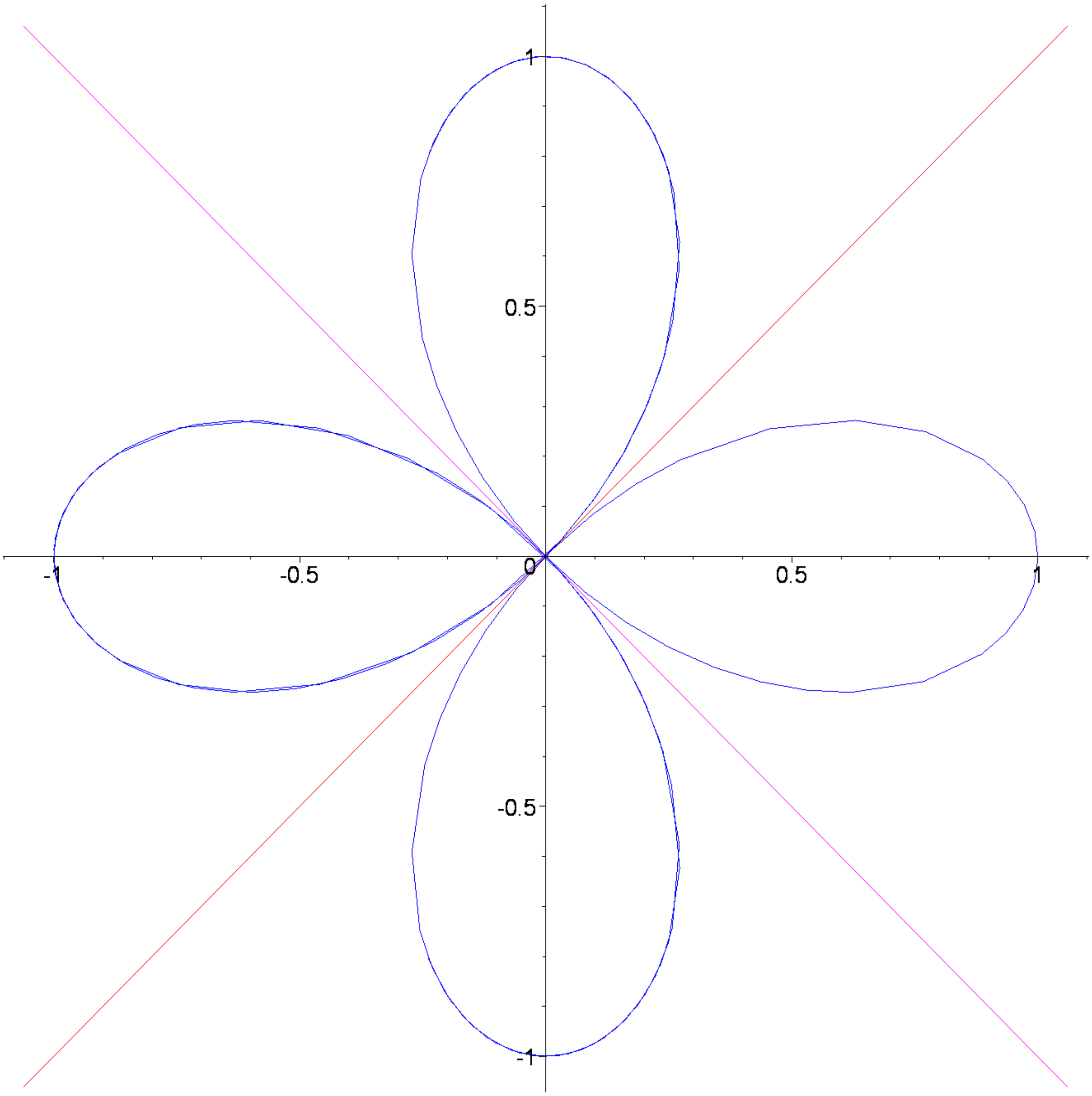
Visualisations sur l'ordinateur

```
> ############## FEUILLE 3, 2008-2009
```

Exercice 1. Montrer que la courbe d’équation polaire r = cos 2\theta, admet
pour tangentes à l’origine les droites d’angles polaires \theta = \pm ¼. Tracer la courbe.

```
> restart:with(plots):
  P1:=plot([cos(2*t)*cos(t),cos(2*t)*sin(t),
  t=-5..5],color=blue):
  P2:=plot([r*cos(Pi/4),r*sin(Pi/4), r=-1.5..1.5],color=red):
  P3:=plot([r*cos(-Pi/4),r*sin(-Pi/4),
  r=-1.5..1.5],color=magenta):
  display(P1,P2,P3);
Warning, the name changecoords has been redefined
```



Exercice 3. Déterminer un repère de Frenet pour les courbes planes suivantes
(i) droite, (ii) cercle, (iii) ellipse, (iv) hyperbole, (v) parabole, (vi) graphe de la fonction sinus.

```
> #Repère de Frenet(i) droite
  restart:x:=t:y:=2*t+3:
> u:=diff(x,t):v:=diff(y,t):[u,v]:
  l:=simplify(sqrt(u^2+v^2)):
> tau:=[simplify(u/l),simplify(v/l)];
```

$$\tau := \left[\frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}\right]$$

```
> eta:=[-simplify(v/l),simplify(u/l)];
```

