom}_{\text{cont}}(Y, \mathbb{C}_p^\times)$ of (continuous) $p$-adic characters (this is a $\mathbb{C}_p$-analytic Lie group).

The group $X_{N,p}$ is isomorphic to a finite union of discs $U = \{z \in \mathbb{C}_p \mid |z|_p < 1\}$.

A $p$-adic $L$-function $L_{(p)} : X_{N,p} \to \mathbb{C}_p$ is a certain meromorphic function on $X_{N,p}$. Such a function often come from a $p$-adic measure $\mu_{(p)}$ on $Y$ (bounded or admissible in the sense of Amice-Vélu, see [Am-V]). The $p$-adic Mellin transform of $\mu_{(p)}$ is given for all $x \in X_{N,p}$ by

$$L_{(p)}(x) = \int_{Y_{N,p}} x(y) d\mu_{(p)}(y), \quad L_{(p)}(x) : X \to \mathbb{C}_p$$
Theorem B (on admissible measures attched to the triple product: fixed balanced weights case). Under the assumptions as above there exist a $\mathbb{C}_p$-valued measure $\tilde{\mu}_{f_1 \otimes f_2 \otimes f_3}^\lambda$ on $Y_{N,p}$, and a $\mathbb{C}_p$-analytic function $D_{(p)}(x, f_1 \otimes f_2 \otimes f_3) : X_p \to \mathbb{C}_p$, given for all $x \in X_{N,p}$ by the integral $D_{(p)}(x, f_1 \otimes f_2 \otimes f_3) = \int_{Y_{N,p}} x(y) d\tilde{\mu}_{f_1 \otimes f_2 \otimes f_3}^\lambda(y)$, and having the following properties:
(i) for all pairs $(r, \chi)$ such that $\chi \in X_{N,p}^{\text{tors}}$, and all three corresponding Dirichlet characters $\chi_j$ have $Np$-complete conductor $(j = 1, 2, 3)$, and $r \in \mathbb{Z}$ is an integer with $0 \leq r \leq k_2 + k_3 - k_1 - 2$, the following equality holds:
$$D_{(p)}(\chi x_p^r, f_1 \otimes f_2 \otimes f_3) = i_p \left( \frac{(\psi_1 \psi_2)(2)C_\chi^{4(k_2+k_3-2-r)}}{G(\chi_1)G(\chi_2)G(\chi_3)G(\psi_1 \psi_2 \chi_1)\lambda_p^{2v}} \right) \Lambda(f_1^\rho \otimes f_2^\rho \otimes f_3^\rho, k_2 + k_3 - 2 - r, \psi_1 \psi_2 \chi)$$
$$\langle f_1^0 \otimes f_2^0 \otimes f_3^0, f_1,0 \otimes f_2,0 \otimes f_3,0 \rangle_{T,N_p}$$
where $v = \text{ord}_p(C_\chi)$, $G(\chi)$ denotes the Gauß sum of a primitive
Dirichlet character $\chi_0$ attached to $\chi$ (modulo the conductor of $\chi_0$),
(ii) if $\text{ord}_p \lambda_p = 0$ then the holomorphic function in (i) is a bounded $\mathbb{C}_p$-analytic function;
(iii) in the general case (but assuming that $\lambda_p \neq 0$) the holomorphic function in (i) belongs to the type $o(\log(x^H_p))$ with $H = [2\text{ord}_p \lambda_p] + 1$ and it can be represented as the Mellin transform of the $H$-admissible $\mathbb{C}_p$-valued measure $\tilde{\mu}_f^\lambda \otimes f_2 \otimes f_3$ (in the sense of Amice-Vélu) on $Y$
(iv) Let $k = k_2 + k_3 - k_1 - 2$. If $H \leq k - 2$ then the function $D_{(p)}$ is uniquely determined by the above conditions (i).
Let us describe now a $p$-adic measures attached to Garrett’s triple product of three Coleman’s families

$$k_j' \mapsto \{ f_{j,k_j'} = \sum_{n=1}^{\infty} a_{n,j}(k_j')q^n \}(j = 1, 2, 3). \quad (5.14)$$

Consider the product of three eigenvalues:

$$\lambda = \lambda_p(k_1', k_2', k_3') = \alpha_{p,1}^{(1)}(k_1')\alpha_{p,2}^{(1)}(k_2')\alpha_{p,3}^{(1)}(k_3')$$

and assume that the slope of this product

$$\sigma = \text{ord}_p(\lambda(k_1', k_2', k_3')) = \sigma(k_1', k_2', k_3') = \sigma_1 + \sigma_2 + \sigma_3$$

is constant and positive for all triplets $(k_1', k_2', k_3')$ in an appropriate $p$-adic neighbourhood of the fixed triplet of weights $(k_1, k_2, k_3)$.

Let $\mathcal{A} = \mathcal{A}(\mathcal{B})$ denote an affinoid algebra $\mathcal{A}$ associated with an analytic space $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3$, a neighbourhood around $(k_1, k_2, k_3) \in X^3$ (with a given $k$ and $\psi \mod N$).
Theorem C (on \( p \)-adic measures for families of triple products)  
Put \( H = [2\text{ord}_p(\lambda)] + 1 \). There exists a sequence of distributions \( \Phi_r \) on \( Y \) with values in \( \mathcal{M} = \mathcal{M}(A) \) giving an \( H \)-admissible measure \( \tilde{\Phi}^\lambda \) with values in \( \mathcal{M}^\lambda \subset \mathcal{M} \) with the following properties:

There exists an \( A \)-linear form \( \ell = \ell_{f_1 \otimes f_2 \otimes f_3, \lambda} : \mathcal{M}(A)^\lambda \to A \) (given by (5.16), such that the composition

\[
\tilde{\mu} = \tilde{\mu}_{f_1 \otimes f_2 \otimes f_3, \lambda} := \ell_{f_1 \otimes f_2 \otimes f_3, \lambda}(\tilde{\Phi}^\lambda)
\]

is an \( H \)-admissible measure with values in \( A \), and for all \( (k'_1, k'_2, k'_3) \) in the affinoid neighborhood \( \mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3 \), as above, satisfying

\[
k'_1 \leq k'_2 + k'_3 - 2
\]

we have that the evaluated integrals

\[
e_{\nu(k'_1, k'_2, k'_3)}\left( (\ell_{f_1 \otimes f_2 \otimes f_3, \lambda})(\tilde{\Phi}^\lambda)(y_p r \chi) \right)
\]

on the arithmetical characters \( y_p r \chi \) coincide with the critical special
values

$$\Lambda^*(f_{1,k_1'} \otimes f_{2,k_2'} \otimes f_{3,k_3'}, k_2' + k_3' - 2 - r, \chi)$$

for $r = 0, \cdots, k_2' + k_3' - k_1' - 2$, and for all Dirichlet characters $\chi \mod Np^v, v \geq 1$, with all three corresponding Dirichlet characters $\chi_j$ given by (5.13), having $Np$-complete conductors. Here the normalisation of $\Lambda^*$ includes at the same time certain Gauss sums, Petersson scalar products, powers of $\pi$ and of $\lambda(k_1', k_2', k_3')$, and a certain finite Euler product.
The $p$-adic Mellin transform and four variable $p$-adic analytic functions

Any $h$-admissible measure $\tilde{\mu}$ on $Y$ with values in a $p$-adic Banach algebra $A$ can be characterized its Mellin transform $\mathcal{L}_{\tilde{\mu}}(x)$ $\mathcal{L}_{\tilde{\mu}} : X \to A$, defined by $\mathcal{L}_{\tilde{\mu}}(x) = \int_Y x(y) d\tilde{\mu}(y)$, where $x \in X$, $\mathcal{L}_{\tilde{\mu}}(x) \in A$.

