On p-adic families of L-functions

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A motivation: why study L-values attached to modular forms?

A popular proceedure in number theory is the following:

Construct a generating function $f = \sum_{n=0}^{\infty} a_n q^n \in \mathbb{C}[[q]]$ of an arithmetical function $n \mapsto a_n$, for example $a_n = p(n)$	$\rightsquigarrow$	Compute f via modular forms, for example $\sum_{n=0}^{\infty} p(n) q^n = (\Delta/q)^{-1/24}$	$\rightsquigarrow$	A number (solution)
Example 1 [Chand70]: (Hardy-Ramanujan) $p(n) = \frac{e^{\pi\sqrt{2/3(n-1/24)}}}{4\sqrt{3}\lambda_n^2} + O(e^{\pi\sqrt{2/3(n-1/24)}}/\lambda_n^3)$, $\lambda_n = \sqrt{n-1/24}$,		$\uparrow$ Good bases, finite dimensions, many relations and identities		$\uparrow$ Values of L-functions, periods, congruences, ...

Other examples: Birch and Swinnerton-Dyer conjecture, ... L-values attached to modular forms

Statement of the problem of Coleman-Mazur

This talk is about the paper [PaTV] by A.P., *Two variable p-adic L functions attached to eigenfamilies of positive slope*, Invent. Math. v. 154, N3 (2003), pp. 551 - 615, and about some further developments.

The Tate field $\mathbb{C}_p$

Fix a prime p, and let $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}}_p$ be the Tate field (the completion of the field of p-adic numbers)

We fix an embedding $i_p : \overline{\mathbb{Q}} \to \mathbb{C}_p$, and view algebraic numbers as p-adic numbers via i_p.

A primitive cusp eigenform f

$f = f_k = \sum_{n\geq 1} a_n q^n \in \mathcal{S}_k(\Gamma_0(N), \psi)$, (where $q = e(z) = \exp(2\pi iz)$, $\mathrm{Im}\,(z) > 0$)

a primitive cusp eigenform $f = f_k$ of weight $k \geq 2$ for $\Gamma_0(N)$ with a Dirichlet character ψ (mod N).

The special values of the L-function attached to f at $s = 1, \cdots, k-1$:

$L_f(s, \chi) = \sum_{n\geq 1} \chi(n) a_n n^{-s}$,

(χ are Dirichlet characters)

where $1 - a_p X + \psi(p) p^{k-1} X^2$ $= (1 - \alpha_p X)(1 - \alpha'_p X)$ is the Hecke polynomial α_p and α'_p are called the Satake parameters of f

Periods of f Following a known theorem of Shimura [Sh59] and Manin [Ma73], there exist two non-zero complex constants $c^+(f), c^-(f) \in \mathbb{C}^\times$ (the *periods* of f) such that for all $s = 1, \cdots, k-1$ and for all Dirichlet characters χ of fixed parity, $(-1)^{k-s}\chi(-1) = \pm 1$, the normalized special values are *algebraic numbers*:

$$L^*(f, s, \chi) = \frac{(2i\pi)^{-s}\Gamma(s) L_f(s, \chi)}{c^\pm(f)} \in \overline{\mathbb{Q}}. \qquad (2.1)$$

A family of slope $\sigma > 0$ of cusp eigenforms f_k of weight $k \geq 2$ containing f

$$k \mapsto f_k = \sum_{n=1}^{\infty} a_n(k) q^n \in \overline{\mathbb{Q}}[\![q]\!] \subset \mathbb{C}_p[\![q]\!]$$

1) the Fourier coefficients $a_n(k)$ of f_k and the Satake p-parameter $\alpha_p(k)$ are given by certain p-adic analytic functions $k \mapsto a_n(k)$ for $(n, p) = 1$

2) the slope is constant and positive: $\mathrm{ord}(\alpha_p(k)) = \sigma > 0$

A model example of a p-adic family (not cusp and $\sigma = 0$): Eisenstein series

$$a_n = \sum_{d|n} d^{k-1}, f_k = E_k$$

the Fourier coefficients $a_n(k)$ and one of the Satake p-parameters $\alpha_p(k) = 1$ are p-adic analytic functions, and $\mathrm{ord}_p(\alpha_p(k)) = \mathrm{ord}_p(1) = 0$

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The existence of families of slope $\sigma > 0$: R.Coleman, [CoPB]

He gave an example with $p = 7$, $f = \Delta$, $k = 12$ $a_7 = \tau(7) = -7 \cdot 2392, \sigma = 1$,

and a program in PARI for computing such families is contained in [CST98] (see also the Web-page of W.Stein, `http://modular.fas.harvard.edu/`)

The Problem, *see [Co-Ma] R. Coleman, B. Mazur, The eigencurve. Galois representations in arithmetic algebraic geometry, (Durham, 1996), London Math. Soc. Lecture Note Ser., 254, at p.6*

Given a p-adic analytic family $k \mapsto f_k = \sum_{n=1}^{\infty} a_n(k) q^n \in \overline{\mathbb{Q}}[\![q]\!]$ *of positive slope* $\sigma > 0$*, to construct a two-variable p-adic L-function interpolating* $L^*(f_k, s, \chi)$ *on* (s, k).

Known cases:

- One-variable case ($k = k$ is fixed, $\sigma > 0$), treated in [Am-Ve] by Y. Amice, J. Vélu, in [Vi76] by M.M. Višik, and in [MTT] by B. Mazur; J. Tate; J. Teitelbaum
- $\sigma = 0$ (H.Hida) ("ordinary families") (see in [Hi93])
- Special values of L-functions attached to families f_k of Yu.I. Manin and M. M.Vishik, [Ma-Vi] : $f_k = \sum_{\mathfrak{a}\subset \mathcal{O}_K} \lambda^{k-1}(\mathfrak{a})q^{N\mathfrak{a}}$ and of N.M.Katz, [Kat]), which are are certain ordinary families they correspond to powers of a grössen-character λ of an imaginary quadratic field K at a *splitting prime* p, (resp. to grössencharacters of type A_0 of the idèle class group $\mathbb{A}_K^*/K^*$ (in the sense of Weil [We56],) of a CM-field K.

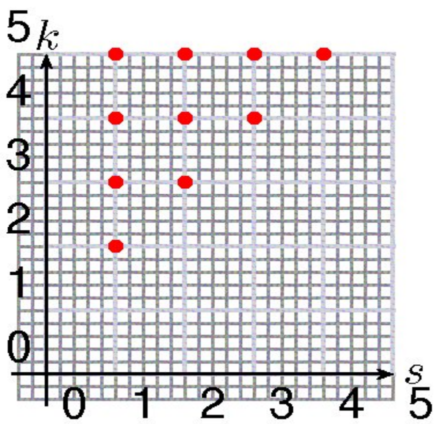
Motivation comes from the conjecture of Birch and Swinnerton-Dyer, see in [Colm03]

For a cusp eigenform $f = f_2$, corresponding to an elliptic curve E by Wiles [Wi], we consider a family containing f.

