Triple products of Coleman's families and their periods (a joint work with S.Boecherer)

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Abstract

For a prime number $p \geq 5$, we consider three classical cusp eigenforms

$$f_j(z) = \sum_{n=1}^{\infty} a_{n,j} e(nz) \in \mathcal{S}_{k_j}(N_j, \psi_j), \ (j = 1, 2, 3)$$

of weights k_1, k_2, k_3, of conductors N_1, N_2, N_3, and of nebentypus characters ψ_j mod N_j $(j = 1, 2, 3)$.

According to H.Hida [Hi86] and R.Coleman [CoPB], one can include each f_j $(j = 1, 2, 3)$ (under suitable assumptions on p and on f_j) into a p-adic analytic family

$$k'_j \mapsto \{f_{j,k'_j} = \sum_{n=1}^{\infty} a_n(f_{j,k'_j}) q^n\}$$

of cusp eigenforms f_{j,k'_j} of weights k'_j in such a way that $f_{j,k_j} = f_j$ $(j = 1, 2, 3)$, and that all their Fourier coefficients $a_n(f_{j,k'_j})$ are given by certain p-adic analytic functions $k'_j \mapsto a_{n,j}(k'_j)$.

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The purpose of this talk is to describe a *four variable* p-adic L function attached to Garrett's triple product of three Coleman's families

$$k'_j \mapsto \left\{ f_{j,k'_j} = \sum_{n=1}^{\infty} a_{n,j}(k')q^n \right\} (j = 1, 2, 3)$$

of cusp eigenforms of three fixed slopes $\sigma_j = v_p(\alpha^{(1)}_{p,j}(k'_j)) \geq 0$ where $\alpha^{(1)}_{p,j} = \alpha^{(1)}_{p,j}(k'_j)$ is an eigenvalue (which depends on k'_j) of Atkin's operator $U = U_p$ acting on Fourier expansions by $U(\sum_{n\geq 0}^{\infty} a_n q^n) = \sum_{n\geq 0}^{\infty} a_{np} q^n$.

Let us consider the product of three eigenvalues:

$$\lambda = \lambda(k'_1, k'_2, k'_3) = \alpha^{(1)}_{p,1}(k'_1)\alpha^{(1)}_{p,2}(k'_2)\alpha^{(1)}_{p,3}(k'_3)$$

and assume that the slope of this product

$$\sigma = v_p(\lambda(k'_1, k'_2, k'_3)) = \sigma(k'_1, k'_2, k'_3) = \sigma_1 + \sigma_2 + \sigma_3$$

is *constant and positive* for all triplets (k'_1, k'_2, k'_3) in an appropriate p-adic neighbourhood of the fixed triplet of weights

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(k_1, k_2, k_3). The each value σ_j is fixed throughout this section.

Our p-adic L-functions are Mellin transforms of certain measures with values in $\mathcal{A}$, where $\mathcal{A} = \mathcal{A}(\mathcal{B})$ denotes an affinoid algebra associated with an affinoid space $\mathcal{B}$ as in [CoPB], where $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3$, is an affinoid neighbourhood around $(k_1, k_2, k_3) \in \mathcal{W}^3$ (with a given integer k and Dirichlet character ψ mod N). We construct such a measure from higher twists of classical Siegel-Eisenstein series, which produce distributions with values in certain Banach $\mathcal{A}$-modules $\mathcal{M} = \mathcal{M}(N; \mathcal{A})$ of triple modular forms with coefficients in the algebra $\mathcal{A}$.

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0 Introduction and Main results

Let p be a prime. The purpose of this talk is to describe a four variable p-adic L function attached to Garrett's triple product of three Coleman's families

$$k'_j \mapsto \left\{ f_{j,k'_j} = \sum_{n=1}^{\infty} a_{n,j}(k')q^n \right\}$$

of cusp eigenforms of three fixed slopes $\sigma_j = v_p(\alpha^{(1)}_{p,j}(k'_j)) \geq 0$ where $\alpha^{(1)}_{p,j} = \alpha^{(1)}_{p,j}(k'_j)$ is an eigenvalue (which depends on k'_j) of Atkin's operator $U = U_p$ acting on Fourier expansions by
$U(\sum_{n\geq 0}^{\infty} a_n q^n) = \sum_{n\geq 0}^{\infty} a_{np} q^n$.

For this purpose we use the theory of p-adic integration with values in spaces of *nearly holomorphic modular forms* (in the sense of Shimura, see [ShiAr]).

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A family of slope $\sigma > 0$ of cusp eigenforms $f_{k'}$ of weight $k' \geq 2$ containing f

$$k' \mapsto f_{k'} = \sum_{n=1}^{\infty} a_n(k')q^n$$
$$\in \overline{\mathbb{Q}}[\![q]\!] \subset \mathbb{C}_p[\![q]\!]$$

A model example of a p-adic family (not cusp and $\sigma = 0$): Eisenstein series
$$a_n = \sum_{d|n} d^{k'-1},\ f_{k'} = E_{k'}$$

1) the Fourier coefficients $a_n(k')$ of $f_{k'}$ and one of the Satake p-parameters $\alpha_p(k')$ are given by certain p-adic analytic functions $k' \mapsto a_n(k')$ for $(n,p) = 1$

2) the slope is constant and positive: $\mathrm{ord}(\alpha_p(k')) = \sigma > 0$

1 Generalities on triple products

Consider three primitive cusp eigenforms

$$f_j(z) = \sum_{n=1}^{\infty} a_{n,j} q^n \in \mathcal{S}_{k_j}(N_j, \psi_j), \ (j = 1, 2, 3) \tag{1.1}$$

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of weights k_1, k_2, k_3, of conductors N_1, N_2, N_3, and of nebentypus characters ψ_j mod N_j, and let χ denote a varying Dirichlet character.

The triple product with a Dirichlet character χ is defined as the following complex L-function (an Euler product of degree eight):

$$L(f_1 \otimes f_2 \otimes f_3, s, \chi) = \prod_{p \nmid N_1 N_2 N_3} L((f_1 \otimes f_2 \otimes f_3)_p, \chi(p) p^{-s}), \tag{1.2}$$

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$$\text{where } L((f_1 \otimes f_2 \otimes f_3)_p, X)^{-1} = \tag{1.3}$$

$$\det\left(1_8 - X \begin{pmatrix} \alpha_{p,1}^{(1)} & 0 \\ 0 & \alpha_{p,1}^{(2)} \end{pmatrix} \otimes \begin{pmatrix} \alpha_{p,2}^{(1)} & 0 \\ 0 & \alpha_{p,2}^{(2)} \end{pmatrix} \otimes \begin{pmatrix} \alpha_{p,3}^{(1)} & 0 \\ 0 & \alpha_{p,3}^{(2)} \end{pmatrix}\right)$$

$$= \prod_{\eta} (1 - \alpha_{p,1}^{(\eta(1))} \alpha_{p,2}^{(\eta(2))} \alpha_{p,3}^{(\eta(3))} X)$$

$$= (1 - \alpha_{p,1}^{(1)} \alpha_{p,2}^{(1)} \alpha_{p,3}^{(1)} X)(1 - \alpha_{p,1}^{(1)} \alpha_{p,2}^{(1)} \alpha_{p,3}^{(2)} X) \cdot \ldots \cdot (1 - \alpha_{p,1}^{(2)} \alpha_{p,2}^{(2)} \alpha_{p,3}^{(2)} X)$$

product taken over all $\eta : \{1, 2, 3\} \to \{1, 2\}$, and

$$1 - a_{p,j} X - \psi_j(p) p^{k_j - 1} X^2 = (1 - \alpha_{p,j}^{(1)}(p) X)(1 - \alpha_{p,j}^{(2)}(p) X),$$

are three Hecke p–polynomials of forms f_j. We always assume that

$$k_1 \geq k_2 \geq k_3, \text{ and } k_1 \leq k_2 + k_3 - 2 \text{ (“balanced” weights)} \tag{1.4}$$

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We use the corresponding normalized L function (see [De79], [Co], [Co-PeRi]), which has the form:

$$\Lambda(f_1 \otimes f_2 \otimes f_3, s, \chi) = \tag{1.5}$$
$$\Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(s-k_3+1)\Gamma_{\mathbb{C}}(s-k_2+1)\Gamma_{\mathbb{C}}(s-k_1+1)L(f_1 \otimes f_2 \otimes f_3, s, \chi),$$

where $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$. The Gamma-factor determines the critical values $s = k_1, \cdots, k_2 + k_3 - 2$ of $\Lambda(s)$, which we explicitely evaluate (like $\zeta(2) = \frac{\pi^2}{6}$).

A functional equation of $\Lambda(s)$ has the form:

$$s \mapsto k_1 + k_2 + k_3 - 2 - s.$$

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According to H.Hida [Hi86] and R.Coleman [CoPB], one can include each f_j $(j = 1, 2, 3)$ (under suitable assumptions on p and on f_j) into a p-adic analytic family

$$\mathbf{f}_j : k'_j \mapsto \{f_{j,k'_j} = \sum_{n=1}^{\infty} a_n(f_{j,k'_j})q^n\}$$

of cusp eigenforms f_{j,k'_j} of weights k'_j in such a way that $f_{j,k_j} = f_j$, and that all their Fourier coefficients $a_n(f_{j,k'_j})$ are given by certain p-adic analytic functions $k'_j \mapsto a_{n,j}(k'_j)$.

Our previous result: a two variable p-adic L-function constructed interpolating on all k' a function $(k', s) \mapsto L^*(f_{k'}, s, \chi)(s = 1, \cdots, k' - 1)$ for such a family (see [PaTV] by A.P., *Two variable p-adic L functions attached to eigenfamilies of positive slope*, Invent. Math. v. 154, N3 (2003), pp. 551 - 615).

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2 Statement of the problem

Given three p-adic analytic families $\mathbf{f}_j$ of positive slope $\sigma_j > 0$, to construct a four-variable p-adic L-function attached to Garrett's triple product of these families

(we may view such function as interpolating the special values $\Lambda(f_{1,k'_1} \otimes f_{2,k'_2} \otimes f_{3,k'_3}, s, \chi)$ at critical points $s = k'_1, \cdots, k'_2 + k'_3 - 2$; we prove that these values are algebraic numbers afters dividing by certain "periods"). However the construction uses directly modular forms, and not the L-values in question, and a comparaison of special values of two functions is done after the construction.

We consider the p-adic weight space $X = \mathrm{Hom}_{cont}(Y, \mathbb{C}_p^*)$. This is an analytic space over $\mathbb{C}_p$, which consists of all continuous characters of the profinite group $Y = (\mathbb{Z}/N\mathbb{Z})^* \times \mathbb{Z}_p^*$, containing all classical weights (k'_j, ψ_j), $j = 1, 2, 3$.

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Consider the product of the Satake parmeters

$$\lambda_p = \alpha_{p,1}^{(1)} \alpha_{p,2}^{(1)} \alpha_{p,3}^{(1)} = \lambda_p(k'_1, k'_2, k'_3)$$

We assume that $|\alpha_{p,j}^{(1)}|_p \le |\alpha_{p,j}^{(2)}|_p$, and that the slope $\sigma = \mathrm{ord}_p(\lambda_p(k'_1, k'_2, k'_3)$ is constant and positive for all triplets (k'_1, k'_2, k'_3) in a p-adic neighbourhood $\mathcal{B} \subset X^3$ of the fixed triplet of weights (k_1, k_2, k_3).

The existence of families of slope $\sigma > 0$ was established by R.Coleman in [CoPB]

He gave an example with $p = 7$, $f = \Delta$, $k = 12$ $a_7 = \tau(7) = -7 \cdot 2392, \sigma = 1$.	A program in PARI for computing such families is contained in [CST98] (see also the Web-page of W.Stein, `http://modular.fas.harvard.edu/`)

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Our method

uses a version of Garrett's integral representation for the triple L-functions of the form: for $r = 0, \cdots, k_2' + k_3' - k_1' - 2$,
$\Lambda(f_{1,k_1'} \otimes f_{2,k_2'} \otimes f_{3,k_3'}, k_2' + k_3' - r, \chi) =$

$$\int\int\int_{(\Gamma_0(N^2p^{2v})\backslash\mathbb{H})^3} \overline{\tilde{f}_{1,k_1'}(z_1)\tilde{f}_{2,k_2'}(z_2)\tilde{f}_{3,k_3'}(z_3)}\mathcal{E}(z_1, z_2, z_3; -r, \chi) \prod_j (\frac{dx_j dy_j}{y_j^2})$$

where $\mathcal{E}$ is the triple modular form of weight (k_1', k_2', k_3'), and of fixed character (ψ_1, ψ_2, ψ_3), obtained from a nearly holomorphic Siegel-Eisenstein series by applying Boecherer's higher twist and Ibukiyama's differential operator;

the theory of p-adic integration with values in $\mathcal{A}$-modules $\mathcal{M}_T(\mathcal{A})$ of triple nearly holomorphic modular forms over p-adic Banach algebras $\mathcal{A}$, which allows to view $\mathcal{E}$ as an element of $\mathcal{M}_T(\mathcal{A})$;

the spectral theory of Atkin's U-operator allows to evaluate the integral using a projection of $\mathcal{M}_T(\mathcal{A})$ to the λ-part $\mathcal{M}_T(\mathcal{A})^\lambda$.

