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Riordan trees and the homotopy sl_2 weight system

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ABSTRACT

The purpose of this paper is twofold. On one hand, we introduce a modification of the dual canonical basis for invariant tensors of the 3-dimensional irreducible representation of $U_q(sl_2)$, given in terms of Jacobi diagrams, a central tool in quantum topology. On the other hand, we use this modified basis to study the so-called homotopy sl_2 weight system, which is its restriction to the space of Jacobi diagrams labeled by distinct integers. Noting that the sl_2 weight system is completely determined by its values on trees, we compute the image of the homotopy part on connected trees in all degrees; the kernel of this map is also discussed.

1. Introduction

The sl_2 weight system W is a $\mathbb{Q}$-algebra homomorphism from the space $\mathcal{B}(n)$ of Jacobi diagrams labeled by $\{1, \dots, n\}$ to the algebra $\mathrm{Inv}\left(S(sl_2)^{\otimes n}\right)$ of invariant tensors of the symmetric algebra $S(sl_2)$. The relevance of this construction lies in low dimensional topology. Jacobi diagrams form the target space for the Kontsevich integral Z, which is universal among finite type and quantum invariants of knotted objects: in particular, by postcomposing Z with the sl_2 weight system and specializing each factor at some finite-dimensional representation of quantum group $U_q(sl_2)$, one recovers the colored Jones polynomial. Hence, while the results of this paper are purely algebraic, we will see that they are motivated by, and have applications to, quantum topology – see Remark 1.4 at the end of this introduction.

An easy preliminary observation on the sl_2 weight system is the following.

Lemma 1.1. *The sl_2 weight system is determined by its values on connected trees, i.e. connected and simply connected Jacobi diagrams.*

(Although this result might be well-known, a proof is given in Section 2.4.)

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Table 1
The dimensions of $\mathcal{C}_n$, $\mathrm{Inv}(sl_2^{\otimes n})$ and $\mathrm{Ker}\, W_n^h$.

n	2	3	4	5	6	7	8	9	k
$\dim \mathcal{C}_n$	1	1	2	6	24	120	720	5040	$(k-2)!$
$\dim \mathrm{Inv}(sl_2^{\otimes n})$	1	1	3	6	15	36	91	232	R_k
$\dim \mathrm{Ker}\, W_n^h$	0	0	0	0	10	84	630	4808	$(k-2)! - R_k + \frac{1+(-1)^k}{2}$

In this paper, we focus on the *homotopy part* $\mathcal{B}^h(n)$ of $\mathcal{B}(n)$, which is generated by diagrams labeled by distinct elements in $\{1, \dots, n\}$. Here, the terminology alludes to the link-homotopy relation on (string) links, which is generated by self crossing changes. It was shown by Habegger and Masbaum [4] that the restriction of the Kontsevich integral to $\mathcal{B}^h(n)$ is a link-homotopy invariant, and is deeply related to Milnor link-homotopy invariants, which are classical invariants generalizing the linking number.

Let us state our main results on the *homotopy sl_2 weight system*, that is, the restriction of the sl_2 weight system to $\mathcal{B}^h(n)$. Owing to Lemma 1.1, we can fully understand this map by studying the restrictions

$$W_n^h \colon \mathcal{C}_n \to \mathrm{Inv}(sl_2^{\otimes n})$$

of the sl_2 weight system to the space $\mathcal{C}_n$ of connected trees with n univalent vertices labeled by distinct elements in $\{1, \dots, n\}$. Here, the target space $\mathrm{Inv}(sl_2^{\otimes n})$ is the invariant part of the n-fold tensor power of the adjoint representation (the 3-dimensional irreducible representation) of sl_2. Recall that the dimension of $\mathcal{C}_n$ is given by $(n-2)!$, while the dimension of $\mathrm{Inv}(sl_2^{\otimes n})$ is known to be the so-called [1] Riordan numbers R_n which can be defined by $R_2 = R_3 = 1$ and $R_n = (n-1)(2R_{n-1} + 3R_{n-2})/(n+1)$. These numbers are also found under the name of Motzkin sums, or ring numbers in the literature.

We have:

Theorem 1.2.

(i) *The weight system map W_n^h is injective if and only if $n \leq 5$.*
(ii) *For n odd and $n = 2$, the weight system map W_n^h is surjective.*
(iii) *For $n \geq 4$ even, W_n^h has a 1-dimensional cokernel, spanned by $c^{\otimes \frac{n}{2}}$, where $c = \frac{1}{2} h \otimes h + e \otimes f + f \otimes e \in \mathrm{Inv}(sl_2^{\otimes 2})$.*

The dimensions of $\mathcal{C}_n$, $\mathrm{Inv}(sl_2^{\otimes n})$ and $\mathrm{Ker}\, W_n^h$ are given in Table 1.

Let $\mathfrak{S}_n$ be the symmetric group in n elements. The spaces $\mathcal{C}_n$ and $\mathrm{Inv}(sl_2^{\otimes n})$ have $\mathfrak{S}_n$-module structures, such that $\mathfrak{S}_n$ acts on $\mathcal{C}_n$ by permuting the labels, and acts on $\mathrm{Inv}(sl_2^{\otimes n})$ by permuting the factors. The sl_2 weight system is a $\mathfrak{S}_n$-module homomorphism, and the characters $\chi_{\mathcal{C}_n}$ and $\chi_{\mathrm{Inv}(sl_2^{\otimes n})}$ are already known (see Lemma 3.7 and Proposition 3.8). Thus, by Theorem 1.2, we can determine the characters $\chi_{\ker(W_n^h)}$ and $\chi_{\mathrm{Im}(W_n^h)}$ of the kernel and the image of W_n^h, respectively, as follows.

Corollary 1.3. (i) *For $n = 2$ or $n > 2$ odd, we have*

$$\chi_{\ker(W_n^h)} = \chi_{\mathcal{C}_n} - \chi_{\mathrm{Inv}(sl_2^{\otimes n})} \quad \text{and} \quad \chi_{\mathrm{Im}(W_n^h)} = \chi_{\mathrm{Inv}(sl_2^{\otimes n})}.$$

(ii) *For $n \geq 4$ even, we have*

$$\chi_{\ker(W_n^h)} = \chi_{\mathcal{C}_n} - \chi_{\mathrm{Inv}(sl_2^{\otimes n})} + \chi_U \quad \text{and} \quad \chi_{\mathrm{Im}(W_n^h)} = \chi_{\mathrm{Inv}(sl_2^{\otimes n})} - \chi_U,$$

where U is the trivial representation.

Although the proof of Theorem 1.2 is mainly combinatorial, it heavily relies on the following algebraic result.

Theorem *(Theorem 3.2). The set*

$$\mathfrak{I}_n := \{W(T);\ T \text{ is a Riordan tree of order } n\}$$

forms a basis for $\mathrm{Inv}(sl_2^{\otimes n})$.