Key property of $h$-admissible measures $\tilde{\mu}$: its Mellin transform $\mathcal{L}_{\tilde{\mu}}$ is analytic with values in $A$.

Let $A = A(B) = A_1 \hat{\otimes} A_2 \hat{\otimes} A_3 = A(B_1) \hat{\otimes} A(B_2) \hat{\otimes} A(B_3)$ denote again the Banach algebra $A$ where $B$ is an affinoid neighbourhood around $(k_1, k_2, k_3) \in X^3$ (with a given integer $k$ and Dirichlet character $\psi \mod N$).
**Theorem D** (on $p$-adic analytic function in four variables)

Put $H = [2\text{ord}_p(\lambda)] + 1$. There exists a $p$-adic analytic function in four variables $(x, s) \in X \times B_1 \times B_2 \times B_3 \subset X \times X \times X \times X$:

$$
\mathcal{L}_{\bar{\mu}} : (x, s) \mapsto ev_s(\mathcal{L}_{\bar{\mu}}(x)) \quad (x \in X, \quad \mathcal{L}_{\bar{\mu}}(x) \in A).
$$

with values in $\mathbb{C}_p$, such that for all $(k_1', k_2', k_3')$ in the affinoid neighborhood as above $\mathcal{B} = B_1 \times B_2 \times B_3$, satisfying $k_1' \leq k_2' + k_3' - 2$, we have that the special values $\mathcal{L}_{\bar{\mu}}(x, s)$ at the arithmetical chracters $x = y_p^r \chi$, and $s = (k_1', k_2', k_3') \in \mathcal{B}$ coincide with the normalized critical special values

$$
\Lambda^*(f_{1,k_1'} \otimes f_{2,k_2'} \otimes f_{3,k_3'}, k_2' + k_3' - 2 - r, \chi) \quad (r = 0, \cdots, k_2' + k_3' - k_1' - 2),
$$

for Dirichlet characters $\chi \mod Np^v, v \geq 1$, such that all three corresponding Dirichlet characters $\chi_j$ given by (5.13), have $Np$-complete conductors where the same normalisation of $\Lambda^*$ as in Theorem C.
Moreover, for any fixed $s = (k'_1, k'_2, k'_3) \in \mathcal{B}$ the function

$$x \mapsto \mathcal{L}_{\tilde{\mu}}(x, s)$$

is $p$-adic analytic of type $o(\log^H(\cdot))$.

Indeed, we obtain the function in question $\mathcal{L}_\mu(x, s)$ by evaluation at $s = ((s_1, \psi_1), (s_2, \psi_2), (s_3, \psi_3)) \in \mathcal{B}$: this is a $p$-adic analytic function in four variables $(x, s) \in X \times \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3 \subset X \times X \times X \times X$:

$$\mathcal{L}_{\tilde{\mu}}(x, s) := ev_s(\mathcal{L}_{\tilde{\mu}})(x) \quad (x \in X, \ s \in \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3, \ \mathcal{L}_{\tilde{\mu}}(x) \in \mathcal{A}).$$
This is a joint work in progress with S. Boecherer, we use:

1) the higher twists of the Siegel-Eisenstein series, studied in [PaSE],
2) Ibukiyama’s differential operators (see [Ibu], [BSY]).

5.2 Scheme of the Proof

1) A crucial point of our construction is the higher twist. We define
the higher twist of the series $F_{\chi,r}$ by the characters (5.13) as the
following formal nearly holomorphic Fourier expansion:

$$F_{\bar{\chi}_1, \bar{\chi}_2, \bar{\chi}_3, \chi, r} = \sum_{\mathcal{J}} \bar{\chi}_1(t_{12}) \bar{\chi}_2(t_{13}) \bar{\chi}_3(t_{23}) \det(\mathcal{J})^{k-2r-\kappa} Q(R, \mathcal{J}; k - 2r, r) a_{\chi, r}(\mathcal{J}) q^{\mathcal{J}}.$$

Here for an arbitrary Dirichlet character $\chi \mod Np^v$ we consider
the Dirichlet characters (5.13):

\[ \chi_1 \mod Np^v = \chi, \ \chi_2 \mod Np^v = \psi_2 \bar{\psi}_3 \chi, \]

\[ \chi_3 \mod Np^v = \psi_1 \bar{\psi}_3 \chi, \psi = \chi^2 \psi_1 \psi_2 \bar{\psi}_3. \]

We use the Siegel-Eisenstein series \( F_{\chi,r} \) which depends on the character \( \chi \), but its precise nebentypus character is \( \psi = \chi^2 \psi_1 \psi_2 \bar{\psi}_3 \), and it is defined by \( F_{\chi,r} = G^* (z, -r; k, (Np^v)^2, \psi) \), where \( z \) denotes a variable in the Siegel upper half space \( \mathbb{H}_3 \), and the normalized series \( G^*(z, s; k, (Np^v)^2, \psi) \) is given by (4.6).

This series depends on \( s = -r \), and for the critical values at integral points \( s \in \mathbb{Z} \) such that \( 2 - k \leq s \leq 0 \), it represents a (nearly) holomorphic Siegel modular form in the sense of Shimura [ShiAr]:

\[ F_{\chi,r} = \sum_{\mathcal{J}} \det(\mathcal{J})^{k-2r-\kappa} Q(R, \mathcal{J}; k - 2r, r)a_{\chi,r}(\mathcal{J}) q^{\mathcal{J}}. \]
2) We use an algebraic version of Ibukiyama’s differential operator, which generalizes the algebraic “pull-back”: it transforms a nearly holomorphic Siegel modular form of weight $k'$ to a nearly holomorphic triple modular form of weight $(k'_1, k'_2, k'_3)$ ($k' = k'_2 + k'_3 - k'_1$).

On a holomorphic Siegel modular form $F = \sum_{\mathcal{J}} a(\mathcal{J})q^{\mathcal{J}}$, this operator has the form

$$\mathcal{L}_{k',\nu'}^{\lambda'}(F) = \sum_{t_1, t_2, t_3 \geq 0} \sum_{\mathcal{J}: t_{11} = t_1, t_{22} = t_2, t_{23} = t_3} \mathcal{P}(k'_1, k'_2, k'_3, \mathcal{J}) a(\mathcal{J}) q_1^{t_1} q_2^{t_2} q_3^{t_3},$$

where $\lambda' = k'_1 - k'_3 \geq \mu' = k'_1 - k'_2 \geq 0$, and $\mathcal{P}(k'_1, k'_2, k'_3; r; \mathcal{J}) = (t_{11}t_{22}t_{33})^{\lambda'}(t_{12}t_{13}t_{23})^{\mu'}$ is certain Ibukiyama’s polynomial.
As a result we obtain a sequence of triple modular distributions \( \Phi_r(\chi) \) with values in the tensor product 
\[ \mathcal{M}_T(\mathcal{A}) = \mathcal{M}(\mathcal{A}) \widehat{\otimes}_A \mathcal{M}(\mathcal{A}) \widehat{\otimes}_A \mathcal{M}(\mathcal{A}) \]
of three Banach \( \mathcal{A} \)-modules of arithmetical nearly holomorphic modular forms (the normalizing factor \( 2^r \) is needed in order to prove certain congruences between \( \Phi_r \)). Note that \( \mathcal{M}_T(\mathcal{A}) \) is again a Banach \( \mathcal{A} \)-module on which \( U_T \) acts as a completely continuous operator.