$$\eta := \left[-\frac{2\sqrt{5}}{5}, \frac{\sqrt{5}}{5}\right]$$

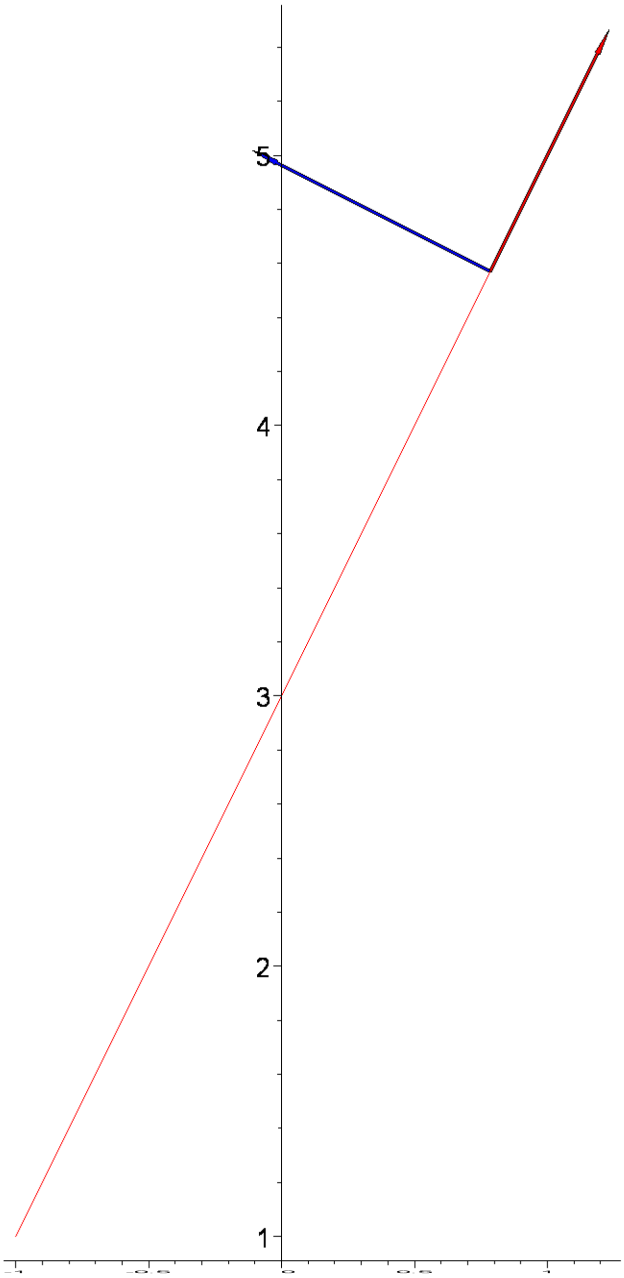
```
> t0:=Pi/4; x0:=subs(t=t0,x):
  y0:=subs(t=t0,y):u0:=subs(t=t0,u):v0:=subs(t=t0,v):
  with(plots):
  b0:=
  plot([x,y,t=-1..1]):
```

$$t0 := \frac{\pi}{4}$$

```
Warning, the name changecoords has been redefined
```

```
> b1 := arrow(<x0,y0>, <u0,v0>, length=[1], width=[0.01,
  relative], head_length=[0.1, relative], color=red):
  b2 := arrow( <x0,y0>, <-v0,u0>,length=[1],width=[0.01,
  relative], head_length=[0.1, relative], color=blue):

  display(b0, b1, b2, scaling=CONSTRAINED);
```



```
> #Repère de Frenet(ii) cercle
  restart:x:=cos(t):y:=sin(t):
> u:=diff(x,t):v:=diff(y,t):[u,v]:
  l:=simplify(sqrt(u^2+v^2)):
> tau:=[simplify(u/l),simplify(v/l)];
```

$$\tau := [-\sin(t), \cos(t)]$$

```
> eta:=[-simplify(v/l),simplify(u/l)];
```

$$\eta := [-\cos(t), -\sin(t)]$$

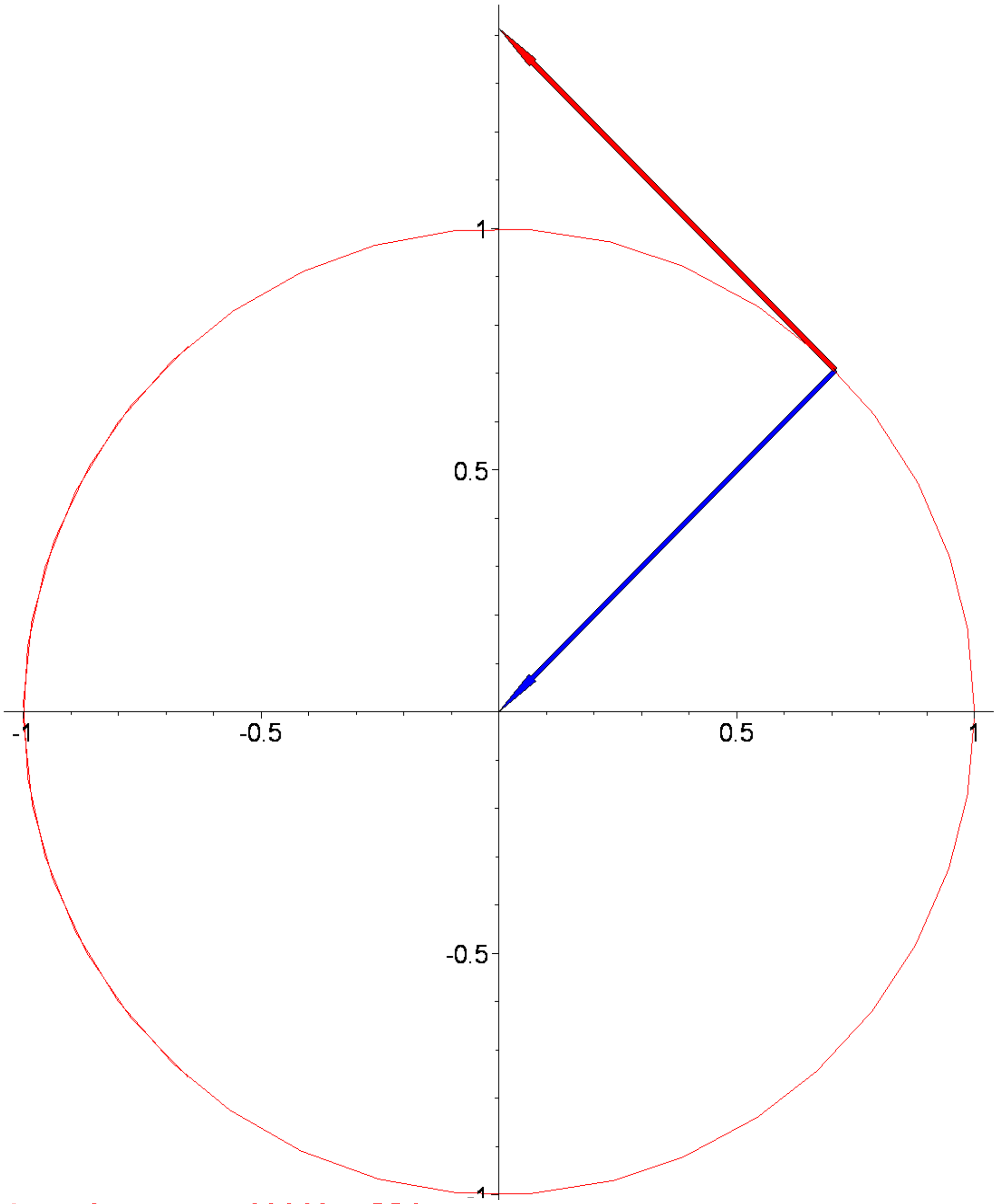
```
> t0:=Pi/4; x0:=subs(t=t0,x):
  y0:=subs(t=t0,y):u0:=subs(t=t0,u):v0:=subs(t=t0,v):
  with(plots):
  b0:=
  plot([x,y,t=-4..4]):
>
```