Here, Riordan trees of order n are a special class of elements of $\mathcal{B}^h(n)$; roughly speaking, a Riordan tree is a disjoint union of linear tree diagrams (i.e. of the shape of Fig. 2), whose label sets comprise a Riordan partition of $\{1,\dots,n\}$ – see Definition 3.1.

Theorem 3.2 is proved using the work of Frenkel and Khovanov [3], who studied graphical calculus for the dual canonical basis of tensor products of finite-dimensional irreducible representations of $U_q(sl_2)$. More precisely, we define a new basis for $\mathrm{Inv}_{U_q}(V_2^{\otimes n})$, the space of $U_q(sl_2)$-invariants of tensor products of the 3-dimensional irreducible representation V_2, by inserting copies of the Jones–Wenzl projector in the dual canonical basis studied in [3]. This basis is actually unitriangular to the Frenkel–Khovanov basis, see Theorem 4.3. The result is a graphical description of invariant tensors in terms of Jacobi diagrams; see e.g. [7,9,11] for related graphical approaches to invariant tensors. We expect that this result and its possible generalizations could also be interesting from an algebraic point of view.

Remark 1.4. Consider the projection Z^h of the Kontsevich integral Z onto the space $\mathcal{B}^{t,h}(n)$ of tree Jacobi diagrams labeled by distinct elements of $\{1,\dots,n\}$. In Proposition 10.6 of [4], Habegger and Masbaum show that, for string links, the leading term of Z^h determines (and is determined by) the first non-vanishing Milnor link-homotopy invariants. The non-injectivity of the map W_n^h for $n \geq 5$ tells us that, expectedly, this is in general no longer the case for quantum invariant $W \circ Z$ – yet, it is remarkable that it still determines the first non-vanishing Milnor link-homotopy invariants of length up to 5. On the other hand, since Z extends to a graded isomorphism on the free abelian group generated by string links, surjectivity of the map W_n^h readily implies surjectivity of the linear extension of $W_n^h \circ Z$ (see also Remark 3.3). By Theorem 1.2, the surjectivity defect is given by $c^{\otimes n}$; it is not hard to check that, for a $2n$-component string link, the coefficient of $c^{\otimes n}$ in $W \circ Z$ is given by a product of linking numbers (this follows from a similar result at the level of the Kontsevich integral Z), and is in particular zero for string links with vanishing linking numbers.

Similar observations can be made for the universal sl_2 invariant, using Theorem 5.5 of [8].

The rest of this paper is organized in three sections. In Section 2 we recall the definitions of Jacobi diagrams and the sl_2 weight system, and give a result which in particular implies Lemma 1.1. Section 3 introduces Riordan trees and the tree basis of $\mathrm{Inv}(sl_2^{\otimes n})$, which are used to prove Theorem 1.2. Finally, in Section 4 we recall a few elements from the graphical calculus developed by Frenkel and Khovanov, and use it to prove Theorem 3.2.

2. Jacobi diagrams and the sl_2 weight system

In this section we give the definitions of the sl_2 weight system W and proof of Lemma 1.1.

2.1. The Lie algebra sl_2 and its symmetric algebra

Recall that the Lie algebra sl_2 is the 3-dimensional Lie algebra over $\mathbb{Q}$ generated by h, e, and f with Lie bracket

$$[h,e] = 2e, \quad [h,f] = -2f, \quad [e,f] = h. \tag{1}$$

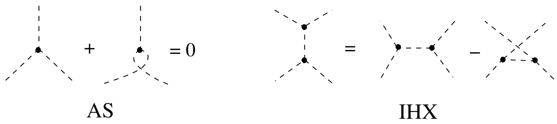


Fig. 1. The AS and IHX relations.

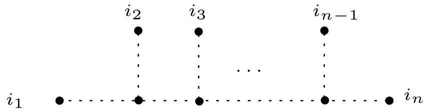


Fig. 2. A linear tree Jacobi diagram.

Let $S = S(sl_2)$ be the symmetric algebra of sl_2. The adjoint action, acting as a derivation, endows S, and more generally $S^{\otimes n}$ for any $n \geq 1$, with a structure of sl_2-modules. Note that $sl_2^{\otimes n}$, the n-fold tensor power of sl_2, is isomorphic to the subspace of $S^{\otimes n}$ having degree one in each factor.

We denote by $\mathrm{Inv}(S^{\otimes n})$ and $\mathrm{Inv}(sl_2^{\otimes n})$ the set of invariant tensors of $S^{\otimes n}$ and $sl_2^{\otimes n}$, respectively (that is, elements that are mapped to zero when acted on by h, e, and f).

2.2. Jacobi diagrams

A *Jacobi diagram* is a finite unitrivalent graph, such that each trivalent vertex is equipped with a cyclic ordering of its three incident half-edges. Each connected component is required to have at least one univalent vertex. An *internal edge* of a Jacobi diagram is an edge connecting two trivalent vertices. The *degree* of a Jacobi diagram is half its number of vertices.

In this paper we call a simply connected (not necessarily connected) Jacobi diagram a *tree*. A tree consisting of a single edge is called a *strut*.

Let $\mathcal{B}(n)$ be the completed $\mathbb{Q}$-space spanned by Jacobi diagrams whose univalent vertices are labeled by elements of $\{1, ..., n\}$, subject to the AS and IHX relations shown in Fig. 1. Here completion is given by the degree. Note that $\mathcal{B}(n)$ has an algebra structure with multiplication given by disjoint union.

Let $\mathcal{B}^h(n) \subset \mathcal{B}(n)$ denote the subspace generated by Jacobi diagrams labeled by distinct[1] elements in $\{1, \dots, n\}$. Note that $\mathcal{B}^h(n)$ is the polynomial algebra on the space $\mathcal{C}^h(n)$ of connected diagrams labeled by distinct elements in $\{1, \dots, n\}$.

As is customary, for each of the spaces defined above we use a subscript k to denote the corresponding subspaces spanned by degree k elements.

We denote by $\mathcal{C}_n$ the space of connected trees where each of the labels $1, \dots, n$ appears exactly once. It is a well-known fact, easily checked using the AS and IHX relations, that a basis for $\mathcal{C}_n$ is given by *linear trees*, i.e. connected trees of the form shown in Fig. 2, where the labels i_1 and i_n are two arbitrarily chosen elements of $\{1, \dots, n\}$, and where $i_2, \dots, i_{n-1}$ are running over all (pairwise distinct) elements of $\{1, \dots, n\} \setminus \{i_1, i_n\}$. This shows that $\dim \mathcal{C}_n = (n-2)!$, as recalled in the introduction.

2.3. The sl_2 weight system

We now define the sl_2 *weight system*, which is a $\mathbb{Q}$-algebra homomorphism

$$W : \mathcal{B}(n) \to \mathrm{Inv}(S^{\otimes n}).$$

[1] The superscript h stands for 'homotopy' since, as noted in the introduction, $\mathcal{B}^h(n)$ is the relevant space for link-homotopy invariants of (string) links.

Recalling that $\mathcal{B}(n)$ is (the completion of) the commutative polynomial algebra on the space of connected diagrams, it is enough to define it on the latter. We closely follow [8, §4.3].