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Here $\mathcal{A} = \mathcal{A}(\mathcal{B})$ is a certain p-adic Banach algebra of functions on an open analytic subspace $\mathcal{B} \subset X^3$ in the product of three copies of the weight space $X = \mathrm{Hom}_{cont}(Y, \mathbb{C}_p^*)$. This is an analytic space over $\mathbb{C}_p$, which consists of all continuous characters of the profinite group $Y = (\mathbb{Z}/N\mathbb{Z})^* \times \mathbb{Z}_p^*$.

The weights k' correspond to certain points in the weight space X.

Let $\mathcal{B}_j \subset X$ denote an open analytic subspace conaining $k_j \in X$, and let $\mathcal{A}_j = \mathcal{A}_j(\mathcal{B}_j)$ be the p-adic Banach algebra attached to $\mathcal{B}_j$. Any series $\mathbf{f}_j = \sum_{n\ge1} a_n q^n \in \mathcal{A}_j[\![q]\!]$ produces a family of q-expansions
$\{f_{j,k_j'} = ev_{k_j'}(\mathbf{f}_j) = \sum_{n\ge1} ev_{k_j'}(a_n)q^n \in \mathbb{C}_p[\![q]\!]\}$, which happen to be classical modular forms in $\overline{\mathbb{Q}}[\![q]\!]$ under a fixed embedding $i_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ and the specialization maps $ev_{k_j'} : \mathcal{A}_j \to \mathbb{C}_p$.

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• We may assume that $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3$, and consider the specialization maps $ev_{\mathbf{k}'} : \mathcal{A} \to \mathbb{C}_p$, where $\mathbf{k}' = (k_1', k_2', k_3') \in \mathcal{B}$.

• We construct an analytic function $\mathcal{L}_\mu : X \to \mathcal{A} = \mathcal{A}(\mathcal{B})$ as the p-adic Mellin transform

$$\mathcal{L}_\mu(x) = \int_Y x(y)\, d\mu(y) \quad (\text{where } x \in X = \mathrm{Hom}_{cont}(Y, \mathbb{C}_p^*), x = x(y)),$$

of a certain measure μ with values in $\mathcal{A}$ on the profinite group Y.

We obtain the function in question $\mathcal{L}_\mu(x, \mathbf{s})$ by evaluation at $\mathbf{s} = ((s_1, \psi_1), (s_2, \psi_2), (s_3, \psi_3)) \in \mathcal{B}$: this is a p-adic analytic function in four variables $(x, \mathbf{s}) \in X \times \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3 \subset X \times X \times X \times X$:

$$\mathcal{L}_\mu(x, \mathbf{s}) := ev_{\mathbf{s}}(\mathcal{L}_\mu(x)) \quad (x \in X,\ \mathbf{s} \in \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3,\ \mathcal{L}_\mu(x) \in \mathcal{A}).$$

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• We check an equality relating the values of the constructed analytic function $\mathcal{L}_\mu(x, \mathbf{s})$ at the arithmetical characters $x = y_p^r \chi \in X$, and at triple weights $\mathbf{s} = (k_1', k_2', k_3') \in \mathcal{B}$, with the normalized critical special values

$$L^*(f_{1,k_1'} \otimes f_{2,k_2'} \otimes f_{3,k_3'}, k_2' + k_3' - 2 - r, \chi) \quad (r = 0, \cdots, k_2' + k_3' - k_1' - 2),$$

for certain Dirichlet characters χ mod $Np^v, v \geq 1$. These are algebraic numbers, embedded into $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}}_p$ (the Tate field of p-adic numbers). The normalisation of L^* includes at the same time Gauss sums, Petersson scalar products, powers of π, $\lambda_p(k_1', k_2', k_3')$, and a certain finite Euler product.

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3 Arithmetical nearly holomorphic modular forms

Let $\mathcal{A}$ be a commutative ring (a subring of $\mathbb{C}$ or a normed $\mathcal{O}$-algebra $\mathcal{A}$ where $\mathcal{O}$ is the ring of integers in a finite extension K of $\mathbb{Q}_p$).

Arithmetical nearly holomorphic modular forms (in the sense of Shimura, [ShiAr] are certain formal series

$$g = \sum_{n=0}^{\infty} a(n;R)q^n \in \mathcal{A}[\![q]\!][R], \text{ with the property}$$

that for $\mathcal{A} = \mathbb{C}$, $z = x + iy \in \mathbb{H}$, $R = (4\pi y)^{-1}$, the series converges to a $\mathcal{C}^\infty$-modular form on $\mathbb{H}$ of a given weight k and Dirichlet character ψ. The coefficients $a(n;R)$ are polynomials in $\mathcal{A}[R]$. If $\deg_R a(n;R) \le r$ for all n, we call g nearly holomorphic of type r (it is annihilated by $(\frac{\partial}{\partial \bar{z}})^{r+1}$, see [ShiAr]).

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We use the notation $\mathcal{M}_{k,r}(N,\psi,\mathcal{A})$ or $\tilde{\mathcal{M}}(N,\psi,\mathcal{A})$ for $\mathcal{A}$-modules of such forms. (In our constructions the weight k varies).

A known example (see the introduction to [ShiAr]) is given by the series

$$-12R + E_2 := -12R + 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n$$

$$= \frac{3}{\pi^2}\lim_{s\to 0} y^s \sum_{m_1,m_2\in\mathbb{Z}}{}' (m_1+m_2z)^{-2}|m_1+m_2z|^{-2s}, (R = (4\pi y)^{-1})$$

where $\sigma_1(n) = \sum_{d|n} d$.

There is the action of the Shimura differential operator

$$\delta_k : \mathcal{M}_{k,r}(N,\psi,\mathcal{A}) \to \mathcal{M}_{k+2,r+1}(N,\psi,\mathcal{A}),$$

given over $\mathbb{C}$ by $\delta_k(f) = (\frac{1}{2\pi i}\frac{\partial}{\partial z} - \frac{k}{4\pi y})f$.

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This operator is a correction of the Ramanujan operator

$$\theta(\sum_{n=0}^{\infty} a_n q^n) = \sum_{n=1}^{\infty} n a_n q^n = \frac{1}{2\pi i}\frac{\partial}{\partial z}(\sum_{n=0}^{\infty} a_n q^n) = q\frac{\partial}{\partial q}(\sum_{n=0}^{\infty} a_n q^n),$$

which does not preserve the modularity. For example $\theta\Delta = E_2\Delta$, where E_2 is a quasimodular form (in the sense of Kaneko and Zagier, see [Ka-Za]).

Notice that $\delta_k f = (\theta - kR)f$, and that Serre's operator $f \mapsto \theta f - \frac{k}{12}E_2 f$ takes $\mathcal{M}_k$ to $\mathcal{M}_{k+2}$.

Note that that the arithmetical twist operator

$$\theta_\chi(\sum_{n=0}^{\infty} a_n q^n) = \sum_{n=1}^{\infty} \chi(n) a_n q^n$$

is a natural analog of the Ramanujan operator.

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Let $\mathcal{A}$ be a commutative ring. The tensor product over $\mathcal{A}$

$$\mathcal{M}_{\mathbf{k},r,T}(N,\psi,\mathcal{A}) = \mathcal{M}_{k_1,r}(N,\psi_1,\mathcal{A}) \otimes \mathcal{M}_{k_2,r}(N,\psi_2,\mathcal{A}) \otimes \mathcal{M}_{k_3,r}(N,\psi_3,\mathcal{A})$$

consists of triple arithmetical modular forms as certain formal series of the form

$$g = \sum_{n_1,n_2,n_3=0}^{\infty} a(n_1,n_2,n_3;R_1,R_2,R_3) q_1^{n_1} q_2^{n_2} q_3^{n_3}$$

$$\in \mathcal{A}[\![q_1,q_2,q_3]\!][R_1,R_2,R_3], \text{ where } z_j = x_j + iy_j \in \mathbb{H},\ R_j = (4\pi y_j)^{-1},$$

with the property that for $\mathcal{A} = \mathbb{C}$, the series converges to a $\mathcal{C}^\infty$-modular form on $\mathbb{H}^3$ of a given weight (k_1,k_2,k_3) and character (ψ_1,ψ_2,ψ_3), $j = 1,2,3$. The coefficients $a(n_1,n_2,n_3;R_1,R_2,R_3)$ are polynomials in $\mathcal{A}[R_1,R_2,R_3]$. Examples of such modular forms come from the restriction to the diagonal of Siegel modular forms of degree 3.

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4 Siegel-Eisenstein series

Recall some definitions concerning Siegel modular forms.

Let $J_{2m} = \begin{pmatrix} 0_m & -1_m \\ 1_m & 0_m \end{pmatrix}$. The symplectic group

$$\mathrm{Sp}_m(\mathbb{R}) = \left\{ g \in \mathrm{GL}_{2m}(\mathbb{R}) | {}^t g \cdot J_{2m} g = J_{2m} \right\},$$

acts on the Siegel upper half plane

$$\mathbb{H}_m = \left\{ z = {}^t z \in M_m(\mathbb{C}) | \mathrm{Im} z > 0 \right\}$$

by $g(z) = (az+b)(cz+d)^{-1}$, where we use the bloc notation $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sp}_{2m}(\mathbb{R})$, and the congruence subgroup $\Gamma_0^m(N) = \{\gamma \in \mathrm{Sp}_m(\mathbb{Z}) \mid \gamma \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)\} \subset \mathrm{Sp}_m(\mathbb{Z})$.

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A Siegel modular form $f \in \mathcal{M}_k(\Gamma_0^m(N), \chi)$ of degree $m > 1$, weight k and a Dirichlet chracter χ mod N is a holomorphic function $f\colon \mathbb{H}_m \to \mathbb{C}$ such that for every $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0^m(N)$ one has

$$f(\gamma(z)) = \chi(\det d) \det(cz+d)^k f(z).$$

The Fourier expansion of f uses the semi-group $B_m = \{\mathcal{T} = {}^t\mathcal{T} \geq 0 | \mathcal{T} \text{ half-integral}\}$:
$f(z) = \sum_{\mathcal{T} \in B_m} a(\mathcal{T}) q^{\mathcal{T}} \in \mathbb{C}[\![q^{B_m}]\!]$(a formal q-expansion $\in \mathbb{C}[\![q^{B_m}]\!]$),
where $q^{\mathcal{T}} = \exp(2\pi i \mathrm{tr}(\mathcal{T} z))$

$$= \prod_{i=1}^{m} q_{ii}^{\mathcal{T}_{ii}} \prod_{i<j} q_{ij}^{2\mathcal{T}_{ij}} \subset \mathbb{C}[\![q_{11}, \ldots, q_{mm}]\!][q_{ij},\ q_{ij}^{-1}]_{i,j=1,\cdots,m}$$

and $q_{ij} = \exp(2\pi(\sqrt{-1} z_{i,j}))$.

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EXAMPLE 4.1 (SIEGEL-EISENSTEIN SERIES) (see in [Nag2], p.408):

$$\begin{aligned} E_4^{(2)}(z) =& 1 + 240q_{11} + 240q_{22} + 2160q_{11}^2 + (240q_{12}^{-2} + 13440q_{12}^{-1} \\ & 30240 + 13440q_{12} + 240q_{12}^2)q_{11}q_{22} + 2160q_{22}^2 + \dots \\ E_6^{(2)}(z) =& 1 - 504q_{11} - 504q_{22} - 16632q_{11}^2 + (-540q_{12}^{-2} + 44352q_{12}^{-1} \\ & 166320 + 44352q_{12} - 504q_{12}^2)q_{11}q_{22} - 16632q_{22}^2 + \dots . \end{aligned}$$

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Arithmetical nearly holomorphic Siegel modular forms

Consider a commutative ring $\mathcal{A}$, the formal variables $q = (q_{i,j})_{i,j=1,\dots,m}$, $R = (R_{i,j})_{i,j=1,\dots,m}$, and the ring of *formal Fourier series*

$$\mathcal{A}[\![q^{B_m}]\!][R_{i,j}] = \left\{ f = \sum_{\mathcal{T} \in B_m} a(\mathcal{T}, R) q^{\mathcal{T}} \;\middle|\; a(\mathcal{T}, R) \in \mathcal{A}[R_{i,j}] \right\} \quad (4.1)$$

(over the complex numbers this notation corresponds to $q^{\mathcal{T}} = \exp(2\pi i \mathrm{tr}(\mathcal{T} z))$, $R = (4\pi \mathrm{Im}(z))^{-1}$).

The formal Fourier expansion of a nearly holomorphic Siegel modular form f with coefficients in $\mathcal{A}$ is a certain element of $\mathcal{A}[\![q^{B_m}]\!][R_{i,j}]$. We call f arithmetical in the sense of Shimura [ShiAr], if $\mathcal{A} = \overline{\mathbb{Q}}$.

4.1 Algebraic differential operators of Maass and Shimura

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Let us consider the Maass differential operator (see [Maa]) Δ_m of degree m, acting on complex $\mathcal{C}^\infty$-functions on $\mathbb{H}_m$ by:

$$\Delta_m = \det(\tilde{\partial}_{ij}), \qquad \tilde{\partial}_{ij} = 2^{-1}(1+\delta_{ij})\partial/\partial_{ij}, \tag{4.2}$$

its algebraic version is the Ramanujan operator of degree m:

$$\Theta_m := \det(\frac{1}{2\pi i}\tilde{\partial}_{ij}) = \det(\theta_{ij}) = \frac{1}{(2\pi i)^m}\Delta_m, \tag{4.3}$$

where $\theta_{ij} = 2^{-1}(1+\delta_{ij})q_{ij}\frac{\partial}{\partial q_{ij}}$, $\Theta_m(q^{\mathcal{T}}) = \det(\mathcal{T})q^{\mathcal{T}}$.