$$t0 := \frac{\pi}{4}$$

```
Warning, the name changecoords has been redefined
```

```
> b1 := arrow(<x0,y0>, <u0,v0>, length=[1], width=[0.01,
  relative], head_length=[0.1, relative], color=red):
  b2 := arrow( <x0,y0>, <-v0,u0>,length=[1],width=[0.01,
  relative], head_length=[0.1, relative], color=blue):

  display(b0, b1, b2, scaling=CONSTRAINED);
```




```
> #Repère de Frenet(iii) ellipse
  restart:x:=2*(cos(t)):y:=sin(t):
> u:=diff(x,t):v:=diff(y,t):[u,v]:
  l:=simplify(sqrt(u^2+v^2)):
> tau:=[simplify(u/l),simplify(v/l)];
```

$$\tau := \left[-\frac{2\sin(t)}{\sqrt{-3\cos(t)^2+4}}, \frac{\cos(t)}{\sqrt{-3\cos(t)^2+4}} \right]$$

```
> eta:=[-simplify(v/l),simplify(u/l)];
```

$$\eta := \left[-\frac{\cos(t)}{\sqrt{-3\cos(t)^2+4}}, -\frac{2\sin(t)}{\sqrt{-3\cos(t)^2+4}} \right]$$

```
> t0:=Pi/4; x0:=subs(t=t0,x):
  y0:=subs(t=t0,y):u0:=subs(t=t0,u):v0:=subs(t=t0,v):
  with(plots):
  b0:=
```

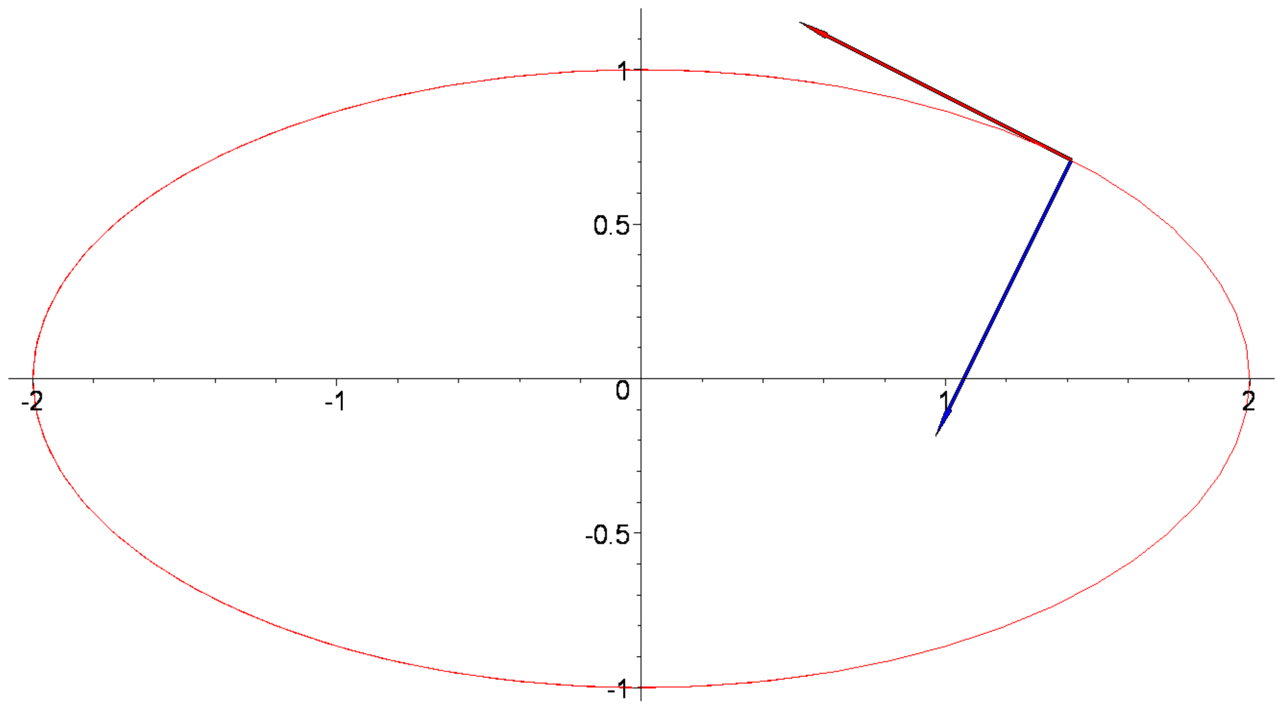
```
plot([x,y,t=-5..5]):
>
```

$$t0 := \frac{\pi}{4}$$

Warning, the name changecoords has been redefined

```
> b1 := arrow(<x0,y0>, <u0,v0>, length=[1], width=[0.01,
  relative], head_length=[0.1, relative], color=red):
  b2 := arrow( <x0,y0>, <-v0,u0>,length=[1],width=[0.01,
  relative], head_length=[0.1, relative], color=blue):

  display(b0, b1, b2, scaling=CONSTRAINED);
```




```
> #Repère de Frenet(iv) parabole
  restart:x:=t:y:=t^2:
> u:=diff(x,t):v:=diff(y,t):[u,v]:
```

```
  l:=simplify(sqrt(u^2+v^2)):
> tau:=[simplify(u/l),simplify(v/l)];
```