We will use the non-degenerate symmetric bilinear form

$$\kappa\colon sl_2 \otimes sl_2 \to \mathbb{Q}$$

given by

$$\kappa(h,h)=2, \quad \kappa(e,f)=1, \quad \kappa(h,e)=\kappa(h,f)=\kappa(e,e)=\kappa(f,f)=0.$$

The bilinear form κ identifies sl_2 with the dual Lie algebra sl_2^*. Note that, under this identification, $\kappa \in (sl_2^{\otimes 2})^* \simeq sl_2^* \otimes sl_2^*$ itself corresponds to the quadratic Casimir tensor

$$c = \frac{1}{2} h \otimes h + f \otimes e + e \otimes f \in \mathrm{Inv}(sl_2^{\otimes 2}), \tag{2}$$

while the Lie bracket $[-,-] \in sl_2^* \otimes sl_2^* \otimes sl_2$ corresponds to the invariant tensor

$$b = \sum_{\sigma \in \mathfrak{S}_3} (-1)^{|\sigma|} \sigma(h \otimes e \otimes f) \tag{3}$$

$$= h \otimes e \otimes f + e \otimes f \otimes h + f \otimes h \otimes e - h \otimes f \otimes e - f \otimes e \otimes h - e \otimes h \otimes f, \tag{4}$$

where σ acts by permutation of the factors.

Let D_{ij} be a strut with vertices labeled by $1 \le i, j \le n$. Rewriting formally (2) as $c = \sum c_1 \otimes c_2$, we set

$$W(D_{ij}) = \sum 1 \otimes \cdots \otimes c_1 \otimes \cdots \otimes c_2 \otimes \cdots \otimes 1 \in \mathrm{Inv}(S^{\otimes n}),$$

where c_1 and c_2 are at the ith and jth position, respectively.

Now, let $m \ge 2$. For a connected diagram $D \in \mathcal{B}_m(n)$, attach a copy of $b \in \mathrm{Inv}(sl_2^{\otimes 3})$ to each trivalent vertex of D, a copy of sl_2 being associated to each of the 3 half-edges at the trivalent vertex following the cyclic ordering. Each internal edge of D is divided into two half-edges, and we contract the two corresponding copies of sl_2 by κ. Fixing an arbitrary total order on the set of univalent vertices of D, we get in this way an element $x_D = \sum x_1 \otimes \cdots \otimes x_{m+1}$ of $\mathrm{Inv}(sl_2^{\otimes m+1})$, the ith factor corresponding to the ith univalent vertex of D. We then define $W(D) \in \mathrm{Inv}(S^{\otimes n})$ by

$$W(D) = \sum y_1 \otimes \cdots \otimes y_n, \tag{5}$$

where y_j is the product of all $x_i \in sl_2$ such that the ith vertex is labeled by j.

It is known that W is well-defined, i.e. is invariant under the AS and IHX relations. The next lemma, due to Chmutov and Varchenko [2], gives another relation satisfied by the sl_2 weight system.

Lemma 2.1. *The sl_2 weight system W factors through the CV relation below*

CV

Note that the CV relation is not degree-preserving. Note also that this relation might involve diagrams with a circular component: the value of W on such component is set to $W(\bigcirc) = 3 = \dim sl_2$.

Remark 2.2. It is worth noting here that the restriction of the sl_2 weight system to $\mathcal{C}_n$ takes values in $\mathrm{Inv}(sl_2^{\otimes n})$. Likewise, the homotopy sl_2 weight system, i.e. its restriction to $\mathcal{B}^h(n)$, takes values in $\mathrm{Inv}(\langle sl_2\rangle^{\otimes n})$, where $\langle sl_2\rangle^{\otimes n} = (\mathbb{Q} \oplus sl_2)^{\otimes n} \subset S(sl_2)^{\otimes n}$ is the subspace of tensors having degree *at most one* in each factor.

2.4. The space $\mathcal{B}_{sl_2}(n)$ of sl_2-Jacobi diagrams

In view of Lemma 2.1, it is natural to consider the following space.

Definition 2.3. The space of sl_2-Jacobi diagrams is the quotient space

$$\mathcal{B}_{sl_2}(n) = \mathcal{B}(n)/CV, \bigcirc_3$$

of $\mathcal{B}(n)$ by the ideal generated by the CV relation and the relation $\bigcirc_3$ that maps a circular component to a factor 3.

Note that the algebra structure on $\mathcal{B}(n)$ descends to $\mathcal{B}_{sl_2}(n)$. This is however no longer a graded algebra (although one could impose such a structure by considering the number of univalent vertices).

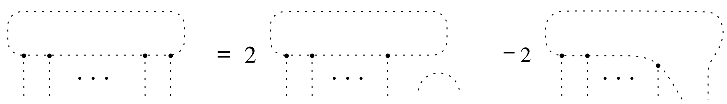
Since the sl_2 weight system factors through $\mathcal{B}_{sl_2}(n)$, it is useful for our study to get some insight in this space.

Proposition 2.4. *As an algebra, $\mathcal{B}_{sl_2}(n)$ is generated by (connected) trees.*

This in particular implies Lemma 1.1 stated in the introduction.

Proof. It suffices to prove that any connected Jacobi diagram in $\mathcal{B}_{sl_2}(n)$ can be expressed as a combination of trees. The proof is by a double induction, on the number of cycles in the diagrams and on the minimal length of the cycles (the length of a cycle is the number of internal edges contained in it).

Consider a connected diagram C with k cycles, and pick a cycle of minimal length l. If the cycle has length $l = 0$, then the diagram C is a loop, which can be replaced by a coefficient 3 by the $\bigcirc_3$ relation. If $l = 1$, then it follows from the AS relation that C is zero. Now, if $l \geq 2$, we can apply the CV relation at some internal edge of the cycle, which gives



, (6)

where the rightmost term is a diagram with $n - 1$ cycles, and where the middle term has a cycle of length $l - 2$. We can thus apply (6) recursively to reduce the length of this cycle, until we obtain a cycle of length either 1 or 0, as above. Then C writes as a combination of diagrams with less than k cycles. This concludes the proof. □

3. Invariant tensors and the homotopy sl_2 weight system

In this section we give a basis for $\mathrm{Inv}(sl_2^{\otimes n})$ in terms of Riordan trees, and use this basis to prove Theorem 1.2. The kernel of the homotopy sl_2 weight system is briefly discussed at the end of the section.

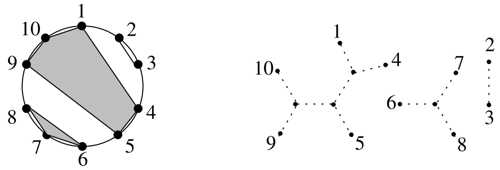


Fig. 3. The Riordan tree associated to the Riordan partition $\{\{1,4,5,9,10\},\{2,3\},\{6,7,8\}\}$.