One can try to approach $k = 2, s = 1$ from the other direction, taking $k \to 2$ instead of $s \to 1$, this leads to a formul linking the derivative over s at $s = 1$ of the p-adic L-function with the derivative over k at $k = 2$ of the p-adic analytic function $\alpha_p(k)$, see in [CST98]:

$$\boxed{L'_{p,f}(1) = \mathcal{L}_p(f)L_{p,f}(1)}$$

with $\mathcal{L}_p(f) = -2\frac{d\alpha_p(k)}{dk}\Big|_{k=2}$



The validity of this formula needs the existence of our two variable L-function!

Our method

is a combination of the Rankin-Selberg method, the theory of p-adic integration with values in p-adic Banach algebras $\mathcal{A}$ and the spectral theory of Atkin's U-operator: $U = U_p : \mathcal{A}[\![q]\!] \to \mathcal{A}[\![q]\!]$ defined by:

$$U\left(\sum_{n\geq 1} a_n q^n\right) = \sum_{n\geq 1} a_{pn} q^n \in \mathcal{A}[\![q]\!].$$

Here $\mathcal{A} = \mathcal{A}(\mathcal{B})$ is a certain p-adic Banach algebra of functions on an open analytic subspace $\mathcal{B} \subset X$ of the weight space $X = \mathrm{Hom}_{cont}(Y, \mathbb{C}_p^*)$. This is an analytic space over $\mathbb{C}_p$, which consists of all continuous characters of the profinite group $Y \cong (\mathbb{Z}/N\mathbb{Z})^* \times \mathbb{Z}_p^*$.
The classical analogue of the weight space is the whole complex plane

$$\mathbb{C} = \mathrm{Hom}_{cont}(\mathbb{R}_+^*, \mathbb{C}^*), s \mapsto (y \mapsto y^s).$$

The weights k correspond to certain points in $\mathcal{B} \subset X$. Any series $f = \sum_{n\geq 1} a_n q^n \in \mathcal{A}[\![q]\!]$ produces a family of q-expansions $\left\{f_k = ev_k(f) = \sum_{n\geq 1} ev_k(a_n) q^n \in \mathbb{C}_p[\![q]\!]\right\}$, which can be classical modular forms in $\overline{\mathbb{Q}}[\![q]\!]$.

1) We construct first an analytic function $\mathcal{L}_\mu : X \to \mathcal{A} = \mathcal{A}(\mathcal{B})$ as the Mellin transform

$$\mathcal{L}_\mu(x) = \int_Y x(y)\, d\mu(y) \ \ (\text{where } x \in X = \mathrm{Hom}_{cont}(Y, \mathbb{C}_p^*), x = x(y)),$$

μ is a certain measure with values in $\mathcal{A}$, on the profinite group Y.

2) For each $s \in \mathcal{B} \subset X$, there is the evaluation homomorphism $ev_s : \mathcal{A}(\mathcal{B}) \to \mathbb{C}_p$; we obtain $\mathcal{L}_\mu(x, s)$ by evaluation of an $\mathcal{A}$-valued integral:

$$\mathcal{L}_\mu(x, s) = ev_s(\mathcal{L}_\mu(x)) = ev_s\left(\int_Y x d\mu\right) \ \ (x \in X, \ \ \mathcal{L}_\mu(x) \in \mathcal{A}).$$

This gives a p-adic analytic L-function in two variables $(x, s) \in X \times \mathcal{B} \subset X \times X$:

$$(x, s) \longmapsto \mathcal{L}_\mu(x, s).$$

3) We check an equality relating the algebraic numbers $L_{f_k}^*(s, \chi)$ $(s = 1, \cdots, k-1)$ with the values $\mathcal{L}_\mu(x, k)$ at certain points $x \in X$ (more precisely, at $x = \chi \cdot y_p^k$).

p-adic integration and the p-adic weight space

Consider the group $Y = \varprojlim_v (\mathbb{Z}/Np^v\mathbb{Z})^\times \cong (\mathbb{Z}/N\mathbb{Z})^\times \times \mathbb{Z}_p^\times$ (a profinite group with a projection $y_p : Y \to \mathbb{Z}_p^\times$)

and the group of p-adic characters
$X = X_N = \mathrm{Hom}_{\mathrm{cont}}(Y, \mathbb{C}_p^\times) \ni \chi, y_p^n$, (the p-adic weight space, which is a $\mathbb{C}_p$-analytic group)

where
$\chi \bmod Np^v\mathbb{Z} : (\mathbb{Z}/Np^v\mathbb{Z})^\times \to \mathbb{C}_p^\times$ (a Dirichlet character)
$y_p : Y \to \mathbb{Z}_p^\times$ (the canonical projection, a p-adic character of Y)

The analytic structure on $X = X_N = \mathrm{Hom}_{\mathrm{cont}}(Y, \mathbb{C}_p^\times)$ over $\mathbb{C}_p$ is given by the decomposition:

$$X \xrightarrow{\sim} \mathrm{Hom}((\mathbb{Z}/Np\mathbb{Z})^\times, \mathbb{C}_p^\times) \times \mathrm{Hom}_{\mathrm{cont}}(\Gamma, \mathbb{C}_p^\times)$$

where $Y \cong (\mathbb{Z}/Np\mathbb{Z})^\times \times \Gamma$, $\Gamma = (1 + p\mathbb{Z}_p)^\times$, is a procyclic group of generator $\gamma = 1 + p$, and we see that X is a finite cover of the p-adic unit disc:

$$X \twoheadrightarrow \mathrm{Hom}_{\mathrm{cont}}(\Gamma, \mathbb{C}_p^\times) \xrightarrow{\sim} \mathfrak{U} = \{t \in \mathbb{C}_p \mid |t-1|_p < 1\} \cong \{\chi_t : \gamma \mapsto t \mid t \in \mathfrak{U}\}.$$

Distributions with values in Banach modules: notation

$(k, \psi) = y_p^k \psi \in X$ is a point in the weight space $X = \mathrm{Hom}_{\mathrm{cont}}(Y, \mathbb{C}_p^\times)$ (we write simply k for (k, ψ))

$\mathcal{A}$ (a p-adic Banach algebra)
V (an $\mathcal{A}$-module)
$\mathcal{C}(Y, \mathcal{A})$ (the $\mathcal{A}$-Banach algebra of continuous functions on Y)
$\cup$
$\mathcal{C}^{loc-const}(Y, \mathcal{A})$ (the $\mathcal{A}$-algebra of locally constant functions on Y)

Definition

a) A distribution $\mathcal{D}$ on Y with values in V is an $\mathcal{A}$-linear form

$$\mathcal{D} : \mathcal{C}^{loc-const}(Y, \mathcal{A}) \to V, \quad \varphi \mapsto \mathcal{D}(\varphi) = \int_Y \varphi d\mathcal{D}.$$

b) A measure μ on Y with values in V is a continuous $\mathcal{A}$-linear form

$$\mu : \mathcal{C}(Y, \mathcal{A}) \to V, \quad \varphi \mapsto \int_Y \varphi d\mu.$$

The integral $\int_Y \varphi d\mu$ can be defined for any continuous function φ, and any bounded distribution μ, using the Riemann sums.