$$\tau := \left[\frac{1}{\sqrt{1+4\,t^2}}, \frac{2\,t}{\sqrt{1+4\,t^2}}\right]$$

```
> eta:=[-simplify(v/l),simplify(u/l)];
```

$$\eta := \left[-\frac{2\,t}{\sqrt{1+4\,t^2}}, \frac{1}{\sqrt{1+4\,t^2}}\right]$$

```
> t0:=Pi/4; x0:=subs(t=t0,x):
  y0:=subs(t=t0,y):u0:=subs(t=t0,u):v0:=subs(t=t0,v):
  with(plots):
  b0:=
  plot([x,y,t=-2..2]):
```

$$t0 := \frac{\pi}{4}$$

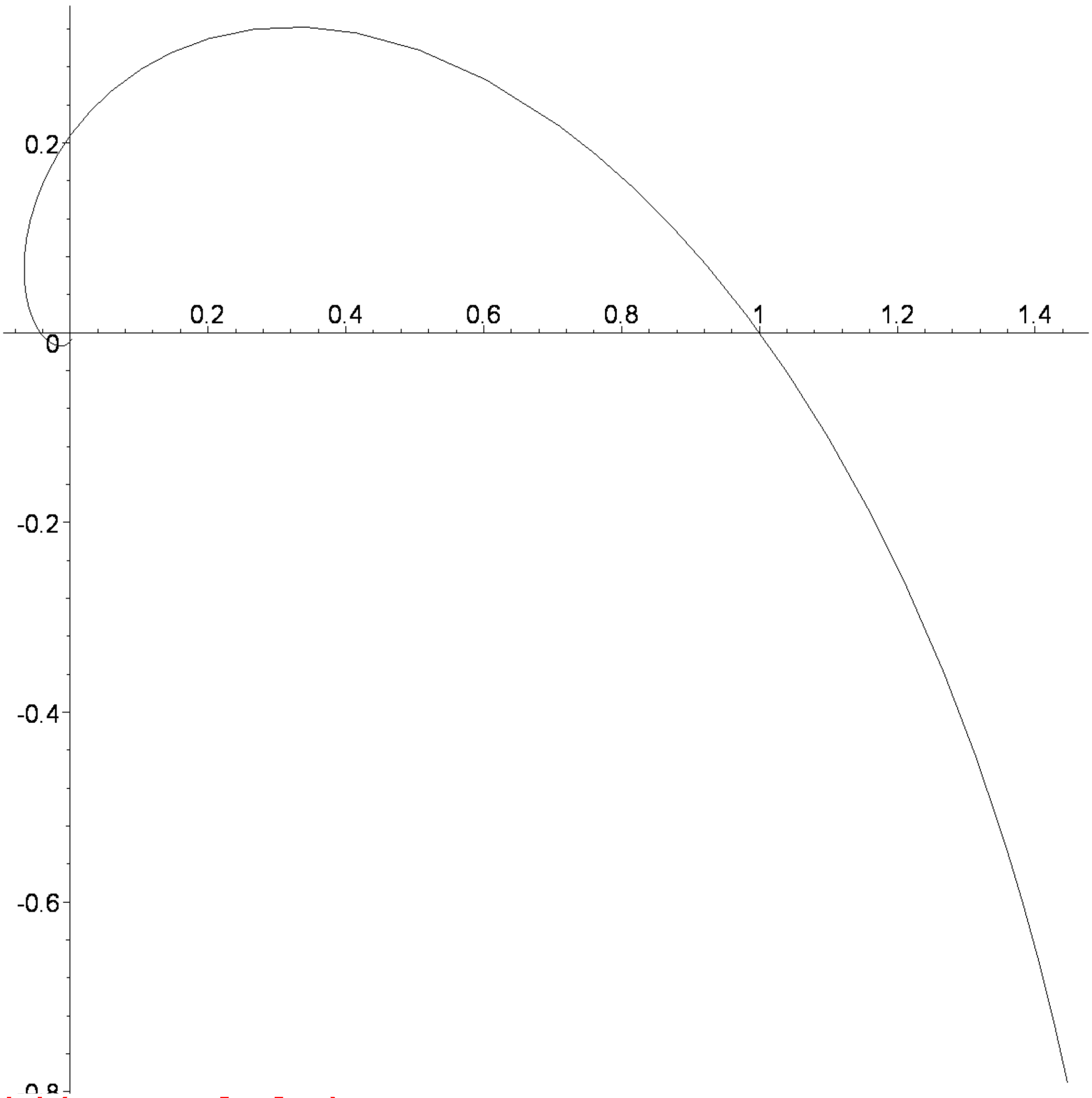
```
Warning, the name changecoords has been redefined
```

```
> b1 := arrow(<x0,y0>, <u0,v0>, length=[1], width=[0.01,
  relative], head_length=[0.1, relative], color=red):
  b2 := arrow( <x0,y0>, <-v0,u0>,length=[1],width=[0.01,
  relative], head_length=[0.1, relative], color=blue):

  #display(b0, b1, b2, scaling=CONSTRAINED);
```

Exercice 4. Calculer la longueur de l'arc de la spirale logarithmique
r = exp(-t) entre les points de paramètre 0 et 1. Calculer la courbure et le
centre ce courbure en tout point.

```
> ######################################     Exercice 4.
  with(plots):plot([exp(-t)*cos(t),exp(-t)*sin(t),
  t=-.5..5],color=black);
```




```
> with(VectorCalculus):
> ArcLength( <exp(-t)*cos(t),exp(-t)*sin(t)>, t=0..2*Pi ) ;
Warning, the assigned names <,> and <|> now have a global binding

Warning, these protected names have been redefined and unprotected: *, +, ., D,
Vector, diff, int, limit, series
```

$$\sqrt{2}\,(\mathbf{e}^{(2\,\pi)} - 1)\,\mathbf{e}^{(-2\,\pi)}$$

```
> SetCoordinates( 'polar' );
  ArcLength( <exp(-t),t>, t=0..2*Pi ) ;
> Curvature( <exp(-t),t> ):
  simplify(%) assuming t::real;
```

$$polar$$

$$\sqrt{2}\,(\mathbf{e}^{(2\,\pi)} - 1)\,\mathbf{e}^{(-2\,\pi)}$$

$$\frac{1}{2}\sqrt{2}\,\mathbf{e}^{t}$$

```
> x:=exp(-t)*cos(t); y:=exp(-t)*sin(t);
```

$$x := e^{(-t)}\cos(t)$$

$$y := e^{(-t)}\sin(t)$$

```
> xp:=diff(x,t);yp:=diff(y,t);
```

$$xp := -e^{(-t)}\cos(t) - e^{(-t)}\sin(t)$$

$$yp := -e^{(-t)}\sin(t) + e^{(-t)}\cos(t)$$

```
> xs:=diff(xp,t);ys:=diff(yp,t);
```

$$xs := 2\,e^{(-t)}\sin(t)$$

$$ys := -2\,e^{(-t)}\cos(t)$$

```
> X:=simplify(x-yp*(xp^2+yp^2)/(xp*ys-yp*xs));
```