3.1. *Tree basis of* $\mathrm{Inv}(sl_2^{\otimes n})$

We now construct a basis for $\mathrm{Inv}(sl_2^{\otimes n})$, as the image by the sl_2 weight system of a certain class of connected tree Jacobi diagrams. For this, we need a couple extra definitions.

On one hand, we call a linear tree *ordered* if, in the notation of Fig. 2, its labels $i_1,\dots,i_n$ satisfy $i_1 < i_2 < \dots < i_n$.

On the other hand, a *Riordan partition* is a partition of $\{1,\dots,n\}$ into parts that contains at least two elements, and whose convex hulls are disjoint when the points are arranged on a circle. For example, $\{\{1,4,5,9,10\},\{2,3\},\{6,7,8\}\}$ is a Riordan partition, as illustrated in Fig. 3, while $\{\{1,4,6\},\{2,3\},\{5,7,8\}\}$ is not.[2] The number of Riordan partitions of $\{1,\dots,n\}$ is given by the Riordan number R_n – see [1, §3.2].

This leads to the following

Definition 3.1. A *Riordan tree* of order n is an element of $\mathcal{B}^h(n)$ such that

- each connected component is an ordered linear tree,
- the partition of $\{1,\dots,n\}$ induced by its connected components is a Riordan partition.

See the right-hand side of Fig. 3 for an example. Note that a Riordan partition uniquely determines a Riordan tree; the number of Riordan trees of order n is thus given by R_n.

Theorem 3.2. *The set*

$$\mathfrak{I}_n := \{W(T);\ T \text{ is a Riordan tree of order } n\}$$

forms a basis for $\mathrm{Inv}(sl_2^{\otimes n})$.

We call this basis the *tree-basis* of $\mathrm{Inv}(sl_2^{\otimes n})$. The proof of Theorem 3.2 is postponed to Section 4, and is somewhat indirect. It uses the graphical calculus for the dual canonical basis for $\mathrm{Inv}(V_2^{\otimes n})$ given by Frenkel and Khovanov in [3]. Although a more direct proof may exist, we hope that the one given in this paper could be interesting from the representation theory point of view.

Remark 3.3. Theorem 3.2 implies immediately that the homotopy sl_2 weight system $W\colon \mathcal{B}^h(n) \to \mathrm{Inv}(\langle sl_2\rangle^{\otimes n})$ is surjective, and Theorem 1.2 can be regarded as a refinement of this observation. (Recall that $\langle sl_2\rangle^{\otimes n}$ was defined in Remark 2.2.)

3.2. *Proof of Theorem 1.2*

The proof of Theorem 1.2 (i) is straightforward using Theorem 3.2: pick a basis for $\mathcal{C}_n$ in terms of linear trees, as outlined at the end of Section 2.2, and write each basis element, using the CV relation, as a linear

[2] A partition satisfying only the second condition is often called non-crossing.

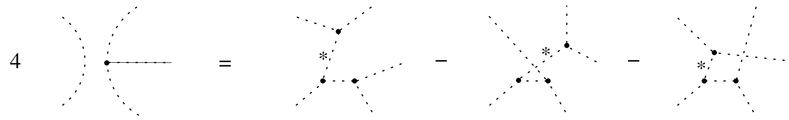


Fig. 4. Relation in $\mathcal{B}_{sl_2}(n)$, given by applying the CV relation at each of the $*$-marked edges on the right-hand side.

combination of Riordan trees of order n. It then suffices to check that, for $n \leq 5$, the matrix obtained in this transformation has rank $(n-2)!$. Non-injectivity for $n \geq 6$ is obvious since the dimension of the target space $\mathrm{Inv}(sl_2^{\otimes n})$ is smaller than that of the domain $\mathcal{C}_n$.

We now turn to the surjectivity results (ii) and (iii). Let $\mathcal{B}^h_Y(n) \subset \mathcal{B}^h(n)$ be the subspace of Jacobi diagrams with at least one trivalent vertex, and let $\mathcal{B}^h_U(n) \subset \mathcal{B}^h(n)$ be the subspace of Jacobi diagrams containing only struts. Set

$$\mathfrak{I}^Y_n := \{W(T);\ T \text{ is a Riordan tree in } \mathcal{B}^h_Y(n)\}, \tag{7}$$

$$\mathfrak{I}^U_n := \{W(T);\ T \text{ is a Riordan tree in } \mathcal{B}^h_U(n)\}. \tag{8}$$

Note that $\mathfrak{I}_n = \mathfrak{I}^Y_n$ for n odd, while $\mathfrak{I}_n = \mathfrak{I}^Y_n \cup \mathfrak{I}^U_n$ for n even.

Based on Theorem 3.2 and this observation, points (ii) and (iii) of Theorem 1.2 follow from the following two lemmas.

Lemma 3.4. *If $T \in \mathcal{B}^h_Y(n)$, then $W(T) \in W(\mathcal{C}_n)$. In particular, $\mathfrak{I}^Y_n \subset W(\mathcal{C}_n)$.*

For $n \geq 2$ even, let $\cup^{\otimes n} = \coprod_{i=1}^{n/2} D_{2i-1,2i}$ denote the tree diagram made of n struts labeled by i and $i+1$ $(1 \leq i \leq n/2)$. Note that $W(\cup^{\otimes n}) = c^{\otimes n} \in \mathfrak{I}^U_n$.

Lemma 3.5.

(i) *We have $W(\cup^{\otimes n}) \not\equiv 0$ modulo $W(\mathcal{C}_n)$.*
(ii) *If $T \in \mathcal{B}^h_U(n)$ with $n \geq 4$ even, then $W(T) \equiv W(\cup^{\otimes n})$ modulo $W(\mathcal{C}_n)$.*

Proof of Lemma 3.4. Let $T \in \mathcal{B}^h_Y(n)$ be a Jacobi diagram, and let k denote the number of connected components of T. If $k > 1$, the equality depicted in Fig. 4 shows how T can be expressed as a combination of tree diagrams with $k-1$ components in $\mathcal{B}^h_{sl_2}(n)$. Since each of these trees contains at least one trivalent vertex, the proof follows by an easy induction on k. □

Remark 3.6. Note that the proof applies more generally to the whole space $\mathcal{B}_{sl_2}(n)$. More precisely, any Jacobi diagram with at least one trivalent vertex decomposes as a combination of connected diagrams in $\mathcal{B}_{sl_2}(n)$. Combining this with Proposition 2.4, we have that $\mathcal{B}_{sl_2}(2k+1)$ is generated, as a vector space, by connected tree Jacobi diagrams and that $\mathcal{B}_{sl_2}(2k)$ is generated by connected trees *and* the disjoint union of n struts $\sqcup_{i=1}^k D_{2i-1,2i}$.

Proof of Lemma 3.5. To show (ii), note that any $T \in \mathcal{B}^h_U(n)$ is obtained from $\cup^{\otimes n}$ by exchanging some labels, which implies that $W(T) - W(\cup^{\otimes n}) \in W\left(\mathcal{B}^h_Y(n)\right)$ by Lemma 2.1. Combining this with Lemma 3.4, we have the assertion.