The important property of the triple modular forms \( \Phi_r(\chi) \): the nebentypus character is fixed and is equal to \((\psi_1, \psi_2, \psi_3)\) (for all \((k'_1, k'_2, k_3)\) and \(\chi\) in question).

Using this property we compute the canonical projection \( \pi_\lambda(\Phi_r(\chi)) \) of the triple modular form \( \Phi_r(\chi) \) onto the \( \lambda \)-characteristic \( \mathcal{A} \)-submodule \( \mathcal{M}_T^\lambda(\mathcal{A}) \) of the triple Atkin’s operator \( U_{T,p} \):

\[ \pi_\lambda : \mathcal{M}_T(\mathcal{A}) \to \mathcal{M}_T^\lambda(\mathcal{A}). \]
We prove that the resulting sequence of modular distributions \( \pi_\lambda(\Phi_r) \) on the profinite group \( Y \) produces a certain \( p \)-adic admissible measure \( \tilde{\Phi}^\lambda \) (in the sense of Amice-Vélu, [Am-V]) with values in a certain locally free \( \mathcal{A} \)-submodule of finite rank

\[
\mathcal{M}_T^\lambda(\mathcal{A}) \subset \mathcal{M}_T(\mathcal{A}) \subset \bigcup_{v \geq 0} \mathcal{M}_T(Np^v, \psi_1, \psi_2, \psi_3; \mathcal{A})
\]

of formal nearly holomorphic triple modular forms of all levels \( Np^v \) and the fixed nebentypus characters \((\psi_1, \psi_2, \psi_3)\). We use congruences between triple modular forms \( \Phi_r(\chi) \in \mathcal{M}_T(\mathcal{A}) \) (they have explicit formal Fourier coefficients).

Then we use a general admissibility criterion saying that these congruences imply \( H \)-admissibility for their projections in \( \mathcal{M}_T^\lambda(\mathcal{A}) \), where \( H = [2\lambda] + 1 \).
3) From $\mathcal{M}_T(\mathcal{A})$ to $\mathcal{A}$: we use a $\overline{\mathbb{Q}}$-valued linear forms of type

$$\mathcal{L} : h \mapsto \frac{\langle \tilde{f}_1 \otimes \tilde{f}_2 \otimes \tilde{f}_3, h \rangle}{\langle \tilde{f}_1, \tilde{f}_1 \rangle \langle \tilde{f}_2, \tilde{f}_2 \rangle \langle \tilde{f}_3, \tilde{f}_3 \rangle}$$

defined on the finite dimensional $\overline{\mathbb{Q}}$-vector characteristic subspace

$$h \in \mathcal{M}_{k'}(\overline{\mathbb{Q}})^{\lambda(k')} \subset \mathcal{M}_{k_1, r^*}(Np, \psi_1; \overline{\mathbb{Q}}) \otimes \mathcal{M}_{k_2, r^*}(Np, \psi_2; \overline{\mathbb{Q}}) \otimes \mathcal{M}_{k_3, r^*}(Np, \psi_3; \overline{\mathbb{Q}}).$$

This map is then extended to an $\mathcal{A}$-linear map

$$\ell = \ell_{f_1 \otimes f_2 \otimes f_3, \lambda} : \mathcal{M}(\mathcal{A})^\lambda \rightarrow \mathcal{A}; \quad (5.16)$$

on the locally free $\mathcal{A}$-module of finite rank $\mathcal{M}(\mathcal{A})^\lambda$. 

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This map produces a sequence of $\mathcal{A}$-valued distributions $\mu_r^\lambda(\chi) \in \mathcal{A}$ in such a way that for all suitable weights $\mathbf{k}' \in \mathcal{B}$ one has

$$ev_{\mathbf{k}'}(\mu_r^\lambda(\chi)) = \mathcal{L}(ev_{\mathbf{k}'}(\pi_\lambda(\Phi_r)(\chi))), \lambda \in \mathcal{A}^\times, \lambda(\mathbf{k}') \in \overline{\mathbb{Q}}^\times,$$

where $\mathbf{k}' = (k'_1, k'_2, k'_3) \in \mathcal{B}$, $ev_{\mathbf{k}'} : \mathcal{B} \to \mathbb{C}_p$ denotes the evaluation map with the property

$$ev_{\mathbf{k}'} : \mathcal{M}(\mathcal{A}) \to \mathcal{M}_{\mathbf{k}'}(\mathbb{C}_p).$$
More precisely, we consider three auxiliary families of modular forms

\[ \tilde{f}_{j,k'}'(z) = \sum_{n=1}^{\infty} \tilde{a}_{n,j,k'} e(nz) \in S_{k'}(\Gamma_0(N_j p^{\nu_j}), \psi_j), \quad (1 \leq j \leq 3, \nu_j \geq 1), \]

with the same eigenvalues as those of (5.14), for all Hecke operators \( T_q \), with \( q \) prime to \( Np \). In our construction we use as \( \tilde{f}_{j,k'}' \) certain “easy transforms” of primitive cusp forms in (1.1). In particular, we choose as \( \tilde{f}_{j} \) certain eigenfunctions \( \tilde{f}_{j,k'} = f_{j,k'}^0 \) of the adjoint Atkin’s operator \( U_p^* \), in this case we denote by \( f_{j,k'}^0 \) the corresponding eigenfunctions of \( U_p \).

The \( \overline{Q} \)-linear form \( \mathcal{L} \) produces a \( \mathbb{C}_p \)-valued admissible measure \( \tilde{\mu}^\lambda = \ell(\tilde{\Phi}^\lambda) \) starting from the modular \( p \)-adic admissible measure \( \tilde{\Phi}^\lambda \) of stage 3), where \( \ell : \mathcal{M}_T(\mathbb{C}_p) \to \mathbb{C}_p \) denotes a \( \mathbb{C}_p \)-linear form, interpolating \( \mathcal{L} \).
4) We show that for any appropriate Dirichlet character \( \chi \mod Np^v \) the integral
\[
\mu^\lambda_r(\chi) = \mathcal{L}(\pi_\lambda(\Phi_r(\chi))) \in \mathcal{A}
\]
evaluated at \((k'_1, k'_2, k'_3) \in \mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3\), coincides (up to a normalisation) with the special \( L \)-value
\[
\mathcal{D}^*(f^\rho_{1,k'_1} \otimes f^\rho_{2,k'_2} \otimes f^\rho_{3,k'_3}, k'_2 + k'_3 - 2 - r, \psi_1 \psi_2 \chi)
\]
under the above assumptions on \( \chi \) and \( r \).
We use a general integral representation of Garrett’s type. The basic idea how a Dirichlet character $\chi$ is incorporated in the integral representation [Ga87, BoeSP] is somewhat similar to the one used in [Boe-Schm], but (surprisingly) more complicated to carry out.