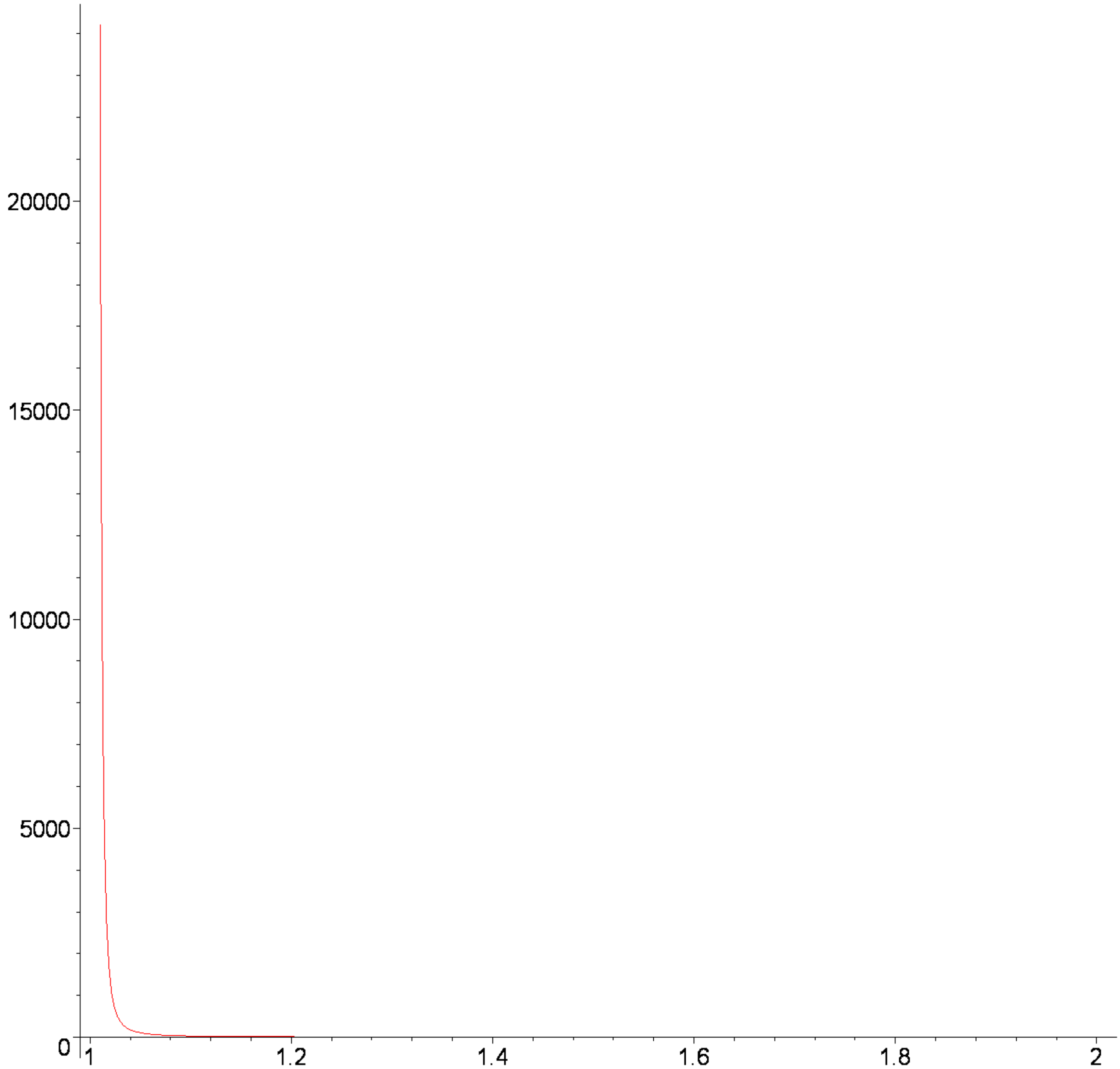
$$X := e^{(-t)}\sin(t)$$

```
> Y:=simplify(y+xp*(xp^2+yp^2)/(xp*ys-yp*xs));
```

$$Y := -e^{(-t)}\cos(t)$$

Exercice 5. Déterminer la limite du rayon de courbure de la courbe paramétrée t -> (1 + t^2,(1 + t)e^(1/t) si t \to 0.

```
> restart:with(plots):
  plot([1+t^2,(1+t)*exp(1/t),t=0.1..1],color=red);
Warning, the name changecoords has been redefined
```




```
> x:=1+t^2;y:=(1+t)*exp(1/t);
```

$$x := 1 + t^2$$

$$y := (1 + t)\, \mathbf{e}^{\left(\frac{1}{t}\right)}$$

```
> xp:=diff(x,t);yp:=diff(y,t);
```

$$xp := 2\, t$$

$$yp := \mathbf{e}^{\left(\frac{1}{t}\right)} - \frac{(1 + t)\, \mathbf{e}^{\left(\frac{1}{t}\right)}}{t^2}$$

```
> xs:=diff(xp,t);ys:=diff(yp,t);
```

$$xs := 2$$

$$ys := -\frac{2\,\mathbf{e}^{\left(\frac{1}{t}\right)}}{t^2} + \frac{2\,(1+t)\,\mathbf{e}^{\left(\frac{1}{t}\right)}}{t^3} + \frac{(1+t)\,\mathbf{e}^{\left(\frac{1}{t}\right)}}{t^4}$$

```
> c:=simplify((xp*ys-yp*xs)/(xp^2+yp^2)^(3/2));
```