We now prove (i). Consider the $\mathbb{C}$-linear map ϕ: $\mathrm{Inv}(sl_2^{\otimes n}) \to \mathbb{C}$ defined (using Theorem 3.2) by

$$\phi(t) = \begin{cases} 0 & \text{for } t \in \mathfrak{I}^Y_n, \\ 1 & \text{for } t \in \mathfrak{I}^{\cup}_n. \end{cases}$$

We prove that $W(\mathcal{C}_n) \subset \mathrm{Ker}(\phi)$, which implies the assertion. It suffices to prove that $W(T) \in \mathrm{Ker}(\phi)$ for a connected tree diagram $T \in \mathcal{C}_n$; actually, as observed at the end of Section 2.2, we may further assume that T is linear.[3] Notice that the number v_T of trivalent vertices of T is its degree minus 1, and that applying the CV relation yields diagrams with $(v_T - 2)$ trivalent vertices. If the degree of T is odd, then by applying the CV relation repeatedly we obtain

$$T = 2^{v_T/2} \sum_{i=1}^{2^{v_T/2}} (-1)^i U_i,$$

where $U_i \in \mathcal{B}_U^h(n)$. Although this expression is not unique, this always yields $\phi(T) = 0$. Now, in the case where T has even degree, successive applications of the CV relation give

$$T = 2^{(v_T-1)/2} \sum_{i=1}^{2^{(v_T-1)/2}} (-1)^i Y_i,$$

where Y_i has a single trivalent vertex (and $\frac{v_T-1}{2} = \frac{n}{2} - 1$ struts). We thus obtain $\phi(T) = 0$, as desired. □

3.3. $\mathfrak{S}_n$-module structure

For a partition λ of n, let V_λ denote the irreducible representation of $\mathfrak{S}_n$ associated to λ. Note that the adjoint representation of sl_2 corresponds to the vector representation V of $SO(3)$, and the invariant part of $sl_2^{\otimes n}$ corresponds to the invariant part of $V^{\otimes n}$. The tensor powers of the vector representation of $GL(3)$ and its restriction to $SO(3)$ are well-studied classically, using e.g. Schur–Weyl duality or Peter–Weyl Theorem. In particular, we have the following.

Lemma 3.7. *As $\mathfrak{S}_n$-modules, we have*

$$\mathrm{Inv}(sl_2^{\otimes n}) \simeq \bigoplus V_\lambda,$$

where the summation is over partitions $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ of n such that each λ_i is odd or each λ_i is even, i.e., such that $\lambda_1 - \lambda_2, \lambda_2 - \lambda_3 \in 2\mathbb{Z}$.

Corollary 1.3 follows from Theorem 1.2 and Lemma 3.7 as follows.

Proof of Corollary 1.3. The fact that $\chi_{\ker(W_n^h)} = \chi_{\mathcal{C}_n} - \chi_{\mathrm{Inv}(sl_2^{\otimes n})}$ and $\chi_{\mathrm{Im}(W_n^h)} = \chi_{\mathrm{Inv}(sl_2^{\otimes n})}$ for $n = 2$ or $n > 2$ odd immediately follows from Theorem 1.2. By Lemma 3.7, the one dimensional representation appearing in the irreducible decomposition of $\mathrm{Inv}(sl_2^{\otimes n})$ is the trivial representation U. Thus we have that $\chi_{\ker(W_n^h)} = \chi_{\mathcal{C}_n} - \chi_{\mathrm{Inv}(sl_2^{\otimes n})} + \chi_U$ and $\chi_{\mathrm{Im}(W_n^h)} = \chi_{\mathrm{Inv}(sl_2^{\otimes n})} - \chi_U$ for $n \geq 4$ even. □

The character $\chi_{\mathcal{C}_n}$ is known as follows.

Proposition 3.8 *(Kontsevich [6, Theorem 3.2]). As a $\mathfrak{S}_n$-module, the character of $\mathcal{C}_n$ is*

$$\chi_{\mathcal{C}_n}(1^n) = (n-2)!, \quad \chi_{\mathcal{C}_n}(1^1 a^b) = (b-1)! a^{b-1} \mu(a), \quad \chi_{\mathcal{C}_n}(a^b) = -(b-1)! a^{b-1} \mu(a), \tag{9}$$

and $\chi_{\mathcal{C}_n}(*) = 0$ for other conjugacy classes. Here, μ is the Möbius function.

[3] This extra assumption is not necessary for the proof, but makes the arguments simpler to verify.

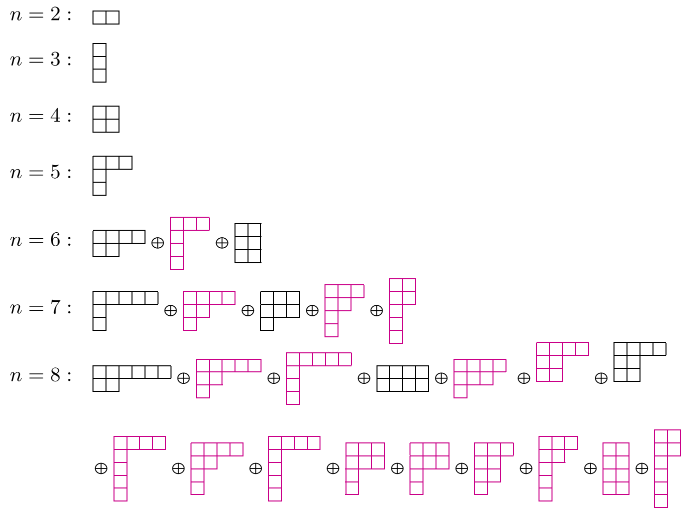


Fig. 5. Irreducible decompositions of $\mathcal{C}_n$, $2 \leq n \leq 8$, as $\mathfrak{S}_n$-modules. The red components are in the kernel of W_n^h. (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

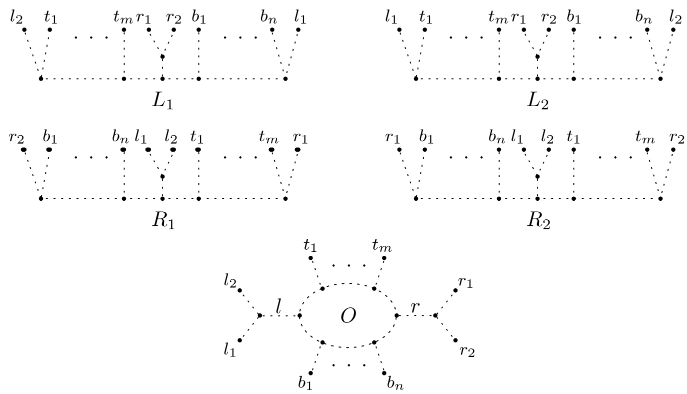


Fig. 6. The diagrams L_1, L_2, R_1, R_2 and O; here $m, n \geq 0$ are such that $k = (m + n + 2)/2$.

