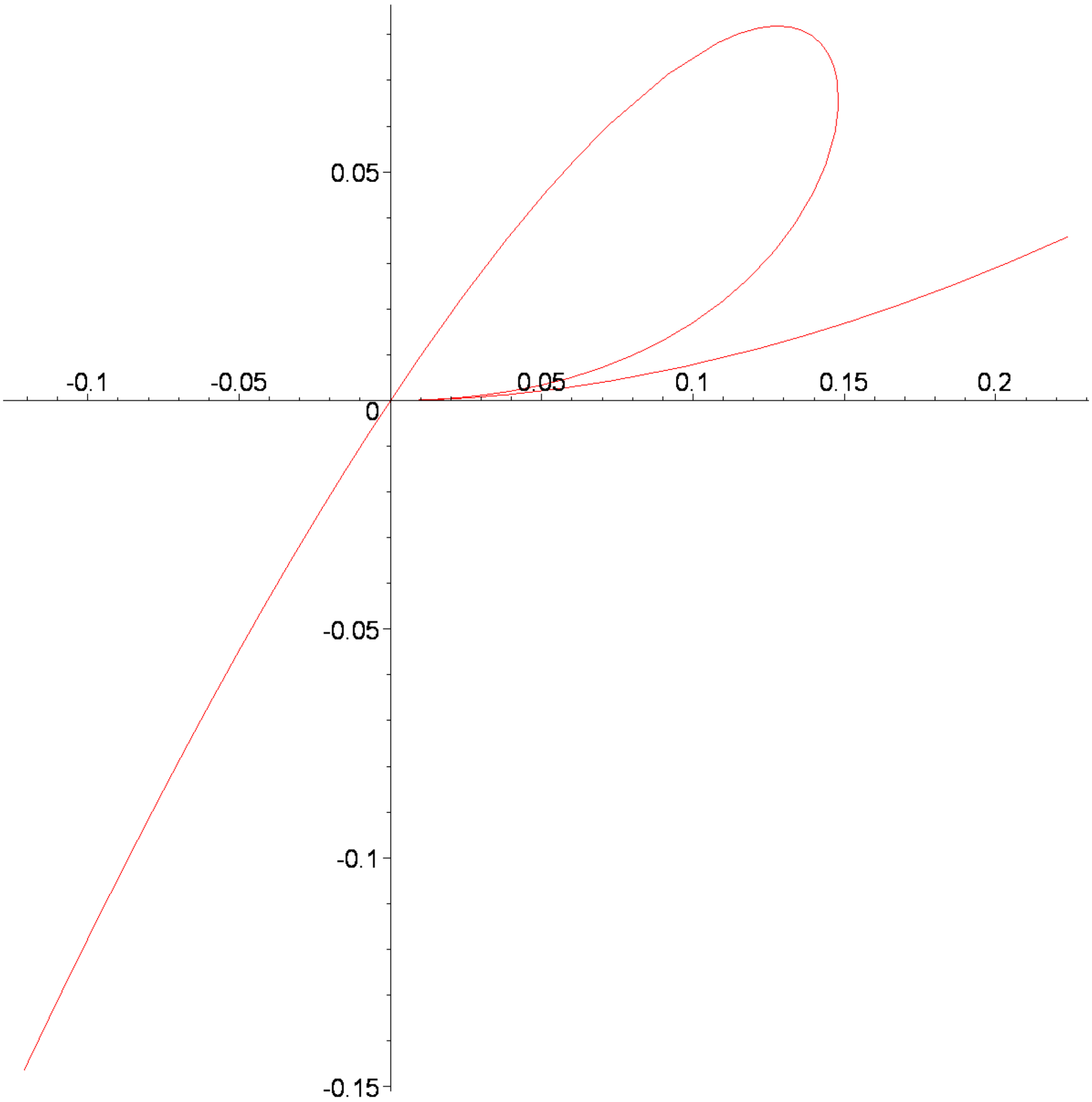
##################### FEUILLE N2 237

####################################### Exercice 1.

Exercice 1. Déterminer les points singuliers de la courbe plane définie sur R par t \mapsto (t^2 +t^3, t^4 +t^5). Déterminer s'il s'agit de points de rebroussement de première ou de seconde espèce.

```
> restart:plot([t^2+t^3,t^4+t^5, t=-1.1...0.4]);
```



####################################### Exercice 2.

Exercice 2. Déterminer les points singuliers de l'astroide A (cf. exercice 4, feuille 1), leur esp`ece, et la tangeante `a l'astroide en ces points. Vérifier que la distance entre les points d'intersection de la tangente en un point régulier M(\theta) \in A avec les axes est constante. En déduire une construction géomérique alternative de l'astroide.

```
> simplify(sin(3*t)-3*sin(t)+4*sin(t)^3);
```

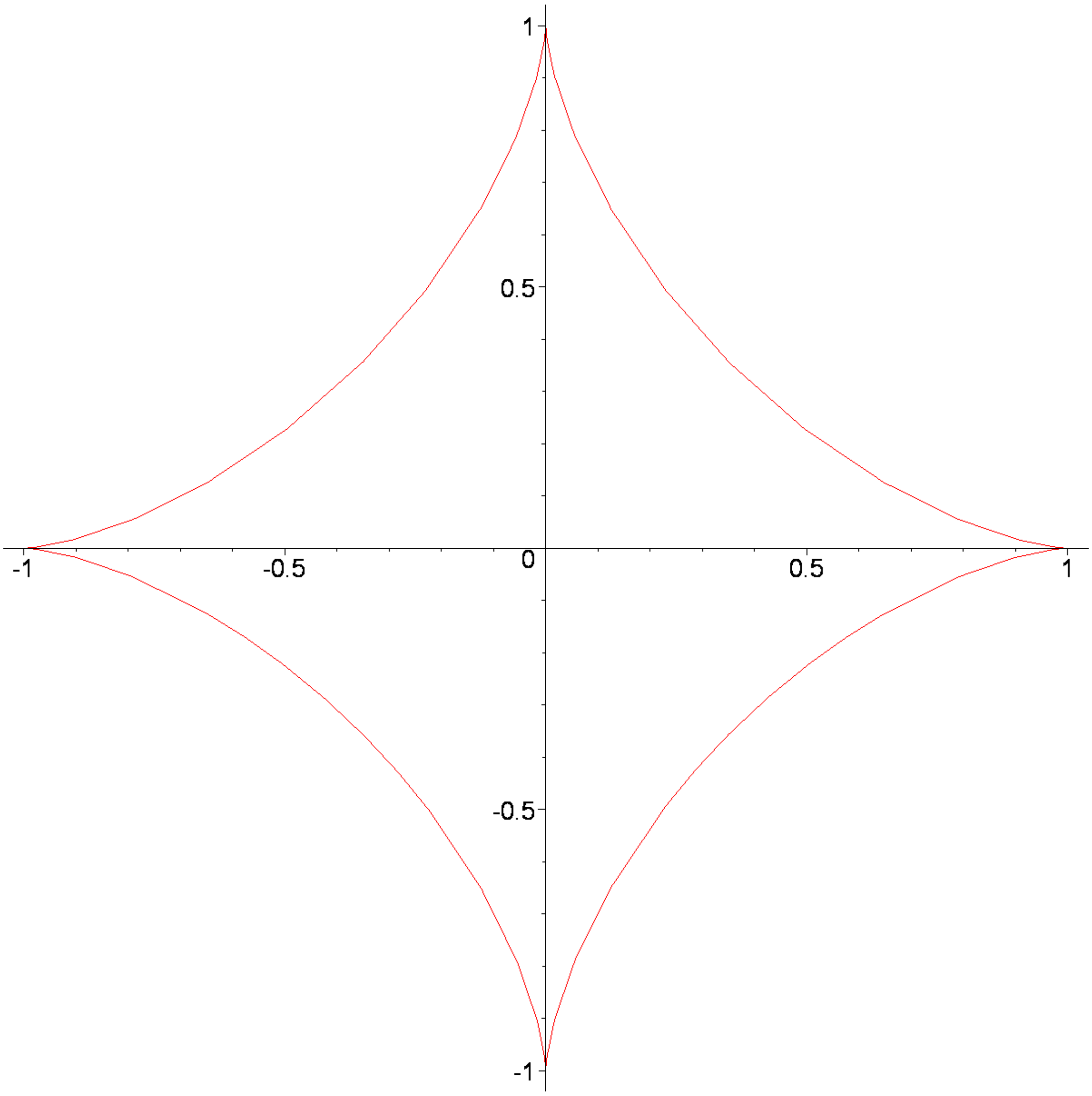
$$0$$

```
> expand(cos(3*t));simplify(cos(3*t)+3*cos(t)-4*cos(t)^3);
```

$$4\cos(t)^3 - 3\cos(t)$$

$$0$$

```
plot([(sin(t))^3, (cos(t))^3, t=0..2*Pi]);
```




```
> [taylor( cos(t)^3,t=0, 5 ),taylor( sin(t)^3,t=0, 5 )];
  [taylor( cos(t)^3,t=Pi/2, 5 ),taylor( sin(t)^3,t=Pi/2, 5 )];
```

$$\left[1-\frac{3}{2}t^2+\frac{7}{8}t^4+\mathrm{O}(t^5),\, t^3+\mathrm{O}(t^5)\right]$$

$$\left[-\left(t-\frac{\pi}{2}\right)^3+\mathrm{O}\left(\left(t-\frac{\pi}{2}\right)^5\right), 1-\frac{3}{2}\left(t-\frac{\pi}{2}\right)^2+\frac{7}{8}\left(t-\frac{\pi}{2}\right)^4+\mathrm{O}\left(\left(t-\frac{\pi}{2}\right)^5\right)\right]$$

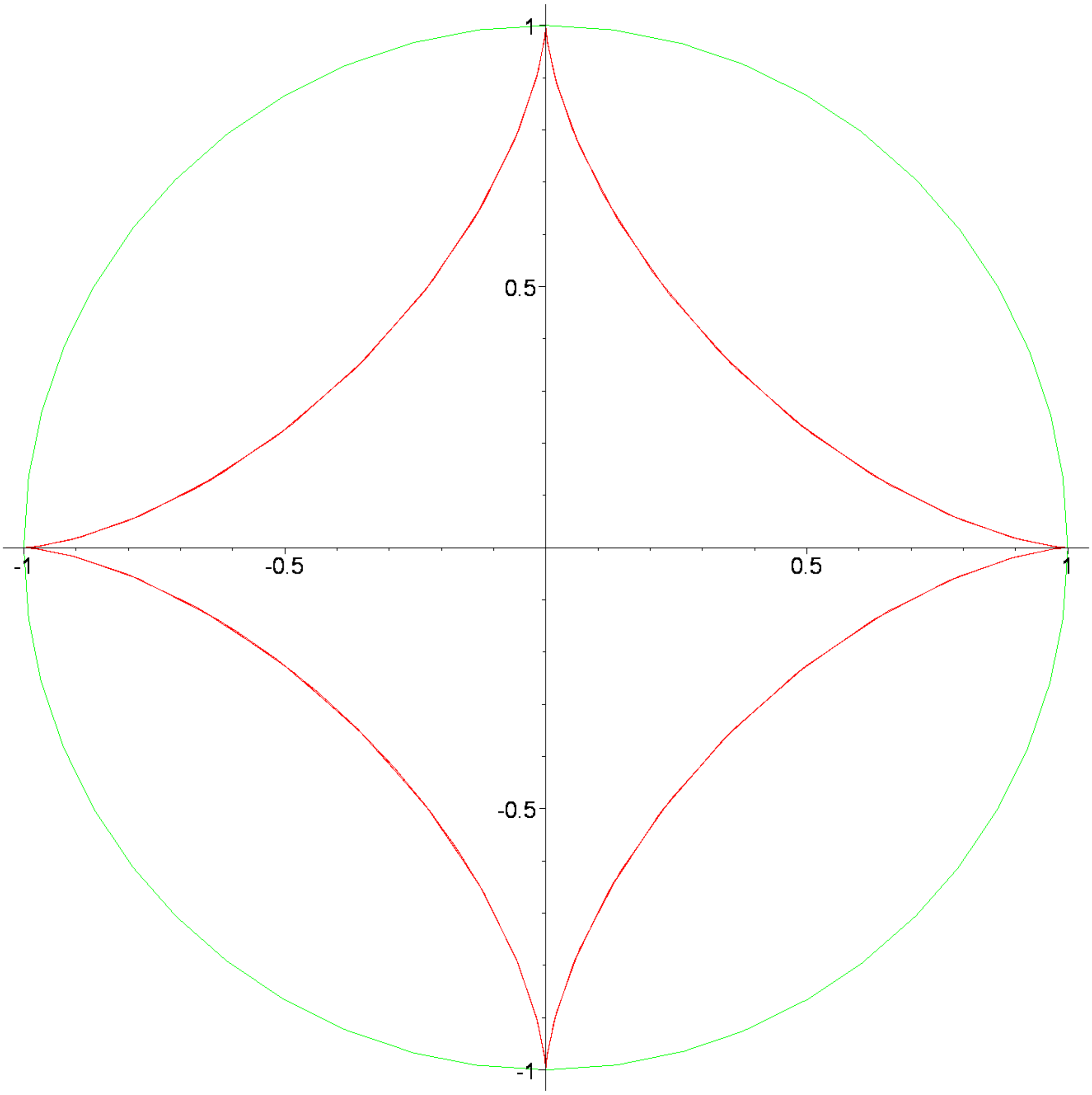
```
> restart:with(VectorCalculus):
  ArcLength( <(sin(t)^3), (cos(t)^3)>, t=0..Pi/2 ) ;
>
```

```
Warning, the assigned names <,> and <|> now have a global binding

Warning, these protected names have been redefined and unprotected: *, +, ., D,
Vector, diff, int, limit, series
```

$$\frac{3}{2}$$

```
> with(plots):
  n:=4;
  P1:=plot([ cos(t), sin(t),t=-Pi..Pi], color=green):
  P2:=plot([(1/n)*((n-1)*cos(t)+cos((n-1)*t)),
  (1/n)*((n-1)*sin(t)-sin((n-1)*t)),t=-n*Pi..n*Pi]):
  display(P1,P2);
Warning, the name changecoords has been redefined
```

$$n := 4$$




```
> restart:n:=4;
  (1/n)*int((((diff((n-1)*cos(t)+cos((n-1)*t),t))^2+
  diff(((n-1)*sin(t)-sin((n-1)*t),t))^2)^(1/2),t=0..2*Pi));
>
```

$$n := 4$$

$$6$$

```
> restart;with(VectorCalculus):
  n:=4;Ln:=ArcLength( <(1/n)*((n-1)*cos(t)+cos((n-1)*t)),
  (1/n)*((n-1)*sin(t)-sin((n-1)*t))>, t=0..2*Pi ) ;
Warning, the assigned names <,> and <|> now have a global binding

Warning, these protected names have been redefined and unprotected: *, +, ., D,
Vector, diff, int, limit, series
```

$$n := 4$$

$$Ln := 6$$

#limaçons de Pascal

Exercice 3.