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The Shimura differential operator (see [Shi76, ShiAr]):

$$\delta_k f(z) = \det(R)^{k+1-\varkappa}\Theta_m\left[\det(R)^{\varkappa-1-k}f\right], \text{ where } R = (4\pi y)^{-1},$$

acts on arithmetic nearly holomorphic Siegel modular forms, and the composition is defined

$$\delta_k^{(r)} = \delta_{k+2r-2}\circ\cdots\circ\delta_k : \widetilde{\mathcal{M}}_k^m(N,\psi;\overline{\mathbb{Q}}) \to \widetilde{\mathcal{M}}_{k+2r}^m(N,\psi;\overline{\mathbb{Q}}), \tag{4.4}$$

where

$$\delta_k f(z) = \left(\frac{-1}{4\pi}\right)^m \det(y)^{-1}\det(z-\bar{z})^{\varkappa-k}\Delta_m\left[\det(z-\bar{z})^{k-\varkappa+1}f(z)\right].$$

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Let $f = \sum_{\mathcal{T} \in B_m} c(\mathcal{T}) q^{\mathcal{T}} \in \mathcal{M}_k^m(N, \psi)$ be a formal holomorphic Fourier expansion. One shows that $\delta_k^{(r)} f$ is given by

$$\delta_k^{(r)} f = \sum_{\mathcal{T} \in B_m} Q(R, \mathcal{T}; k, r) c(\mathcal{T}) q^{\mathcal{T}}.$$

Here we use a universal polynomial (4.5) which can be defined for all $k \in \mathbb{C}$, and it expresses the action of the Shimura operator on the exponential (of degree m):

$$\delta_k^{(r)}(q^{\mathcal{T}}) = Q(R, \mathcal{T}; k, r) q^{\mathcal{T}}.$$

If $m = 1$, r arbitrary (see [Shi76]),

$$\delta_k^{(r)} = \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} \frac{\Gamma(k+r)}{\Gamma(k+j)} R^{r-j} \theta^j.$$

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If $r = 1$, m arbitrary, one has (see [Maa]):

$$\delta_k f(z) = \sum_{\mathcal{T} \in B_m} c(\mathcal{T}) \sum_{l=0}^{m} (-1)^{m-l} c_{m-l}(k+1-\varkappa) \mathrm{tr}\left({}^t\rho_{m-l}(R) \cdot \rho_l^\star(\mathcal{T})\right) q^{\mathcal{T}}$$

where $R = (4\pi y)^{-1} = (R_{i,j}) \in \mathrm{M}_m(\mathbb{R})$, $c_m(\alpha) = \dfrac{\Gamma_m(\alpha + \kappa)}{\Gamma_m(\alpha + \kappa - 1)}$,

$$\Gamma_m(s) = \pi^{m(m-1)/4} \prod_{j=0}^{m-1} \Gamma(s - (j/2)).$$

Here we use the natural representation $\rho_r : \mathrm{GL}_m(\mathbb{C}) \longrightarrow \mathrm{GL}(\wedge^r \mathbb{C}^m)$ $(0 \le r \le m)$ of the group $\mathrm{GL}_m(\mathbb{C})$ on the vector space $\Lambda^r \mathbb{C}^m$. Thus $\rho_r(z)$ is a matrix of size $\binom{m}{r} \times \binom{m}{r}$ composed of the subdeterminants of z of degree r. Put $\rho_r^\star(z) = \det(z) \rho_{m-r}({}^t z)^{-1}$.

Then the representations ρ_r and $\rho_r^\star$ turn out to be polynomial representations so that for each $z \in \mathrm{M}_m(\mathbb{C})$ the linear operators $\rho_r(z)$, $\rho_r^\star(z)$ are well defined.

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In general (see [CourPa], Theorem 3.14) one has:

$$Q(R,\mathfrak{T}) = Q(R,\mathfrak{T};k,r) \tag{4.5}$$
$$= \sum_{t=0}^{r} \binom{r}{t} \det(\mathfrak{T})^{r-t} \sum_{|L|\le mt-t} R_L(\kappa-k-r)Q_L(R,\mathfrak{T}),$$
$$Q_L(R,\mathfrak{T}) = \mathrm{tr}\left({}^t\rho_{m-l_1}(R)\rho^\star_{l_1}(\mathfrak{T})\right)\cdot\ldots\cdot\mathrm{tr}\left({}^t\rho_{m-l_t}(R)\rho^\star_{l_t}(\mathfrak{T})\right)).$$

In (4.5), L goes over all the multi-indices $0 \le l_1 \le \cdots \le l_t \le m$, such that $|L| = l_1 + \cdots + l_t \le mt - t$, and $R_L(\beta) \in \mathbb{Z}[1/2][\beta]$ in (4.5) are polynomials in β of degree $(mt - |L|)$ (used with $\beta = \kappa - k - r$).

Note the differentiation rule of degree m (see [Sh83], p.466):

$$\Delta(fg) = \sum_{r=0}^{m} \mathrm{tr}\left({}^t\rho_r(\tilde{\partial}/\partial z)f \cdot \rho^\star_{m-r}(\tilde{\partial}/\partial z)g\right).$$

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Example 4.2 (Siegel-Eisenstein series of higher level)

$$G^*(z,s;k,\chi,N) \tag{4.6}$$
$$= \det(y)^s \sum_{c,d} \chi(\det c)\det(cz+d)^{-k}|\det(cz+d)|^{-2s}\,\cdot$$
$$\cdot\,\tilde{\Gamma}(k,s)L_N(k+2s,\chi)\left(\prod_{i=1}^{[m/2]} L_N(2k+4s-2i,\chi^2)\right),\ \textit{where}$$

(c,d) runs over all "non-associated coprime symmetric pairs" with $\det(c)$ coprime to N, $\kappa = (m+1)/2$, and for m odd the Γ-factor has the form: (2.134)
$\tilde{\Gamma}(k,s) = i^{mk}2^{-m(k+1)}\pi^{-m(s+k)}\Gamma_m(k+s)$.

We use this series with $m = 3$, $\kappa = \dfrac{m+1}{2} = 2$, $[m/2] = 1$.

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THEOREM 4.3 (SIEGEL, SHIMURA [SH83], P. FEIT [FEI86]) *Let m be an odd integer such that $2k > m$, and $N > 1$ be an integer, then:*

For an integer s such that $s = -r$, $0 \le r \le k - \kappa$, there is the following Fourier expansion

$$G^\star(z, -r) = G^\star(z, -r; k, \chi, N) = \sum_{A_m \ni \mathcal{T} \ge 0} a(\mathcal{T}, R) q^{\mathcal{T}}, \qquad (4.7)$$

where for $s > (m + 2 - 2k)/4$ in (4.7) the only non-zero terms occur for positive definite $\mathcal{T} > 0$, polynomials $Q(R, \mathcal{T}; k - 2r, r)$ are given by (4.5) , and for all $\mathcal{T} > 0$, $\mathcal{T} \in A_m$, where

$$a(\mathcal{T}, R) = M(\mathcal{T}, \chi, k - 2r) \cdot \det(\mathcal{T})^{k-2r-\kappa} Q(R, \mathcal{T}; k - 2r, r), \qquad (4.8)$$

$$M(\mathcal{T}, k - 2r, \chi) = \prod_{\ell \mid \det(2\mathcal{T})} M_\ell(\mathcal{T}, \chi(\ell)\ell^{-k+2r}) \qquad (4.9)$$

is a finite Euler product, in which $M_\ell(\mathcal{T}, x) \in \mathbb{Z}[x]$. ∎

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5 Distributions and admissible measures

Notation

$\mathcal{A}$	(a p-adic Banach algebra)
V	(an $\mathcal{A}$-module)
$\mathcal{C}(Y, \mathcal{A})$	(the $\mathcal{A}$-Banach algebra
$\cup$	of continuous functions on Y)
$\mathcal{C}^{loc-const}(Y, \mathcal{A})$	(the $\mathcal{A}$-algebra
	of locally constant functions on Y)

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Definition 5.1 *a) A* distribution $\mathcal{D}$ *on* Y *with values in* V *is an* $\mathcal{A}$*-linear form*

$$\mathcal{D} : \mathcal{C}^{loc-const}(Y, \mathcal{A}) \to V, \quad \varphi \mapsto \mathcal{D}(\varphi) = \int_Y \varphi d\mathcal{D}.$$

b) A measure μ *on* Y *with values in* V *is a continuous* $\mathcal{A}$*-linear form*

$$\mu : \mathcal{C}(Y, \mathcal{A}) \to V, \quad \varphi \mapsto \int_Y \varphi d\mu.$$

The integral $\int_Y \varphi d\mu$ can be defined for any continuous function φ, and any bounded distribution μ, using the Riemann sums.

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Admissible measures of Amice-Vélu

Let h be a positive integer. A more delicate notion of an h-admissible measure was introduced in [Am-V] by Y. Amice, J. Vélu (see also [MTT], [V]):

Definition 5.2

a) For $h \in \mathbb{N}, h \geq 1$ *let* $\mathcal{P}^h(Y, \mathcal{A})$ *denote the* $\mathcal{A}$*-module of* locally polynomial functions *of degree* $< h$ *of the variable* $y_p : Y \to \mathbb{Z}_p^\times \hookrightarrow \mathcal{A}^\times$*; in particular,*

$$\mathcal{P}^1(Y, \mathcal{A}) = \mathcal{C}^{loc-const}(Y, \mathcal{A})$$

(the $\mathcal{A}$*-submodule of* locally constant functions*). Let also denote* $\mathcal{C}^{loc-an}(Y, \mathcal{A})$ *the* $\mathcal{A}$*-module of* locally analytic functions*, so that*

$$\mathcal{P}^1(Y, \mathcal{A}) \subset \mathcal{P}^h(Y, \mathcal{A}) \subset \mathcal{C}^{loc-an}(Y, \mathcal{A}) \subset \mathcal{C}(Y, \mathcal{A}).$$

b) Let V *be a normed* $\mathcal{A}$*-module with the norm* $|\cdot|_{p,V}$*. For a given*

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positive integer h an h-admissible measure on Y with values in V is an $\mathcal{A}$-module homomorphism

$$\tilde{\Phi} : \mathcal{P}^h(Y, \mathcal{A}) \to V$$

such that for fixed $a \in Y$ and for $v \to \infty$ the following growth condition *is satisfied:*

$$\left| \int_{a+(Np^v)} (y_p - a_p)^{h'} d\tilde{\Phi} \right|_{p,V} = o(p^{-v(h'-h)}) \quad (5.10)$$

$$\text{for all} \quad h' = 0, 1, \dots, h-1, a_p := y_p(a)$$

The condition (5.10) allows to integrate only the locally-analytic functions: there exists a unique extension of $\tilde{\Phi}$ to $\mathcal{C}^{loc-an}(Y, \mathcal{A}) \to V$ (via the embedding $\mathcal{P}^h(Y, \mathcal{A}) \subset \mathcal{C}^{loc-an}(Y, \mathcal{A})$).

This condition allows to integrate locally-analytic functions on Y along $\tilde{\Phi}$ using Taylor's expansions!

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5.1 U_p–Operator and the method of canonical projection

We construct an h-admissible measure $\widetilde{\Phi}^\lambda : \mathcal{P}^h(Y, \mathcal{A}) \to \mathcal{M}(\mathcal{A})$ out of a sequence of distributions $\Phi_r : \mathcal{P}^1(Y, \mathcal{A}) \to \mathcal{M}(\mathcal{A})$ on local monomials y_p^r of each degree r by

$$\int_{(a)_v} y_p^r \, d\widetilde{\Phi}^\lambda = \pi_\lambda(\Phi_r((a)_v)), \text{ where } \Phi_r((a)_v)) \in M = \mathcal{M}(\mathcal{A}).$$

Here Φ_r take values in an $\mathcal{A}$-module

$$M = \mathcal{M}(\mathcal{A}) \subset \mathcal{A}[\![q_1, q_2, q_3]\!][R_1, R_2, R_3]$$

of nearly holomorphic triple modular forms over $\mathcal{A}$ (for $0 \le r \le h-1$, $h = [2\text{ord}_p \lambda_p] + 1$)

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Here $\mathcal{A}$ is an $\mathbb{C}_p$-algebra, and $\lambda \in \mathcal{A}^\times$ is a fixed non-zero eigenvalue of triple Atkin's operator $U_T = U_{T,p}$, acting on $\mathcal{M}(\mathcal{A})$,

$$\pi_\lambda : \mathcal{M}(\mathcal{A}) \to \mathcal{M}(\mathcal{A})^\lambda$$

is the canonical projection operator onto the maximal $\mathcal{A}$-submodule $\mathcal{M}(\mathcal{A})^\lambda$ over which the operator $U_T - \lambda I$ is nilpotent (we call $\mathcal{M}(\mathcal{A})^\lambda$ the λ-characteristic submodule of $\mathcal{M}(\mathcal{A})$).
The projector π_λ is defined by its kernel:
$\mathrm{Ker}\,\pi_\lambda := \bigcap_{n\geq 1} \mathrm{Im}\,(U_T - \lambda I)^n$.