Thus by Corollary 1.3 we can calculate the character $\chi_{\ker(W_n^h)}$ explicitly. See Fig. 5 for the low degree cases.

3.4. Generating the kernel

It follows from Theorem 1.2 that the dimension of the kernel of the weight system map W_k^h is given by $(k-2)! - R_k + \frac{1+(-1)^k}{2}$. In this short section, we investigate some typical elements of this kernel. More precisely, we consider 1-*loop relators* of degree k, which are linear combinations of elements of $\mathcal{C}_{k+1}$ of the form

$$L_1 - L_2 - R_1 + R_2,$$

where L_1, L_2, R_1, R_2 are degree k tree Jacobi diagrams as shown in Fig. 6.

Let us explain why these are indeed mapped to zero by W_k^h. Denote by O the element of $\mathcal{B}_{k+1}^h(k+1)$ represented in Fig. 6. We call such an element a 2-*forked wheel*. Now, by applying the CV relation at the internal edge l of O (see the figure), we have that

$$W_k^h(O) = 2W_k^h(L_1) - 2W_k^h(L_2),$$

while applying CV at internal edge r yields

$$W_k^h(O) = 2W_k^h(R_1) - 2W_k^h(R_2),$$

thus showing that $L_1 - L_2 - R_1 + R_2$ is in the kernel of W_k^h.

Notice that, in degree ≤ 5, all 1-loop relators are trivial, which agrees with the fact that the weight system map is injective. Computations performed using a code in `Scilab` allowed us to check that, up to degree $k = 8$, the kernel of the weight system map W_k^h is generated by 1-loop relators of degree k.[4] It would be interesting to see up to what degree this statement still holds, and what are the additional kernel elements when it doesn't.

4. The dual canonical basis and the sl_2 weight system

In this section, we review the graphical calculus used by Frenkel and Khovanov in [3] to describe tensor products of finite-dimensional irreducible representations of quantum group $U_q(sl_2)$. This graphical calculus for invariant tensors appeared originally in the work of Rumer, Teller and Weyl [10], and was later adapted to the quantum setting in [3].

More precisely, we first recall in Section 4.1 some basic facts on $U_q(sl_2)$ and its representations, in Section 4.2 we recall the graphical calculus for the dual canonical basis for invariant tensors of 3-dimensional irreducible representations of sl_2, and in Section 4.3 we show that a simple modification of this basis is well-behaved with respect to the universal sl_2 weight system.

4.1. Quantum group $U_q(sl_2)$ and finite-dimensional irreducible representations

Let $U_q = U_q(sl_2)$ be the algebra over $\mathbb{C}(q)$ with generators K, K^{-1}, E, F and relations

$$KK^{-1} = K^{-1}K = 1, \quad KE = q^2EK, \quad KF = q^{-2}FK, \quad EF - FE = \frac{K - K^{-1}}{q - q^{-1}},$$

for q a non-zero complex indeterminate.

For $n \geq 0$, denote by V_n the fundamental $(n+1)$-dimensional irreducible representation of U_q, with basis

$$\{v_i;\ -n \leq i \leq n,\ i = n \pmod 2\}$$

such that the action of U_q is given by

$$Ev_i = \left[\frac{n+i+2}{2}\right] v_{i+2}, \quad Fv_i = \left[\frac{n-i+2}{2}\right] v_{i-2}, \quad K^{\pm 1}v_i = q^{\pm i}v_i$$

where $[m] = (q^m - q^{-m})/(q - q^{-1})$ and $v_{n+2} = v_{-n-2} = 0$.

Let $\langle\ ,\ \rangle\colon V_n \otimes V_n \to \mathbb{C}(q)$ be the symmetric bilinear pairing defined by

$$\langle v_{n-2k}, v_{n-2l}\rangle = \frac{[n]!}{[k]![n-k]!}\delta_{k,l};\ 0 \leq k, l \leq n,$$

where $[k]! := \prod_{i \leq k}[i]$, and let $\{v^i;\ -n \leq i \leq n,\ i = n \pmod 2\}$ be the dual basis with respect to this pairing. In particular, for $n = 1$, the dual basis is simply given by $v^i = v_i$ $(i = \pm 1)$, while for $n = 2$, we have $v^2 = v_2$, $v^0 = \frac{1}{[2]}v_0$ and $v^{-2} = v_{-2}$.

[4] The authors are indebted to Raphaël Rossignol for writing this code.

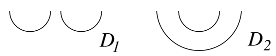


Fig. 7. Diagrams representing the dual canonical basis of $\mathrm{Inv}_{U_q}(V_1^{\otimes 2})$.

We also define the bilinear pairing $\langle\ ,\ \rangle$ of $V_{n_1} \otimes \ldots \otimes V_{n_m}$ and $(V_{n_1} \otimes \ldots \otimes V_{n_m})^* = V_{n_m} \otimes \ldots \otimes V_{n_1}$ by[5]

$$\langle v_{k_1} \otimes \ldots \otimes v_{k_m}, v^{k'_m} \otimes \ldots \otimes v^{k'_1} \rangle = \prod_{i=1}^{m} \delta_{k_i,k'_i}.$$

We refer the reader to Chapters 2 and 3 of the book [5] for a much more detailed treatment of this subject.

4.2. Graphical representations of the dual canonical basis for invariant tensor products

In what follows, we will only deal with 1 and 2-dimensional representations, which is sufficient for the purpose of this paper. We thus only give a very partial overview of the work in [3], where we refer the reader for further reading. We will mostly follow the notation of [3].

Let $\delta_1 \colon \mathbb{C} \to V_1 \otimes V_1$ denote the map defined by

$$\delta_1(\mathbf{1}) = v^1 \otimes v^{-1} - q^{-1} v^{-1} \otimes v^1.$$

In [3, Thm. 1.9], Frenkel and Khovanov showed that the intersection of the dual canonical basis of $V_1^{\otimes 2m}$ and the space $\mathrm{Inv}_{U_q}(V_1^{\otimes 2m})$ of invariant tensors forms a basis of $\mathrm{Inv}_{U_q}(V_1^{\otimes 2m})$:

$$\{(\delta_1)^{2(m-1)}_{i_{m-1}} (\delta_1)^{2(m-2)}_{i_{m-2}} \cdots (\delta_1)^2_{i_1} (\delta_1).\mathbf{1};\ 0 \le i_j \le j \text{ for each index } 1 \le j \le m-1\},$$

where $(\delta_1)^k_l : V_1^{\otimes k} \to V_1^{\otimes k+2}$ is defined by $(\delta_1)^k_l = \mathbf{1}^{\otimes l} \otimes \delta_1 \otimes \mathbf{1}^{\otimes (k-l)}$ $(0 \le l \le k)$.