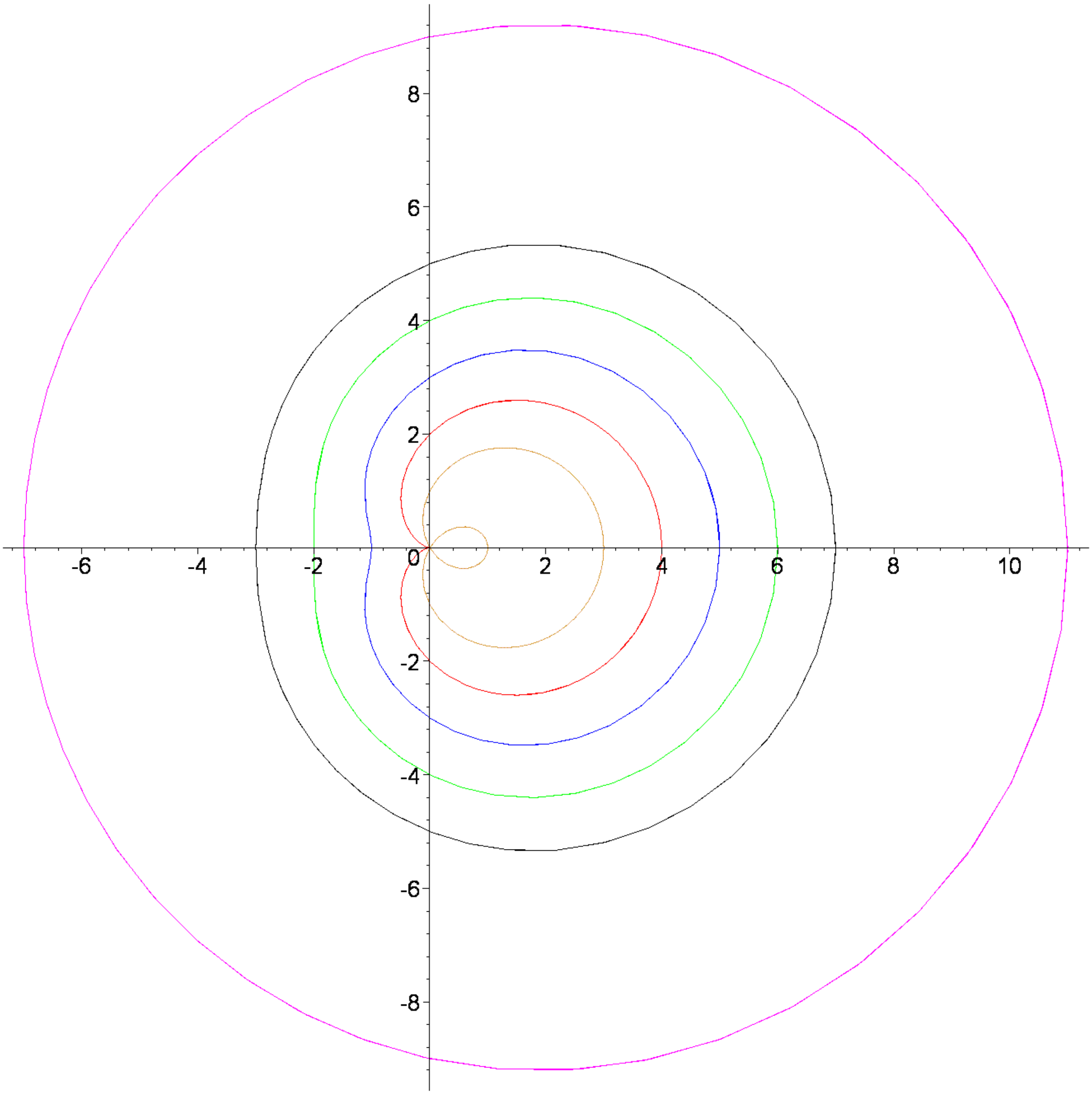
Exercice 3. Soit C un cercle de centre (1, 0) et de rayon 1.

a) Déterminer une équation polaire de C.

b) Soit D une droite passant par l'origine qui coupe C en un point P.
On construit sur D deux points M et N distincts tels que d(P;M) = d(P;N) = a,
où a est un réel strictement positif fixé.
Déterminer une équation polaire de
l'ensemble \Gamma_a décrit par les points M et N si l'on varie D ({\it limaçons de Pascal}).

c) Déterminer, lorsque a décrit]0;\infty[,
l'ensemble des points des courbes \Gamma_a,
dont la tangente est verticale.

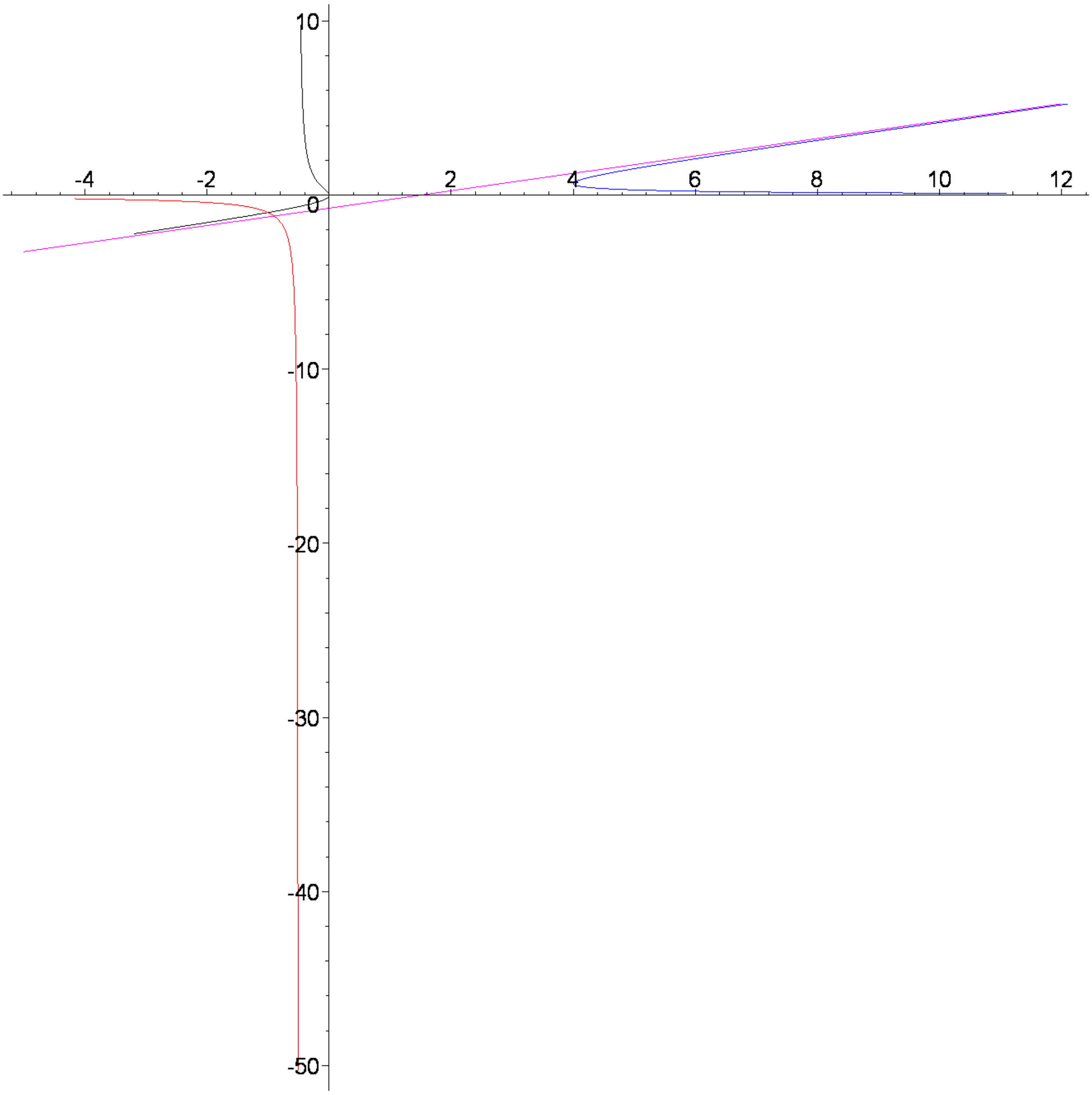
```
> with(plots):P1:=polarplot([2*cos(t)+1,t,t=0..4*Pi],color=gold):
  P2:=polarplot([2*cos(t)+2,t,t=0..4*Pi],color=red):
  P3:=polarplot([2*cos(t)+3,t,t=0..4*Pi],color=blue):
  P4:=polarplot([2*cos(t)+4,t,t=0..4*Pi],color=green):
  P5:=polarplot([2*cos(t)+5,t,t=0..4*Pi],color=black):
  P6:=polarplot([2*cos(t)-9,t,t=0..4*Pi],color=magenta):
  display(P1,P2,P3,P4,P5, P6);
Warning, the name changecoords has been redefined
```

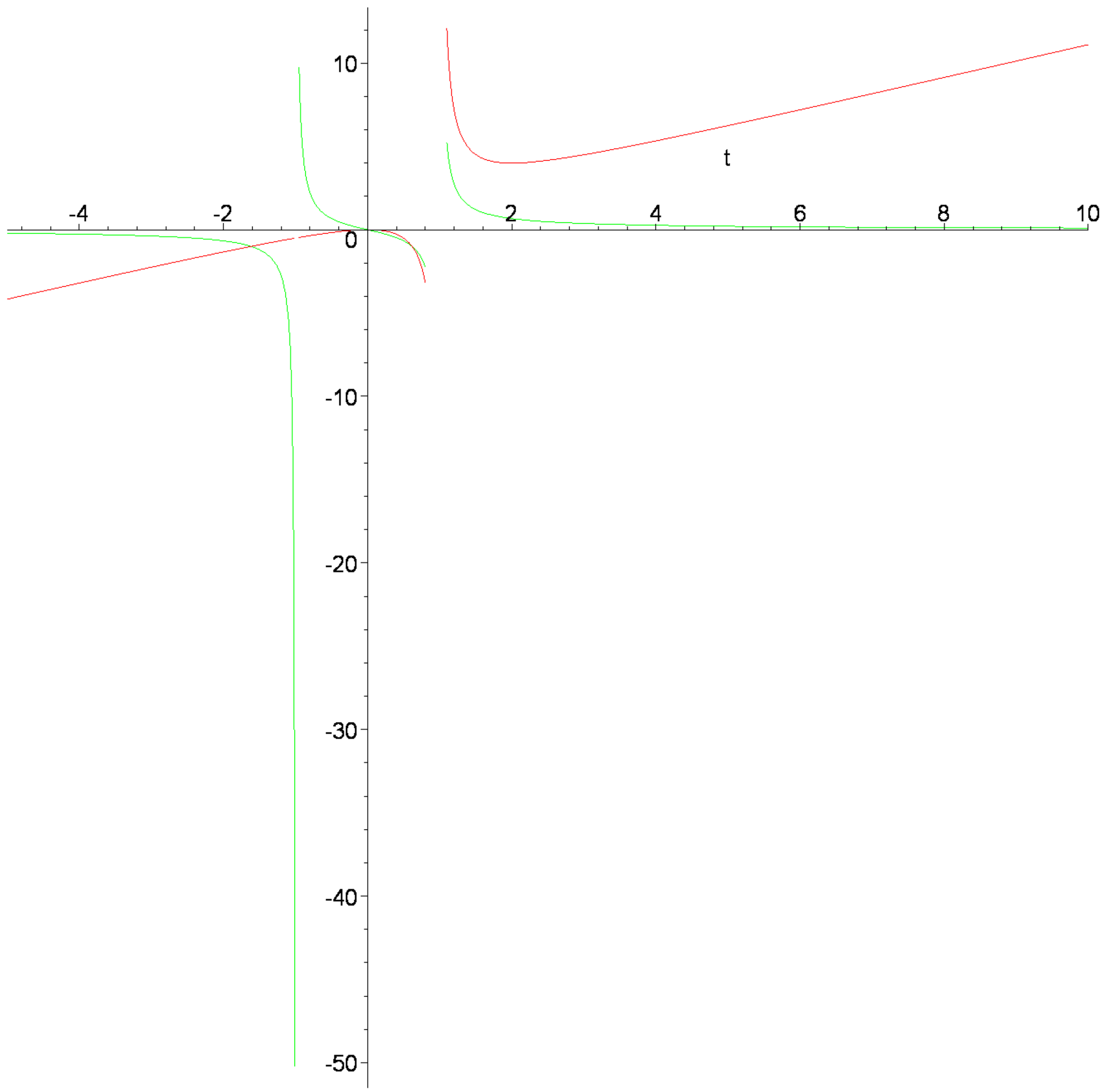


Exercice 4. Etudier les branches in nies des courbes planes dé nies par :
1) t \mapsto (t^2/(t-1) , t/(t^2-1)),
2) (Examen de juin 2006) r(\theta)=1+1/(\theta-\pi/4)

```
> ######################################     Exercice 4,1).
  #étudier les branches infinies
  with(plots):
  B1:=plot([t^2/(t-1), t/(t^2-1), t=-5..-1.01],color=red):
  B11:=plot([t^2/(t-1), t/(t^2-1)], t=-5..-1.01):
  B2:=plot([t^2/(t-1), t/(t^2-1), t=-0.95..0.8],color=black):
  B21:=plot([t^2/(t-1), t/(t^2-1)], t=-0.95..0.8):
  B3:=plot([t^2/(t-1), t/(t^2-1), t=1.1..10],color=blue):
  B31:=plot([t^2/(t-1), t/(t^2-1)], t=1.1..10):
  B:=plot([t,(t/2)-3/4,t=-5..12],color=magenta): # asymptote
  oblique
  display(B1,B2,B3, B);
```

```
display(B11,B21,B31);
####################################################################
```






```
> eliminate( {x-t^2/(t-1), y-t/(t^2-1)}, t);
```