Note however that the existence of a $\mathcal{A}$-valued admissible measure $\tilde{\mu}^\lambda = \ell(\tilde{\Phi}^\lambda)$ already established at stage 4), does not depend on this technical computation.
6 Criterion of admissibility

**Theorem 6.1** Let $0 < |\alpha|_p < 1$ and suppose that there exists a positive integer $\kappa$ such that the following conditions are satisfied: for all $r = 0, 1, \cdots, h - 1$ with $h = [\text{ord}_p \alpha] + 1$, and $v \geq 1$,

$$\Phi_r(a + (Np^v)) \in \mathcal{M}(Np^{\kappa v})$$  \hspace{1cm} (the level condition) \hspace{1cm} (6.1)

and the following congruence holds: for all $t = 0, 1, \cdots, h - 1$

$$U^{\kappa v} \sum_{r=0}^{t} \binom{t}{r} (-a_p)^{t-r} \Phi_r(a + (Np^v)) \equiv 0 \mod p^{vt} \hspace{1cm} (6.2)$$

\hspace{1cm} (the divisibility condition)

Then the linear form given by $\tilde{\Phi}^\alpha(\delta_{a+(Np^v)y_p^r}) := \pi_{\alpha}(\Phi_r(a + (Np^v)))$ on local monomials (for all $r = 0, 1, \cdots, h - 1$), is an $h$-admissible measure: $\tilde{\Phi}^\alpha : \mathcal{P}^h(Y, \mathbb{Q}) \rightarrow \mathcal{M}^\alpha \subset \mathcal{M}$
Proof uses the commutative diagram:

\[
\begin{array}{ccc}
\mathcal{M}(Np^{v+1}, \psi; \mathcal{A}) & \xrightarrow{\pi_{\alpha,v}} & \mathcal{M}^\alpha(Np^{v+1}, \psi; \mathcal{A}) \\
\downarrow U^v & & \downarrow \lambda U^v \\
\mathcal{M}(Np, \psi; \mathcal{A}) & \xrightarrow{\pi_{\alpha,0}} & \mathcal{M}^\alpha(Np, \psi; \mathcal{A}) = \mathcal{M}^\alpha(Np^{v+1}, \psi; \mathcal{A}).
\end{array}
\]

The existence of the projectors $\pi_{\alpha,v}$ comes from Coleman’s Theorem A.4.3 [CoPB].

On the right: $U$ acts on the locally free $\mathcal{A}$-module $\mathcal{M}^\alpha(Np^{v+1}, \mathcal{A})$ via
the matrix:
\[
\begin{pmatrix}
\alpha & \cdots & \cdots & \ast \\
0 & \alpha & \cdots & \ast \\
0 & 0 & \ddots & \cdots \\
0 & 0 & \cdots & \alpha
\end{pmatrix}
\]
where \( \alpha \in \mathcal{A}^\times \)

\( \Longrightarrow U^v \) is an isomorphism over \( \text{Frac}(\mathcal{A}) \),

and one controls the denominators of the modular forms of all levels \( v \) by the relation:

\[
\pi_{\alpha,v}(h) = U^{-v} \pi_{\alpha,0}(U^v h) =: \pi_{\alpha}(h)
\]  

(6.3)

The equality (6.3) can be used as the definition of \( \pi_{\alpha} \). The growth condition (see section 5) for \( \pi_{\alpha}(\Phi_r) \) is deduced from the congruences (6.2) between modular forms, using the relation (6.3).
7 Main congruence for the higher twists

The purpose of this section is to show that the admissibility criterion of Theorem 6.1 with $\kappa = 2$ is satisfied by the sequence of triple modular distributions $\Phi_r$.

We need to check that the nearly holomorphic triple modular forms $\Phi_r(\chi)$ are of level $N^2\chi^{2v}$, nebentypus $(\psi_1, \psi_2, \psi_3)$, and satisfy the congruences

$$\left| U_T^{2v} \left( \sum_{r'=0}^{r} \binom{r}{r'} (-a_p^0)^{r-r'} \Phi_{r'}((a)_v) \right) \right|_p \leq Cp^{-vr} \quad (7.1)$$

and for all $r = 0, 1, \cdots, k - 2$. 
7.1 Special Fourier coefficients of the higher twist of the Siegel-Eisenstein distributions

Let us use the Fourier expansions of $\Phi_r(\chi)|U_p^{2v}$ as follows

$$\Phi_r(\chi)|U_p^{2v} = \sum_{t_1, t_2, t_3 \geq 0} a(p^{2v}t_1, p^{2v}t_2, p^{2v}t_3; p^{2v}R_1, p^{2v}R_2, p^{2v}R_3, r)q_1^{t_1}q_2^{t_2}q_3^{t_3}$$

with

$$a(p^{2v}t_1, p^{2v}t_2, p^{2v}t_3; p^{2v}R_1, p^{2v}R_2, p^{2v}R_3, r) = \sum_{\mathcal{J}: \text{diag}(\mathcal{J}) = (p^{2v}t_1, p^{2v}t_2, p^{2v}t_3)} \bar{\chi}(t_{12}t_{13}t_{23})\bar{\psi}_2\psi_3(t_{13})\bar{\psi}_1\psi_3(t_{23}) \times$$

$$\det(\mathcal{J})^{k-2r-\kappa}Q(p^{2v}\text{diag}(R_1, R_2, R_3), \mathcal{J}; k - 2r, r)2^r a_{\chi, r}(\mathcal{J})$$

$$= \sum_{\mathcal{J}: \text{diag}(\mathcal{J}) = (p^{2v}t_1, p^{2v}t_2, p^{2v}t_3)} v_{\chi, r}(\mathcal{J}, \text{diag}(R_1, R_2, R_3)),$$
where

\[
v_{\chi, r}(\mathcal{J}, \text{diag}(R_1, R_2, R_3)) = \bar{\chi}(t_{12} t_{13} t_{23}) \bar{\psi}_2 \psi_3(t_{13}) \bar{\psi}_1 \psi_3(t_{23}) \times (7.3)
\]

\[
\times \det(\mathcal{J})^{k-2r-\kappa} Q(p^{2v} \text{diag}(R_1, R_2, R_3), \mathcal{J}; k-2r, r) 2^r a_{\chi, r}(\mathcal{J})
\]

\[
= \chi^{(p)}(2) \bar{\chi}^{(p)}(\mathcal{J}) \chi^\circ(t_{12} t_{13} t_{23}) \bar{\psi}_2 \psi_3(t_{13}) \bar{\psi}_1 \psi_3(t_{23}) \times
\]

\[
\times \det(\mathcal{J})^{k-2r-\kappa} Q(p^{2v} \text{diag}(R_1, R_2, R_3), \mathcal{J}; k-2r, r) 2^r a_{\chi, r}(\mathcal{J}).
\]

Let us notice that, for any \( \mathcal{J} \) with \( \text{diag}(\mathcal{J}) = (p^{2v} t_1, p^{2v} t_2, p^{2v} t_3) \) one has \( \det(\mathcal{J}) \equiv 2t_{12} t_{13} t_{23} \mod p^{2v} \), \( \chi^{(p)}(2t_{12} t_{13} t_{23}) = \chi^{(p)}(\det(\mathcal{J})) = \chi(\det(\mathcal{J}) \chi^\circ(\det(\mathcal{J})), \]

\[
2^r a_{\chi, r}(\mathcal{J}) = \int_Y y_p^r \chi(y) d\mathcal{F}_\mathcal{J},
\]

with \( \chi = \chi^{(p)} \chi^\circ, \chi^{(p)} \mod p^v, \chi^\circ \mod N \), and \( p \nmid N \),

for a bounded measure \( \mathcal{F}_\mathcal{J} \) on \( Y \) with values in \( \overline{\mathbb{Q}} \).
It follows that

\[ v_{\chi,r}(\mathcal{T}, \text{diag}(R_1, R_2, R_3)) = \]