Admissible measures of Amice-Vélu

A more delicate notion of an h-admissible measure was introduced in [Am-Ve] by Y. Amice, J. Vélu (see also [MTT], [Vi76]):

Definition

a) For $h \in \mathbb{N}, h \geq 1$ let $\mathcal{P}^h(Y, \mathcal{A})$ denote the $\mathcal{A}$-module of locally polynomial functions *of degree $< h$ of the variable $y_p : Y \to \mathbb{Z}_p^\times \hookrightarrow \mathcal{A}^\times$; in particular,*

$$\mathcal{P}^1(Y, \mathcal{A}) = \mathcal{C}^{loc-const}(Y, \mathcal{A})$$

(the $\mathcal{A}$-submodule of locally constant functions*). Let also denote $\mathcal{C}^{loc-an}(Y, \mathcal{A})$ the $\mathcal{A}$-module of* locally analytic functions, *so that*

$$\mathcal{P}^1(Y, \mathcal{A}) \subset \mathcal{P}^h(Y, \mathcal{A}) \subset \mathcal{C}^{loc-an}(Y, \mathcal{A}) \subset \mathcal{C}(Y, \mathcal{A}).$$

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Admissible measures of Amice-Vélu (continued)

b) Let V be a normed $\mathcal{A}$-module with the norm $|\cdot|_{p,V}$. For a given positive integer h an h-admissible measure on Y with values in V is an $\mathcal{A}$-module homomorphism

$$\tilde{\Phi} : \mathcal{P}^h(Y, \mathcal{A}) \to V$$

such that for fixed $a \in Y$ and for $v \to \infty$ the following growth condition *is satisfied:*

$$\left| \int_{a+(Np^v)} (y_p - a_p)^{h'} d\tilde{\Phi} \right|_{p,V} = o(p^{-v(h'-h)}) \qquad (3.2)$$

$$\text{for all} \quad h' = 0, 1, \ldots, h-1, a_p := y_p(a)$$

The condition (3.2) allows to integrate only the locally-analytic functions: there exists a unique extension of $\tilde{\Phi}$ to $\mathcal{C}^{loc-an}(Y, \mathcal{A}) \to V$ (via the embedding $\mathcal{P}^h(Y, \mathcal{A}) \subset \mathcal{C}^{loc-an}(Y, \mathcal{A})$). The integral is defined using generalized Riemann sums: take the beginning of the Taylor expansion of a locally-analytic function $\phi \in \mathcal{C}^{loc-an}(Y, \mathcal{A})$ (of order $h-1$) instead of just values of a function ϕ.

The p-adic Mellin transform and two variable p-adic analytic functions

Any h-admissible measure $\tilde{\mu}$ on Y with values in a p-adic Banach algebra $\mathcal{A}$ can be caracterized by the logarithmic growth $o(\log^h(\cdot))$ of its *Mellin transform* $\mathcal{L}_{\tilde{\mu}}(x)$ (see [Am-Ve], [Vi76], [HaH]):

$$\mathcal{L}_{\tilde{\mu}} : X \to \mathcal{A}, \text{ defined by } \mathcal{L}_{\tilde{\mu}}(x) = \int_Y x(y) d\tilde{\mu}(y),$$

where $x \in X, \ \mathcal{L}_{\tilde{\mu}}(x) \in \mathcal{A}, \ X \subset \mathcal{C}^{loc-an}(Y, \mathcal{A})^\times$
Key property of h-admissible measures $\tilde{\mu}$: its Mellin transform $\mathcal{L}_{\tilde{\mu}}$ is analytic with values in $\mathcal{A}$.

Then we obtain the function $\mathcal{L}_\mu(x,s)$ by evaluation at (s, ψ): this is a p-adic analytic function in two variables $(x,s) \in X \times \mathcal{B} \subset X \times X$:

$$\mathcal{L}_{\tilde{\mu}}(x,s) = ev_s(\mathcal{L}_{\tilde{\mu}}) \quad (x \in X, \ \ \mathcal{L}_{\tilde{\mu}}(x) \in \mathcal{A}).$$

Example ([Am-Ve], [MTT], [Vi76])

For a primitive cusp eigenform $f = f_k = \sum_{n\geq 1} a_n q^n \in \mathcal{S}_k(\Gamma_0(N), \psi)$ *of weight* $k \geq 2$ *for* $\Gamma_0(N)$ *with a Dirichlet character* ψ *and positive slope* $\sigma = \mathrm{ord}_p(\alpha)$ *define the integer* $h = [\sigma] + 1$ *(where* $\sigma < k-1$*, and* $1 - a_p X + \psi(p) p^{k-1} X^2 = (1 - \alpha_p X)(1 - \alpha'_p X)$ *as above). Then there exists an h-admissible* $\mathbb{C}_p$*-valued measure* $\tilde{\mu} = \tilde{\mu}_{\alpha,f}(y)$ *on* Y *such that for all couples* (j, χ) *with* $0 \leq j \leq k-2$*, and for any nontrivial primitive Dirichlet character* χ mod p^v *satisfying* $\chi\xi(-1) = (-1)^{k-1-j}$*, there is the following equality (in* $\mathbb{C}_p$*):*

$$\int_Y \chi(y) y_p^j \, d\tilde{\mu} = i_p\left(\frac{p^{vj} G(\chi)}{\alpha^v} L_f^*(1+j, \bar{\chi})\right) \ \left(= \mathcal{L}_{\tilde{\mu}}(\chi y_p^j)\right), \quad (3.3)$$

where $G(\chi)$ *is the Gauss sum of the character* χ mod p^v*, and* $L_f^*(1+j, \bar{\chi})$ *is given by a choice of periods (2.1).* In other words, the complex L-values (3.3) attached to f coincide with the values $\mathcal{L}_{\tilde{\mu}}(\chi\, y_p^j)$ of the p-adic Mellin transform of $\tilde{\mu}$.

Coleman's families: notation

The proof of the existence of families of slope $\sigma > 0$ by R.Coleman, [CoPB], uses the following ideas: let us consider

$[K : \mathbb{Q}_p] < \infty$ – a finite extension of $\mathbb{Q}_p$ contaning all the Fourier coefficients $i_p(a_n)$ of f

$\mathcal{A} = \mathcal{A}_K(\mathcal{B})$ – the K-Banach algebra of rigid-analytic functions

$ev_k : \mathcal{A} \to K$ – the evaluation map defined for all $(k, \psi) \in \mathcal{B}$ (a neigbourhood around $(k, \psi) \in X$).