$$\left[\{ t = \frac{y\,(x+1)}{y\,x-1} \}, \{ 2\,y^2\,x + y^2 - y\,x^2 + y\,x + x \} \right]$$

```
> ######################################################################
> #Repères de Frenet
  restart:#assume((t>-5),(t<1)):
  x:=t^2/(t-1):y:=t/(t^2-1):
> u:=simplify(diff(x,t)):v:=simplify(diff(y,t)):
  l:=simplify(sqrt(u^2+v^2)):
> tau:=[simplify(u),simplify(v)];
```

$$\tau := \left[\frac{t\,(t-2)}{(t-1)^2}, -\frac{t^2+1}{(t^2-1)^2} \right]$$

```
> eta:=[-simplify(v),simplify(u)];
```

$$\eta := \left[\frac{t^2+1}{(t^2-1)^2}, \frac{t(t-2)}{(t-1)^2}\right]$$

```
> t1:=-2; x1:=subs(t=t1,x):
  y1:=subs(t=t1,y):
  u1:=subs(t=t1,u):
  v1:=subs(t=t1,v):
  with(plots):
  b10:=
  plot([t^2/(t-1), t/(t^2-1), t=-5..-1.5],color=red):
```

$$t1 := -2$$

```
Warning, the name changecoords has been redefined
```

```
> with(plots):b11 := arrow(<x1,y1>, <u1,v1>, length=[1],
  width=[0.05, relative], head_length=[0.1, relative],
  color=red):
  b12 := arrow( <x1,y1>, <-v1,u1>,length=[1],width=[0.05,
  relative], head_length=[0.1, relative], color=blue):
```

```
> b20:=plot([t^2/(t-1), t/(t^2-1), t=-0.95..0.8],color=black):
  t2:=-1/2; x2:=subs(t=t2,x):
  y2:=subs(t=t2,y):
  u2:=subs(t=t2,u):
  v2:=subs(t=t2,v):

  b21 := arrow(<x2,y2>, <u2,v2>, length=[1], width=[0.05,
  relative], head_length=[0.1, relative], color=red):
  b22 := arrow( <x2,y2>, <-v2,u2>,length=[1],width=[0.05,
  relative], head_length=[0.1, relative], color=blue):
```

$$t2 := \frac{-1}{2}$$

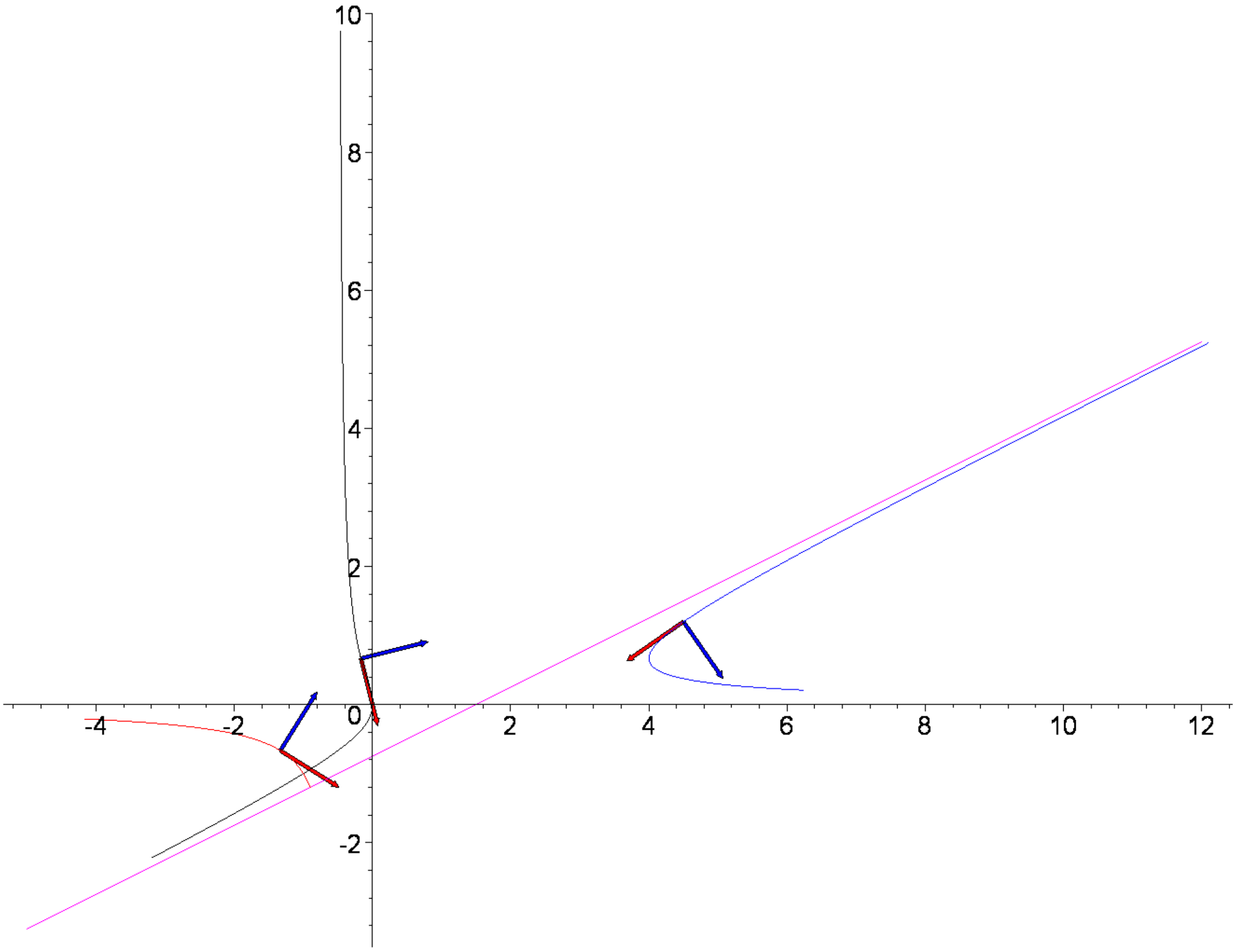
```
> b30:=plot([t^2/(t-1), t/(t^2-1), t=1.1..5],color=blue):
  t3:=1.5; x3:=subs(t=t3,x):
  y3:=subs(t=t3,y):
  u3:=subs(t=t3,u);
  v3:=subs(t=t3,v):

  b31 := arrow(<x3,y3>, <u3,v3>, length=[1], width=[0.05,
  relative], head_length=[0.1, relative], color=red):
  b32 := arrow( <x3,y3>, <-v3,u3>,length=[1],width=[0.05,
  relative], head_length=[0.1, relative], color=blue):
```

$$t3 := 1.5$$

$u3 := -3.000000000$

```
> B:=plot([t,(t/2)-3/4,t=-5..12],color=magenta): # asymptote
  oblique
```

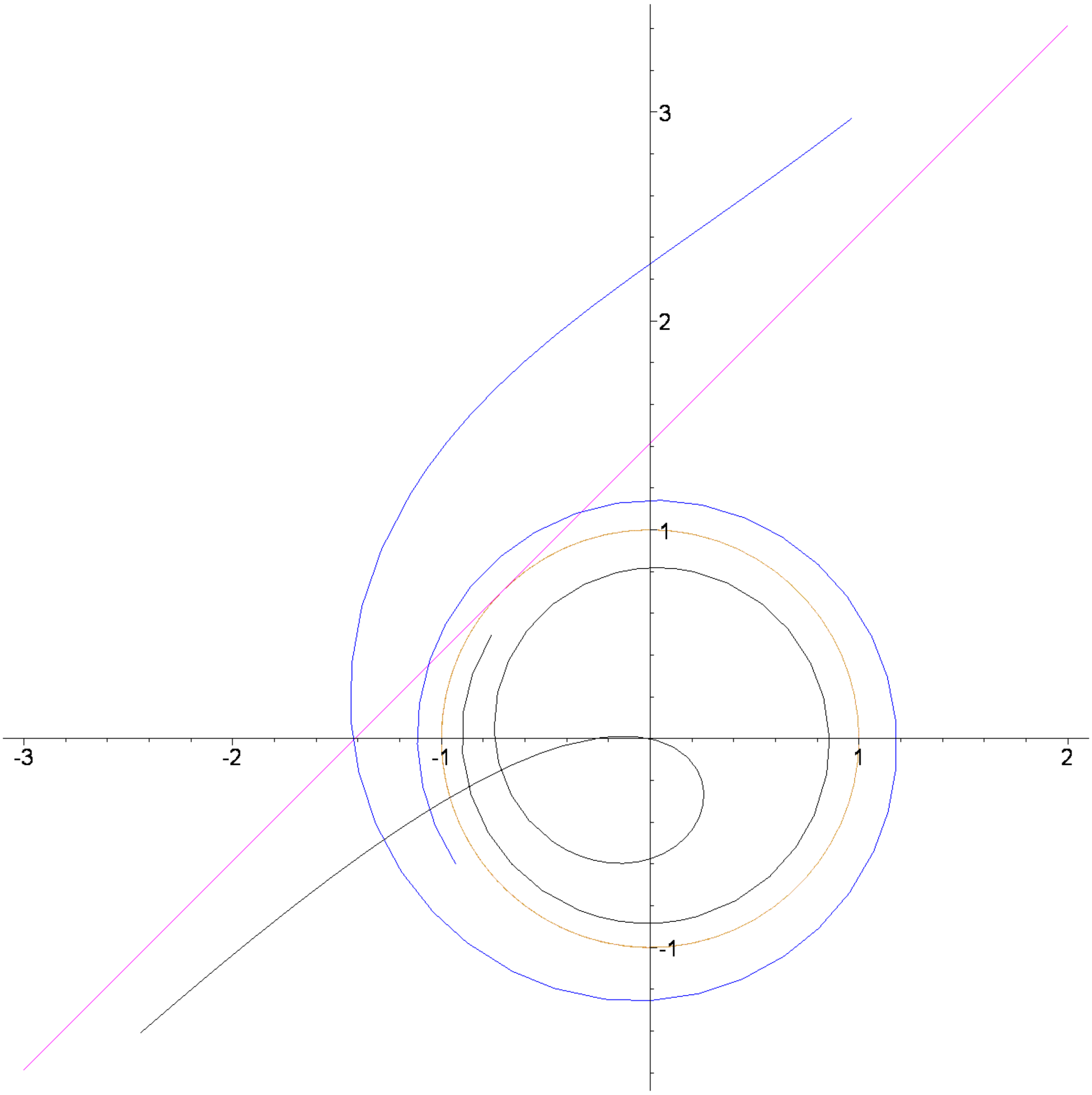
```
> display(b10, b11, b12, b20, b21, b22, b30, b31, b32,
  B,scaling=CONSTRAINED);
```



Exercice 4.
2) (Examen de juin 2006) r(\theta)=1+1/(\theta-\pi/4)

```
> ######################################     Exercice 4,2).
  B4:=polarplot([1+(1/(t-Pi/4)),t,t=-10..Pi/6],color=black):
  B5:=polarplot([1+(1/(t-Pi/4)),t,t=Pi/2.5..10],color=blue):
  B6:=plot([t,t+sqrt(2),t=-3..2],color=magenta): # asymptote
  oblique
  B7:=plot([cos(t),sin(t),t=-5..5],color=gold):
```

```
display(B4,B5, B6, B7);
```




```
> evalf(-1-4/(3*Pi));evalf(-sqrt(2));
```

$$-1.424413181$$

$$-1.414213562$$

```
> k:=0;evalf(1+1k:=0;evalf(1+1/((k*Pi)-(Pi/4)));
```

$$k := 0$$

$$-0.273239544$$

```
> k:=-1;evalf(1+1/((k*Pi)-(Pi/4)));
```

$$k := -1$$

$$0.7453520911$$

```
> k:=-1/2;evalf(1+1/((k*Pi)-(Pi/4)));
```

$$k := \frac{-1}{2}$$

0.5755868186

```
> k:=-1;evalf(1+1/((k*Pi)-(Pi/4)));
```

$k := -1$

0.7453520911

```
> k:=-3/2;evalf(1+1/((k*Pi)-(Pi/4)));
```

$k := \frac{-3}{2}$

0.8181086365

```
> f:=diff((1+(1/(t-Pi/4)))*sin(t),t);
```

$$f := -\frac{\sin(t)}{\left(t-\frac{\pi}{4}\right)^2} + \left(1 + \frac{1}{t-\frac{\pi}{4}}\right)\cos(t)$$

```
> evalf(solve(f=0));
```