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Triple modular forms are certain formal series

$$g = \sum_{n_1,n_2,n_3=0}^{\infty} a(n_1, n_2, n_3; R_1, R_2, R_3) q_1^{n_1} q_2^{n_2} q_3^{n_3}$$

$\in \mathcal{A}[\![q_1, q_2, q_3]\!][R_1, R_2, R_3]$, where $z_j = x_j + iy_j \in \mathbb{H}$, $R_j = (4\pi y_j)^{-1}$,

with the property that for $\mathcal{A} = \mathbb{C}$, the series converges to a $\mathcal{C}^\infty$-modular form on $\mathbb{H}^3$ of a given weight (k_1, k_2, k_3) and character (ψ_1, ψ_2, ψ_3), $j = 1, 2, 3$. The coefficients $a(n_1, n_2, n_3; R_1, R_2, R_3)$ are polynomials in $\mathcal{A}[R_1, R_2, R_3]$, and the triple Atkin's operator is given by
$U_T(g) = \sum_{n_1,n_2,n_3=0}^{\infty} a(pn_1, pn_2, pn_3; pR_1, pR_2, pR_3) q_1^{n_1} q_2^{n_2} q_3^{n_3}$.

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Eigenfunctions of U_p and of U_p^*.

Recall that for a primitive cusp eigenform $f_j = \sum_{n=1}^{\infty} a_n(f)q^n$, there is an eigenfunction $f_{j,0} = \sum_{n=1}^{\infty} a_n(f_{j,0})q^n \in \overline{\mathbb{Q}}[\![q]\!]$ of $U = U_p$ with the eigenvalue $\alpha = \alpha_{p,j}^{(1)} \in \overline{\mathbb{Q}}$ $(U(f_0) = \alpha f_0)$ given by

$$f_{j,0} = f_j - \alpha_{p,j}^{(2)} f_j | V_p = f_j - \alpha_{p,j}^{(2)} p^{-k/2} f_j | \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \tag{5.11}$$

$$\sum_{n=1}^{\infty} a_n(f_{j,0})n^{-s} = \sum_{\substack{n=1 \\ p \nmid n}}^{\infty} a_n(f_j)n^{-s}(1 - \alpha_{p,j}^{(1)} p^{-s})^{-1}.$$

Moreover, there is an eigenfunction f_j^0 of U_p^* given by

$$f_j^0 = f_{j,0}^{\rho}\Big|_k \begin{pmatrix} 0 & -1 \\ Np & 0 \end{pmatrix}, \text{ where } f_{j,0}^{\rho} = \sum_{n=1}^{\infty} \overline{a(n, f_0)} q^n. \tag{5.12}$$

Therefore, $U_T(f_{1,0} \otimes f_{2,0} \otimes f_{3,0}) = \lambda(f_{1,0} \otimes f_{2,0} \otimes f_{3,0})$.

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Let us describe in more detail critical values of the L function $L(f_1 \otimes f_2 \otimes f_3, s, \chi)$.

For an arbitrary Dirichlet character χ mod Np^v consider the following Dirichlet characters:

$$\chi_1 \bmod Np^v = \chi, \ \chi_2 \bmod Np^v = \psi_2 \bar{\psi}_3 \chi, \tag{5.13}$$

$$\chi_3 \bmod Np^v = \psi_1 \bar{\psi}_3 \chi, \boldsymbol{\psi} = \chi^2 \psi_1 \psi_2 \overline{\psi}_3;$$

later on we impose the condition that the conductors of the corresponding primitive characters $\chi_{0,1}, \chi_{0,2}, \chi_{0,3}$ are Np-completes (i.e. have the same prime divisors as resp. those of Np).

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Theorem A (algebraic properties of the triple product) *Assume that* $k_2 + k_3 - k_1 \geq 2$*, then for all pairs* (χ, r) *such that the corresonding Dirichlet characters* χ_j *have* Np*-complete conductors, and* $0 \leq r \leq k_2 + k_3 - k_1 - 2$*, we have that*

$$\frac{\Lambda(f_1^\rho \otimes f_2^\rho \otimes f_3^\rho, k_2 + k_3 - 2 - r, \psi_1\psi_2\chi)}{\langle f_1^\rho \otimes f_2^\rho \otimes f_3^\rho, f_1^\rho \otimes f_2^\rho \otimes f_3^\rho \rangle_T} \in \overline{\mathbb{Q}}$$

where

$$\langle f_1^\rho \otimes f_2^\rho \otimes f_3^\rho, f_1^\rho \otimes f_2^\rho \otimes f_3^\rho \rangle_T := \langle f_1^\rho, f_1^\rho \rangle_N \langle f_2^\rho, f_2^\rho \rangle_N f_3^\rho, f_3^\rho \rangle_N$$
$$= \langle f_1, f_1 \rangle_N \langle f_2, f_2 \rangle_N f_3, f_3 \rangle_N.$$

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Fix a positive integer N, a Dirichlet character ψ mod N and consider the commutative profinite group

$Y = Y_{N,p} = \varprojlim_v (\mathbb{Z}/Np^v\mathbb{Z})^*$ and its group $X_{N,p} = \mathrm{Hom}_{cont}(Y, \mathbb{C}_p^\times)$ of (continuouos) p-adic characters (this is a $\mathbb{C}_p$-analytic Lie group). The group $X_{N,p}$ is isomorphic to a finite union of discs $U = \{z \in \mathbb{C}_p \mid |z|_p < 1\}$.

A p-adic L-function $L_{(p)} : X_{N,p} \to \mathbb{C}_p$ is a certain meromorphic function on $X_{N,p}$. Such a function often come from a p-adic measure $\mu_{(p)}$ on Y (*bounded* or *admissible* in the sense of Amice-Vélu, see [Am-V]). The p-adic Mellin transform of $\mu_{(p)}$ is given for all $x \in X_{N,p}$ by

$$L_{(p)}(x) = \int_{Y_{N,p}} x(y) \mathrm{d}\mu_{(p)}(y), L_{(p)}(x) : X \to \mathbb{C}_p$$

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Theorem B (on admissible measures attched to the triple product:fixed balanced weights case). *Under the assumptions as above there exist a $\mathbb{C}_p$-valued measure $\tilde\mu^\lambda_{f_1\otimes f_2\otimes f_3}$ on $Y_{N,p}$, and a $\mathbb{C}_p$-analytic function*
$\mathcal{D}_{(p)}(x, f_1\otimes f_2\otimes f_3) : X_p \to \mathbb{C}_p$, *given for all $x \in X_{N,p}$ by the integral $\mathcal{D}_{(p)}(x, f_1\otimes f_2\otimes f_3) = \int_{Y_{N,p}} x(y)\mathrm{d}\tilde\mu^\lambda_{f_1\otimes f_2\otimes f_3}(y)$, and having the following properties:*
(i) *for all pairs (r,χ) such that $\chi \in X^{\mathrm{tors}}_{N,p}$, and all three corresonding Dirichlet characters χ_j have Np-complete conductor $(j = 1, 2, 3)$, and $r \in \mathbb{Z}$ is an integer with $0 \le r \le k_2 + k_3 - k_1 - 2$, the following equality holds:*

$$\mathcal{D}_{(p)}(\chi x_p^r, f_1\otimes f_2\otimes f_3) = i_p\Big(\frac{(\psi_1\psi_2)(2)C_\chi^{4(k_2+k_3-2-r)}}{G(\chi_1)G(\chi_2)G(\chi_3)G(\psi_1\psi_2\chi_1)\lambda_p^{2v}}$$

$$\frac{\Lambda(f_1^\rho\otimes f_2^\rho\otimes f_3^\rho, k_2+k_3-2-r, \psi_1\psi_2\chi)}{\langle f_1^0\otimes f_2^0\otimes f_3^0, f_{1,0}\otimes f_{2,0}\otimes f_{3,0}\rangle_{T,Np}}\Big)$$

where $v = \mathrm{ord}_p(C_\chi)$, $G(\chi)$ denotes the Gauß sum of a primitive

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Dirichlet character χ_0 attached to χ (modulo the conductor of χ_0),
(ii) *if $\mathrm{ord}_p\lambda_p = 0$ then the holomorphic function in (i) is a* bounded $\mathbb{C}_p$-analytic function*;*
(iii) *in the general case (but assuming that $\lambda_p \ne 0$) the holomorphic function in (i) belongs to the* type $o(\log(x_p^H))$ with $H = [2\mathrm{ord}_p\lambda_p] + 1$ *and it can be represented as the Mellin transform of the H-admissible $\mathbb{C}_p$-valued measure $\tilde\mu^\lambda_{f_1\otimes f_2\otimes f_3}$ (in the sense of Amice-Vélu) on Y*
(iv) *Let $k = k_2 + k_3 - k_1 - 2$. If $H \le k - 2$ then the function $\mathcal{D}_{(p)}$ is uniquely determined by the above conditions (i).*

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Let us describe now a p-adic measures attached to Garrett's triple product of three Coleman's families

$$k'_j \mapsto \{f_{j,k'_j} = \sum_{n=1}^{\infty} a_{n,j}(k')q^n\}(j = 1, 2, 3). \tag{5.14}$$

Consider the product of three eigenvalues:

$$\lambda = \lambda_p(k'_1, k'_2, k'_3) = \alpha^{(1)}_{p,1}(k'_1)\alpha^{(1)}_{p,2}(k'_2)\alpha^{(1)}_{p,3}(k'_3)$$

and assume that the slope of this product

$$\sigma = \mathrm{ord}_p(\lambda(k'_1, k'_2, k'_3)) = \sigma(k'_1, k'_2, k'_3) = \sigma_1 + \sigma_2 + \sigma_3$$

is constant and positive for all triplets (k'_1, k'_2, k'_3) in an appropriate p-adic neighbourhood of the fixed triplet of weights (k_1, k_2, k_3).

Let $\mathcal{A} = \mathcal{A}(\mathcal{B})$ denote an affinoid algebra $\mathcal{A}$ associated with an analytic space $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3$, a neighbourhood around $(k_1, k_2, k_3) \in X^3$ (with a given k and ψ mod N).

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Theorem C (on p-adic measures for families of triple products) *Put $H = [2\mathrm{ord}_p(\lambda)] + 1$. There exists a sequence of distributions Φ_r on Y with values in $\mathcal{M} = \mathcal{M}(\mathcal{A})$ giving an H-admissible measure $\tilde{\Phi}^\lambda$ with values in $\mathcal{M}^\lambda \subset \mathcal{M}$ with the following properties:*

There exists an $\mathcal{A}$-linear form $\ell = \ell_{\mathbf{f}_1\otimes\mathbf{f}_2\otimes\mathbf{f}_3,\lambda} : \mathcal{M}(\mathcal{A})^\lambda \to \mathcal{A}$ (given by (5.16), such that the composition

$$\tilde{\mu} = \tilde{\mu}_{\mathbf{f}_1\otimes\mathbf{f}_2\otimes\mathbf{f}_3,\lambda} := \ell_{\mathbf{f}_1\otimes\mathbf{f}_2\otimes\mathbf{f}_3,\lambda}(\tilde{\Phi}^\lambda)$$

is an H-admissible measure with values in $\mathcal{A}$, and for all (k'_1, k'_2, k'_3) in the affinoid neighborhood $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3$, as above, satisfying $k'_1 \le k'_2 + k'_3 - 2$ we have that the evaluated integrals

$$ev_{(k'_1,k'_2,k'_3)}\left((\ell_{\mathbf{f}_1\otimes\mathbf{f}_2\otimes\mathbf{f}_3,\lambda})(\tilde{\Phi}^\lambda)(y_p^r\chi)\right)$$

on the arithmetical chracters $y_p^r\chi$ coincide with the critical special

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values

$$\Lambda^*(f_{1,k_1'} \otimes f_{2,k_2'} \otimes f_{3,k_3'}, k_2' + k_3' - 2 - r, \chi)$$

for $r = 0, \cdots, k_2' + k_3' - k_1' - 2$, *and for all Dirichlet characters* $\chi \bmod Np^v, v \geq 1$, *with all three corresonding Dirichlet characters* χ_j *given by (5.13), having* Np*-complete conductors. Here the normalisation of* Λ^* *includes at the same time certain Gauss sums, Petersson scalar products, powers of* π *and of* $\lambda(k_1', k_2', k_3')$, *and a certain finite Euler product.*

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The p-adic Mellin transform and four variable p-adic analytic functions

Any h-admissible measure $\tilde{\mu}$ on Y with values in a p-adic Banach algebra $\mathcal{A}$ can be caracterized its *Mellin transform* $\mathcal{L}_{\tilde{\mu}}(x)$ $\mathcal{L}_{\tilde{\mu}} : X \to \mathcal{A}$, defined by $\mathcal{L}_{\tilde{\mu}}(x) = \int_Y x(y) d\tilde{\mu}(y)$, where $x \in X,\ \mathcal{L}_{\tilde{\mu}}(x) \in \mathcal{A}$,

Key property of h-admissible measures $\tilde{\mu}$: its Mellin transform $\mathcal{L}_{\tilde{\mu}}$ is analytic with values in $\mathcal{A}$.