$$c := -2\,t\,\mathbf{e}^{\left(\frac{1}{t}\right)}\,(-4\,t-1+t^3-t^2) \Bigg/ \left(\left(4\,t^6+\mathbf{e}^{\left(\frac{2}{t}\right)}\,t^4-\mathbf{e}^{\left(\frac{2}{t}\right)}\,t^2-2\,\mathbf{e}^{\left(\frac{2}{t}\right)}\,t^3+\mathbf{e}^{\left(\frac{2}{t}\right)}+2\,\mathbf{e}^{\left(\frac{2}{t}\right)}\,t\right) \sqrt{\frac{4\,t^6+\mathbf{e}^{\left(\frac{2}{t}\right)}\,t^4-\mathbf{e}^{\left(\frac{2}{t}\right)}\,t^2-2\,\mathbf{e}^{\left(\frac{2}{t}\right)}\,t^3+\mathbf{e}^{\left(\frac{2}{t}\right)}+2\,\mathbf{e}^{\left(\frac{2}{t}\right)}\,t}{t^4}}\right)$$

```
> limit(c,t=infinity);
```

$$0$$

```
> limit(c,t=0);
```

$$0$$

Exercice 6. Soit la parabole x = t; y = t^2. Calculer la longueur de l'arc entre t = 0 et t = 1.

```
> restart:with(VectorCalculus):simplify(ArcLength( <t,t^2>, t=0..1
  )) ;
Warning, the assigned names <,> and <|> now have a global binding

Warning, these protected names have been redefined and unprotected: *, +, ., D,
Vector, diff, int, limit, series
```

$$\frac{\sqrt{5}}{2} - \frac{1}{4}\ln(-2+\sqrt{5})$$

```
> int(sqrt(1+4*t^2), t);
```

$$\frac{t\sqrt{1+4\,t^2}}{2} + \frac{1}{4}\operatorname{arcsinh}(2\,t)$$

```
> simplify(Curvature( <t,t^2>, t )) ;
```

$$\frac{2\,\operatorname{csgn}((t^2+4\,|t|^4)\,\bar{t}^2)}{(1+4\,t^2)^{(3/2)}}$$

```
> x:=t;y:=t^2;
```

$$x := t$$

$$y := t^2$$

```
> xp:=diff(x,t);yp:=diff(y,t);
```

$$xp := 1$$

$$yp := 2\,t$$

```
> xs:=diff(xp,t);ys:=diff(yp,t);
```

$$xs := 0$$

$$ys := 2$$

```
> c:=(xp*ys-yp*xs)/(xp^2+yp^2)^(3/2);
```

$$c := \frac{2}{(1+4\,t^2)^{(3/2)}}$$

```
> X:=simplify(x-yp*(xp^2+yp^2)/(xp*ys-yp*xs));
```

$$X := -4\,t^3$$

```
> Y:=simplify(y+xp*(xp^2+yp^2)/(xp*ys-yp*xs));
```

$$Y := 3\,t^2 + \frac{1}{2}$$

Exercice 7. Calculer la courbure de la branche de l'hyperbole y =1/x et x > 0. En quel(s) point(s) est-elle maximale ?

```
> restart:x:=t;y:=1/t;
```

$$x := t$$

$$y := \frac{1}{t}$$

```
> xp:=diff(x,t);yp:=diff(y,t);
```

$$xp := 1$$

$$yp := -\frac{1}{t^2}$$

```
> xs:=diff(xp,t);ys:=diff(yp,t);
```

$$xs := 0$$

$$ys := \frac{2}{t^3}$$

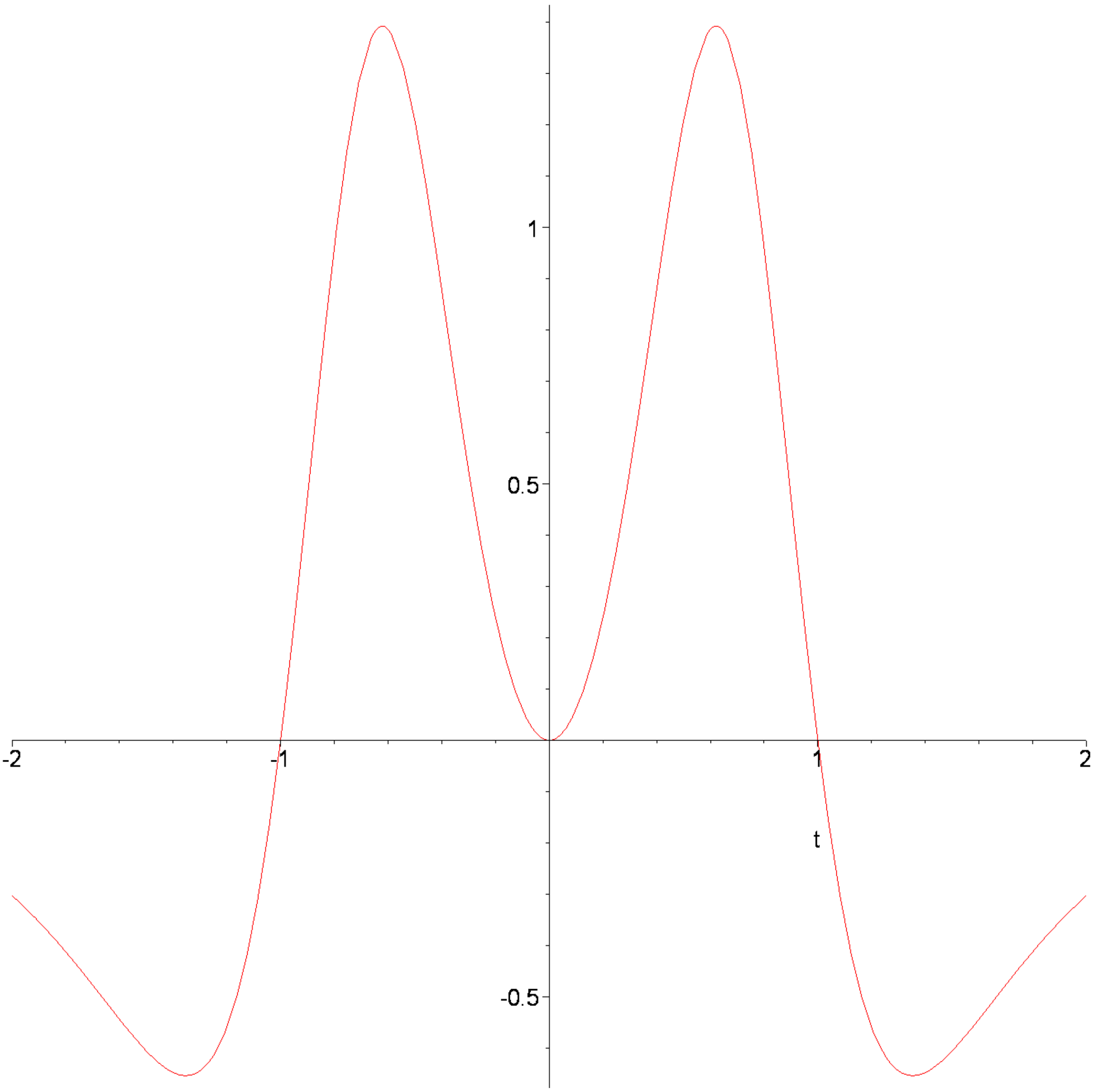
```
> c:=simplify((xp*ys-yp*xs)/(xp^2+yp^2)^(3/2));
```