-0.1006362855

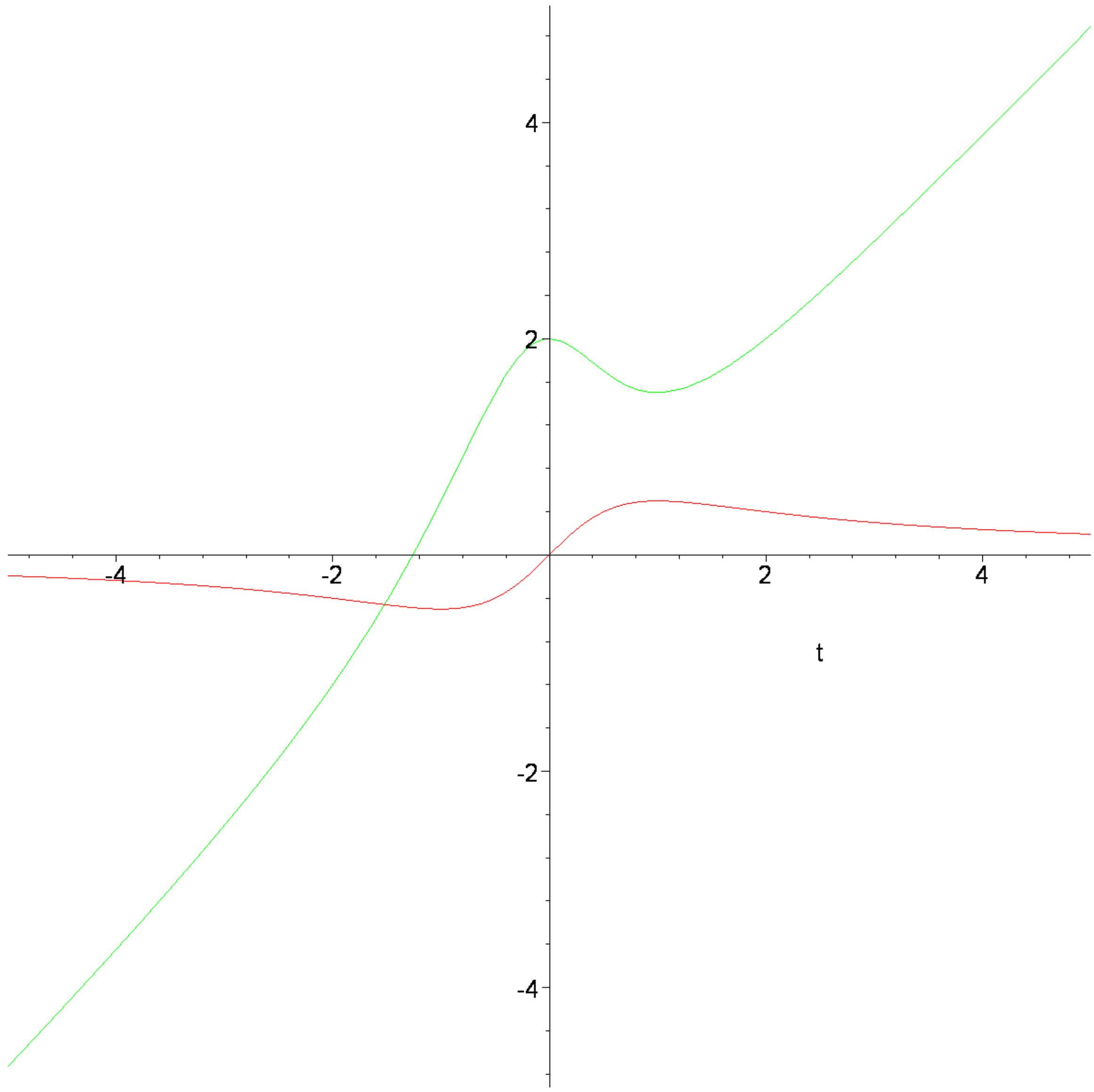
```
> evalf(Pi/4-1/2);evalf(Pi/4-1);
```

0.2853981635

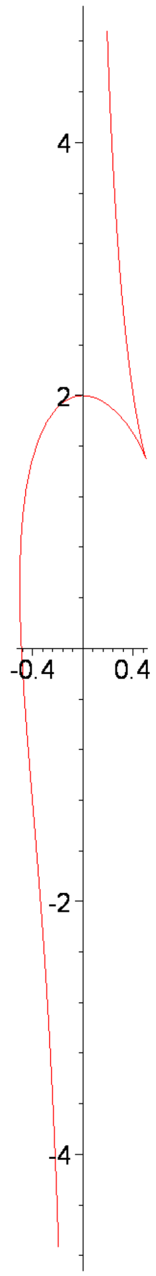
-0.2146018365

Exercice 5.

```
> ######################################     Exercice 5,1).
  restart:plot([t/(1+t^2),(2+t^3)/(1+t^2)], t=-5...5);
  plot([t/(1+t^2),(2+t^3)/(1+t^2), t=-5...5]);
  evalf(2^(1/3));evalf(eval(t/(1+t^2),t=-2^(1/3)));
  #eliminate( {t/(1+t^2)-x,(2+t^3)/(1+t^2)-y}, t);evalf(%);
```



4
2
-4
-2
2
4
t
-2
-4



1.259921050

-0.4869446308

```
> diff(t/(1+t^2),t);solve(diff(t/(1+t^2),t)=0);
  simplify(diff((2+t^3)/(1+t^2),t));solve(diff((2+t^3)/(1+t^2),t)=
  0);
>
```

$$\frac{1}{1+t^2} - \frac{2\,t^2}{(1+t^2)^2}$$

-1, 1

$$\frac{t\,(3\,t+t^3-4)}{(1+t^2)^2}$$

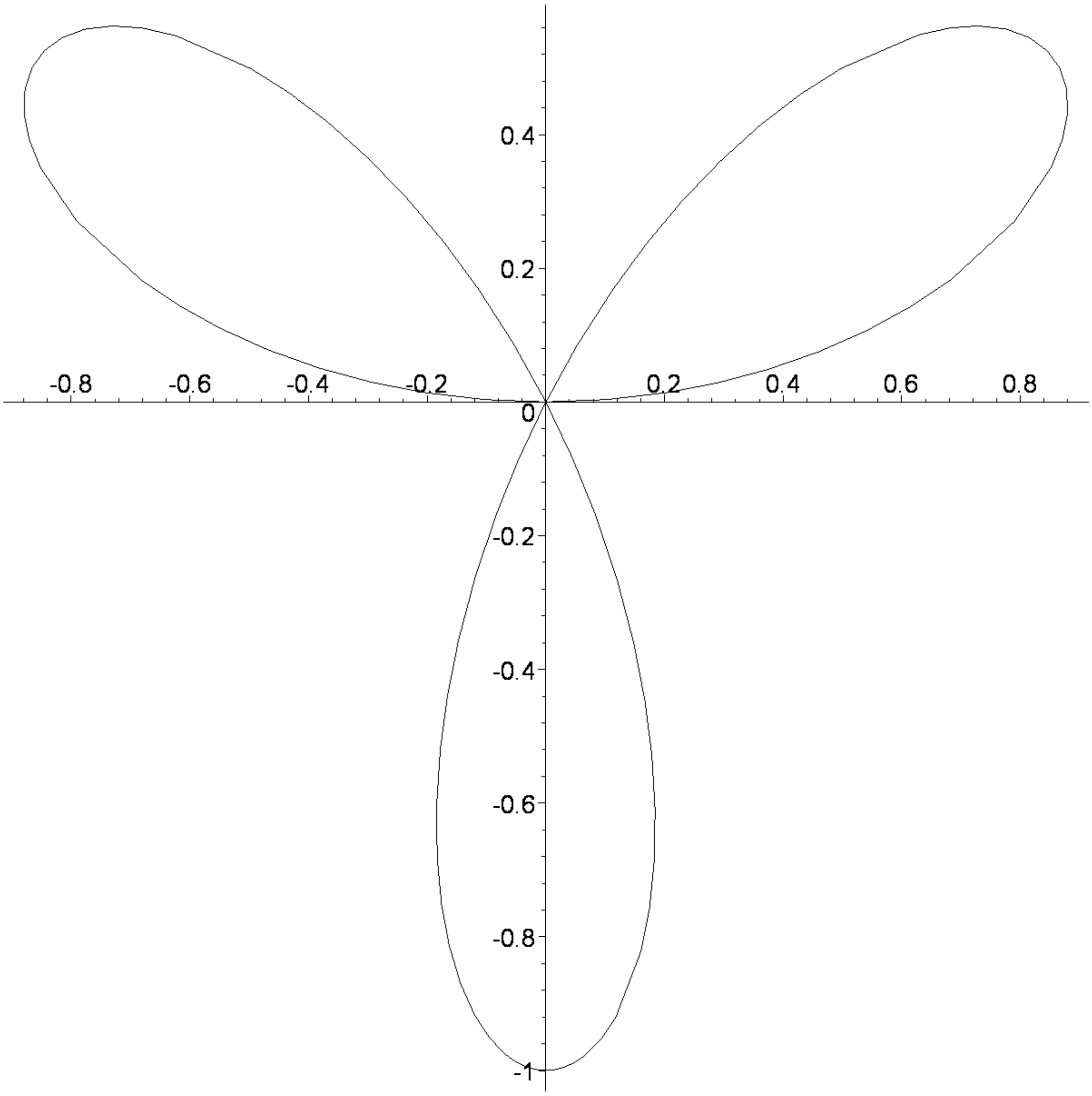
$$0,\, 1,\, -\frac{1}{2}+\frac{1}{2}\,I\,\sqrt{15},\, -\frac{1}{2}-\frac{1}{2}\,I\,\sqrt{15}$$

```
> ################ Point de rebroussement
  taylor( t/(1+t^2),t=1, 5 );taylor((2+t^3)/(1+t^2),t=1, 5 );
```

$$\frac{1}{2}-\frac{1}{4}(t-1)^2+\frac{1}{4}(t-1)^3-\frac{1}{8}(t-1)^4+O((t-1)^5)$$

$$\frac{3}{2}+\frac{3}{4}(t-1)^2-\frac{1}{4}(t-1)^3-\frac{1}{8}(t-1)^4+O((t-1)^5)$$

```
> ######################################     Exercice 5, 2).
  # (Rosace à trois boucles)
  polarplot([sin(3*t),t,t=0..Pi],color=black);
```



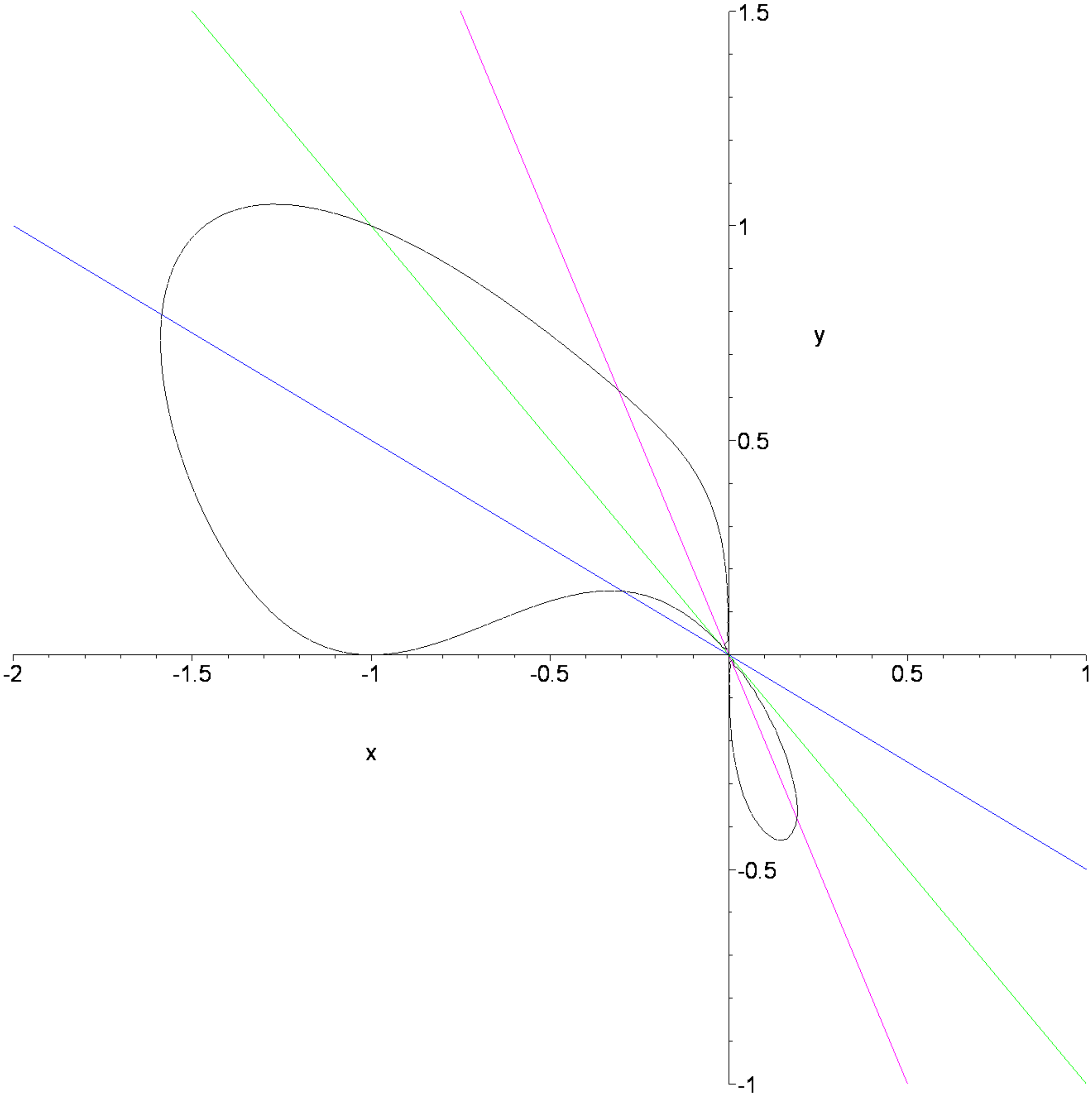

```
> #restart:with(VectorCalculus):
  #SetCoordinates( 'polar' );
> #ArcLength( <sin(3*t), t>,t=0..Pi/3 ) ;
>
```

Exercice 6.