\[ \chi^{(p)}(2)\bar{\chi}(\det(\mathcal{T})) \det(\mathcal{T})^{-r} \chi^{\circ}(\det(\mathcal{T}) \bar{\psi}_2 \psi_3(t_{13}) \bar{\psi}_1 \psi_3(t_{23}) \cdot \det(\mathcal{T})^{k-r-\kappa} Q(p^{2v} \text{diag}(R_1, R_2, R_3), \mathcal{T}; k - 2r, r) 2^r a_{\chi,r}(\mathcal{T}) \]

\[ = \det(\mathcal{T})^{k-r-\kappa} Q(p^{2v} \text{diag}(R_1, R_2, R_3), \mathcal{T}; k - 2r, r) \chi^{\circ}(2) \]

\[ \int_Y \chi y_p^r d\mathcal{F}_{\mathcal{I}; \chi^{\circ}, \psi_1, \psi_2, \psi_3}, \]

where \( \mathcal{F}_{\mathcal{I}; \chi^{\circ}, \psi_1, \psi_2, \psi_3} \) denotes the bounded measure defined by the equality:

\[ \int_Y \chi y_p^r d\mathcal{F}_{\mathcal{I}; \chi^{\circ}, \psi_1, \psi_2, \psi_3} \]

\[ = \chi^{(p)}(2)\chi^{\circ}(2) 2^r \bar{\chi}(\det(\mathcal{T})) \det(\mathcal{T})^{-r} \chi^{\circ}(\det(\mathcal{T}) \bar{\psi}_2 \psi_3(t_{13}) \bar{\psi}_1 \psi_3(t_{23}) a_{\chi,r}(\mathcal{T}). \]
7.2 Main congruence for the Fourier expansions

Let us use the orthogonality relations for Dirichlet characters in order to prove the admissibility of the distributions given by the sequence $\pi_\lambda(\Phi_r(\chi))$ using the Fourier expansions (7.2). According to the admissibility criterion of Theorem 6.1 we need to check the following main congruence:

\[
\left| \sum_{r'=0}^{r} \binom{r}{r'} (-a^0_p)^{r-r'} \right| \leq C p^{-vr},
\]

where we use the notation (7.4) for $v_{\chi,r'}(\mathcal{T}, \text{diag}(R_1, R_2, R_3))$, implying that the coefficients

\[
i_p(v_{\chi,r'}(\mathcal{T}, \text{diag}(R_1, R_2, R_3)))
\]
in (7.3) are given as sums of the following expressions:

\[ B_r(\chi, \mathcal{T}) = \overline{\chi}(2) \det(\mathcal{T})^{k-r-\kappa} \int_Y \chi y_p^T d\mathcal{F}_{\mathcal{T};\chi^0,\psi_1,\psi_2,\psi_3}. \]  

(7.7)

\[ \cdot \sum_{t=0}^{r} \binom{r}{t} \det(\mathcal{T})^{r-t} \sum_{|L| \leq mt-t} R_L(k - k + r) Q_L(p^{2\nu} \text{diag}(R_1, R_2, R_3), \mathcal{T}), \]

where \( \mathcal{F}_{\mathcal{T};\chi^0,\psi_1,\psi_2,\psi_3} \) denotes the bounded measure defined by (7.5). Using the expressions (7.7), the main congruence (7.6) is reduced to proving the congruence for the numbers \( B_r(\chi, \mathcal{T}) \): there exists a non-zero integer \( C_k \) such that

\[ C_k \cdot \sum_{r'=0}^{r} \binom{r}{r'} (-a_p^0)^{r-r'} \frac{1}{\varphi(Np^\nu)} \sum_{\chi \mod Np^\nu} \chi^{-1}(a) B_{r'}(\chi, \mathcal{T}) \equiv 0 \mod p^{vr} \]

(7.8)

\[ \iff C_k \cdot A \equiv 0 \mod Np^{vr}, \]
where we use the notation

\[
A = A_r(T; \chi^\circ, \psi_1, \psi_2, \psi_3) = \sum_{r'=0}^{r} \binom{r}{r'} (-a_p^0)^{r-r'} \frac{1}{\varphi(Np^v)} \sum_{\chi \mod Np^v} \chi^{-1}(a) \cdot \chi^\circ(2) \det(T)^{k-r'-\kappa} \int_Y \chi y_p^{r'} d\mathcal{F}_T; \chi^\circ, \psi_1, \psi_2, \psi_3 \sum_{t=0}^{r'} \binom{r'}{t} \det(T)^{r'-t} \sum_{|L| \leq mt-t} R_L(\kappa - k + r') Q_L(p^{2v} \text{diag}(R_1, R_2, R_3), T). 
\]

Note that \( R_L(\kappa - k + r') \) is a polynomial of degree \( mt - |L| = 3t - |L| \) in \( \kappa - k + r' \) (see (4.5)), hence in \( r' \), and \( \binom{r'}{t} \) is a polynomial of degree \( t \) in \( r' \). One can therefore write

\[
\binom{r'}{t} R_L(\kappa - k + r) = \sum_{n=0}^{4t-|L|} \mu_n \frac{(r' + n + 1)!}{(r' + 1)!}.
\]
Here the coefficients $\mu_n$ are certain fixed rational numbers (independent of $r'$).

Using the orthogonality relations for Dirichlet characters mod $Np^v$, we see that the sum over $r'$ in (7.9), denoted by $C = C_r(t, L, \mathcal{T}; \chi^\circ, \psi_1, \psi_2, \psi_3)$, takes the form

$$C_r(t, L, \mathcal{T}; \chi^\circ, \psi_1, \psi_2, \psi_3) = \overline{\chi^\circ}(2) \text{det}(\mathcal{T})^{k-t-\kappa}$$

$$\int_{y \equiv a \mod p^v} \sum_{n=0}^{4t-|L|} \mu_n \sum_{r'=0}^{r} \binom{r}{r'} (-a)^{r-r'} \frac{(r' + n + 1)!}{(r' + 1)!} y^{r'} d\mathcal{F}_{\mathcal{T}}(y)$$

$$y^{-n} \frac{\partial^n}{\partial y^n} (y^{n+1}(y - a)^r)$$

where $\mathcal{F}_{\mathcal{T}}(y) = \mathcal{F}_{\mathcal{T}; \chi^\circ, \psi_1, \psi_2, \psi_3}(y)$

Note that we write $\chi = \chi^\circ \chi^{(p)}$, fix $\chi^\circ$, and sommate over all characters $\chi^{(p)}$ mod $p^v$. We have therefore $(y - a)^r \equiv 0 \mod (p^v)^r$
in the integration domain $y \equiv a \mod p^v$, implying the congruence
\[
c_k C_r(t, L, \mathcal{T}; \chi^\circ, \psi_1, \psi_2, \psi_3) \equiv 0 \pmod{(p^v)^{r-n}} \tag{7.10}
\]
\[
\implies \equiv 0 \pmod{(p^v)^{r-4t+|L|}},
\]
where $c_k \in \mathbb{Q}^*$ is a nonzero constant coming from the denominators of the fixed rational numbers $\mu_n$, and of the bounded distributions $\mathcal{F}_{\mathcal{T}; \chi^\circ, \psi_1, \psi_2, \psi_3}$. 
7.3 Proof of the Main congruence