Let $\mathcal{A} = \mathcal{A}(\mathcal{B}) = \mathcal{A}_1 \hat{\otimes} \mathcal{A}_2 \hat{\otimes} \mathcal{A}_3 = \mathcal{A}(\mathcal{B}_1) \hat{\otimes} \mathcal{A}(\mathcal{B}_2) \hat{\otimes} \mathcal{A}(\mathcal{B}_3)$ denote again the Banach algebra $\mathcal{A}$ where $\mathcal{B}$ is an affinoid neighbourhood around $(k_1, k_2, k_3) \in X^3$ (with a given integer k and Dirichlet character $\psi \bmod N$).

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THEOREM D (ON p-ADIC ANALYTIC FUNCTION IN FOUR VARIABLES *Put* $H = [2\mathrm{ord}_p(\lambda)] + 1$. *There exists a p-adic analytic function in four variables* $(x, \mathbf{s}) \in X \times \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3 \subset X \times X \times X \times X$:

$$\mathcal{L}_{\tilde{\mu}} : (x, \mathbf{s}) \longmapsto ev_{\mathbf{s}}(\mathcal{L}_{\tilde{\mu}(x)}) \quad (x \in X, \quad \mathcal{L}_{\tilde{\mu}}(x) \in \mathcal{A}).$$

with values in $\mathbb{C}_p$, *such that for all* (k_1', k_2', k_3') *in the affinoid neighborhood as above* $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3$, *satisfying* $k_1' \leq k_2' + k_3' - 2$, *we have that the special values* $\mathcal{L}_{\tilde{\mu}}(x, \mathbf{s})$ *at the arithmetical chracters* $x = y_p^r \chi$, *and* $\mathbf{s} = (k_1', k_2', k_3') \in \mathcal{B}$ *coincide with the normalized critical special values*

$$\Lambda^*(f_{1,k_1'} \otimes f_{2,k_2'} \otimes f_{3,k_3'}, k_2' + k_3' - 2 - r, \chi) \quad (r = 0, \cdots, k_2' + k_3' - k_1' - 2),$$

for Dirichlet characters $\chi \bmod Np^v, v \geq 1$, *such that all three corresonding Dirichlet characters* χ_j *given by (5.13), have* Np*-complete conductors where the same normalisation of* Λ^* *as in Theorem C.*

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Moreover, for any fixed $\mathbf{s} = (k_1', k_2', k_3') \in \mathcal{B}$ *the function*

$$x \longmapsto \mathcal{L}_{\tilde{\mu}}(x, \mathbf{s})$$

is p-adic analytic of type $o(\log^H(\cdot))$.

Indeed, we obtain the function in question $\mathcal{L}_\mu(x, \mathbf{s})$ by evaluation at $\mathbf{s} = ((s_1, \psi_1), (s_2, \psi_2), (s_3, \psi_3)) \in \mathcal{B}$: this is a p-adic analytic function in four variables $(x, \mathbf{s}) \in X \times \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3 \subset X \times X \times X \times X$:

$$\mathcal{L}_{\tilde{\mu}}(x, \mathbf{s}) := ev_{\mathbf{s}}(\mathcal{L}_{\tilde{\mu}})(x) \quad (x \in X, \ \mathbf{s} \in \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3, \ \mathcal{L}_{\tilde{\mu}}(x) \in \mathcal{A}).$$

This is a joint work in progress with S.Boecherer, we use:

1) the higher twists of the Siegel-Eisenstein series, studied in [PaSE],

2) Ibukiyama's differential operators (see [Ibu], [BSY]).

5.2 Scheme of the Proof

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1) A crucial point of our construction is the higher twist . We define the higher twist of the series $F_{\chi,r}$ by the characters (5.13) as the following formal nearly holomorphic Fourier expansion:

$$F_{\chi,r}^{\bar\chi_1,\bar\chi_2,\bar\chi_3} = \sum_{\mathcal{T}} \bar\chi_1(t_{12})\bar\chi_2(t_{13})\bar\chi_3(t_{23})\det(\mathcal{T})^{k-2r-\kappa}Q(R,\mathcal{T};k-2r,r)a_{\chi,r}(\mathcal{T})q^{\mathcal{T}}. \quad (5.15)$$

Here for an arbitrary Dirichlet character χ mod Np^v we consider

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the Dirichlet characters (5.13):

$$\chi_1 \bmod Np^v = \chi,\ \ \chi_2 \bmod Np^v = \psi_2\bar\psi_3\chi,$$
$$\chi_3 \bmod Np^v = \psi_1\bar\psi_3\chi, \boldsymbol{\psi} = \chi^2\psi_1\psi_2\overline{\psi}_3.$$

We use the Siegel-Eisenstein series $F_{\chi,r}$ which depends on the character χ, but its precise nebentypus character is $\boldsymbol{\psi} = \chi^2\psi_1\psi_2\overline{\psi}_3$, and it is defined by $F_{\chi,r} = G^\star(z,-r;k,(Np^v)^2,\boldsymbol{\psi})$, where z denotes a variable in the Siegel upper half space $\mathbb{H}_3$, and the normalized series $G^\star(z,s;k,(Np^v)^2,\boldsymbol{\psi})$ is given by (4.6).

This series depends on $s=-r$, and for the critical values at integral points $s\in\mathbb{Z}$ such that $2-k\le s\le 0$, it represents a (*nearly*) *holomorphic* Siegel modular form in the sense of Shimura [ShiAr]:

$$F_{\chi,r} = \sum_{\mathcal{T}} \det(\mathcal{T})^{k-2r-\kappa}Q(R,\mathcal{T};k-2r,r)a_{\chi,r}(\mathcal{T})q^{\mathcal{T}}.$$

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2) We use an algebraic version of Ibukiyama's differential operator, which generalizes the algebraic "pull-back": it transforms a nearly holomorphic Siegel modular form of weight k'

to a nearly holomorphic triple modular form of weight (k'_1, k'_2, k'_3) $(k' = k'_2 + k'_3 - k'_1)$.

On a holomorphic Siegel modular form $F = \sum_{\mathcal{T}} a(\mathcal{T})q^{\mathcal{T}}$, this operator has the form

$$\mathcal{L}_{k'}^{\lambda',\nu'}(F) = \sum_{t_1,t_2,t_3 \geq 0} \sum_{\substack{\mathcal{T}: t_{11}=t_1, \\ t_{22}=t_2, t_{23}=t_3}} \mathcal{P}(k'_1, k'_2, k'_3, \mathcal{T})a(\mathcal{T})q_1^{t_1}q_2^{t_2}q_3^{t_3},$$

where $\lambda' = k'_1 - k'_3 \geq \mu' = k'_1 - k'_2 \geq 0$, and $\mathcal{P}(k'_1, k'_2, k'_3; r; \mathcal{T})$ $= (t_{11}t_{22}t_{33})^{\lambda'}(t_{12}t_{13}t_{23})^{\mu'}$ is certain Ibukiyama's polynomial.

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As a result we obtain a sequence of triple modular distributions $\Phi_r(\chi)$ with values in the tensor product $\mathcal{M}_T(\mathcal{A}) = \mathcal{M}(\mathcal{A})\widehat{\otimes}_{\mathcal{A}}\mathcal{M}(\mathcal{A})\widehat{\otimes}_{\mathcal{A}}\mathcal{M}(\mathcal{A})$ of three Banach $\mathcal{A}$-modules of arithmetical nearly holomorphic modular forms (the normalizing factor 2^r is neeeded in order to prove certain congruences between Φ_r). Note that $\mathcal{M}_T(\mathcal{A})$ is again a Banach $\mathcal{A}$-module on which U_T acts as a completely continuous operator.

The important property of the triple modular forms $\Phi_r(\chi)$: the nebentypus character is fixed and is equal to (ψ_1, ψ_2, ψ_3) (for all (k'_1, k'_2, k_3) and χ in question).

Using this property we compute the canonical projection $\pi_\lambda(\Phi_r(\chi))$ of the triple modular form $\Phi_r(\chi)$ onto the λ-characteristic $\mathcal{A}$-submodule $\mathcal{M}_T{}^{\lambda}(\mathcal{A})$ of the triple Atkin's operator $U_{T,p}$:

$$\pi_\lambda : \mathcal{M}_T(\mathcal{A}) \to \mathcal{M}_T{}^{\lambda}(\mathcal{A}).$$

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We prove that the resulting sequence of modular distributions $\pi_\lambda(\Phi_r)$ on the profinite group Y produces a certain p-adic admissible measure $\tilde{\Phi}^\lambda$ (in the sense of Amice-Vélu, [Am-V]) with values in a certain locally free $\mathcal{A}$-submodule of finite rank

$$\mathcal{M}_T{}^\lambda(\mathcal{A}) \subset \mathcal{M}_T(\mathcal{A}) \subset \bigcup_{v\geq 0} \mathcal{M}_T(Np^v, \psi_1, \psi_2, \psi_3; \mathcal{A})$$

of formal nearly holomorphic triple modular forms of all levels Np^v and the fixed nebentypus characters (ψ_1, ψ_2, ψ_3). We use congruences between triple modular forms $\Phi_r(\chi) \in \mathcal{M}_T(\mathcal{A})$ (they have explicit formal Fourier coefficients).

Then we use a general admissibility criterion saying that these congruences imply H-admissibility for their projections in $\mathcal{M}_T^\lambda(\mathcal{A})$, where $H = [2\lambda] + 1$.

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3) From $\mathcal{M}_T^\lambda(\mathcal{A})$ to $\mathcal{A}$: we use a $\overline{\mathbb{Q}}$-valued linear forms of type

$$\mathcal{L}\ : h \longmapsto \frac{\left\langle \tilde{f}_1 \otimes \tilde{f}_2 \otimes \tilde{f}_3, h \right\rangle}{\left\langle \tilde{f}_1, \tilde{f}_1 \right\rangle \left\langle \tilde{f}_2, \tilde{f}_2 \right\rangle \left\langle \tilde{f}_3, \tilde{f}_3 \right\rangle}$$

defined on the finite dimensional $\overline{\mathbb{Q}}$-vector characteristic subspace

$h \in \mathcal{M}_{\mathbf{k}'}(\overline{\mathbb{Q}})^{\lambda(\mathbf{k}')} \subset$

$\mathcal{M}_{k_1,r^*}(Np, \psi_1; \overline{\mathbb{Q}}) \otimes \mathcal{M}_{k_2,r^*}(Np, \psi_2; \overline{\mathbb{Q}}) \otimes \mathcal{M}_{k_3,r^*}(Np, \psi_3; \overline{\mathbb{Q}})$.

This map is then extended to an $\mathcal{A}$-linear map

$$\ell = \ell_{\mathbf{f}_1 \otimes \mathbf{f}_2 \otimes \mathbf{f}_3, \lambda} : \mathcal{M}(\mathcal{A})^\lambda \to \mathcal{A}; \qquad (5.16)$$

on the locally free $\mathcal{A}$-module of finite rank $\mathcal{M}(\mathcal{A})^\lambda$.

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This map produces a sequence of $\mathcal{A}$-valued distributions $\mu_r^\lambda(\chi) \in \mathcal{A}$ in such a way that for all suitable weights $\mathbf{k}' \in \mathcal{B}$ one has

$$ev_{\mathbf{k}'}(\mu_r^\lambda(\chi)) = \mathcal{L}(ev_{\mathbf{k}'}(\pi_\lambda(\Phi_r)(\chi))), \lambda \in \mathcal{A}^\times, \lambda(\mathbf{k}') \in \overline{\mathbb{Q}}^\times,$$

where $\mathbf{k}' = (k'_1, k'_2, k'_3) \in \mathcal{B}$, $ev_{\mathbf{k}'} : \mathcal{B} \to \mathbb{C}_p$ denotes the evaluation map with the property

$$ev_{\mathbf{k}'} : \mathcal{M}(\mathcal{A}) \to \mathcal{M}_{\mathbf{k}'}(\mathbb{C}_p).$$

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More precisely, we consider three auxilliary families of modular forms

$$\tilde{f}_{j,k'_j}(z) = \sum_{n=1}^{\infty} \tilde{a}_{n,j,k'_j} e(nz) \in S_{k'_j}(\Gamma_0(N_j p^{\nu_j}), \psi_j), \ (1 \le j \le 3, \nu_j \ge 1), \tag{5.17}$$

with the same eigenvalues as those of (5.14), for all Hecke operators T_q, with q prime to Np. In our construction we use as $\tilde{f}_{j,k'_j}$ certain "easy transforms" of primitive cusp forms in (1.1). In particular, we choose as $\tilde{f}_j$ certain eigenfunctions $\tilde{f}_{j,k'_j} = f^0_{j,k'_j}$ of the adjoint Atkin's operator U_p^*, in this case we denote by $f_{j,k'_j,0}$ the corresponding eigenfunctions of U_p.

The $\overline{\mathbb{Q}}$-linear form $\mathcal{L}$ produces a $\mathbb{C}_p$-valued admissible measure $\tilde{\mu}^\lambda = \ell(\tilde{\Phi}^\lambda)$ starting from the modular p-adic admissible measure $\tilde{\Phi}^\lambda$ of stage 3), where $\ell : \mathcal{M}_T(\mathbb{C}_p) \to \mathbb{C}_p$ denotes a $\mathbb{C}_p$-linear form, interpolating $\mathcal{L}$.