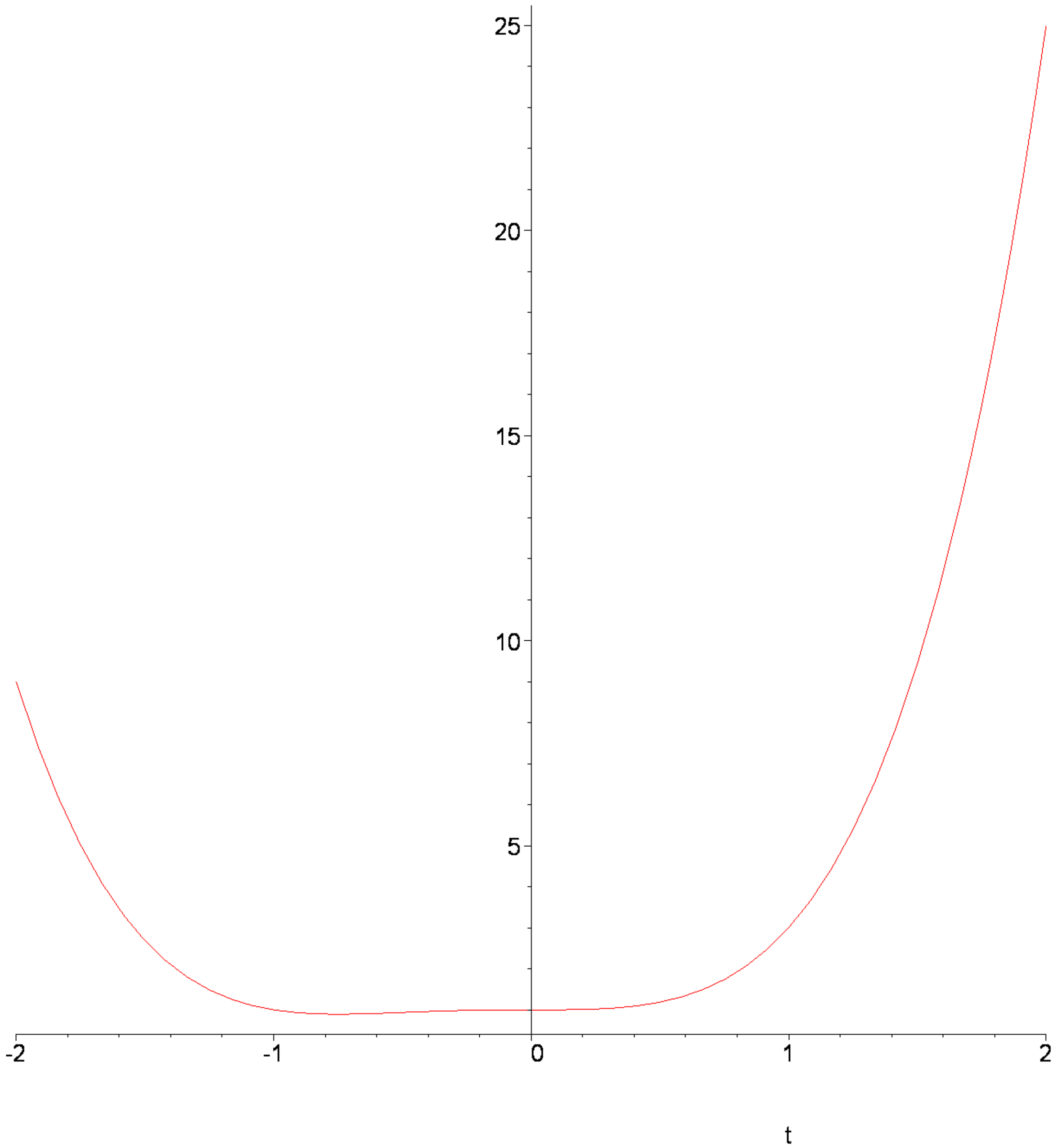
```
> ###########################################
  ######################################     Exercice 6.
  restart:with(plots):
```

```
I1:=implicitplot(x*(y + x) + 2*x^3 + x^4 + y^4=0,
x=-2..1,y=-1..1.5,color=black , numpoints=10000):
I2:=
plot(-x, x=-2..1,y=-1..1.5,color=green):
I3:=
plot(-x/2, x=-2..1,y=-1..1.5,color=blue):
I4:=
plot(-x*2, x=-2..1,y=-1..1.5,color=magenta):
display(I1,I2, I3, I4);
```

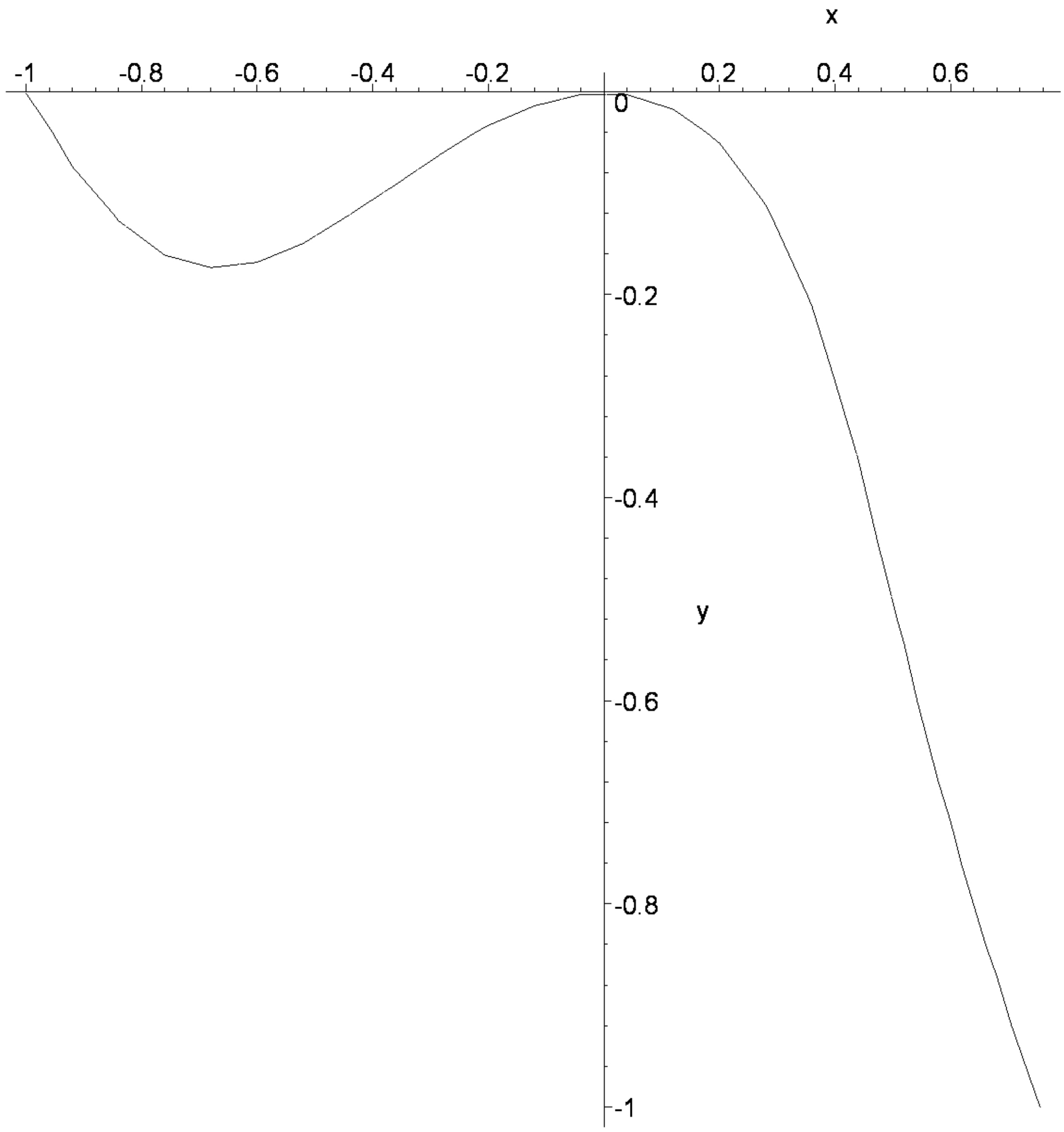
Warning, the name changecoords has been redefined