$\mathcal{M}(N; \mathcal{A})^\dagger = \bigcup_{v \geq 1} \mathcal{M}(Np^v, \psi; \mathcal{A})^\dagger \subset \mathcal{A}[\![q]\!]$ – a Banach $\mathcal{A}$-module of overconvergent families of modular forms: this module is generated by some $g = \sum_{n=0}^{\infty} b_n q^n \in \mathcal{A}[\![q]\!]$ such that $ev_k(g) \in K[\![q]\!]$ are classical cusp eigenforms for all k with (k, ψ) in a neighbourhood $\mathcal{B}$ of $(k, \psi) \in X$.

Coleman proved:

- The operator U acts as a completely continuous operator on each $\mathcal{A}$-submodule $\mathcal{M}(Np^v; \mathcal{A})^\dagger \subset \mathcal{A}[\![q]\!]$ (i.e. U is a limit of finite-dimensional operators)

$\Longrightarrow$ there exists the Fredholm determinant $P_U(T) = \det(Id - T \cdot U) \in \mathcal{A}[\![T]\!]$

- there is a version of the Riesz theory: for any inverse root $\alpha \in \mathcal{A}^*$ of $P_U(T)$ there exists an eigenfunction g, $Ug = \alpha g$

such that $ev_k(g) \in K[\![q]\!]$ are classical cusp eigenforms for all k such that (k, ψ) is in a neigbourhood $\mathcal{B}$ around $(k, \psi) \in X$ (see in [CoPB])

Definition

a) *A function* $g \in \mathcal{M}(Np^v; \mathcal{A})^\dagger \subset \mathcal{A}[\![q]\!]$ *is called Coleman's family if* $Ug = \alpha g$*, and the functions* $ev_k(g) \in K[\![q]\!]$ *are cusp eigenforms for all* k *such that* (k, ψ) *is in a neigbourhood* $\mathcal{B}$ *around* (k, ψ) *in the p-adic weight space* X*, and* $\mathrm{ord}_p(\alpha(k)) = \sigma > 0$ *is constant and positive, where* $\alpha(k) = ev_k(\alpha) \in K \cap i_p(\overline{\mathbb{Q}})$
b) *Let* $f_k \in \overline{\mathbb{Q}}[\![q]\!]$ *denote the primitive cusp eigenform attached to* $ev_k(g) \in K[\![q]\!]$*. Then the family* $\{f_k\}$ *of classical primitive cusp forms is also called Coleman's family.*

Remark

Hida's families correspond to $\sigma = 0$*, they were constructed in [Hi86] (see also [Hi93]).*
There exist analogues of Hida's families in the Siegel modular case.

Recall that by [Ra52], [Za77] and [Sh77], the numbers

$$\frac{L_f(1+j, \bar{\chi}) L_f(k-1, \bar{\xi})}{\pi^{k+r} \langle f_k, f_k \rangle_{Np}} \text{ are algebraic for all } j \in \mathbb{Z} \text{ with } 0 \le j \le k-2,$$

$\chi\xi(-1) = (-1)^{k-1-j}$ (here $\langle f_k, f_k \rangle_{Np}$ denotes the Petersson scalar product.

Main Theorem

Consider a nonzero analytic function $\alpha = \alpha(s) \in \mathcal{A}^\times$ *defined in a neigbourhood* $\mathcal{B}$ *of* $(k, \psi) \in X$*, and consider Coleman's family*

$$f = \left\{ f_k = \sum_{n=1}^{\infty} a_n(k) q^n \right\} \in \mathcal{A}[\![q]\!]$$

with coefficients in the algebra $\mathcal{A} = \mathcal{A}(\mathcal{B})$*, where* $\alpha \in \mathcal{A}^\times$ *is the corresponding eigenvalue of* U*. Suppose that the slope* $\mathrm{ord}_p(\alpha) = \sigma > 0$ *is fixed for all* $\alpha = \alpha(k)$ *with* (k, ψ) *in* $\mathcal{B}$*, and define the integer* $h = [\sigma] + 1$*.*
Then there exists an h-admissible measure $\tilde{\mu} = \mu_{\alpha, f}$ *with values in* $\mathcal{A}$ *on the group* Y*, determind by the following property:*

Main theorem (continued)

for all couples (j,χ) with $0 \le j \le k-2$, $k > 2\sigma+2$, any primitive Dirichlet character χ mod p^v satisfying $\chi\xi(-1) = (-1)^{k-1-j}$, the following equality holds:

$$ev_k\left(\int_Y \chi(y)\, y_p^j\, d\tilde{\mu}\right) = i_p\left(R_k \cdot \frac{p^{vj}G(\chi)}{\alpha_p(k)^v} L^*_{f_k}(1+j,\bar{\chi}),\right) \quad (5.4)$$

where $G(\chi)$ is the Gauss sums of χ mod p^v, and $R_k \in \mathbb{Q}^\times$ is an elementary factor coming from an explicit choice of periods $c^\pm(f_k)$. The choice of periods: fix two Dirichlet characters ξ mod p of different parity then

$$c^\pm(f_k) = \frac{(-2i\pi)^{k-1}\langle f_k, f_k\rangle_{Np}}{\Gamma(k-1)L_{f_k}(k-1,\bar{\xi})}, \text{ where } \xi(-1) = \pm(-1)^{k-1}. \quad (5.5)$$

A key ingredient in our construction

is the use of a linear form

$$\ell_{\alpha(k)} : \mathcal{M}_k(Np,\psi,\overline{\mathbb{Q}}) \to \overline{\mathbb{Q}},$$

such that $\alpha(k) \in \mathbb{Q}^\times$, $\ell_{\alpha(k)}(U_p h) = \alpha \ell_{\alpha(k)}(h)$ for all $h \in \mathcal{M}_k(Np,\psi,\overline{\mathbb{Q}})$, and $1 - a_p X + \psi(p)p^{k-1}X^2 = (1-\alpha(k)X)(1-\alpha(k)'X)$ for a primitive cusp eigenform $f = \sum_{n=1}^{\infty} a_n q^n \in \mathcal{S}_k(\Gamma_0(N),\psi,\overline{\mathbb{Q}})$ of weight $k \ge 2$ for $\Gamma_0(N)$ with a Dirichlet character ψ (mod N). One can define such linear form by

$\ell_\alpha : h \longmapsto \dfrac{\langle f^0, h\rangle}{\langle f^0, f_0\rangle}$, where

f_0 is an eigenfunction of U_p: $f_0|U_p = \alpha(k)f_0$, and

f^0 is the corresponding eigenfunction of U_p^*: $f^0|U_p^* = \overline{\alpha(k)}f^0$,

Functions f_0 and f^0

Recall that for any primitive cusp eigenform $f = \sum_{n=1}^{\infty} a_n(f)q^n$, there is an eigenfunction of $U = U_p$ with the eigenvalue $\alpha = \alpha_p^{(1)} \in \overline{\mathbb{Q}}$ $(U(f_0) = \alpha f_0)$ given by

$$f_0 = \sum_{n\geq 1} a_n q^n - \alpha' \sum_{n\geq 1} a_n q^{pn} = \sum_{n\geq 1} a(f_0, n)q^n \in \mathcal{S}_k(\Gamma_0(Np), \psi, \overline{\mathbb{Q}}), \text{ and}$$

$$f^0 = f_0^{\rho}\Big|_k \begin{pmatrix} 0 & -1 \\ Np & 0 \end{pmatrix}, \quad f_0^{\rho} = \sum_{n\geq 1} \overline{a}(f_0, n)q^n \in \mathcal{S}_k(\Gamma_0(Np), \overline{\psi}, \overline{\mathbb{Q}})$$

is an eigenfunction of the adjoint operator U_p^*, is explicitely computed in [Go-Ro].