Now the expression (7.9) transforms to

\[ A_r(\mathcal{T}) = \sum_{t=0}^{r} \sum_{|L| \leq 2t} \det(\mathcal{T})^t \]  \hspace{1cm} (7.11)

\[ C(t, L, \mathcal{T}) \det(\mathcal{T})^{k-2r-\kappa} Q_L(p^{2v} \text{diag}(R_1, R_2, R_3), \mathcal{T}), \]

where \( Q_L(p^{2v} \text{diag}(R_1, R_2, R_3), \mathcal{T}) \) is a homogeneous polynomial of degree \( 3t - |L| \) in the variables \( R_{ij} \) implying the congruence

\[ Q_L(p^{2v} \text{diag}(R_1, R_2, R_3), \mathcal{T}) \equiv 0 \pmod{(p^{2v})^{(3t-|L|)}}. \] \hspace{1cm} (7.12)
On the other hand we know from the description (4.5) of the polynomial

\[
Q(R, \mathcal{I}) = Q(R, \mathcal{I}; k - 2r, r) = \\
\sum_{t=0}^{r} \binom{r}{t} \det(\mathcal{I})^{r-t} \sum_{|L| \leq 2t} R_L(\kappa - k + r)Q_L(R, \mathcal{I}),
\]

\[
Q_L(R, \mathcal{I}) = \text{tr} \left( t^{\rho_{3-l_1}}(R)\rho_{l_1}^*(\mathcal{I}) \right) \cdot \ldots \cdot \text{tr} \left( t^{\rho_{3-l_t}}(R)\rho_{l_t}^*(\mathcal{I}) \right),
\]

that \(2t - |L| \geq 0\) so we obtain the desired congruence as follows

\[
\begin{cases}
    c_k C_r(t, L, \mathcal{I}) \equiv 0 \pmod{(p^v)^{r-4t+|L|}} \\
    Q_L(p^{2v} \text{diag}(R_1, R_2, R_3), \mathcal{I}) \equiv 0 \pmod{(p^{2v})^{3t-|L|}}
\end{cases}
\]  

(7.13)

\[
\Rightarrow c_k A_r(\mathcal{I}) \equiv 0 \pmod{p^{vr}},
\]

since \(v(r - 4t + |L|) + 2v(3t - |L|) = vr + 2vt - v|L| \geq vr\), proving (7.6).
8 Computation of $p$-adic integrals and $L$-values

8.1 Construction of $p$-adic measures

Let $\mathcal{M} = \mathcal{M}_T(A) = \bigcup_{v \geq 0} \mathcal{M}_{r^*}(Np^v, \psi_1; A) \otimes_A \mathcal{M}_{k,r^*}(Np^v, \psi_2; A) \otimes_A \mathcal{M}_{k,r^*}(Np^v, \psi_3; A)$ be the $A$-module of nearly holomorphic triple modular forms with formal Fourier coefficients in the $\mathbb{C}_p$-Banach algebra $A = A(\mathcal{B})$, where $k' = (k'_1, k'_2, k'_3) \in \mathcal{B}$, $ev_{k'} : \mathcal{B} \to \mathbb{C}_p$ denotes the evaluation map with the property

$$ev_{k'} : \mathcal{M}(A) \to \mathcal{M}_{k'}(\mathbb{C}_p).$$
Let us define an $\mathcal{A}$-valued measure
\[
\tilde{\mu}^\lambda(y; f_1 \otimes f_2 \otimes f_3) : \mathcal{C}^{loc-an}(Y, \mathcal{A}) \to \mathcal{A}
\]
by applying a certain trilinear form $\ell_{T,\lambda} : \mathcal{M}(Np^v; \mathcal{A}) \to \mathcal{A}$
\[
\tilde{\mu}^\lambda(y; f_1 \otimes f_2 \otimes f_3) = \ell_{T,\lambda}(\tilde{\Phi}^\lambda)
\] (8.1)
to the $h$-admissible measure $\tilde{\Phi}^\lambda$ of Theorem 6.1 on $Y$ with values in
$\mathcal{M}(\mathcal{A})^\lambda \subset \mathcal{M}(Np; \mathcal{A})$. That $h$-admissible measure was defined as an
$\mathcal{A}$-linear map $\tilde{\Phi}^\lambda : \mathcal{P}^h(Y, \mathcal{A}) \to \mathcal{M}(\mathcal{A})^\lambda$ satisfying for any $(a)_\nu \subset Y$
and for all $r = 0, 1, \ldots, h - 1$ the following equality:
\[
\int_{(a)_\nu} y_p^r \, d\tilde{\Phi}^\lambda = \pi^\lambda(\Phi_r((a)_\nu)) \in \mathcal{M}(Np),
\]
where $h = [2\text{ord}_p \lambda(p)] + 1$, hence
\[
\int_{(a)_\nu} y_p^r \, d\tilde{\mu}^\lambda(y; f_1 \otimes f_2 \otimes f_3) = \ell_{T,\lambda} \left( \int_{(a)_\nu} y_p^r \, d\tilde{\Phi}^\lambda(y) \right). \quad (8.2)
\]
8.2 Evaluation of the integral

\[ ev_{k'}(\int \chi(y) y^r_p \, d\mu^\lambda(y; f_1 \otimes f_2 \otimes f_3)) = \int \chi(y) y^r_p \, d\bar{\mu}^\lambda(y; f_{1,k_1} \otimes f_{2,k_2} \otimes f_{3,k_3}) \]  

for \( r \in \mathbb{N} \), \( 0 \leq r \leq k_2' + k_3' - k_1' - 2 \). The result is given in terms of Garrett's triple \( L \) function

\[ \mathcal{D}^*(f_{1,k_1}^\rho \otimes f_{2,k_2}^\rho \otimes f_{3,k_3}^\rho, k_2' + k_3' - 2 - r, \psi_1 \psi_2 \chi) \]. Let us use the action of the involution \( W_{N_j} \) of the exact level \( N_j \) of
\[ f_j: \]
\[
\left. f_{j,k_j'} \right|_k W_{N_j} = \begin{pmatrix} 0 & -1 \\ N_j & 0 \end{pmatrix} = \gamma_j, k_j' \cdot f_{j,k_j}',
\]
\[
\left. f_{j,k_j'}^\rho \right|_k W_{N_j} = \begin{pmatrix} 0 & -1 \\ N_j & 0 \end{pmatrix} = \tilde{\gamma}_j, k_j' \cdot f_{j,k_j}',
\]

where \( f_{j,k_j'}^\rho (z) = \sum_{n=1}^{\infty} \tilde{a}_{n,j,k_j'} e(nz) \in S_{k_j'}(N_j, \bar{\psi}_j), \quad (8.4) \)

\( (j = 1, 2, 3) \) and \( \gamma_j, k_j' \) is the corresponding root number. \quad (8.5)
Recall the notation (5.11) and (5.12):

\[ f_{j,k_j',0} = f_{j,k_j} - \alpha^{(2)}_{p,j,k_j} f_{j,k_j}' | V_p = f_{j,k_j} - \alpha^{(2)}_{p,j,k_j} p^{-k/2} f_{j,k_j}' \left( \begin{array}{c} p \\ 0 \\ 1 \end{array} \right) \]

\[ f_{j,k_j}',0 = \sum_{n=1}^{\infty} a(n,f_0)q^n, \quad f_{j,k_j}' = f_{j,k_j}',0 |_k W_{Np} = f_{j,k_j}',0 |_k \left( \begin{array}{c} 0 \\ 0 \\ Np \\ 0 \end{array} \right). \]