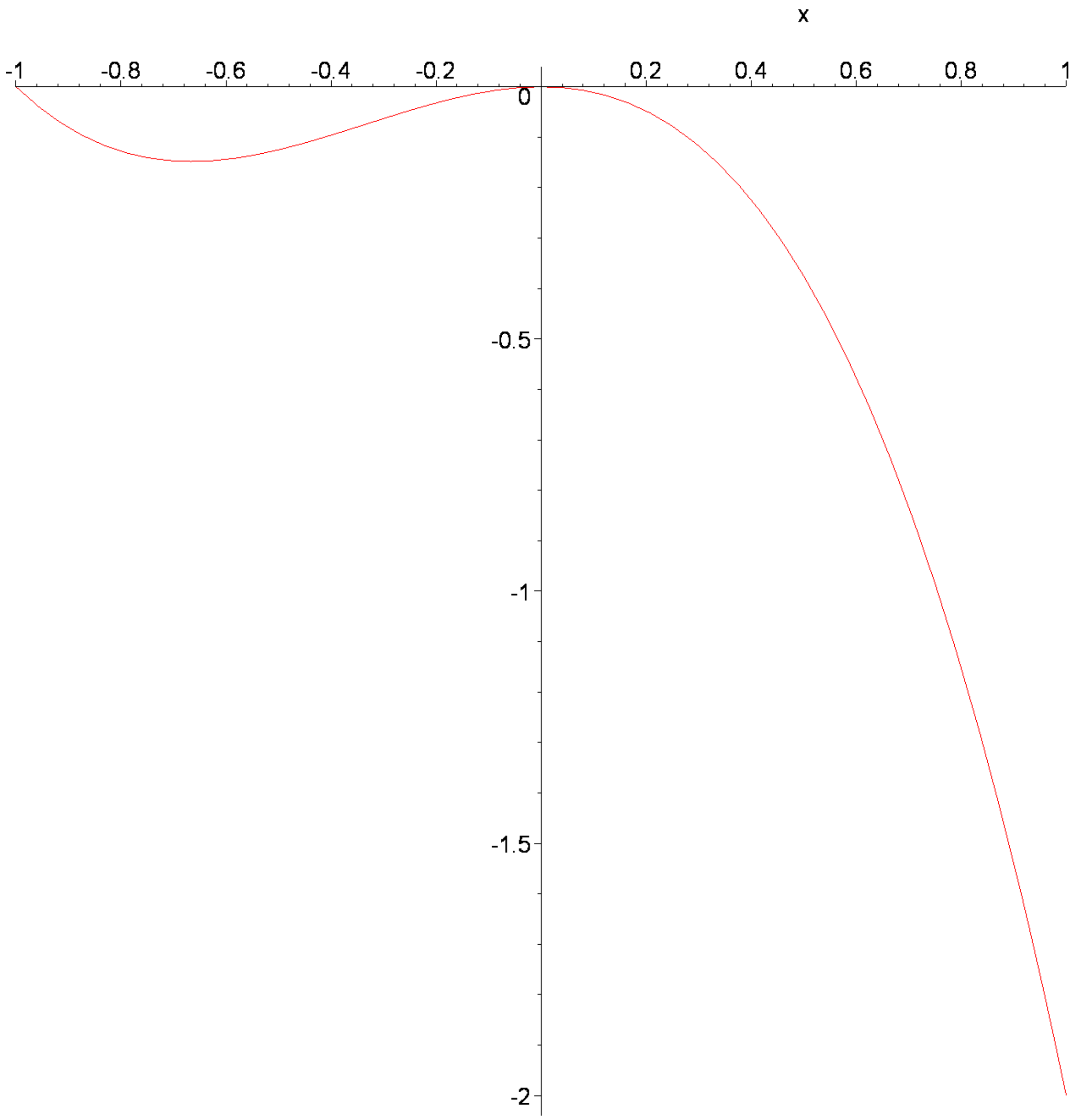

```
> plot(t^4+t^3+1, t=-2..2);
```



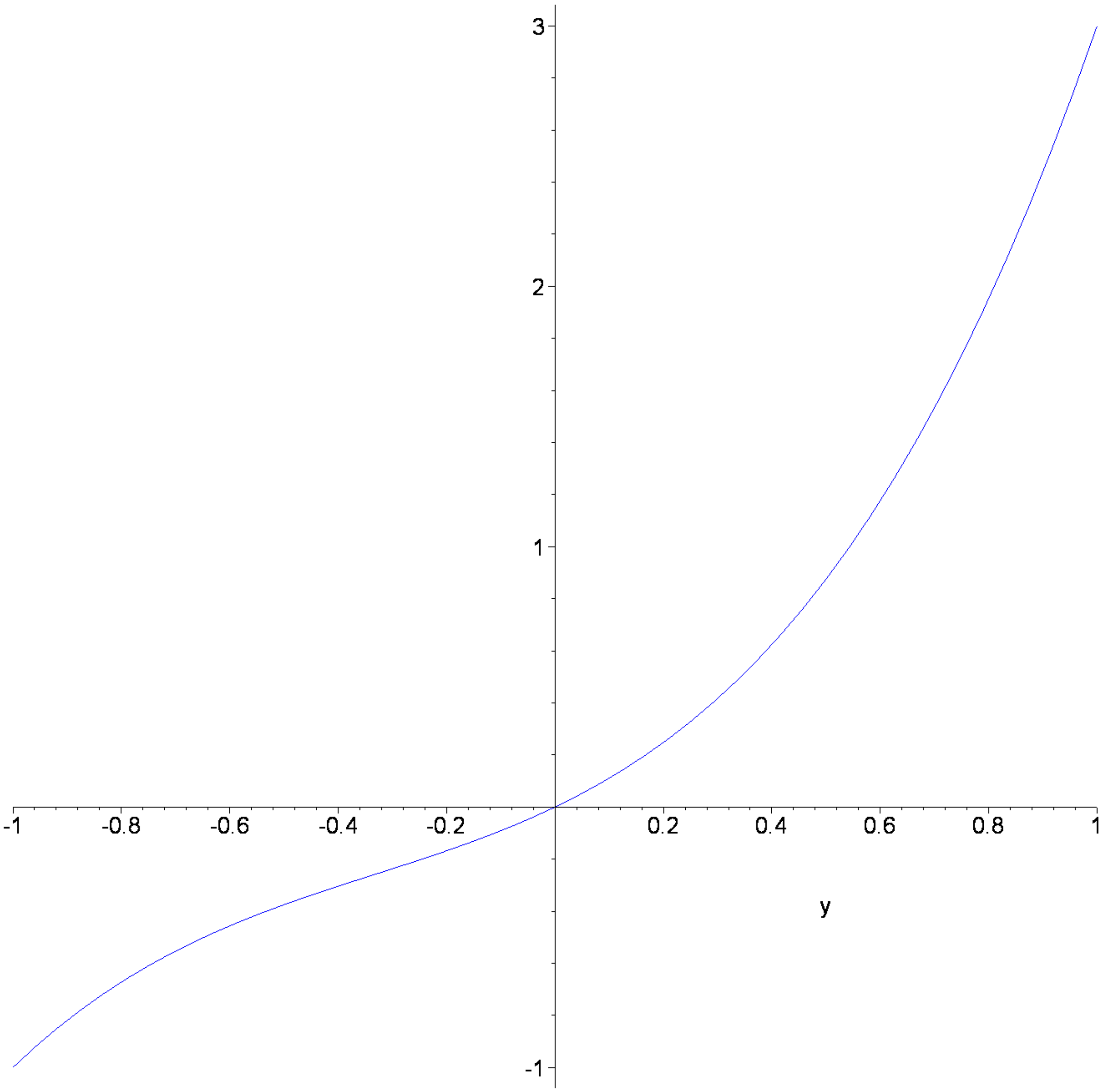

```
> ########################################    Exercice 7.
  implicitplot(x^3 + y^3 + x^2 + y^2 + y=0, x=-1..1,
  y=-1..1,color=black);
```



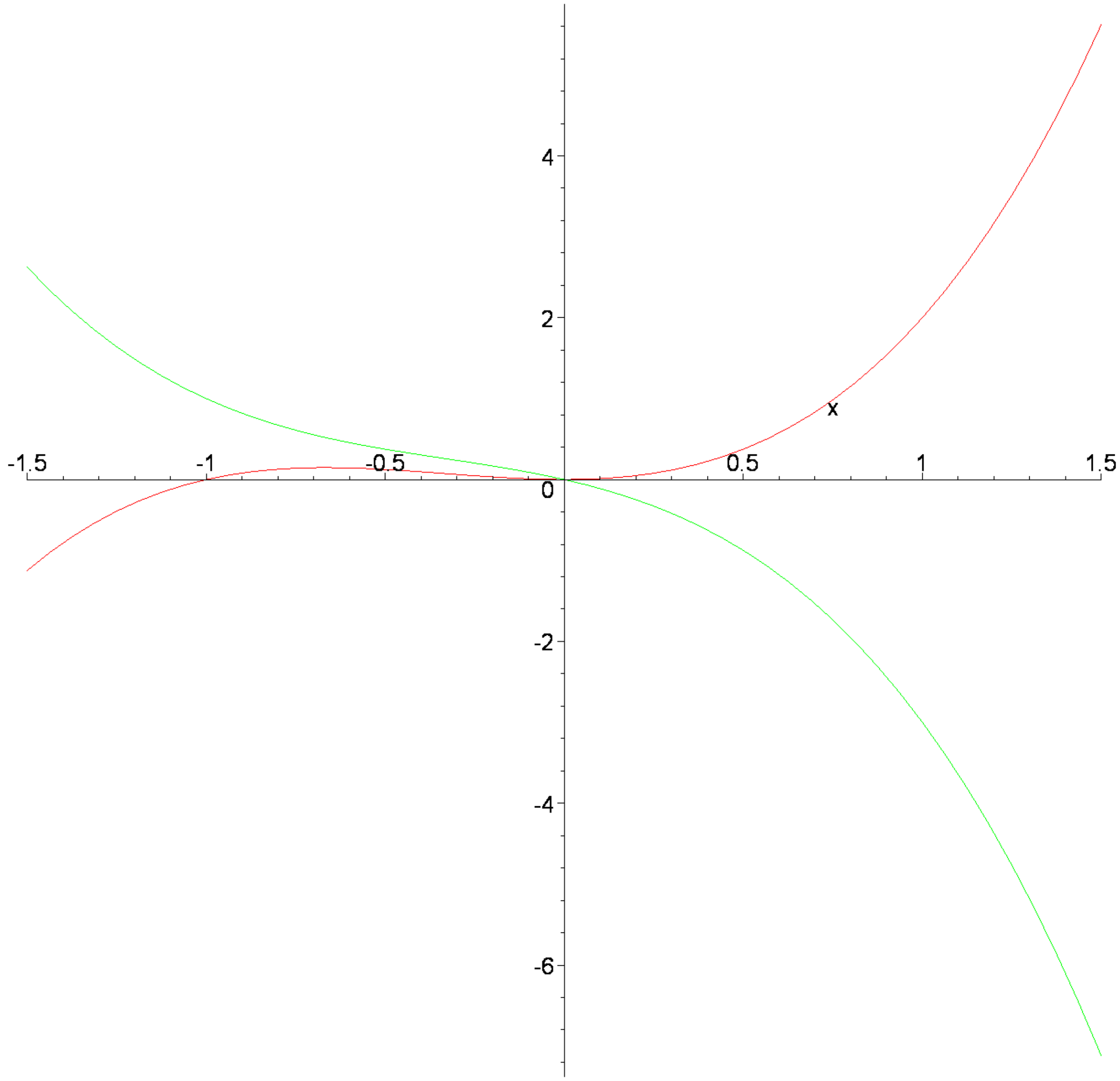

```
> plot(-x^3-  x^2 , x=-1..1,color=red);
```




```
>
> plot(y^3 +y^2 + y, y=-1..1,color=blue);
```




```
> plot([x^3 + x^2 , -x^3 - x^2 - x, ], x=-1.5..1.5);
```




```
> #################################################
  #Repère de Frenet
  restart;F:=x*(y + x) + 2*x^3 + x^4 + y^4;Fx:=diff(F,
  x);Fy:=diff(F, y);
  y0:=1; x0:=-1;

>
>
   with(plots):b0:=implicitplot(F=0, x=-2..2,y=-2..2,color=black ,
  numpoints=10000):
  #y0:=1; x0:=sqrt(y0^2+1);
  Fx0:=eval( Fx,[x=x0, y=y0]);
  Fy0:=eval( Fy,[x=x0, y=y0]);
  l0:=(Fx0^2+Fy0^2)^(1/2);
  b1 := arrow(<x0,y0>, <-Fy0/l0,Fx0/l0>, length=[1],width=[0.01,
```

```
relative], head_length=[0.1, relative],  color=red):
b2 := arrow( <x0,y0>,
<-Fx0/l0,-Fy0/l0>,length=[1],width=[0.01,relative],
color=blue):

display(b0, b1, b2, scaling=CONSTRAINED);
```

$$F := x\,(y+x) + 2\,x^3 + x^4 + y^4$$

$$Fx := y + 2\,x + 6\,x^2 + 4\,x^3$$

$$Fy := x + 4\,y^3$$

$$y0 := 1$$

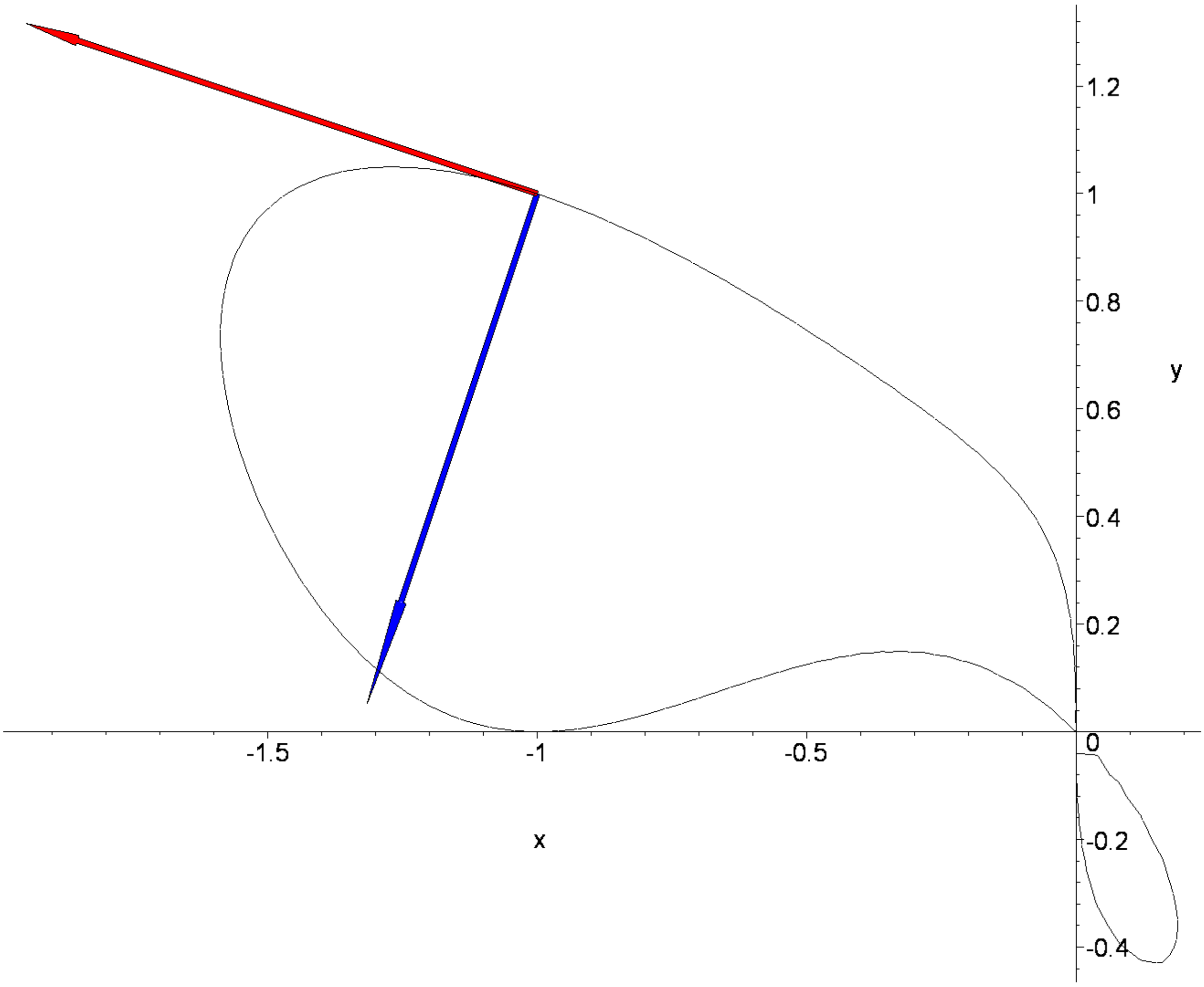
$$x0 := -1$$

```
Warning, the name changecoords has been redefined
```