The answer to the question of Coleman–Mazur

is given by the function (5.6) of the following theorem:

Theorem

Under the assumptions and notations of Theorem 5.2 there exists a unique p-adic analytic function on $X \times \mathcal{B}$ (of two variables x, s),

$$\mathcal{L}_{\alpha,f}(\cdot_1, \cdot_2, \xi, f) : X \times \mathcal{B} \to \mathbb{C}_p \qquad (5.6)$$

such that
i) for any fixed $(s, \psi) \in \mathcal{B}$, the function $\mathcal{L}_{\alpha,f}(x, s; \xi, f)$ of the variable x is $\mathbb{C}_p$-analytic and has the logarithmic growth $o(\log^h(x))$,
ii) for each couple (χ, j) with $0 \leq j \leq k-2$, $k > 2\sigma + 2$ and any primitive Dirichlet character $\chi \bmod p^v \in X^{\mathrm{tors}}$ with values in $K^\times$ satisfying $v \geq 2$, $\chi\xi(-1) = (-1)^{k-1-j}$, the special value $\mathcal{L}(\chi y_p^j, k; \xi, f_k)$ is given by the image under i_p of the algebraic number $R_k \cdot \dfrac{p^{vj} G(\chi)}{\alpha_p(k)^v} L_{f_k}^*(1+j, \bar{\chi})$, where $G(\chi)$ is the Gauss sums of $\chi \bmod p^v$, and $R_k \in \mathbb{Q}^\times$ is an elementary factor given by the explicit choice of periods $c^\pm(f_k)$, as in (5.5).

Construction of the admissible measure $\tilde{\mu}$

Recall the Definition 3.2: an *h-admissible measure* on a profinite group Y with values in an $\mathcal{A}$-module V is an $\mathcal{A}$-module homomorphism

$$\tilde{\mu} : \mathcal{P}^h(Y, \mathcal{A}) \to V,$$

satisfying a certain growth condition (3.2).
This means that $\tilde{\mu}$ is given by a sequence $\{\mu_j\}$ of certain distributions on Y, in such a way that for $j = 0, 1, \cdots, h-1$ and for all compact open subsets $U \subset Y$ one has

$$\int_U y_p^j d\tilde{\mu} = \mu_j(U). \tag{6.7}$$

Recall: the growth condition (3.2)

is needed in order to define an h-admissible measure $\tilde{\mu}$ out of a sequence $\{\mu_j\}$ of distributions on Y, in such a way that

$$\int_U y_p^j d\tilde{\mu} = \mu_j(U)$$

for $j = 0, 1, \cdots, h-1$ and for all compact open subsets $U \subset Y$.
This condition has the form: for $t = 0, 1, \cdots, h-1$

$$\left| \int_{a+(Np^v)} (y_p - a_p)^t d\tilde{\mu} \right|_p \tag{6.8}$$

$$= \left| \sum_{j=0}^{t} \binom{t}{j} (-a_p)^{t-j} \mu_j(a + (Np^v)) \right|_p = o(p^{v(h-t)}) \text{ for } v \to \infty.$$

In this condition the elements $\mu_j(a + (Np^v))$ belong to a p-adic Banach algebra $\mathcal{A}$.

We construct the sequence μ_j out of products of Eisenstein series:

$$\mu_j = \ell_\alpha(\pi_\alpha(\Phi_j)), \quad (j = 0, 1, \cdots, h-1), h = [\sigma] + 1.$$

- Φ_j is a sequence of modular distributions on Y with values in a certain $\mathcal{A}$-module $\mathcal{M} = \mathcal{M}_N(\psi; \mathcal{A})$ of modular forms with coefficients in $\mathcal{A}$ (it has infinite rank):

$$\mathcal{M}_N(\psi; \mathcal{A}) := \bigcup_{v \geq 0} \mathcal{M}(Np^v, \psi; \mathcal{A}),$$

(our modular forms $\Phi_j(\chi)$ are products of certain families of classical Eisenstein series in $\mathcal{A}[\![q]\!]$)
- π_α is the canonical projector onto the characteristic $\mathcal{A}$-submodule $\mathcal{M}^\alpha = \mathcal{M}^\alpha(\mathcal{A})$ of Atkin's operator $U\left(\sum_{n\geq 0} b_n q^n\right) = \sum_{n\geq 0} b_{pn} q^n$
(Key point: the $\mathcal{A}$-module $\mathcal{M}^\alpha(\mathcal{A})$ is locally free of finite rank)
- $\ell_\alpha \in \mathrm{Hom}_\mathcal{A}(\mathcal{M}^\alpha, \mathcal{A})$ is a $\mathcal{A}$-linear form (given by the Petersson scalar product with $h \in \mathcal{M}^\alpha$, as in Section 3: $h \longmapsto \dfrac{\langle f^0, h\rangle}{\langle f^0, f_0\rangle}$, normalized by the equality $\ell_\alpha(g) = 1$ for Coleman's eigenfunction $g = f_0 \in \mathcal{M}^\alpha$.

Main congruence: criterion of admissibility

Theorem

Let $0 < |\alpha|_p < 1$ and suppose that the following conditions are satisfied: for all $r = 0, 1, \cdots, h-1$ with $h = [\mathrm{ord}_p \alpha] + 1$, and $v \geq 1$,

$$\Phi_r(a + (Np^v)) \in \mathcal{M}(Np^v)^\dagger \qquad \text{(the level condition)} \tag{6.9}$$

and the following p-adic congruence holds: for all $t = 0, 1, \cdots, h-1$

$$U^v \sum_{r=0}^{t} \binom{t}{r} (-a_p)^{t-r} \Phi_r(a + (Np^v)) \equiv 0 \bmod p^{vt} \tag{6.10}$$

(the divisibility condition)

Consider the linear form $\tilde\Phi^\alpha(\delta_{a+(Np^v)} y_p^r) := \pi_\alpha(\Phi_r(a + (Np^v))$ (defined on local monomials of degree $r = 0, 1, \cdots, h-1$).
Then $\tilde\Phi^\alpha$ is an h-admissible measure: $\tilde\Phi^\alpha : \mathcal{P}^h(Y, \overline{\mathbb{Q}}) \to \mathcal{M}^\alpha \subset \mathcal{M}$