**Proposition 8.1** Under the notations and assumptions as in Theorem C, the value of the integral (8.3) at \( k' = (k'_1, k'_2, k'_3) \) is given for \( 0 \leq r \leq k'_2 + k'_3 - k'_1 - 2 \) by the image under \( i_p \) of the following algebraic number

\[
T(k') \cdot \lambda^{-2v} \mathcal{L}_{Np}(-r) \cdot \mathcal{D}^*(f^\rho_{1,k'_1} \otimes f^\rho_{2,k'_2} \otimes f^\rho_{3,k'_3}, k'_2 + k'_3 - k'_1 - 2 - r, \psi_1 \psi_2 \chi) / \langle f^0_{1,k'_1} \otimes f^0_{2,k'_2} \otimes f^0_{3,k'_3}, f_{1,k'_1,0} \otimes f_{2,k'_2,0} \otimes f_{3,k'_3,0} \otimes \rangle T,N^2p^{2v},
\]
where

\[ T(k') = \]
\[ 2^{-r} \frac{((Np)^3/N_1N_2N_3)^{k/2} \gamma_{1,k'_1} \gamma_{2,k'_2} \gamma_{3,k'_3}}{N_1,1 N_1,2 N_1,3 G(\chi_1,0) G(\chi_2,0) G(\chi_3,0)} \times \]
\[ \times (Np^2v)^{k-2r} \frac{N^2p^{2v} \varphi(N^2p^{2v}) \varphi(Np^v)}{[\Gamma_0(N^2p^{2v}) : \Gamma(N^2p^{2v})]^3}, \]

\( \gamma_{j,k'_j} \) is the corresponding root number and the factor \( \mathcal{L}_{Np}(-r) \), given by (8.13).

### 8.3 Evaluation of the trilinear form

In the rest of this section we write for simplicity \( k_j, f_j \) and \( \lambda \) instead of \( k'_j, f_{k'_j}, \lambda(k') \).

In order to compute the \( p \)-adic integral, the next step of the proof uses computations similar to those in [Hi85], §4 and §7. More
precisely let us write the integral in the form

\[ \int_Y \chi(y) \, y_p^r \, d\tilde{\mu}_\lambda(y; f_1 \otimes f_2 \otimes f_3) = \sum_{a \in Y_v} \chi(a) \int_{(a)_v} y_p^r \, d\ell_{T,\lambda}(\tilde{\Phi}^\lambda(y)) = \]

\[ = \ell_{T,\lambda} \left( \sum_{a \in Y_v} \chi(a) \int_{(a)_v} y_p^r \, d\tilde{\Phi}^\lambda(y) \right) = \ell_{T,\lambda} \left( \sum_{a \in Y_v} \chi(a) \Phi^\lambda_r((a)_v) \right), \]

(8.6)

where \((a)_v = (a + (Np^v)) \subset Y\), and by definition (8.1)

\[ \tilde{\mu}^\lambda(y; f_1 \otimes f_2 \otimes f_3) = \ell_{T,\lambda}(\tilde{\Phi}^\lambda(y)), \]

(8.7)

\[ \int_{(a)_v} y_p^r \, d(\tilde{\Phi}^\lambda) = \Phi^\lambda_r((a)_v) \in \mathcal{M}_T^\lambda(Np) \]

(8.8)

for \(r = 0, 1, \ldots, h - 1\). Moreover \(\Phi_r((a)_v)\) is a triple modular form of level \(N^2p^{2v}\) as a value of a higher twist of a Siegel-Eisenstein
distributions, hence

\[
\Phi^\Lambda_r(\chi) = U_T^{-2v} \left[ \pi_{\lambda,T,1} U_T^{2v} \left( 2^r \mathcal{L}_{k_2+k_3-k_1-r}^\lambda,\nu \left( F_{\chi,r} \bar{\chi}_1,\bar{\chi}_2,\bar{\chi}_3 \right) \right) \right].
\] (8.9)

Taking into account the equalities (8.9), the integral (8.6) transforms to the following

\[
\int_Y \chi(y) y^r_p \, d\bar{\mu}^\lambda(y; f_1 \otimes f_2 \otimes f_3) = \ell_{T,\lambda} \left( \sum_{a \in Y_v} \chi(a) \Phi^\Lambda_r((a)_v) \right)
\] (8.10)

\[
= \ell_{T,\lambda} \left( U_T^{-2v} \left[ \pi_{\lambda,T,1} U_T^{2v} \left( 2^r \mathcal{L}_{k_2+k_3-k_1-r}^\lambda,\nu \left( F_{\chi,r} \bar{\chi}_1,\bar{\chi}_2,\bar{\chi}_3 \right) \right) \right] \right)
\]

Notice that then it follows that the sum in the right hand side of the equality (8.10) can be expressed through the functions:

\[
\int_Y \chi(y) y^r_p \, d\bar{\mu}^\lambda(y; f_1 \otimes f_2 \otimes f_3)(y) = \ell_{T,\lambda} \left( U_T^{-2v} \left[ \pi_{\lambda,T,1} U_T^{2v} \left( 2^r \mathcal{L}_{k_2+k_3-k_1-r}^\lambda,\nu \left( F_{\chi,r} \bar{\chi}_1,\bar{\chi}_2,\bar{\chi}_3 \right) \right) \right] \right)
\] (8.11)
where we use $\mathcal{L}^{\lambda,\nu}_{k_2+k_3-k_1}$ (Ibukiyama’s operator with $\lambda = k_1 - k_3 \geq \mu = k_1 - k_2 \geq 0$).

The Fourier expansion of the function

$$g = \mathcal{L}^{\lambda,\nu}_{k_2+k_3-k_1}(F_{\chi,r}^{\bar{\chi}_1,\bar{\chi}_2,\bar{\chi}_3})$$

can be computed:

$$\mathcal{E}(z_1, z_2, z_3; -r, k_1, k_2, k_3, Np^v, \psi, \chi_1, \chi_2, \chi_3)$$

$$= N_{1,1}N_{1,2}N_{1,3}(\bar{\chi}_1\bar{\chi}_2\bar{\chi}_3)(2)G(\chi_{0,1})G(\chi_{0,2})G(\chi_{0,3}) \cdot g.$$

Thus it represents a nearly holomorphic triple modular form in the $\overline{\mathbb{Q}}$-module

$$\mathcal{M}(\overline{\mathbb{Q}}) = \mathcal{M}_T(N^2p^{2v}, \psi_1 \otimes \psi_2 \otimes \psi_3; \overline{\mathbb{Q}}) \subset$$

$$\mathcal{M}_{k,r^*}(N^2p^{2v}, \psi_1; \overline{\mathbb{Q}}) \otimes \mathcal{M}_{k,r^*}(N^2p^{2v}, \psi_2; \overline{\mathbb{Q}}) \otimes \mathcal{M}_{k,r^*}(N^2p^{2v}, \psi_3; \overline{\mathbb{Q}}).$$
Then we have:

\[ \mathcal{L}_{T,\lambda} : \mathcal{M}_T(N^2p^{2v}; \mathbb{C}) \to \mathbb{C}, \]

\[ g \mapsto \frac{\langle f_1^0 \otimes f_2^0 \otimes f_3^0, \lambda^{-2v}U_T^{2v}g \rangle_{T,N^2p}}{\langle f_1^0 \otimes f_2^0 \otimes f_3^0, f_1,0 \otimes f_2,0 \otimes f_3,0 \rangle_{T,N^2p}}, \]

\[ \ell_{T,\lambda}(U_T^{-2v}[\pi_{\lambda,T,1}U_T^{2v}(g)]) = i_p \left( \frac{\langle f_1^0 \otimes f_2^0 \otimes f_3^0, \lambda^{-2v}U_T^{2v}g \rangle_{T,N^2p}}{\langle f_1^0 \otimes f_2^0 \otimes f_3^0, f_1,0 \otimes f_2,0 \otimes f_3,0 \rangle_{N^2p}} \right) \]

\[ = i_p \left( \lambda^{-2v}p^{3 \cdot 2v(k-1)} \cdot \frac{\langle V^{2v}(f_1^0 \otimes f_2^0 \otimes f_3^0), g \rangle_{T,N^2p^{2v+1}}}{\langle f_1^0 \otimes f_2^0 \otimes f_3^0, f_1,0 \otimes f_2,0 \otimes f_3,0 \rangle_{T,N^2p}} \right). \]