$$c := \frac{2\,t}{(t^4+1)\sqrt{\frac{t^4+1}{t^4}}}$$

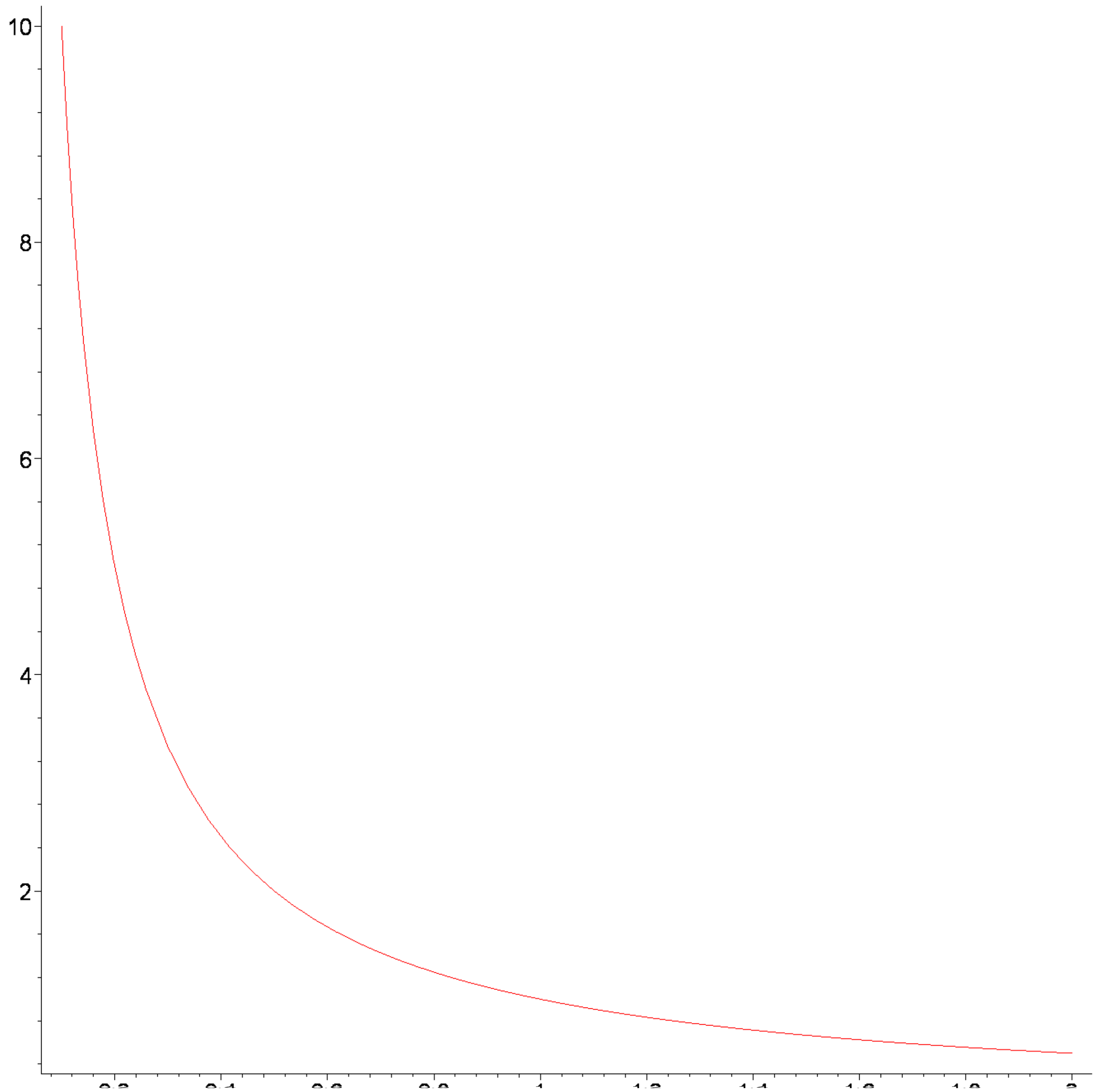
```
> simplify(diff(c,t));solve(diff(c,t));with(plots):plot(diff(c,t),
  t=-2..2);plot([t,1/t,t=.1..2]);
```

$$-\frac{6\,(t^4-1)}{(t^4+1)^2\sqrt{\frac{t^4+1}{t^4}}}$$

$$-1,\ 1,\ I,\ -I$$



1
0.5
-0.5
-2
-1
1
2
t




```
> 
> solve(diff(c,t));
```

$$-1, 1, I, -I$$

```
> simplify(subs(t=1,c));
```

$$\frac{\sqrt{2}}{2}$$

```
> ################### Ex 9
  restart:x:=t;y:=exp(t);
```

$$x := t$$

$$y := e^t$$

```
> xp:=diff(x,t);yp:=diff(y,t);
```

$$xp := 1$$

$$yp := e^t$$

```
> xs:=diff(xp,t);ys:=diff(yp,t);
```

$$xs := 0$$

$$ys := e^t$$

```
> c:=(xp*ys-yp*xs)/(xp^2+yp^2)^(3/2);
```

$$c := \frac{e^t}{(1+(e^t)^2)^{(3/2)}}$$

```
> u/(1+u^2)^(3/2);diff(u/(1+u^2)^(3/2),
  u);solve(diff(u/(1+u^2)^(3/2), u));
```

$$\frac{u}{(1+u^2)^{(3/2)}}$$

$$\frac{1}{(1+u^2)^{(3/2)}} - \frac{3\,u^2}{(1+u^2)^{(5/2)}}$$

$$-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}$$

```
> t1:=solve(diff(c,t));
```