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4) We show that for any appropriate Dirichlet character χ mod Np^v the integral

$$\mu_r^\lambda(\chi) = \mathcal{L}(\pi_\lambda(\Phi_r(\chi))) \in \mathcal{A}$$

evaluated at $(k'_1, k'_2, k'_3) \in \mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3$, coincides (up to a normalisation) with the special L-value

$$\mathcal{D}^*(f^\rho_{1,k'_1} \otimes f^\rho_{2,k'_2} \otimes f^\rho_{3,k'_3}, k'_2 + k'_3 - 2 - r, \psi_1\psi_2\chi)$$

under the above assumptions on χ and r).

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We use a general integral representation of Garrett's type. The basic idea how a Dirichlet character χ is incorporated in the integral representation [Ga87, BoeSP] is somewhat similar to the one used in [Boe-Schm], but (surprisingly) more complicated to carry out.

Note however that the existence of a $\mathcal{A}$-valued admissible measure $\tilde{\mu}^\lambda = \ell(\tilde{\Phi}^\lambda)$ already established at stage 4), does not depend on this technical computation.

6 Criterion of admissibility

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THEOREM 6.1 *Let* $0 < |\alpha|_p < 1$ *and suppose that there exists a positive integer* $\varkappa$ *such that the following conditions are satisfied: for all* $r = 0, 1, \cdots, h-1$ *with* $h = [\varkappa \mathrm{ord}_p \alpha] + 1$*, and* $v \geq 1$*,*

$$\Phi_r(a + (Np^v)) \in \mathcal{M}(Np^{\varkappa v}) \qquad \text{(the level condition)} \tag{6.1}$$

and the following congruence holds: for all $t = 0, 1, \cdots, h-1$

$$U^{\varkappa v} \sum_{r=0}^{t} \binom{t}{r} (-a_p)^{t-r} \Phi_r(a + (Np^v)) \equiv 0 \bmod p^{vt} \tag{6.2}$$

(the divisibility condition)

Then the linear form given by $\tilde{\Phi}^\alpha(\delta_{a+(Np^v)} y_p^r) := \pi_\alpha(\Phi_r(a + (Np^v))$ *on local monomials (for all* $r = 0, 1, \cdots, h-1$*), is an* h*-admissible measure:* $\tilde{\Phi}^\alpha : \mathcal{P}^h(Y, \overline{\mathbb{Q}}) \to \mathcal{M}^\alpha \subset \mathcal{M}$

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Proof uses the commutative diagram:

$$\begin{array}{ccccc} \mathcal{M}(Np^{v+1}, \psi; \mathcal{A}) & \xrightarrow{\pi_{\alpha,v}} & \mathcal{M}^\alpha(Np^{v+1}, \psi; \mathcal{A}) & & \\ U^v \downarrow & & \downarrow \wr\, U^v & & \\ \mathcal{M}(Np, \psi; \mathcal{A}) & \xrightarrow[\pi_{\alpha,0}]{} & \mathcal{M}^\alpha(Np, \psi; \mathcal{A}) & = & \mathcal{M}^\alpha(Np^{v+1}, \psi; \mathcal{A}). \end{array}$$

The existence of the projectors $\pi_{\alpha,v}$ comes from Coleman's Theorem A.4.3 [CoPB].

On the right: U acts on the locally free $\mathcal{A}$-module $\mathcal{M}^\alpha(Np^{v+1}, \mathcal{A})$ via the matrix:

$$\begin{pmatrix} \alpha & \cdots & \cdots & * \\ 0 & \alpha & \cdots & * \\ 0 & 0 & \ddots & \cdots \\ 0 & 0 & \cdots & \alpha \end{pmatrix} \text{ where } \alpha \in \mathcal{A}^\times$$

Slide 64 $\Longrightarrow U^v$ is an isomorphism over $\mathrm{Frac}(\mathcal{A})$,

and one controls the denominators of the modular forms of all levels v by the relation:

$$\pi_{\alpha,v}(h) = U^{-v}\pi_{\alpha,0}(U^v h) =: \pi_\alpha(h) \tag{6.3}$$

The equality (6.3) can be used as the definition of π_α. The growth condition (see section 5) for $\pi_\alpha(\Phi_r)$ is deduced from the congruences (6.2) between modular forms, using the relation (6.3).

7 Main congruence for the higher twists

The purpose of this section is to show that the admissibility criterion of Theorem 6.1 with $\varkappa = 2$ is satisfied by the sequence of triple modular distributions Φ_r.

Slide 65 We need to check that the nearly holomorphic triple modular forms $\Phi_r(\chi)$ are of level $N^2\chi^{2v}$, nebentypus (ψ_1, ψ_2, ψ_3), and satisfy the congruences

$$\left| U_T^{2v}\Big(\sum_{r'=0}^{r}\binom{r}{r'}(-a_p^0)^{r-r'}\Phi_{r'}\big((a)_v\big)\Big)\right|_p \le Cp^{-vr} \tag{7.1}$$

and for all $r = 0, 1, \cdots, k-2$.

7.1 Special Fourier coefficients of the higher twist of the Siegel-Eisenstein distributions

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Let us use the Fourier expansions of $\Phi_r(\chi)|U_p^{2v}$ as follows

$$\Phi_r(\chi)|U_p^{2v} = \sum_{t_1,t_2 t_3\geq 0} a(p^{2v}t_1, p^{2v}t_2, p^{2v}t_3; p^{2v}R_1, p^{2v}R_2, p^{2v}R_3, r) q_1^{t_1} q_2^{t_2} q_3^{t_3} \tag{7.2}$$

with

$$\begin{aligned} &a(p^{2v}t_1, p^{2v}t_2, p^{2v}t_3; p^{2v}R_1, p^{2v}R_2, p^{2v}R_3, r) \\ &\quad = \sum_{\mathcal{T}:\mathrm{diag}(\mathcal{T})=(p^{2v}t_1,p^{2v}t_2,p^{2v}t_3)} \bar\chi(t_{12}t_{13}t_{23})\bar\psi_2\psi_3(t_{13})\bar\psi_1\psi_3(t_{23})\times \\ &\quad \times \det(\mathcal{T})^{k-2r-\kappa} Q(p^{2v}\mathrm{diag}(R_1,R_2,R_3),\mathcal{T};k-2r,r)2^r a_{\chi,r}(\mathcal{T}) \\ &\quad = \sum_{\mathcal{T}:\mathrm{diag}(\mathcal{T})=(p^{2v}t_1,p^{2v}t_2,p^{2v}t_3)} v_{\chi,r}(\mathcal{T},\mathrm{diag}(R_1,R_2,R_3)), \end{aligned}$$

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where

$$\begin{aligned} v_{\chi,r}(\mathcal{T},\mathrm{diag}(R_1,R_2,R_3)) &= \bar\chi(t_{12}t_{13}t_{23})\bar\psi_2\psi_3(t_{13})\bar\psi_1\psi_3(t_{23})\times \\ &\quad \times \det(\mathcal{T})^{k-2r-\kappa} Q(p^{2v}\mathrm{diag}(R_1,R_2,R_3),\mathcal{T};k-2r,r)2^r a_{\chi,r}(\mathcal{T}) \\ &= \chi^{(p)}(2)\bar\chi^{(p)}(\mathcal{T})\chi^\circ(t_{12}t_{13}t_{23})\bar\psi_2\psi_3(t_{13})\bar\psi_1\psi_3(t_{23})\times \\ &\quad \times \det(\mathcal{T})^{k-2r-\kappa} Q(p^{2v}\mathrm{diag}(R_1,R_2,R_3),\mathcal{T};k-2r,r)2^r a_{\chi,r}(\mathcal{T}). \end{aligned} \tag{7.3}$$

Let us notice that, for any $\mathcal{T}$ with $\mathrm{diag}(\mathcal{T}) = (p^{2v}t_1, p^{2v}t_2, p^{2v}t_3)$ one has $\det(\mathcal{T}) \equiv 2t_{12}t_{13}t_{23} \bmod p^{2v}$, $\chi^{(p)}(2t_{12}t_{13}t_{23}) = \chi^{(p)}(\det(\mathcal{T})) = \chi(\det(\mathcal{T})\overline{\chi^\circ(\det(\mathcal{T})}$,

$$2^r a_{\chi,r}(\mathcal{T}) = \int_Y y_p^r \chi(y) d\mathcal{F}_\mathcal{T},$$

with $\chi = \chi^{(p)}\chi^\circ, \chi^{(p)} \bmod p^v, \chi^\circ \bmod N,$ and $p \nmid N$,

for a bounded measure $\mathcal{F}_\mathcal{T}$ on Y with values in $\overline{\mathbb{Q}}$.

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It follows that

$$v_{\chi,r}(\mathcal{T}, \mathrm{diag}(R_1, R_2, R_3)) =$$
$$\chi^{(p)}(2)\bar{\chi}(\det(\mathcal{T}))\det(\mathcal{T})^{-r}\overline{\chi^\circ(\det(\mathcal{T})}\bar{\psi}_2\psi_3(t_{13})\bar{\psi}_1\psi_3(t_{23})\cdot$$
$$\cdot \det(\mathcal{T})^{k-r-\kappa}Q(p^{2v}\mathrm{diag}(R_1, R_2, R_3), \mathcal{T}; k-2r, r)2^r a_{\chi,r}(\mathcal{T}) \tag{7.4}$$
$$= \det(\mathcal{T})^{k-r-\kappa}Q(p^{2v}\mathrm{diag}(R_1, R_2, R_3), \mathcal{T}; k-2r, r)\overline{\chi^\circ}(2)$$
$$\int_Y \chi y_p^r \mathrm{d}\mathcal{F}_{\mathcal{T};\chi^\circ,\psi_1,\psi_2,\psi_3},$$

where $\mathcal{F}_{\mathcal{T};\chi^\circ,\psi_1,\psi_2,\psi_3}$ denotes the bounded measure defined by the equality:

$$\int_Y \chi y_p^r \mathrm{d}\mathcal{F}_{\mathcal{T};\chi^\circ,\psi_1,\psi_2,\psi_3} \tag{7.5}$$
$$= \chi^{(p)}(2)\chi^\circ(2)2^r\bar{\chi}(\det(\mathcal{T}))\det(\mathcal{T})^{-r}\overline{\chi^\circ(\det(\mathcal{T})}$$
$$\bar{\psi}_2\psi_3(t_{13})\bar{\psi}_1\psi_3(t_{23})a_{\chi,r}(\mathcal{T}).$$

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7.2 Main congruence for the Fourier expansions

Let us use the orthogonality relations for Dirichlet characters in order to prove the admissibility of the distributions given by the sequence $\pi_\lambda(\Phi_r(\chi))$ using the Fourier expansions (7.2). According to the admissibility criterion of Theorem 6.1 we need to check the following *main congruence*:

$$\Big|\sum_{r'=0}^{r}\binom{r}{r'}(-a_p^0)^{r-r'} \tag{7.6}$$
$$\frac{1}{\varphi(Np^v)}\sum_{\chi \bmod Np^v}\chi^{-1}(a)v_{\chi,r'}(\mathcal{T}, p^{2v} \quad \mathrm{diag}(R_1, R_2, R_3))\Big|_p \le Cp^{-vr},$$

where we use the notation (7.4) for $v_{\chi,r'}(\mathcal{T}, \mathrm{diag}(R_1, R_2, R_3))$, implying that the coefficients

$$i_p(v_{\chi,r'}(\mathcal{T}, \mathrm{diag}(R_1, R_2, R_3)))$$

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in (7.3) are given as sums of the following expressions:

$$B_r(\chi,\mathcal{T}) = \overline{\chi^\circ}(2)\det(\mathcal{T})^{k-r-\kappa}\int_Y \chi y_p^r d\mathcal{F}_{\mathcal{T};\chi^\circ,\psi_1,\psi_2,\psi_3}\cdot \tag{7.7}$$

$$\cdot\sum_{t=0}^{r}\binom{r}{t}\det(\mathcal{T})^{r-t}\sum_{|L|\le mt-t} R_L(\kappa-k+r)Q_L(p^{2v}\mathrm{diag}(R_1,R_2,R_3),\mathcal{T}),$$

where $\mathcal{F}_{\mathcal{T};\chi^\circ,\psi_1,\psi_2,\psi_3}$ denotes the bounded measure defined by (7.5). Using the expressions (7.7), the main congruence (7.6) is reduced to proving the congruence for the numbers $B_r(\chi,\mathcal{T})$: there exists a non-zero integer C_k such that

$$C_k\cdot\sum_{r'=0}^{r}\binom{r}{r'}(-a_p^0)^{r-r'}\frac{1}{\varphi(Np^v)}\sum_{\chi \mathrm{mod} Np^v}\chi^{-1}(a)B_{r'}(\chi,\mathcal{T})\equiv 0 \bmod p^{vr} \tag{7.8}$$

$$\iff C_k\cdot A\equiv 0 \bmod Np^{vr},$$

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where we use the notation

$$\begin{aligned} A =& A_r(\mathcal{T};\chi^\circ,\psi_1,\psi_2,\psi_3) \qquad (7.9)\\ &=\sum_{r'=0}^{r}\binom{r}{r'}(-a_p^0)^{r-r'}\frac{1}{\varphi(Np^v)}\sum_{\chi \bmod Np^v}\chi^{-1}(a)\cdot\\ &\cdot\overline{\chi^\circ}(2)\det(\mathcal{T})^{k-r'-\kappa}\int_Y\chi y_p^{r'}d\mathcal{F}_{\mathcal{T};\chi^\circ,\psi_1,\psi_2,\psi_3}\sum_{t=0}^{r'}\binom{r'}{t}\det(\mathcal{T})^{r'-t}\\ &\sum_{|L|\le mt-t}R_L(\kappa-k+r')Q_L(p^{2v}\mathrm{diag}(R_1,R_2,R_3),\mathcal{T}).\end{aligned}$$

Note that $R_L(\kappa-k+r')$ is a polynomial of degree $mt-|L|=3t-|L|$ in $\kappa-k+r'$ (see (4.5)), hence in r', and $\binom{r'}{t}$ is a polynomial of degree t in r'. One can therefore write

$$\binom{r'}{t}R_L(\kappa-k+r)=\sum_{n=0}^{4t-|L|}\mu_n\frac{(r'+n+1)!}{(r'+1)!}.$$

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Here the coefficients μ_n are certain fixed rational numbers (independent of r').