Graphically, $V_1^{\otimes 2m}$ is represented by $2m$ fixed points on the x-axis of the real plane, and an element of the dual canonical basis of $\mathrm{Inv}_{U_q}(V_1^{\otimes 2m})$ is represented by a union of m non-intersecting arcs embedded in the lower half-plane and connecting these points, each arc corresponding to a copy of the map δ_1. For example, the dual canonical basis of $\mathrm{Inv}_{U_q}(V_1^{\otimes 4})$ consists of two elements $(\delta_1)^2_0.(\delta_1).\mathbf{1}$ and $(\delta_1)^2_1.(\delta_1).\mathbf{1}$, which are represented by the two diagrams D_1 and D_2 in Fig. 7, respectively.

Now, it follows from [3, Thm. 1.11] that these basis elements induce a basis $\mathcal{B}^0_m$ for $\mathrm{Inv}_{U_q}(V_2^{\otimes m})$, by taking their image under $\pi_2^{\otimes m}$, where $\pi_2 \colon V_1 \otimes V_1 \to V_2$ is defined by

$$\pi_2(v^1 \otimes v^1) = v^2, \quad \pi_2(v^{-1} \otimes v^{-1}) = v^{-2}, \tag{10}$$

$$\pi_2(v^1 \otimes v^{-1}) = q^{-1} v^0, \quad \pi_2(v^{-1} \otimes v^1) = v^0. \tag{11}$$

The map π_2 is graphically represented by a box with two incident points (corresponding to the two copies of V_1) on its lower horizontal edge, see Fig. 8.

Since $\pi_2 \circ \delta_1 = 0$, a diagram containing a box whose incident points are connected by an arc is equal to zero. If there is no such box, then this defines a non-trivial element of $\mathrm{Inv}_{U_q}(V_2^{\otimes m})$.

In summary, an element of the dual canonical basis $\mathcal{B}^0_m$ of $\mathrm{Inv}_{U_q}(V_2^{\otimes m})$ is graphically incarnated by m horizontally aligned boxes, whose incident edges are connected by m non-intersecting arcs, such that each arc is incident to two distinct boxes.

[5] The action of U_q on tensor powers of irreducible representations is defined via the comultiplication map Δ in the Hopf algebra structure of U_q; the dual action with respect to $\langle\ ,\ \rangle$ is likewise given by $u(x \otimes y) = \bar{\Delta}(u)(x \otimes y)$, where $\bar{\Delta}(u) = (\sigma \otimes \sigma)\Delta(\sigma(u))$ with the bar involution $\sigma \colon U_q \to U_q$. See e.g. [5, Chap. 3] or [3, § 1] for the details.

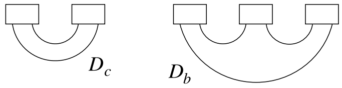


Fig. 8. Diagrams representing the dual canonical bases for $\mathrm{Inv}_{U_q}(V_2^{\otimes 2})$ and $\mathrm{Inv}_{U_q}(V_2^{\otimes 3})$.

Remark 4.1. Arranging the n boxes on a circle, these diagrams representing elements of $\mathcal{B}^0_m$ naturally appear to be in one-to-one correspondence with (convex hulls of) Riordan partitions of $\{1, \dots, n\}$. This agrees with the fact that the dimension of $\mathrm{Inv}(V_2^{\otimes m})$ is given by the Riordan number R_m.

Example 4.2. We conclude with a couple of examples. For $m = 2$, $\mathrm{Inv}_{U_q}(V_2^{\otimes 2})$ is spanned by D_c in Fig. 8, which represents the element

$$\begin{aligned}\tilde{c} :&= (\pi_2 \otimes \pi_2)(\delta_1)^2_1.(\delta_1).\mathbf{1}\\ &= (\pi_2 \otimes \pi_2)\left(v^1 \otimes v^1 \otimes v^{-1} \otimes v^{-1} - q^{-1}v^{-1} \otimes v^1 \otimes v^{-1} \otimes v^1\right.\\ &\quad \left.- q^{-1}v^1 \otimes v^{-1} \otimes v^1 \otimes v^{-1} + q^{-2}v^{-1} \otimes v^{-1} \otimes v^1 \otimes v^1\right)\\ &= v^2 \otimes v^{-2} - (q^{-1} + q^{-3})v^0 \otimes v^0 + q^{-2}v^{-2} \otimes v^2.\end{aligned}$$

Similarly, $\mathrm{Inv}_{U_q}(V_2^{\otimes 3})$ has dimension 1 with basis given by the diagram D_b represented in Fig. 8. We leave it to the reader to verify that this diagram represents the element

$$\begin{aligned}\tilde{b} :=\ & v^2 \otimes v^0 \otimes v^{-2} + q^{-2}v^0 \otimes v^{-2} \otimes v^2 + q^{-2}v^{-2} \otimes v^2 \otimes v^0 + q^{-5}v^0 \otimes v^0 \otimes v^0\\ & - q^{-2}v^2 \otimes v^{-2} \otimes v^0 - q^{-2}v^0 \otimes v^2 \otimes v^{-2} - q^{-2}v^{-2} \otimes v^0 \otimes v^2 - q^{-1}v^0 \otimes v^0 \otimes v^0.\end{aligned}$$

In the rest of this paper, we will use the term *FK diagrams* to refer to this graphical calculus of Frenkel and Khovanov, and we will consider such diagrams up to planar isotopy (fixing only the m boundary boxes corresponding to the m copies of V_2 in $\mathrm{Inv}(V_2^{\otimes m})$). We will also often blur the distinction between an invariant tensor and the FK diagram representing it.

4.3. The tree basis of $\mathrm{Inv}_{U_q}(V_2^{\otimes n})$

In this section, we modify the dual canonical basis $\mathcal{B}^0_m$ of $\mathrm{Inv}_{U_q}(V_2^{\otimes n})$ recalled above and prove that, at $q = 1$, this new basis corresponds to the tree basis of $\mathrm{Inv}(sl_2^{\otimes n})$ defined in Section 3.1.

The only new ingredient is the *Jones–Wenzl projector*

$$p_2 \colon V_1 \otimes V_1 \to V_1 \otimes V_1$$

defined by

$$p_2(v^1 \otimes v^1) = v^1 \otimes v^1, \quad p_2(v^1 \otimes v^{-1}) = \frac{1}{[2]}\left(q^{-1}v^1 \otimes v^{-1} + v^{-1} \otimes v^1\right), \tag{12}$$

$$p_2(v^{-1} \otimes v^{-1}) = v^{-1} \otimes v^{-1}, \quad p_2(v^{-1} \otimes v^1) = \frac{1}{[2]}\left(v^1 \otimes v^{-1} + qv^{-1} \otimes v^1\right). \tag{13}$$

See Fig. 9 for a graphical definition. There, a vertical strand represents the identity of $\mathrm{End}(V_1)$, while an arc connecting two lower boundary points represents the map $\varepsilon_1 \colon V_1 \otimes V_1 \to V_0$ defined by

$$\varepsilon_1(v^1 \otimes v^{-1}) = -q \quad ; \quad \varepsilon_1(v^{-1} \otimes v^1) = 1 \quad ; \quad \varepsilon_1(v^1 \otimes v^1) = \varepsilon_1(v^{-1} \otimes v^{-1}) = 0.$$

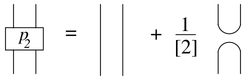


Fig. 9. The Jones–Wenzl projector $p_2 \in \mathrm{End}(V_1^{\otimes 2})$.