The scalar products in (8.12) can be computed but we omit here the details. This implies Proposition 8.1 for certain modular forms.
\[ f_{j,2v}(z) = \sum_{n=1}^{\infty} a_{j,n,2v} e(nz) \] as above:

\[ \mathcal{D}^*(f_1^0 \otimes f_2^0 \otimes f_3^0, 2k - 2 - r, \psi_1 \psi_2 \chi_1) \]

\[ (Np^{2v})^{k-2r} \frac{N^2p^{2v} \varphi(N^2p^{2v}) \varphi(Np^v)}{[\Gamma_0(N^2p^{2v}) : \Gamma(N^2p^{2v})]^3} \mathcal{L}_{Np}(-r) = \]

\[ \left\langle \tilde{f}_{1,2v} \otimes \tilde{f}_{2,2v} \otimes \tilde{f}_{3,2v}, \mathcal{E}(z_1, z_2, z_3; -r, k, N^2p^{2v}, \psi, \chi_1, \chi_2, \chi_3) \right\rangle_{T,N^2p^{2v}}, \]

where

\[ \mathcal{L}_{Np}(s) = \mathcal{L}_{Np}(s; \tilde{f}_{1,2v} \otimes \tilde{f}_{2,2v} \otimes \tilde{f}_{3,2v}) \]

\[ := \sum_{n | N^\infty} G_N(\overline{\psi_1 \psi_2 \chi_1}, 2n) \frac{a_{n,1,2v} a_{n,2,2v} a_{n,3,2v}}{n^{2s+2k-2}}. \]
8.4 Proof of Theorem B

Let us use Proposition 8.1 and (8.13):

\[ 2^{-r} \int_Y \chi(y) y^r_p \, d\tilde{\mu}_\lambda(y; f_1 \otimes f_2 \otimes f_3)(y) \]

\[ = 2^{-r} \ell_{T,\lambda} \left( U_T^{-2v} \left[ \pi_{\lambda,T,1} U_T^{2v}(g) \right] \right) \]

\[ = \frac{((Np)^3 / N_1 N_2 N_3)^{k/2} \tilde{\gamma}_1 \tilde{\gamma}_2 \tilde{\gamma}_3 (\chi_1 \chi_2 \chi_3)(2) p^{3 \cdot v(k-2)}}{\lambda^{2v} N_{1,1} N_{2,1} N_{3,1} G(\chi_1,0) G(\chi_2,0) G(\chi_3,0) \times \times (Np^{2v})^{k-2r} \frac{N^2 p^{2v} \varphi(N^2 p^{2v}) \varphi(Np^v)}{[\Gamma_0(N^2 p^{2v}) : \Gamma(N^2 p^{2v})]^3} \mathcal{L}_{NP}(-r) \times \times \mathcal{D}^*(f_1^0 \otimes f_2^0 \otimes f_3^0, 2k - 2 - r, \psi_1 \psi_2 \chi_1) \langle f_1^0 \otimes f_2^0 \otimes f_3^0, f_{1,0} \otimes f_{2,0} \otimes f_{3,0} \rangle_{T,N^2 p} \]

Let us show that under the assumptions as above there exist an admissible \( \mathbb{C}_p \)-valued measure \( \tilde{\mu}_\lambda^{f_1 \otimes f_2 \otimes f_3} \) on \( Y_{N,p} \), and a \( \mathbb{C}_p \)-analytic
function
\[ D_p(x, f_1 \otimes f_2 \otimes f_3) : X_p \to \mathbb{C}_p, \]
given for all \( x \in X_{N,p} \) by the integral
\[ D_p(x, f_1 \otimes f_2 \otimes f_3) = \int_{Y_{N,p}} x(y) d\tilde{\mu}_f(y), \]
and having the following properties: for all pairs \((r, \chi)\) such that for \( \chi \in X^\text{tors}_p \) the corresponding Dirichlet characters \( \chi_j \) are \( Np \)-complete, and \( r \in \mathbb{Z} \) with \( 0 \leq r \leq k - 2 \), the following equality holds:
\[
D_p(\chi x_p^r, f_1 \otimes f_2 \otimes f_3) = \frac{(\psi_1 \psi_2)(2) C_\chi^4(2k-3-r)}{G(\chi_1) G(\chi_2) G(\chi_3) G(\psi_1 \psi_2 \chi_1) \lambda(p)^{2v}} D^*(f_1^p \otimes f_2^p \otimes f_3^p, 2k - 2 - r, \psi_1 \psi_2 \chi) \langle f_1^p \otimes f_2^p \otimes f_3^p, f_1^p \otimes f_2^p \otimes f_3^p \rangle_T \]
where \( v = \text{ord}_p(C_\chi) \).
\[ \chi_1 \mod Np^v = \chi, \ \chi_2 \mod Np^v = \psi_2 \bar{\psi}_3 \chi, \chi_3 \mod Np^v = \psi_1 \bar{\psi}_3 \chi, \]

\[ G(\chi) \text{ denotes the Gauß sum of a primitive Dirichlet character } \chi_0 \text{ attached to } \chi \text{ (modulo the conductor of } \chi_0). \]

Indeed, we may write

\[ \mathcal{D}(p)(x, f_1 \otimes f_2 \otimes f_3) = C \cdot x(2) \int_Y x(y) d\tilde{\mu}^\lambda(y; f_1 \otimes f_2 \otimes f_3) \]

with an appropriate constant where \( v = \text{ord}_p(C_\chi). \)

Moreover, it follows from the properties of the constructed measure

\[ \tilde{\mu}^\lambda_{f_1 \otimes f_2 \otimes f_3}(y) := C \cdot \tilde{\mu}_\lambda(2^{-1} y; f_1 \otimes f_2 \otimes f_3) \]

that

(ii) if \( \text{ord}_p \lambda(p) = 0 \) then the holomorphic functions in (i), (ii) are bounded \( \mathbb{C}_p \)-analytic functions: it suffices to use the binomial equality with \( r = 1 \) in order to show that in this case the measure \( \tilde{\Phi}^\lambda \) is just bounded because of \( |\lambda(p)|_p = 1 \);
(iii) in the general case (but assuming that $\lambda(p) \neq 0$) the holomorphic functions in (i) belong to the type $o(\log(x_p^h))$ with $h = [2\text{ord}_p \lambda(p)] + 1$ and they can be represented as the Mellin transform of the $h$-admissible measure $\tilde{\mu}_1^\lambda \otimes f_2 \otimes f_3$ (in the sense of Amice-Vélu);

(iv) if $h = [2\text{ord}_p \lambda] + 1 \leq k - 2$ then the function $D_{(p)}$ is uniquely determined by the above conditions (i).
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