Using the orthogonality relations for Dirichlet characters $\bmod Np^v$, we see that the sum over r' in (7.9), denoted by $C = C_r(t, L, \mathcal{T}; \chi^\circ, \psi_1, \psi_2, \psi_3)$, takes the form

$$C_r(t, L, \mathcal{T}; \chi^\circ, \psi_1, \psi_2, \psi_3) = \overline{\chi^\circ}(2) \det(\mathcal{T})^{k-t-\kappa}$$

$$\int_{y \equiv a \bmod p^v} \sum_{n=0}^{4t-|L|} \mu_n \underbrace{\sum_{r'=0}^{r} \binom{r}{r'} (-a)^{r-r'} \frac{(r'+n+1)!}{(r'+1)!} y^{r'}}_{y^{-n} \frac{\partial^n}{\partial y^n} \left(y^{n+1}(y-a)^r\right)} \mathrm{d}\mathcal{F}_{\mathcal{T}}(y)$$

where $\mathcal{F}_{\mathcal{T}}(y) = \mathcal{F}_{\mathcal{T};\chi^\circ,\psi_1,\psi_2,\psi_3}(y)$

Note that we write $\chi = \chi^\circ \chi^{(p)}$, fix χ°, and sommate over all characters $\chi^{(p)} \bmod p^v$. We have therefore $(y-a)^r \equiv 0 \bmod (p^v)^r$

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in the integration domain $y \equiv a \bmod p^v$, implying the congruence

$$c_k C_r(t, L, \mathcal{T}; \chi^\circ, \psi_1, \psi_2, \psi_3) \equiv 0 \pmod{(p^v)^{r-n}} \qquad (7.10)$$

$$\Longrightarrow \qquad \equiv 0 \pmod{(p^v)^{r-4t+|L|}},$$

where $c_k \in \mathbb{Q}^*$ is a nonzero constant coming from the denominators of the fixed rational numbers μ_n, and of the bounded distributions $\mathcal{F}_{\mathcal{T};\chi^\circ,\psi_1,\psi_2,\psi_3}$.

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7.3 Proof of the Main congruence

Now the expression (7.9) transforms to

$$A_r(\mathfrak{T}) = \sum_{t=0}^{r} \sum_{|L|\le 2t} \det(\mathfrak{T})^t \qquad (7.11)$$
$$C(t,L,\mathfrak{T}) \det(\mathfrak{T})^{k-2r-\kappa} Q_L(p^{2v}\mathrm{diag}(R_1,R_2,R_3),\mathfrak{T}),$$

where $Q_L(p^{2v}\mathrm{diag}(R_1,R_2,R_3),\mathfrak{T})$ is a homogeneuos polynomial of degree $3t-|L|$ in the variables R_{ij} implying the congruence

$$Q_L(p^{2v}\mathrm{diag}(R_1,R_2,R_3),\mathfrak{T}) \equiv 0 \pmod{(p^{2v})^{(3t-|L|)}}. \qquad (7.12)$$

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On the other hand we know from the description (4.5) of the polynomial

$$Q(R,\mathfrak{T}) = Q(R,\mathfrak{T};k-2r,r) =$$
$$\sum_{t=0}^{r} \binom{r}{t} \det(\mathfrak{T})^{r-t} \sum_{|L|\le 2t} R_L(\kappa-k+r) Q_L(R,\mathfrak{T}),$$
$$Q_L(R,\mathfrak{T}) = \mathrm{tr}\left({}^t\rho_{3-l_1}(R)\rho^\star_{l_1}(\mathfrak{T})\right)\cdot\ldots\cdot\mathrm{tr}\left({}^t\rho_{3-l_t}(R)\rho^\star_{l_t}(\mathfrak{T})\right),$$

that $2t-|L|\ge 0$ so we obtain the desired congruence as follows

$$\begin{cases} c_k C_r(t,L,\mathfrak{T}) \equiv 0 \pmod{(p^v)^{r-4t+|L|}} \\ Q_L(p^{2v}\mathrm{diag}(R_1,R_2,R_3),\mathfrak{T}) \equiv 0 \pmod{(p^{2v})^{(3t-|L|)}} \end{cases} \qquad (7.13)$$
$$\Rightarrow c_k A_r(\mathfrak{T}) \equiv 0 \pmod{p^{vr}},$$

since $v(r-4t+|L|)+2v(3t-|L|) = vr+2vt-v|L| \ge vr$, proving (7.6). ∎

8 Computation of p-adic integrals and L-values

8.1 Construction of p-adic measures

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Let $\mathcal{M} = \mathcal{M}_T(\mathcal{A}) =$
$\bigcup_{v\geq 0} \mathcal{M}_{r^*}(Np^v, \psi_1; \mathcal{A}) \otimes_{\mathcal{A}} \mathcal{M}_{k,r^*}(Np^v, \psi_2; \mathcal{A}) \otimes_{\mathcal{A}} \mathcal{M}_{k,r^*}(Np^v, \psi_3; \mathcal{A})$
be the $\mathcal{A}$-module of nearly holomorphic triple modular forms with formal Fourier coefficients in the $\mathbb{C}_p$-Banach algebra $\mathcal{A} = \mathcal{A}(\mathcal{B})$, where $\mathbf{k}' = (k'_1, k'_2, k'_3) \in \mathcal{B}$, $ev_{\mathbf{k}'} : \mathcal{B} \to \mathbb{C}_p$ denotes the evaluation map with the property

$$ev_{\mathbf{k}'} : \mathcal{M}(\mathcal{A}) \to \mathcal{M}_{\mathbf{k}'}(\mathbb{C}_p).$$

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Let us define an $\mathcal{A}$-valued measure

$$\tilde{\mu}^\lambda(y; \mathbf{f}_1 \otimes \mathbf{f}_2 \otimes \mathbf{f}_3) : \mathcal{C}^{loc-an}(Y, \mathcal{A}) \to \mathcal{A}$$

by applying a certain trilinear form $\ell_{T,\lambda} : \mathcal{M}(Np^v; \mathcal{A}) \to \mathcal{A}$

$$\tilde{\mu}^\lambda(y; \mathbf{f}_1 \otimes \mathbf{f}_2 \otimes \mathbf{f}_3) = \ell_{T,\lambda}(\tilde{\Phi}^\lambda) \tag{8.1}$$

to the h-admissible measure $\tilde{\Phi}^\lambda$ of Theorem 6.1 on Y with values in $\mathcal{M}(\mathcal{A})^\lambda \subset \mathcal{M}(Np; \mathcal{A})$. That h-admissible measure was defined as an $\mathcal{A}$-linear map $\tilde{\Phi}^\lambda : \mathcal{P}^h(Y, \mathcal{A}) \to \mathcal{M}(\mathcal{A})^\lambda$ satisfying for any $(a)_\nu \subset Y$ and for all $r = 0, 1, \ldots, h-1$ the following equality:

$$\int_{(a)_\nu} y_p^r \, d\tilde{\Phi}^\lambda = \pi_\lambda(\Phi_r((a)_\nu)) \in \mathcal{M}(Np),$$

where $h = [2\mathrm{ord}_p \lambda(p)] + 1$, hence

$$\int_{(a)_\nu} y_p^r \, \mathrm{d}\tilde{\mu}^\lambda(y; f_1 \otimes f_2 \otimes f_3) = \ell_{T,\lambda}\left(\int_{(a)_\nu} y_p^r \, \mathrm{d}\tilde{\Phi}^\lambda(y)\right). \tag{8.2}$$

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8.2 Evaluation of the integral

$$ev_{\mathbf{k}'}\left(\int_Y \chi(y)\, y_p^r \, d\tilde{\mu}^\lambda(y; \mathbf{f}_1 \otimes \mathbf{f}_2 \otimes \mathbf{f}_3)\right) \quad (8.3)$$
$$= \int_Y \chi(y)\, y_p^r \, d\tilde{\mu}^\lambda(y; f_{1,k_1'} \otimes f_{2,k_2'} \otimes f_{3,k_3'})$$

for $r \in \mathbb{N}$, $0 \le r \le k_2' + k_3' - k_1' - 2$. The result is given in terms of Garrett's triple L function
$\mathcal{D}^\star(f^\rho_{1,k_1'} \otimes f^\rho_{2,k_2'} \otimes f^\rho_{3,k_3'}, k_2' + k_3' - 2 - r, \psi_1\psi_2\chi)$. Let us use the action of the involution $W_{N_j} = \begin{pmatrix} 0 & -1 \\ N_j & 0 \end{pmatrix}$ of the exact level N_j of

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f_j:

$$f_{j,k_j'}\big|_k W_{N_j} = \begin{pmatrix} 0 & -1 \\ N_j & 0 \end{pmatrix} = \gamma_{j,k_j'} \cdot f^\rho_{j,k_j'},$$
$$f^\rho_{j,k_j'}\big|_k W_{N_j} = \begin{pmatrix} 0 & -1 \\ N_j & 0 \end{pmatrix} = \bar{\gamma}_{j,k_j'} \cdot f_{j,k_j'},$$

$$\text{where } f^\rho_{j,k_j'}(z) = \sum_{n=1}^{\infty} \bar{a}_{n,j,k_j'} e(nz) \in \mathcal{S}_{k_j'}(N_j, \bar{\psi}_j), \quad (8.4)$$

$(j = 1, 2, 3)$ and $\gamma_{j,k_j'}$ is the corresponding root number. (8.5)

Recall the notation (5.11) and (5.12):

$$f_{j,k'_j,0} = f_{j,k'_j} - \alpha^{(2)}_{p,j,k'_j} f_{j,k'_j}|V_p = f_{j,k'_j} - \alpha^{(2)}_{p,j,k'_j} p^{-k/2} f_{j,k'_j}| \begin{pmatrix} p\,0 \\ 0\,1 \end{pmatrix}$$

$$f^{\rho}_{j,k'_j,0} = \sum_{n=1}^{\infty} \overline{a(n,f_0)} q^n, \;\; f^{0}_{j,k'_j} = f^{\rho}_{j,k'_j,0}|_k \; W_{Np} = f^{\rho}_{j,k'_j,0}\Big|_k \begin{pmatrix} 0 & -1 \\ Np & 0 \end{pmatrix}.$$

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PROPOSITION 8.1 *Under the notations and assumptions as in Theorem C, the value of the integral (8.3) at* $\mathbf{k}' = (k'_1, k'_2, k'_3)$ *is given for* $0 \le r \le k'_2 + k'_3 - k'_1 - 2$ *by the image under* i_p *of the following algebraic number*

$$T(\mathbf{k}') \cdot \lambda^{-2v} \mathfrak{L}_{Np}(-r)$$
$$\frac{\mathcal{D}^{\star}(f^{\rho}_{1,k'_1} \otimes f^{\rho}_{2,k'_2} \otimes f^{\rho}_{3,k'_3}, k'_2 + k'_3 - k'_1 - 2 - r, \psi_1\psi_2\chi)}{\langle f^{0}_{1,k'_1} \otimes f^{0}_{2,k'_2} \otimes f^{0}_{3,k'_3}, f_{1,k'_1,0} \otimes f_{2,k'_2,0} \otimes f_{3,k'_3,0} \otimes \rangle_{T,N^2p^{2v}}},$$

where

$$T(\mathbf{k}') =$$
$$2^{-r} \frac{((Np)^3/N_1N_2N_3)^{k/2} \bar{\gamma}_{1,k'_1} \bar{\gamma}_{2,k'_2} \bar{\gamma}_{3,k'_3} (\chi_1\chi_2\chi_3)(2) p^{3\cdot v(k'_2+k'_3-k'_1-2)}}{N_{1,1}N_{1,2}N_{1,3} G(\chi_{1,0}) G(\chi_{2,0}) G(\chi_{3,0})} \times$$
$$\times (Np^{2v})^{k-2r} \frac{N^2p^{2v}\varphi(N^2p^{2v})\varphi(Np^v)}{[\Gamma_0(N^2p^{2v}) : \Gamma(N^2p^{2v})]^3},$$

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γ_{j,k'_j} *is the corresponding root number and the factor* $\mathfrak{L}_{Np}(-r)$, *given by (8.13).*

8.3 Evaluation of the trilinear form

In the rest of this section we write for simplicity k_j , f_j and λ instead of k'_j, $f_{k'_j,j}$ and $\lambda(\mathbf{k}')$.