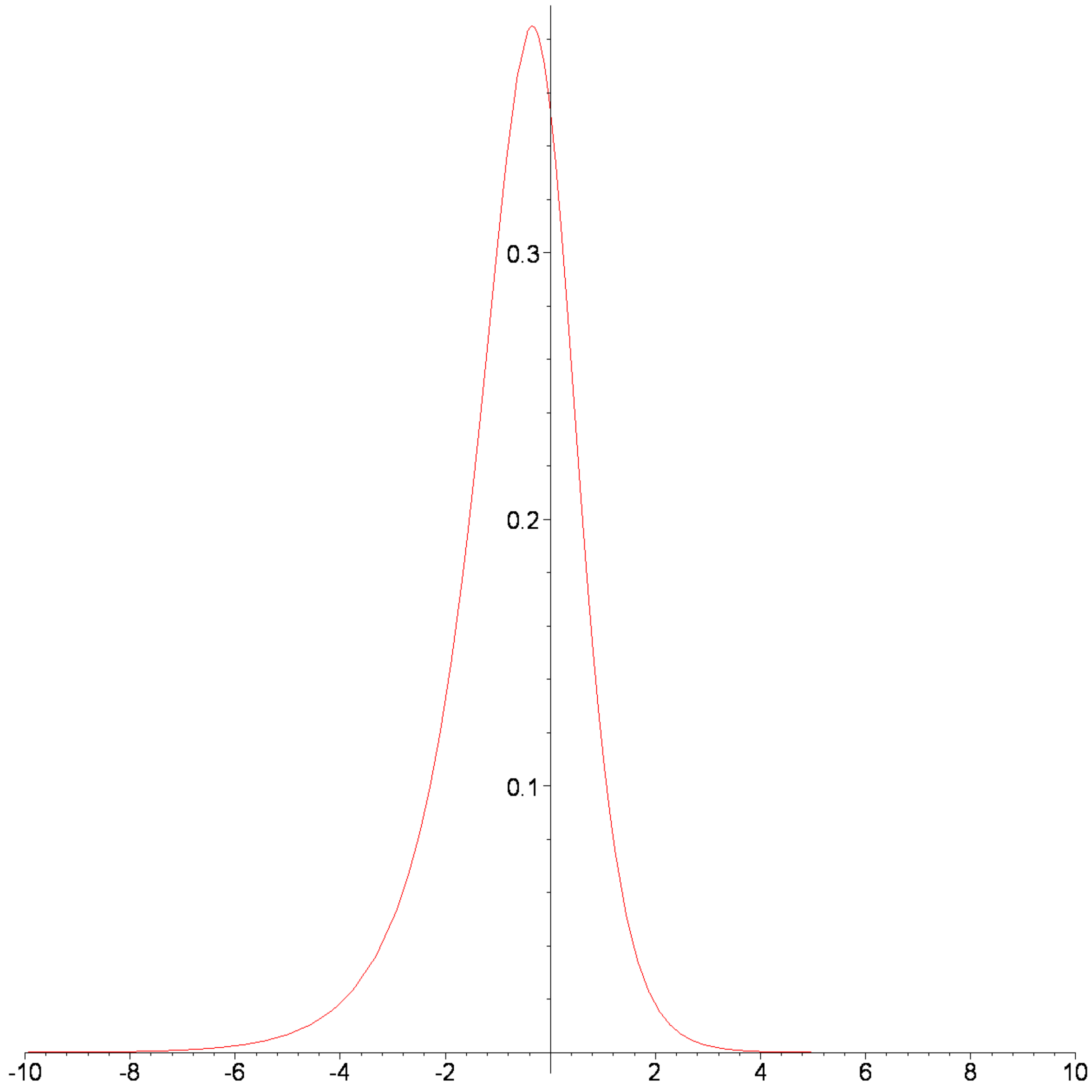
$$t1 := -\frac{1}{2}\ln(2)$$

```
> simplify(subs(t=t1,c));
```

$$\frac{2\sqrt{3}}{9}$$

```
> solve(diff(c,t));plot(c, t=-10..10);
```

$$-\frac{1}{2}\ln(2)$$




```
> #######################   Ex 10 i)
  restart:simplify(diff(ln(x^2+y^2)/2+arctan(y/x), x));
> simplify(diff(ln(x^2+y^2)/2+arctan(y/x), y));
```

$$\frac{x-y}{x^2+y^2}$$

$$\frac{y+x}{x^2+y^2}$$

```
> Phi(x,y):=ln(x^2+y^2)/2+arctan(y/x);
```

$$\Phi(x, y) := \frac{1}{2}\ln(x^2+y^2)+\arctan\left(\frac{y}{x}\right)$$

```
> assume(a>0):f:=subs({x=a, y=a}, Phi(x,y))-subs({x=a, y=-a},
  Phi(x,y));
>
```

$$f := \arctan(1) - \arctan(-1)$$

```
> combine(f,arctan);
```

$$\frac{\pi}{2}$$

```
> #######################  Ex 10 iii)
  assume(a>0,b>0):x:=a*cos(t);y:=b*sin(t);
```

$$x := a\sim \cos(t)$$

$$y := b\sim \sin(t)$$

```
> xp:=diff(x,t);yp:=diff(y,t);
```

$$xp := -a\sim \sin(t)$$

$$yp := b\sim \cos(t)$$

```
> y^2*xp+x^2*yp;
```

$$-b\sim^2 \sin(t)^3 a\sim + a\sim^2 \cos(t)^3 b\sim$$

```
> int((y^2*xp+x^2*yp),t=0..2*Pi);
```

$$0$$