Proof uses the commutative diagram:

$$\begin{array}{ccccc}
\mathcal{M}(Np^{v+1},\psi;\mathcal{A})^{\dagger} & \xrightarrow{\pi_{\alpha,v}} & \mathcal{M}^{\alpha}(Np^{v+1},\psi;\mathcal{A})^{\dagger} & & \\
U^{v}\downarrow & & \downarrow\wr\, U^{v} & & \\
\mathcal{M}(Np,\psi;\mathcal{A})^{\dagger} & \xrightarrow[\pi_{\alpha,0}]{} & \mathcal{M}^{\alpha}(Np,\psi;\mathcal{A})^{\dagger} & = \mathcal{M}^{\alpha}(Np^{v+1},\psi;\mathcal{A})^{\dagger}. &
\end{array}$$

The existence of the projectors $\pi_{\alpha,v}$ comes from Coleman's Theorem A.4.3 [CoPB].
On the right: $\mathcal{M}^{\alpha}(Np^{v+1},\psi;\mathcal{A})^{\dagger}$ does not depend on v (a version of Hida's Control Theorem), and U acts on the locally free $\mathcal{A}$-module $\mathcal{M}^{\alpha}(Np^{v+1},\mathcal{A})^{\dagger}$ via the matrix:

$$\begin{pmatrix} \alpha & \cdots & \cdots & * \\ 0 & \alpha & \cdots & * \\ 0 & 0 & \ddots & \cdots \\ 0 & 0 & \cdots & \alpha \end{pmatrix} \text{ where } \alpha \in \mathcal{A}^{\times}$$

$\Longrightarrow U^{v}$ is an isomorphism over $\mathrm{Frac}(\mathcal{A})$,

One controls the denominators

of the modular forms of all levels v by the relation:

$$\pi_{\alpha,v}(h) = U^{-v}\pi_{\alpha,0}(U^{v}h) =: \pi_{\alpha}(h) \tag{6.11}$$

The equality (6.11) can be used as the definition of π_{α}. The growth condition (3.2) for $\pi_{\alpha}(\Phi_r)$ is deduced from the congruences (6.10) between modular forms, using the relation (6.11).

Recall: then we obtain the function $\mathcal{L}_{\mu}(x,s)$ by evaluation at (s,ψ): this is a p-adic analytic function in two variables $(x,s) \in X \times \mathcal{B} \subset X \times X$:

$$\boxed{\mathcal{L}_{\tilde{\mu}}(x,s) = ev_s(\mathcal{L}_{\tilde{\mu}}) \quad (x \in X, \quad \mathcal{L}_{\tilde{\mu}}(x) \in \mathcal{A}).}$$

Modular Eisenstein distributions Φ_j

Consider again two auxiliary Dirichlet characters $\xi \mod p$, $\xi(-1) = \pm 1$, and use the method of Rankin-Selberg for the convolution

$$D(s, f, g) = L_N(2s + 2 - k - l, \psi\overline{\xi\chi}) \sum_{n=1}^{\infty} a_n b_n n^{-s}, \text{ where} \tag{7.12}$$

$$b_n = \sigma_{l-1,\bar{\chi},\bar{\xi}}(n) = \sum_{d|n, d>0} \bar{\chi}(d)\bar{\xi}(n/d)d^{l-1},$$

are the Fourier coefficients of an Eisenstein series $g = \sum_{n=0}^{\infty} b_n q^n$ of weight l (and of Dirichlet character $\bar{\chi}\bar{\xi}$) if $\chi\xi(-1) = (-1)^l$, so that

$$L_g(s) = \sum_{n=1}^{\infty} b_n n^{-s} = L(s - l + 1, \bar{\chi})L(s, \bar{\xi}).$$

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Rankin's lemma (cf. [Ra52]) gives

$$D(s, f, g) = L_N(2s + 2 - k - l, \psi\overline{\xi\chi}) \sum_{n=1}^{\infty} a_n b_n n^{-s} \tag{7.13}$$
$$= L_f(s - l + 1, \bar{\chi})L_f(s, \bar{\xi}),$$

and evaluation at $s = k - 1$ is expressed through the Rankin-Selberg integral of f with the product of two Eisenstein series of weights $k - 1 - j$ and $1 + j$:

$$\langle f, E_{k-1-j}(\xi, \chi)E_{1+j}(\psi\overline{\xi\chi})\rangle_{Np^v}.$$

One defines the modular distributions Φ_j on the group $Y = \varprojlim_v (\mathbb{Z}/Np^v\mathbb{Z})^\times$ in such a way that the modular form $\Phi_j(\chi) \in \mathcal{A}[\![q]\!]$ is the product of these Eisenstein series with variable coefficients in $\mathcal{A}$:

$$ev_k(\Phi_j(\chi)) := (-1)^j E_{k-1-j}(\xi, \chi)E_{1+j}(\psi\overline{\xi\chi}) =: \Phi_{j,k}(\chi).$$

Main congruence

Explicitely, the Fourier coefficients of Φ_j (for $j = 0, \ldots, k-2$) are given by

$$\Phi_j(a + (Np^v)) \qquad (7.14)$$
$$= \sum_{b \bmod Np^v} \psi\bar{\xi}(b) \sum_{n \geq 0}^{\infty} \sum_{n_1+n_2=n} A_j(n_1, ab) B_j(n_2, b) q^n \in \mathcal{A}[\![q]\!], \text{ where}$$

$$A_j(n_1, ab)(k) = \sum_{\substack{d_1 | n_1 \\ (n_1/d_1) \equiv ab \bmod Np^v}} \xi(d_1) \mathrm{sgn}\,(d_1) d_1^{k-2-j} \qquad (7.15)$$
$$B_j(n_2, b)(k) = \sum_{\substack{d_2 | n_2 \\ d_2 \equiv b \bmod Np^v}} \mathrm{sgn}\,(d_2)(n_2/d_2)^j \quad \text{for} \quad n_2 > 0.$$

(Note that the last series has constant coefficients). One verifies coefficient-by-coefficient that the consructed modular distributions Φ_j satisfy the level condition and the divisibility condition (6.9), (6.10):

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Main congruence (continued)

$$U^v \sum_{j=0}^{t} \binom{t}{j} (-a_p)^{t-j} \Phi_j(a + (Np^v)) \qquad (7.16)$$
$$= \sum_{j=0}^{t} \binom{t}{j} (-a_p)^{t-j} \sum_{n \geq 0}^{\infty} \sum_{n_1+n_2=p^v n} (-1)^j A_j(n_1, ab) B_j(n_2, b) q^n$$
$$\stackrel{?}{\equiv} 0 (\bmod p^{tv}).$$