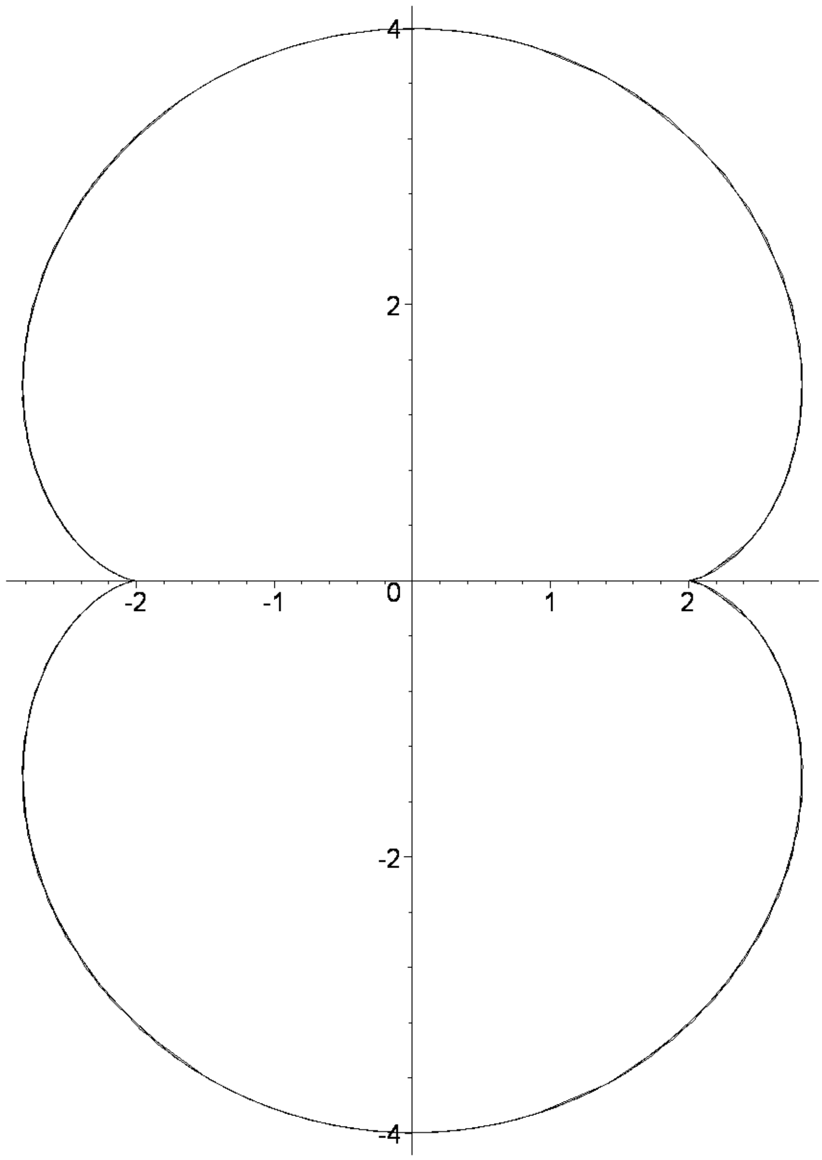
$$Fx0 := 1$$

$$Fy0 := 3$$

$$l0 := \sqrt{10}$$




```
> ########################################    Exercice 8.
  #la néphroïde
  restart:plot([3*cos(t) - cos(3*t),3*sin(t) - sin(3*t),
  t=-10..10],color=black);
```




```
> u:=diff(3*cos(t) - cos(3*t),t);v:=diff(3*sin(t) - sin(3*t),t);
```

$$u := -3\sin(t) + 3\sin(3\,t)$$

$$v := 3\cos(t) - 3\cos(3\,t)$$

```
> xs:=diff(u,t);ys:=diff(v,t);
```

$$xs := -3\cos(t) + 9\cos(3\,t)$$

$$ys := -3\sin(t) + 9\sin(3\,t)$$

```
> simplify(u);simplify(v);simplify(xs);simplify(ys);simplify(u^2+v^2);simplify(u*ys-v*xs);
```

$$6\,(-1 + 2\cos(t)^2)\sin(t)$$

$$12\sin(t)^2\cos(t)$$

$$6\cos(t)\,(-5 + 6\cos(t)^2)$$

$$12\,(-1 + 3\cos(t)^2)\sin(t)$$

$$36\sin(t)^2$$

$$72\sin(t)^2$$

```
> solve(u=0);solve(v=0);
```

$$\pi, 0, \frac{\pi}{4}, -\frac{\pi}{4}, \frac{3\pi}{4}, -\frac{3\pi}{4}$$

$$\frac{\pi}{2}, 0, \pi$$

```
> taylor(3*cos(t) - cos(3*t),t=0,5);taylor(3*sin(t) -
  sin(3*t),t=0,5);
```

$$2 + 3t^2 - \frac{13}{4}t^4 + O(t^5)$$

$$4t^3 + O(t^5)$$

```
> taylor(3*cos(t) - cos(3*t),t=Pi,5);taylor(3*sin(t) -
  sin(3*t),t=Pi,5);
```

$$-2 - 3(t-\pi)^2 + \frac{13}{4}(t-\pi)^4 + O((t-\pi)^5)$$

$$-4(t-\pi)^3 + O((t-\pi)^5)$$

```
> x:=3*cos(t) - cos(3*t);y:=3*sin(t) - sin(3*t);
```

$$x := 3\cos(t) - \cos(3t)$$

$$y := 3\sin(t) - \sin(3t)$$

```
> x1:=subs(t=Pi/4,x);y1:=subs(t=Pi/4,y);
```

$$x1 := 3\cos\left(\frac{\pi}{4}\right) - \cos\left(\frac{3\pi}{4}\right)$$

$$y1 := 3\sin\left(\frac{\pi}{4}\right) - \sin\left(\frac{3\pi}{4}\right)$$

```
> evalf(x1);evalf(y1);
>
```

$$2.828427124$$

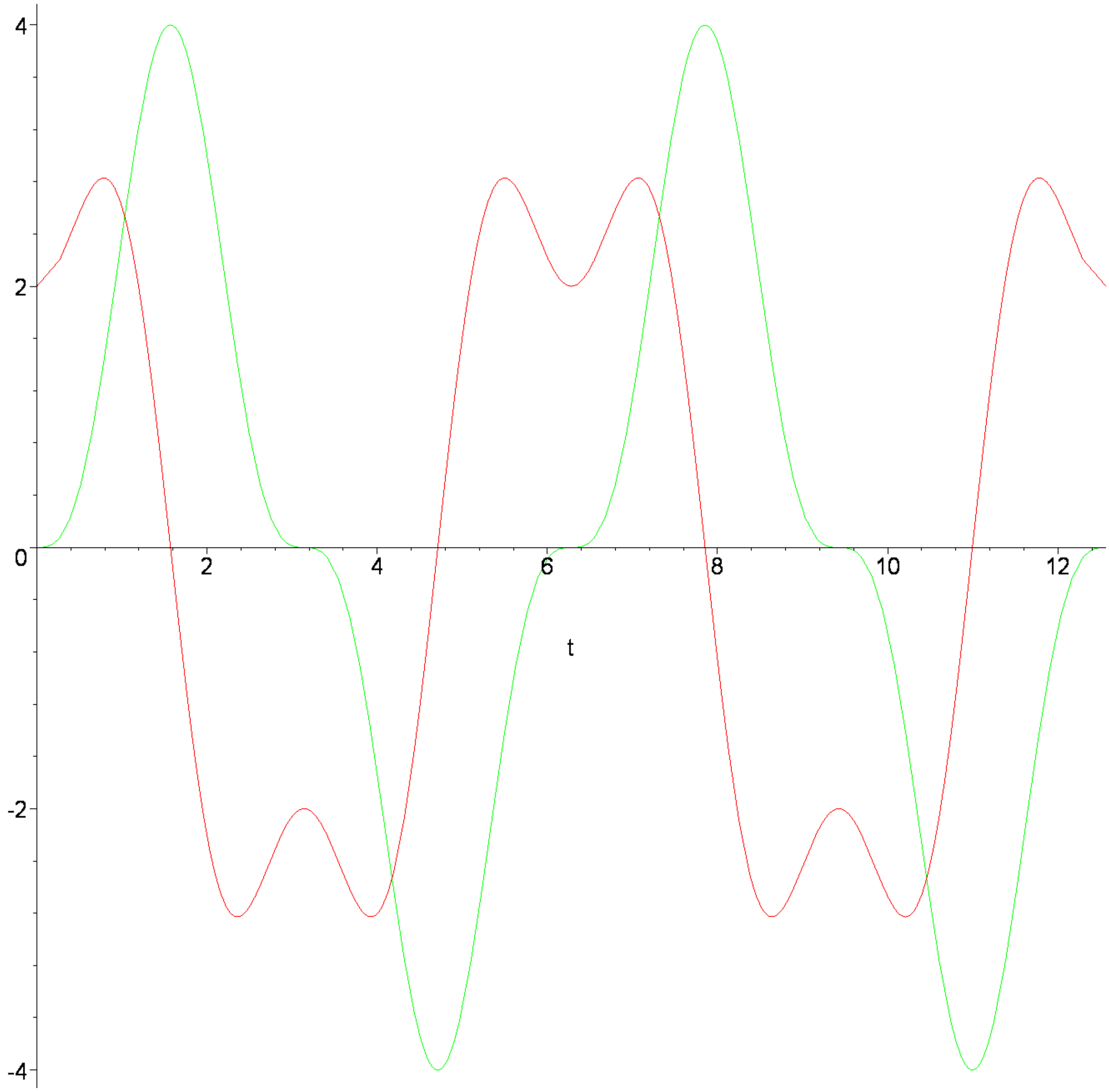
$$1.414213562$$

```
> simplify(3*cos(t) - cos(3*t));simplify(3*sin(t) - sin(3*t));
```

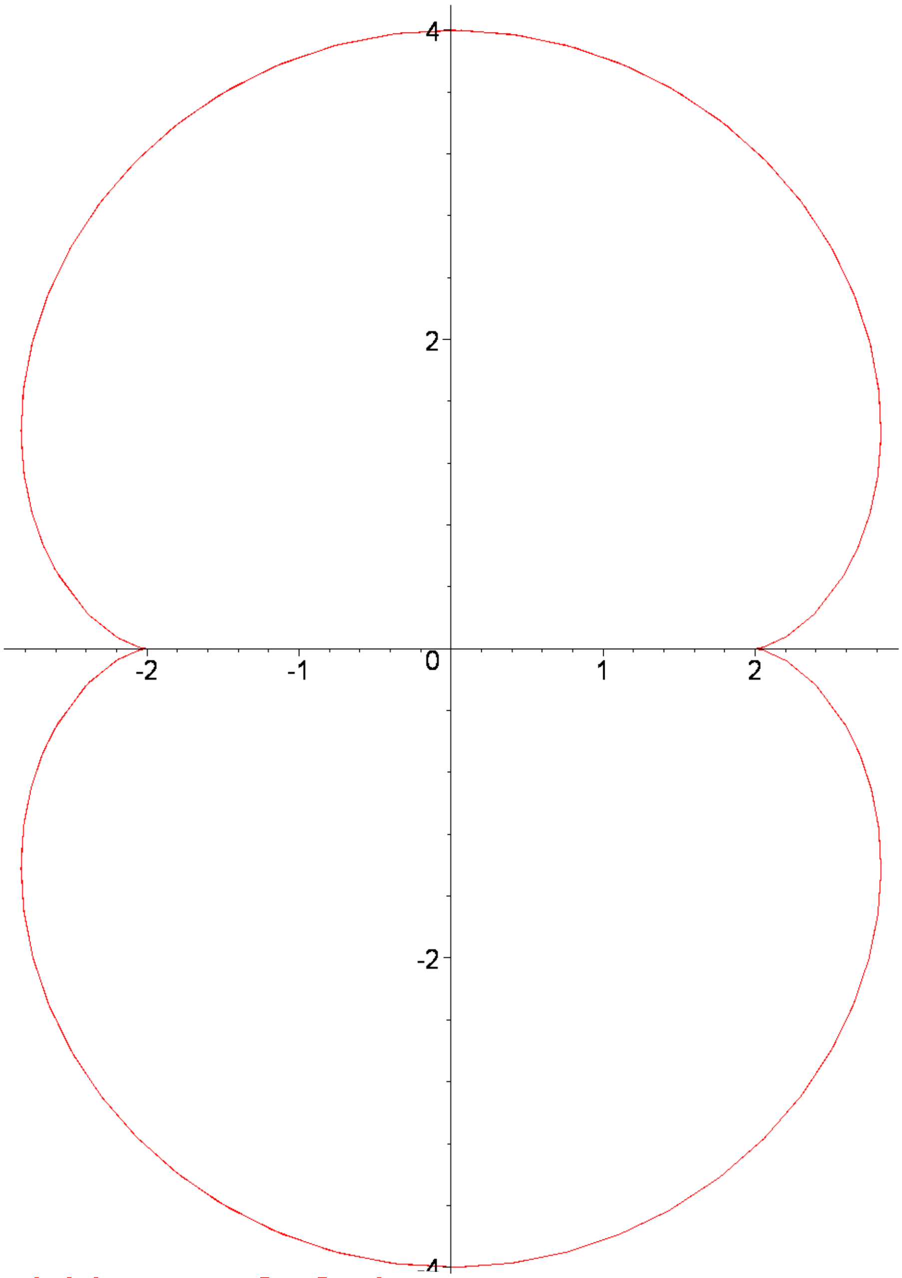
$$-2\cos(t)(-3 + 2\cos(t)^2)$$

$$4\sin(t)^3$$

```
> plot([x,y],t=0..4*Pi);plot([x,y,t=0..4*Pi]);
```



4
2
0
-2
-4
2
4
6
8
10
12
t




```
> restart:with(VectorCalculus):
> ArcLength( <3*cos(t) - cos(3*t),3*sin(t) - sin(3*t)>, t=0..2*Pi
  ) ;
```