In order to compute the p-adic integral, the next step of the proof uses computations similar to those in [Hi85], §4 and §7. More

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precisely let us write the integral in the form

$$\int_Y \chi(y)\, y_p^r \, \mathrm{d}\tilde{\mu}_\lambda(y; f_1 \otimes f_2 \otimes f_3) = \sum_{a \in Y_v} \chi(a) \int_{(a)_v} y_p^r \, \mathrm{d}\ell_{T,\lambda}(\tilde{\Phi}^\lambda)(y)) =$$

$$= \ell_{T,\lambda}\left(\sum_{a \in Y_v} \chi(a) \int_{(a)_v} y_p^r \, \mathrm{d}\tilde{\Phi}^\lambda(y) \right) = \ell_{T,\lambda}\left(\sum_{a \in Y_v} \chi(a) \Phi_r^\lambda((a)_v) \right), \tag{8.6}$$

where $(a)_v = (a + (Np^v)) \subset Y$, and by definition (8.1)

$$\tilde{\mu}^\lambda(y; f_1 \otimes f_2 \otimes f_3) = \ell_{T,\lambda}(\tilde{\Phi}^\lambda)(y), \tag{8.7}$$

$$\int_{(a)_v} y_p^r \, \mathrm{d}\big(\tilde{\Phi}^\lambda\big) = \Phi_r^\lambda((a)_v) \in \mathcal{M}_T^\lambda(Np) \tag{8.8}$$

for $r = 0, 1, \ldots, h-1$. Moreover $\Phi_r\big((a)_v\big)$ is a triple modular form of level $N^2 p^{2v}$ as a value of a higher twist of a Siegel-Eisenstein

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distributions, hence

$$\Phi_r^\lambda(\chi) = U_T^{-2v}\left[\pi_{\lambda,T,1} U_T^{2v}\left(2^r \mathcal{L}^{\lambda,\nu}_{k_2+k_3-k_1-r}(F_{\chi,r}^{\bar{\chi}_1,\bar{\chi}_2,\bar{\chi}_3})\right)\right]. \tag{8.9}$$

Taking into account the equalities (8.9), the integral (8.6) transforms to the following

$$\int_Y \chi(y)\, y_p^r \, \mathrm{d}\tilde{\mu}^\lambda(y; f_1 \otimes f_2 \otimes f_3) = \ell_{T,\lambda}\left(\sum_{a \in Y_v} \chi(a) \Phi_r^\lambda((a)_v) \right) \tag{8.10}$$

$$= \ell_{T,\lambda}\Big(U_T^{-2v}\Big[\pi_{\lambda,T,1} U_T^{2v}\big(2^r \mathcal{L}^{\lambda,\nu}_{k_2+k_3-k_1-r}(F_{\chi,r}^{\bar{\chi}_1,\bar{\chi}_2,\bar{\chi}_3})\big)\Big]\Big)$$

Notice that then it follows that the sum in the right hand side of the equality (8.10) can be expressed through the functions:

$$\int_Y \chi(y)\, y_p^r \, \mathrm{d}\tilde{\mu}^\lambda(y; f_1 \otimes f_2 \otimes f_3)(y)$$

$$\ell_{T,\lambda}\Big(U_T^{-2v}\Big[\pi_{\lambda,T,1} U_T^{2v}\big(2^r \mathcal{L}^{\lambda,\nu}_{k_2+k_3-k_1-r}(F_{\chi,r}^{\bar{\chi}_1,\bar{\chi}_2,\bar{\chi}_3})\big)\Big]\Big) \tag{8.11}$$

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where we use $\mathcal{L}^{\lambda,\nu}_{k_2+k_3-k_1}$ (Ibukiyama's operator with $\lambda = k_1 - k_3 \geq \mu = k_1 - k_2 \geq 0$).

The Fourier expansion of the function

$$g = \mathcal{L}^{\lambda,\nu}_{k_2+k_3-k_1}(F^{\bar\chi_1,\bar\chi_2,\bar\chi_3}_{\chi,r})$$

can be computed:

$$\begin{aligned}&\mathcal{E}(z_1, z_2, z_3; -r, k_1, k_2, k_3, Np^v, \boldsymbol{\psi}, \chi_1, \chi_2, \chi_3)\\ &= N_{1,1}N_{1,2}N_{1,3}(\bar\chi_1\bar\chi_2\bar\chi_3)(2)G(\chi_{0,1})G(\chi_{0,2})G(\chi_{0,3}) \cdot g.\end{aligned}$$

Thus it represents a *nearly holomorphic* triple modular form in the $\overline{\mathbb{Q}}$-module

$$\mathcal{M}(\overline{\mathbb{Q}}) = \mathcal{M}_T(N^2p^{2v}, \psi_1 \otimes \psi_2 \otimes \psi_3; \overline{\mathbb{Q}}) \subset$$
$$\mathcal{M}_{k,r^*}(N^2p^{2v}, \psi_1; \overline{\mathbb{Q}}) \otimes \mathcal{M}_{k,r^*}(N^2p^{2v}, \psi_2; \overline{\mathbb{Q}}) \otimes \mathcal{M}_{k,r^*}(N^2p^{2v}, \psi_3; \overline{\mathbb{Q}}).$$

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Then we have:

$$\begin{aligned}&\mathcal{L}_{T,\lambda} : \mathcal{M}_T(N^2p^{2v}; \mathbb{C}) \to \mathbb{C}, \\ &g \mapsto \frac{\langle f^0_1 \otimes f^0_2 \otimes f^0_3, \lambda^{-2v}U^{2v}_T g\rangle_{T,N^2p}}{\langle f^0_1 \otimes f^0_2 \otimes f^0_3, f_{1,0} \otimes f_{2,0} \otimes f_{3,0}\rangle_{T,N^2p}},\end{aligned} \tag{8.12}$$

$$\ell_{T,\lambda}\left(U^{-2v}_T\left[\pi_{\lambda,T,1}U^{2v}_T(g)\right]\right) = i_p\left(\frac{\langle f^0_1 \otimes f^0_2 \otimes f^0_3, \lambda^{-2v}U^{2v}_T(g)\rangle_{T,N^2p}}{\langle f^0_1 \otimes f^0_2 \otimes f^0_3, f_{1,0} \otimes f_{2,0} \otimes f_{3,0}\rangle_{N^2p}}\right)$$
$$= i_p\left(\lambda^{-2v}p^{3\cdot 2v(k-1)} \cdot \frac{\langle V^{2v}(f^0_1 \otimes f^0_2 \otimes f^0_3), g\rangle_{T,N^2p^{2v+1}}}{\langle f^0_1 \otimes f^0_2 \otimes f^0_3, f_{1,0} \otimes f_{2,0} \otimes f_{3,0}\rangle_{T,N^2p}}\right).$$

The scalar products in (8.12) can be computed but we omit here the details. This implies Proposition 8.1 for certain modular forms

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$$\tilde{f}_{j,2v}(z) = \sum_{n=1}^{\infty} a_{j,n,2v} e(nz) \text{ as above:}$$

$$\mathcal{D}^\star(f_1^\rho \otimes f_2^\rho \otimes f_3^\rho, 2k-2-r, \psi_1\psi_2\chi_1) \tag{8.13}$$

$$(Np^{2v})^{k-2r} \frac{N^2 p^{2v} \varphi(N^2p^{2v})\varphi(Np^v)}{[\Gamma_0(N^2p^{2v}) : \Gamma(N^2p^{2v})]^3} \mathfrak{L}_{Np}(-r) =$$

$$\left\langle \tilde{f}_{1,2v} \otimes \tilde{f}_{2,2v} \otimes \tilde{f}_{3,2v}, \mathcal{E}(z_1, z_2, z_3; -r, k, N^2p^{2v}, \boldsymbol{\psi}, \chi_1, \chi_2, \chi_3) \right\rangle_{T, N^2p^{2v}},$$

where

$$\mathfrak{L}_{Np}(s) = \mathfrak{L}_{Np}(s; \tilde{f}_{1,2v} \otimes \tilde{f}_{2,2v} \otimes \tilde{f}_{3,2v})$$
$$:= \sum_{n | N^\infty} G_N(\overline{\psi_1\psi_2\chi_1}, 2n) \frac{a_{n,1,2v} a_{n,2,2v} a_{n,3,2v}}{n^{2s+2k-2}}.$$

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8.4 Proof of Theorem B

Let us use Propostion 8.1 and (8.13):

$$2^{-r} \int_Y \chi(y)\, y_p^r \, \mathrm{d}\tilde{\mu}^\lambda(y; f_1 \otimes f_2 \otimes f_3)(y) \tag{8.14}$$
$$= 2^{-r} \ell_{T,\lambda} \left(U_T^{-2v} \left[\pi_{\lambda,T,1} U_T^{2v}(g) \right] \right)$$

$$= \frac{((Np)^3/N_1N_2N_3)^{k/2} \bar{\gamma}_1 \bar{\gamma}_2 \bar{\gamma}_3 (\chi_1\chi_2\chi_3)(2) p^{3\cdot v(k-2)}}{\lambda^{2v} N_{1,1} N_{2,1} N_{3,1} G(\chi_{1,0}) G(\chi_{2,0}) G(\chi_{3,0})} \times$$
$$\times (Np^{2v})^{k-2r} \frac{N^2p^{2v}\varphi(N^2p^{2v})\varphi(Np^v)}{[\Gamma_0(N^2p^{2v}) : \Gamma(N^2p^{2v})]^3} \mathfrak{L}_{Np}(-r) \times$$
$$\times \frac{\mathcal{D}^\star(f_1^\rho \otimes f_2^\rho \otimes f_3^\rho, 2k-2-r, \psi_1\psi_2\chi_1)}{\langle f_1^0 \otimes f_2^0 \otimes f_3^0, f_{1,0} \otimes f_{2,0} \otimes f_{3,0} \rangle_{T, N^2p}}$$

Let us show that under the assumptions as above there exist an admissible $\mathbb{C}_p$-valued measure $\tilde{\mu}^\lambda_{f_1 \otimes f_2 \otimes f_3}$ on $Y_{N,p}$, and a $\mathbb{C}_p$-analytic

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function

$$\mathcal{D}_{(p)}(x, f_1 \otimes f_2 \otimes f_3) : X_p \to \mathbb{C}_p,$$

given for all $x \in X_{N,p}$ by the integral

$$\mathcal{D}_{(p)}(x, f_1 \otimes f_2 \otimes f_3) = \int_{Y_{N,p}} x(y) \mathrm{d}\tilde{\mu}^{\lambda}_{f_1\otimes f_2 \otimes f_3}(y),$$

and having the following properties: for all pairs (r, χ) such that for $\chi \in X_p^{\mathrm{tors}}$ the corresonding Dirichlet characters χ_j are Np-complete, and $r \in \mathbb{Z}$ with $0 \le r \le k-2$, the following equality holds:

$$\mathcal{D}_{(p)}(\chi x_p^r, f_1 \otimes f_2 \otimes f_3) = \tag{8.15}$$

$$i_p\Big(\frac{(\psi_1\psi_2)(2) C_\chi^{4(2k-3-r)}}{G(\chi_1)G(\chi_2)G(\chi_3)G(\psi_1\psi_2\chi_1)\lambda(p)^{2v}} \frac{\mathcal{D}^\star(f_1^\rho \otimes f_2^\rho \otimes f_3^\rho, 2k-2-r, \psi_1\psi_2\chi)}{\langle f_1^\rho \otimes f_2^\rho \otimes f_3^\rho, f_1^\rho \otimes f_2^\rho \otimes f_3^\rho \rangle_T}\Big)$$

where $v = \mathrm{ord}_p(C_\chi)$,

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$\chi_1 \bmod Np^v = \chi,\ \chi_2 \bmod Np^v = \psi_2\bar{\psi}_3\chi, \chi_3 \bmod Np^v = \psi_1\bar{\psi}_3\chi,$
$G(\chi)$ denotes the Gauß sum of a primitive Dirichlet character χ_0 attached to χ (modulo the conductor of χ_0).

Indeed, we may write

$$\mathcal{D}_{(p)}(x, f_1 \otimes f_2 \otimes f_3) = C \cdot x(2) \int_Y x(y) \mathrm{d}\tilde{\mu}^{\lambda}(y; f_1 \otimes f_2 \otimes f_3)$$

with an appropriate constant where $v = \mathrm{ord}_p(C_\chi)$.

Moreover, it follows from the properties of the constructed measure

$$\tilde{\mu}^{\lambda}_{f_1\otimes f_2\otimes f_3}(y) := C \cdot \tilde{\mu}_\lambda(2^{-1}y; f_1 \otimes f_2 \otimes f_3)$$

that

(ii) if $\mathrm{ord}_p\lambda(p) = 0$ then the holomorphic functions in (i), (ii) are bounded $\mathbb{C}_p$-analytic functions: it suffices to use the binomial equality with $r = 1$ in order to show that in this case the measure $\tilde{\Phi}^\lambda$ is just bounded because of $|\lambda(p)|_p = 1$);

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(iii) in the general case (but assuming that $\lambda(p) \neq 0$) the holomorphic functions in (i) belong to the type $o(\log(x_p^h))$ with $h = [2\mathrm{ord}_p \lambda(p)] + 1$ and they can be represented as the Mellin transform of the h-admissible measure $\tilde{\mu}^{\lambda}_{f_1 \otimes f_2 \otimes f_3}$ (in the sense of Amice-Vélu);

(iv) if $h = [2\mathrm{ord}_p \lambda] + 1 \leq k - 2$ then the function $\mathcal{D}_{(p)}$ is uniquely determined by the above conditions (i). ∎

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