Let us fix n_1 et n_2 with $n_1 + n_2 = p^v n$, $d_1 | n_1$ and $d_2 | n_2$ with $(n_1/d_1) \equiv ab \bmod Np^v$ et $d_2 \equiv b \bmod Np^v$, and write only the terms which depend on j:

$$\sum_{j=0}^{t} \binom{t}{j} (-a)^{t-j} (-1)^j d_1^{k-2-j} \left(\frac{n_2}{d_2}\right)^j = d_1^{k-2} \left(-a - \left(\frac{n_2}{d_1 d_2}\right)\right)^t \qquad (7.17)$$
$$\equiv d_1^{k-2} d_2^{-t} \left(-a d_2 + \left(\frac{n_1}{d_1}\right)\right)^t \equiv 0 \bmod p^{vt}.$$

The congruence (7.17) is then satisfied because $p \nmid d_2$ and

$$-a - \left(\frac{n_2}{d_1 d_2}\right) \equiv \bmod p^v.$$

Algebraic $\mathcal{A}$-linear form $\ell_\alpha : \mathcal{M}^\alpha(\mathcal{A})^\dagger \to \mathcal{A}$

Let us describe a linear form ℓ_α on the locally free module $\mathcal{M}_N(\psi;\mathcal{A})^\alpha = \pi_\alpha(\mathcal{M}_N(\psi;\mathcal{A}))$ of finite rank.

Let use a basis $\{g^i\}$ of $\mathcal{M}^\alpha(\mathcal{A})^\dagger$ over the field of fractions Frac($\mathcal{A}$), such that $g^1 = g$ is fixed Coleman's eigenvector as above, and g^i are eigenfunctions of all Hecke operators T_l, $(l \nmid Np)$.
Define $\ell_\alpha(h) = x_1,$ where $h = \sum_i x_i g^i, x \in \mathcal{A}$
(the first coordinate of $h \in \mathcal{M}^\alpha(\mathcal{A})^\dagger$. An explicit evaluation in terms of the Peterson product shows:

$$ev_k(\ell_\alpha(h)) = \ell_{\alpha(k)}(h_k), \text{ where } h_k = ev_k(h) \in \mathcal{M}_k(N,\psi).$$

The R.H.S. can be computed for classical modular forms h_k through the (normalized) Petersson scalar product, moreover, $\ell_\alpha(g) = 1$.

Proof of Main Theorem

Take the admissible measure $\tilde{\mu}_\alpha := \ell_{\alpha,f}(\tilde{\Phi}^\alpha),$ with $\tilde{\Phi}^\alpha$ constructed by the admissibility criterium of Theorem 6.1 out of products of Eisenstein series Φ_j and the linear form $\ell_{\alpha,f}$ (the Petersson product over $\mathcal{A}$). Let us compute the integrals

$$ev_k\left(\int_Y \chi y_p^j \, d\tilde{\mu}_{\alpha,f}\right) = ev_k(\ell_\alpha(\pi_\alpha(\Phi_j(\chi))) = \quad (9.18)$$
$$ev_k(\ell_\alpha(U^{-v}\pi_{\alpha,0}U^v\Phi_j(\chi)))$$
$$= \ell_{\alpha(k)}(\pi_{\alpha(k)}\Phi_{j,k}(\chi)) = \alpha(k)^{-v}\frac{\langle f_k^0, U^v\Phi_{j,k}(\chi)\rangle}{\langle f_k^0, (f_k)_0\rangle}$$

for primitive Dirichlet characters χ mod p^v, using the relation (6.11): $\pi_\alpha(h) = U^{-v}\pi_{\alpha,0}(U^v h)$, where
$\Phi_{j,k} = ev_k(\Phi_j) = (-1)^j E_{k-1-j}(\xi,\chi)E_{1+j}(\psi\overline{\xi\chi})$.

Proof of Main Theorem (continued)

The value of the p-adic integral (9.18) can be computed using the Rankin–Selberg convolution:

$$L_{f_k}(s-l+1,\bar{\chi})L_{f_k}(s,\bar{\xi}) = L_N(2s+2-k-l,\psi\overline{\xi\chi})\sum_{n=1}^{\infty} a_n(k)b_n n^{-s}, \quad (9.19)$$

where $b_n = \sigma_{l-1,\bar{\chi},\bar{\xi}}(n) = \sum_{d|n,d>0}\bar{\chi}(d)\bar{\xi}(n/d)d^{l-1}$, are the Fourier coefficients of an Eisenstein series $g = \sum_{n=0}^{\infty} b_n q^n$ of weight l with character $\bar{\chi}\bar{\xi}$) (if $\chi\xi(-1) = (-1)^l$).
Put $s = k-1$, $l = k-1-j$, $j = 0,\cdots k-2$ with $k > 2+j$, into (9.19):

$$L_{f_k}(1+j,\bar{\chi})L_{f_k}(k-1,\bar{\xi}) = L_N(1+j,\psi\overline{\xi\chi})\sum_{n=1}^{\infty} a_n(k)b_n n^{-k+1}.$$

Proof of Main Theorem (continued)

Using this equality, the R.H.S. of (9.18): is then computed using the Rankin–Selberg integral in the form:

$$ev_k(\ell_\alpha(\pi_\alpha(\Phi_j(\chi)))= t_k\cdot\frac{p^{\nu j}G(\chi)}{\alpha(k)^\nu}L^*_{f_k}(1+j,\bar{\chi}),$$

$$\text{where } c^{\pm}(f_k) = \frac{(-2i\pi)^{k-1}\langle f_k,f_k\rangle}{\Gamma(k-1)L_{f_k}(k-1,\bar{\xi})},$$

$G(\chi)$ denotes the Gauss sum of the character χ mod p^v, and $t_k \in \mathbb{Q}^\times$ is an explicit elementary constant. Then one applies directly theorem 6.1 (the admissibility criterion) with $\varkappa = 1$, and the congruences (7.16) in order to obtain the required h-admissibles measures $\tilde{\mu} = \mu_{f,\alpha}$ in the form $\mu_{f,\alpha} = \ell_{f,\alpha}(\tilde{\Phi}^\alpha)$ (given by the sequence of the distributions $\Phi_j^\alpha = \pi_\alpha(\Phi_j)$).

Conclusion

After having an admissible measure $\tilde{\Phi}^\alpha$ with values in modular forms over the algebra $\mathcal{A}$, we then construct the required h-admissibles measures $\tilde{\mu} = \tilde{\mu}_{f,\alpha}$ in the form $\tilde{\mu}_{f,\alpha} = \ell_\alpha(\tilde{\Phi}^\alpha)$, as explained above.