Warning, the assigned names <,> and <|> now have a global binding

Warning, these protected names have been redefined and unprotected: *, +, ., D, Vector, diff, int, limit, series

$$24$$

```
> assume((t>0),(t<Pi)):
  Curvature( <3*cos(t) - cos(3*t),3*sin(t) - sin(3*t)> ):
> simplify(%);
>
>
```

$$\frac{1}{3}\frac{1}{\sin(t\sim)}$$

```
> #########################################################################
```

```
> #Repère de Frenet
restart:assume((t>0),(t<Pi)):x:=3*cos(t) - cos(3*t);y:=3*sin(t) - sin(3*t);
```

$$x := 3\cos(t\sim) - \cos(3\,t\sim)$$

$$y := 3\sin(t\sim) - \sin(3\,t\sim)$$

```
>
```

```
> u:=diff(3*cos(t) - cos(3*t),t):v:=diff(3*sin(t) - sin(3*t),t):[u,v];
l:=simplify(sqrt(u^2+v^2));
```

$$[-3\sin(t\sim) + 3\sin(3\,t\sim),\ 3\cos(t\sim) - 3\cos(3\,t\sim)]$$

$$l := 6\sin(t\sim)$$

```
> tau:=[simplify(u/l),simplify(v/l)];
```

$$\tau := [2\cos(t\sim)^2 - 1,\ 2\sin(t\sim)\cos(t\sim)]$$

```
> eta:=[-simplify(v/l),simplify(u/l)];
```

$$\eta := [-2\sin(t\sim)\cos(t\sim),\ 2\cos(t\sim)^2 - 1]$$

```
> t0:=Pi/4; x0:=subs(t=t0,x);
y0:=subs(t=t0,y);
u0:=subs(t=t0,u);
v0:=subs(t=t0,v);
with(plots):
b0:=
plot([x,y,t=0..4*Pi]):
```

$$t0 := \frac{\pi}{4}$$

$$x0 := 3\cos\left(\frac{\pi}{4}\right) - \cos\left(\frac{3\,\pi}{4}\right)$$

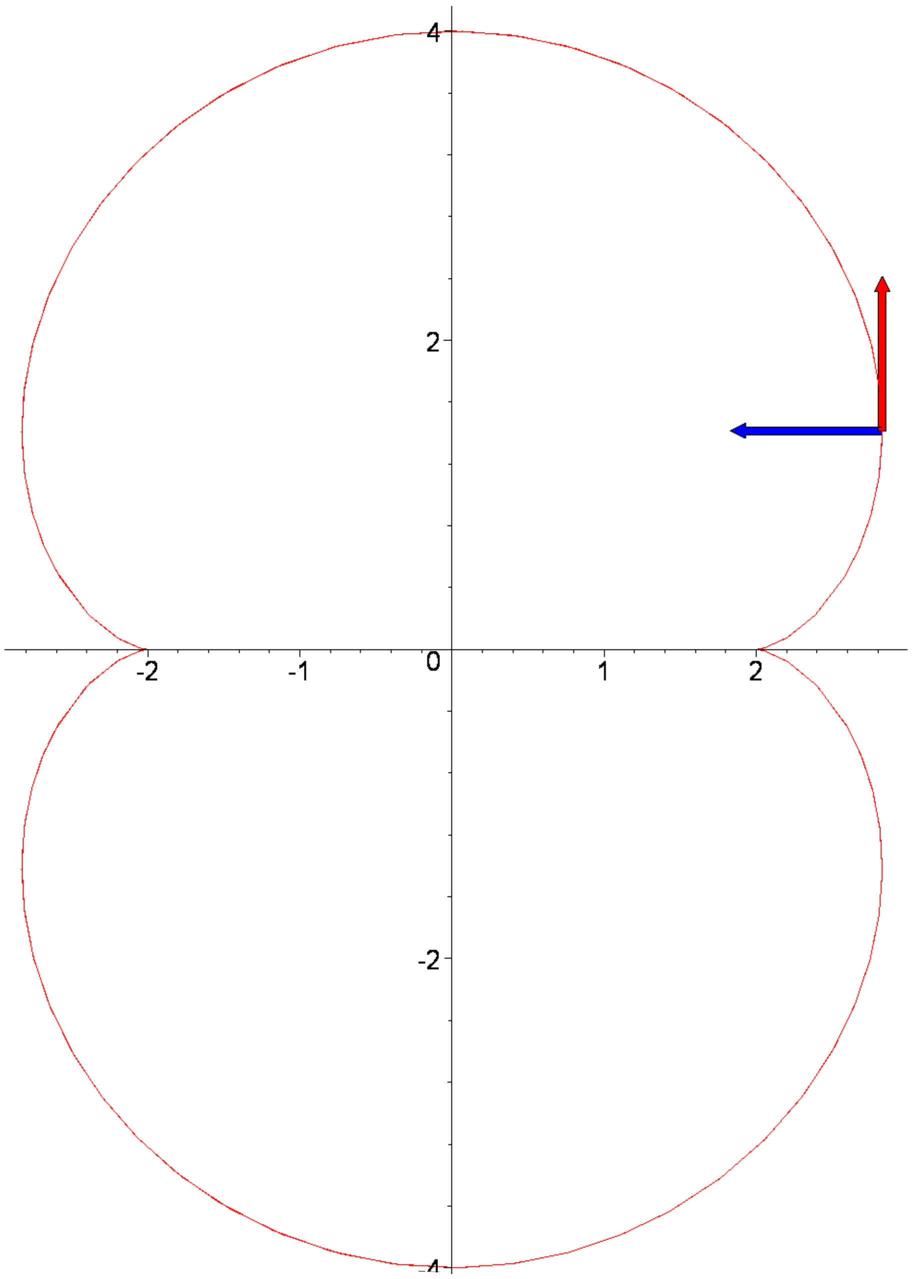
$$y0 := 3\sin\left(\frac{\pi}{4}\right) - \sin\left(\frac{3\,\pi}{4}\right)$$

$$u0 := -3\sin\left(\frac{\pi}{4}\right) + 3\sin\left(\frac{3\,\pi}{4}\right)$$

$$v0 := 3\cos\left(\frac{\pi}{4}\right) - 3\cos\left(\frac{3\,\pi}{4}\right)$$

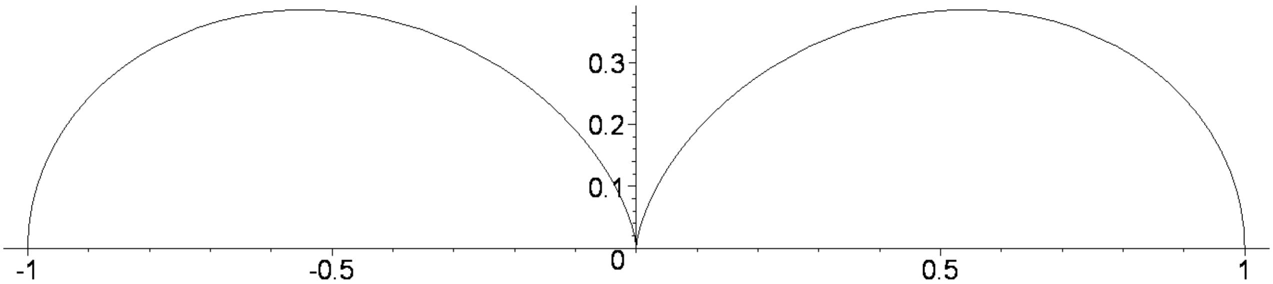
```
Warning, the name changecoords has been redefined
```

```
> b1 := arrow(<x0,y0>, <u0,v0>, length=[1], width=[0.05, relative], head_length=[0.1, relative], color=red):
b2 := arrow( <x0,y0>, <-v0,u0>,length=[1],width=[0.05, relative], head_length=[0.1, relative], color=blue):

display(b0, b1, b2, scaling=CONSTRAINED);
```




```
> ##################################################################
###
```

```
> #######################################     Exercice 8, suite
with(plots):
polarplot([cos(t)^2,t,t=0..Pi],color=black);
```




```
> with(VectorCalculus):SetCoordinates( 'polar' );
  ArcLength( <cos(t)^2,t>, t=0..2*Pi );
  simplify(%) assuming t::real;
> Curvature( <cos(t)^2,t> ):
  simplify(%) assuming t::real;
```

Warning, the assigned names <,> and <|> now have a global binding

Warning, these protected names have been redefined and unprotected: *, +, ., D, Vector, diff, int, limit, series

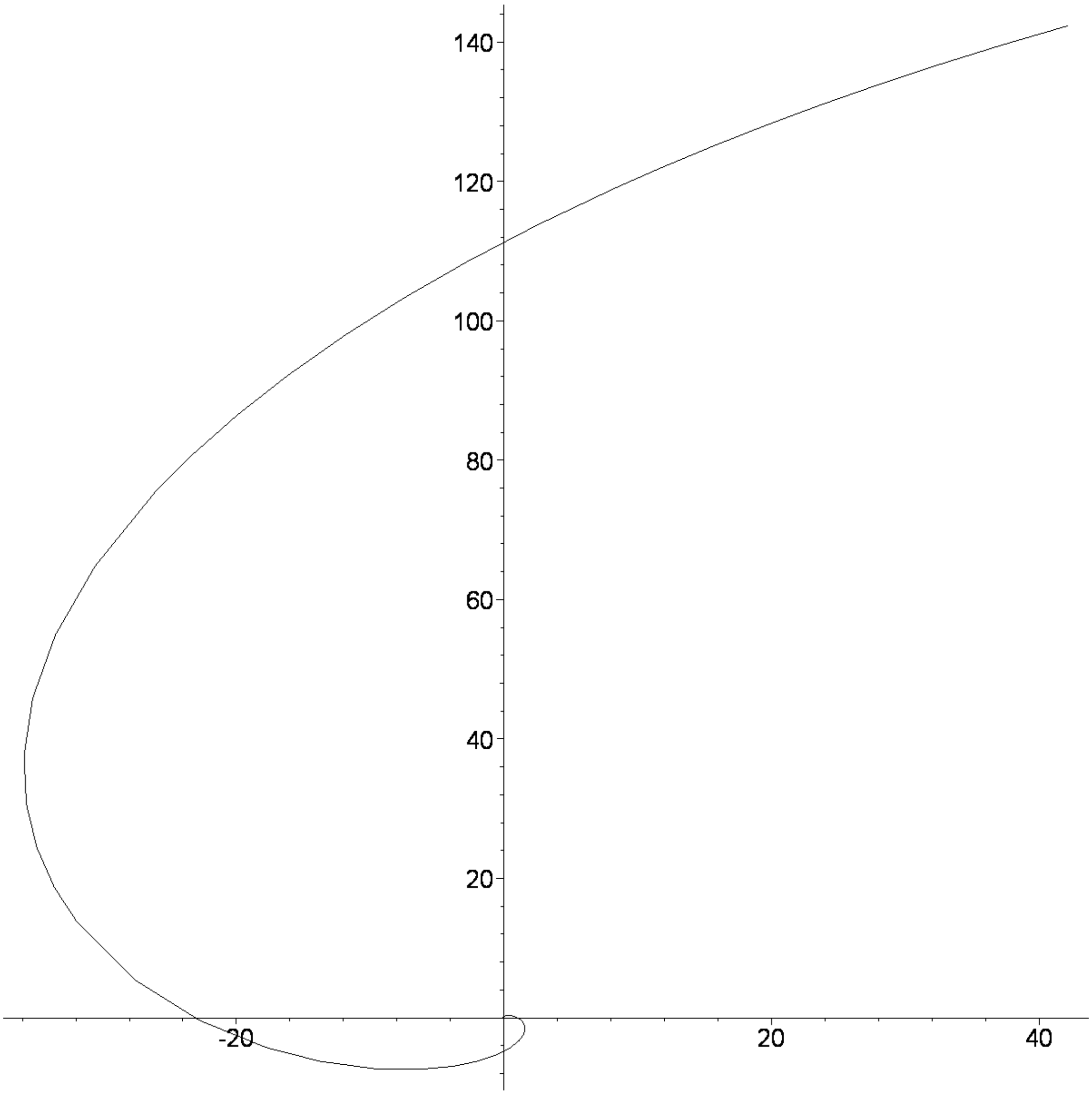
$$polar$$

$$\int_0^{2\pi} \sqrt{9\cos(t)^4 \sin(t)^2 + \left(-2\cos(t)\sin(t)^2 + \cos(t)^3\right)^2}\, dt$$

$$\int_0^{2\pi} \sqrt{-3\cos(t)^2+4}\,|\cos(t)|\,dt$$

$$-\frac{3\,(\cos(t)^2-2)}{(-3\cos(t)^2+4)^{(3/2)}\,|\cos(t)|}$$

```
> ######################################     Exercice 9.
  with(plots):plot([exp(-t)*cos(t),exp(-t)*sin(t),
  t=-5..5],color=black);
```




```
> with(VectorCalculus):
> ArcLength( <exp(-t)*cos(t),exp(-t)*sin(t)>, t=0..2*Pi ) ;
Warning, computation interrupted
> SetCoordinates( 'polar' );
  ArcLength( <exp(-t),t>, t=0..2*Pi ) ;
> Curvature( <exp(-t),t> ):
  simplify(%) assuming t::real;
```

$$polar$$

$$\sqrt{2}\,(\mathbf{e}^{(2\,\pi)}-1)\,\mathbf{e}^{(-2\,\pi)}$$

$$\frac{1}{2}\sqrt{2}\,\mathbf{e}^{t}$$

>