Indeed, we obtain the function in question $\mathcal{L}_\mu(x, \mathbf{s})$ by evaluation at $\mathbf{s} = (s, \psi) \in \mathcal{B}$: this is a p-adic analytic function in two variables $(x, \mathbf{s}) \in X \times \mathcal{B} \subset X \times X$:

$$\mathcal{L}_{\tilde{\mu}}(x, \mathbf{s}) := ev_{\mathbf{s}}(\mathcal{L}_{\tilde{\mu}})(x) \quad (x \in X,\ \mathbf{s} \in \mathcal{B},\ \mathcal{L}_{\tilde{\mu}}(x) \in \mathcal{A}).$$

Here $\mathcal{A} = \mathcal{A}(\mathcal{B})$ denote again the Banach algebra $\mathcal{A}$ and $\mathcal{B}$ is an affinoid neighbourhood around $\mathbf{s} = (s, \psi) \in \mathcal{B}$ (with a given Dirichlet character ψ mod N).

A further development: Garrett's triple products

Our data: three primitive cusp eigenforms

$$f_j(z) = \sum_{n=1}^{\infty} a_{n,j} q^n \in \mathcal{S}_{k_j}(N_j, \psi_j),\ (j = 1, 2, 3) \qquad (10.20)$$

of weights k_1, k_2, k_3, of conductors N_1, N_2, N_3, and of nebentypus characters ψ_j mod N_j, $N := \mathrm{LCM}(N_1, N_2, N_3)$.

Let p be a prime, $p \nmid N$. We view $f_j \in \overline{\mathbb{Q}}[\![q]\!] \overset{i_p}{\hookrightarrow} \mathbb{C}_p[\![q]\!]$ via a fixed embedding $\overline{\mathbb{Q}} \overset{i_p}{\hookrightarrow} \mathbb{C}_p$, $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}}_p$ is Tate's field.
Let χ denote a variable Dirichlet character mod $Np^v, v \geq 0$.
We view k_j as a variable weight in the weight space
$X = X_{Np^v} = \mathrm{Hom}_{cont}(Y, \mathbb{C}_p^*)$, $Y = (\mathbb{Z}/N\mathbb{Z})^* \times \mathbb{Z}_p^* \ni (y_0, y_p)$.
The space X is a p-adic analytic space first used in Serre's [Se2] *Formes modulaires et fonctions zêta p-adiques*. Denote by $(k, \chi) \in X$ the homomorphism $(y_0, y_p) \mapsto \chi(y_0)\chi(y_p \bmod p^v) y_p^k$. We write simply k_j for the couple $(k_j, \psi_j) \in X$.

A four variable p-adic L function

attached to Garrett's triple product of three Coleman's families

$$k_j \mapsto \left\{ f_{j,k_j} = \sum_{n=1}^{\infty} a_{n,j}(k_j) q^n \right\}$$

of cusp eigenforms of three constant slopes $\sigma_j = \mathrm{ord}_p(\alpha^{(1)}_{p,j}(k_j)) \geq 0$ where $\alpha^{(1)}_{p,j}(k_j)$, $\alpha^{(2)}_{p,j}(k_j)$ are the Satake parameters given as inverse roots of the Hecke p–polynomial $1 - a_{p,j}X - \psi_j(p)p^{k_j-1}X^2 = (1 - \alpha^{(1)}_{p,j}(p)X)(1 - \alpha^{(2)}_{p,j}(p)X)$.
We assume that $\mathrm{ord}_p(\alpha^{(1)}_{p,j}(k_j)) \leq \mathrm{ord}_p(\alpha^{(2)}_{p,j}(k_j))$.
This extends a previous result: (see [PaTV], where a two variable p-adic L-function was constructed interpolating on all k a function $(k, s) \mapsto L^*(f_k, s, \chi)$ $(s = 1, \cdots, k-1)$ for such a family.
We use the theory of p-adic integration with values in spaces of nearly holomorphic modular forms (in the sense of Shimura, see [Sh2000]).

Generalities on triple products

The triple product with a Dirichlet character χ is defined as the following complex L-function (an Euler product of degree eight):

$$L(f_1 \otimes f_2 \otimes f_3, s, \chi) = \prod_{p \nmid N} L((f_1 \otimes f_2 \otimes f_3)_p, \chi(p)p^{-s}), \qquad (10.21)$$

where $L((f_1 \otimes f_2 \otimes f_3)_p, X)^{-1} =$ (10.22)

$$\det\left(1_8 - X \begin{pmatrix} \alpha^{(1)}_{p,1} & 0 \\ 0 & \alpha^{(2)}_{p,1} \end{pmatrix} \otimes \begin{pmatrix} \alpha^{(1)}_{p,2} & 0 \\ 0 & \alpha^{(2)}_{p,2} \end{pmatrix} \otimes \begin{pmatrix} \alpha^{(1)}_{p,3} & 0 \\ 0 & \alpha^{(2)}_{p,3} \end{pmatrix}\right)$$

$$= \prod_{\eta} (1 - \alpha^{(\eta(1))}_{p,1} \alpha^{(\eta(2))}_{p,2} \alpha^{(\eta(3))}_{p,3} X)$$

$$= (1 - \alpha^{(1)}_{p,1}\alpha^{(1)}_{p,2}\alpha^{(1)}_{p,3}X)(1 - \alpha^{(1)}_{p,1}\alpha^{(1)}_{p,2}\alpha^{(2)}_{p,3}X) \cdot \ldots \cdot (1 - \alpha^{(2)}_{p,1}\alpha^{(2)}_{p,2}\alpha^{(2)}_{p,3}X),$$

product taken over all 8 maps $\eta : \{1, 2, 3\} \to \{1, 2\}$.

Critical values and functional equation

We use the corresponding normalized L function (see [De79], [Co], [Co-PeRi]), which has the form:

$$\Lambda(f_1 \otimes f_2 \otimes f_3, s, \chi) = \qquad (10.23)$$
$$\Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(s-k_3+1)\Gamma_{\mathbb{C}}(s-k_2+1)\Gamma_{\mathbb{C}}(s-k_1+1)L(f_1 \otimes f_2 \otimes f_3, s, \chi),$$

where $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$.
The Gamma-factor determines the critical values $s = k_1, \cdots, k_2 + k_3 - 2$ of $\Lambda(s)$, which we explicitely evaluate (like in the classical formula $\zeta(2) = \frac{\pi^2}{6}$). A functional equation of $\Lambda(s)$ has the form:

$$s \mapsto k_1 + k_2 + k_3 - 2 - s.$$

Statement of the problem

Given three p-adic analytic families $\mathbf{f}_j$ of slope $\sigma_j \geq 0$, to construct a four-variable p-adic L-function attached to Garrett's triple product of these families

We show that this function interpolates the special values

$$(s, k_1, k_2, k_2) \longmapsto \Lambda(f_{1,k_1} \otimes f_{2,k_2} \otimes f_{3,k_3}, s, \chi)$$

at critical points $s = k_1, \cdots, k_2 + k_3 - 2$ for balanced weights $k_1 \leq k_2 + k_3 - 2$; we prove that these values are algebraic numbers afters dividing by certain "periods".

However, our construction uses directly modular forms, and not the L-values in question.
A comparison of special values of two functions is done after the construction.

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