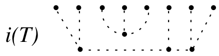


Fig. 10. The embedding $i(T)$ for the Riordan partition $\{\{1, 2, 6, 7, 8\}; \{3, 4, 5\}\}$.

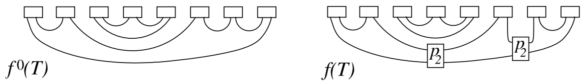


Fig. 11. The FK diagrams for $f^0(T)$ and $f(T)$, for the Riordan partition $\{\{1, 2, 6, 7, 8\}; \{3, 4, 5\}\}$.

Let T be a Riordan tree of order n. We now define two elements $f^0(T)$ and $f(T)$ of $\mathrm{Inv}_{U_q}(V_2^{\otimes n})$ using the graphical calculus introduced in the previous section. Consider a proper embedding $i(T)$ of T in the lower-half plane, such that the j-labeled vertex is sent to the point $(j; 0)$ and such that the cyclic ordering at each trivalent vertex agrees with the orientation of the plane. An example is given in Fig. 10. Note that the Riordan property ensures that such an embedding exists.

We first describe the diagram defining $f^0(T) \in \mathrm{Inv}_{U_q}(V_2^{\otimes n})$. First, replace each point $(j; 0)$ by a box representing a copy of V_2 $(1 \leq j \leq n)$. Next, consider an annular neighborhood of $i(T)$ in the lower-half plane; the boundary of this neighborhood is a collection of disjoint arcs connecting the n boxes, thus providing an FK diagram for $f^0(T)$. See Fig. 11. Note that we have the following reformulation for the dual canonical basis of Frenkel–Khovanov:

$$\mathfrak{B}_n^0 := \{f^0(T);\ f \text{ is a Riordan tree of order } n\}.$$

Now, to obtain the diagram defining $f(T)$ we simply insert a copy of the Jones–Wenzl projector p_2 in the pairs of arcs of $f^0(T)$ induced by each internal edge of T in the above procedure – see the example of Fig. 11.

We have

Theorem 4.3. *The set*

$$\mathfrak{B}_n^{JW} := \{f(T);\ f \textit{ is a Riordan tree of order } n\}$$

forms a basis for $\mathrm{Inv}_{U_q}(V_2^{\otimes n})$.

Proof. Since there is a natural one-to-one correspondence between the set $\mathfrak{B}_n^{JW}$ and the basis $\mathfrak{B}_n^0$, it is enough to prove the independency of the elements in $\mathfrak{B}_n^{JW}$.

So suppose that

$$\sum_{T \in \mathbf{Rio}_n} \alpha_T f(T) = 0,$$

where the sum runs over the set $\mathbf{Rio}_n$ of Riordan trees of order n, and where $\alpha_T \in \mathbb{C}$. Using the formula for the Jones–Wenzl projector p_2 given by Fig. 9, one can express each $f(T)$ as a linear combination

$$f(T) = f^0(T) + \sum_{T' \subset T} \frac{1}{[2]^{i_T - i_{T'}}} f^0(T'),$$

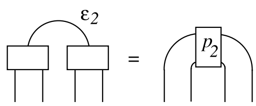


Fig. 12. Graphical definition of the contraction map ε_2.

where the sum runs over all subtrees T' obtained from T by deleting internal edges, and where i_t denotes the number of internal edges of a Riordan tree $t \in \mathbf{Rio}_n$. By substituting this identity in $\sum_{T\in\mathbf{Rio}_n} \alpha_T f(T) = 0$, we have that there exists complex numbers $\alpha'_T \in \mathbb{C}$ such that $\sum_{f\in\mathbf{Rio}_n} \alpha'_T f^0(T) = 0$, and a lower triangular matrix A whose diagonal entries are all 1 such that $(\alpha'_{T_1}, \dots, \alpha'_{T_l})^t = A(\alpha_{T_1}, \dots, \alpha_{T_l})^t$ for a suitably chosen order $\{T_1, \dots, T_l\}$ on $\mathbf{Rio}_n$. Since $\mathfrak{B}^0_n$ is a basis of $\mathrm{Inv}_{U_q}(V_2^{\otimes n})$, we have $(\alpha'_{f_1}, \dots, \alpha'_{f_l}) = 0$, which implies that $\alpha_T = 0$ for all $T \in \mathbf{Rio}_n$. This concludes the proof. □

It turns out that this simple modification of the dual canonical basis of $\mathrm{Inv}_{U_q}(V_2^{\otimes n})$ is directly related to the tree basis introduced in Section 3.1, as we now explain.

Let ρ: $\mathrm{Inv}_{U_q}(V_2^{\otimes n}) \to \mathrm{Inv}(sl_2^{\otimes n})$ be the $\mathbb{C}$-linear map such that

$$\rho(q) = 1, \quad \rho(v^0) = \frac{1}{2}h, \quad \rho(v^2) = -e, \quad \rho(v^{-2}) = f.$$

Proposition 4.4. *Let T be a Riordan tree. If $\deg(T)$ and $\mathrm{tri}(T)$ denote the degree and number of trivalent vertices of T respectively, then we have*

$$\rho\left(f(T)\right) = \frac{(-1)^{\deg(T)}}{2^{\mathrm{tri}(T)}} W(T).$$

It follows immediately that the tree basis of Section 3.1 indeed is a basis for $\mathrm{Inv}(sl_2^{\otimes n})$, as claimed in Theorem 3.2.

Proof. The assertion follows essentially from the definitions. To see this, let us slightly reformulate the definition of $f(T)$, still in terms of FK diagrams but in a spirit that is closer to that of $W(T)$. For each strut component of $i(T)$, pick a copy of the diagram D_c of Fig. 8, and take a copy of the diagram D_b for each trivalent vertex so that a copy of V_2 is associated to each of the incident half-edges following the cyclic ordering. For each internal edge of $i(T)$, we contract the two corresponding copies of V_2 by the map ε_2: $V_2 \otimes V_2 \to \mathbb{C}$ defined by

$$\varepsilon_2(v^2 \otimes v^{-2}) = q^2, \quad \varepsilon_2(v^0 \otimes v^0) = -\frac{1}{q^{-1} + q^{-3}}, \tag{14}$$

$$\varepsilon_2(v^{-2} \otimes v^2) = 1, \quad \varepsilon_2(v^i \otimes v^j) = 0, \quad \text{if} \quad i + j \neq 0. \tag{15}$$

As observed in [3], we have the identity

$$\varepsilon_2 \circ (\pi_2 \otimes \pi_2) = \varepsilon_1 \circ (1 \otimes \varepsilon_1 \otimes 1) \circ (p_2 \otimes p_2).$$

This formula, as illustrated in Fig. 12 above, simply means that the map ε_2 is the insertion of a copy of p_2 at each internal edge. (Recall that p_2 is a projector, i.e. $p_2 \circ p_2 = p_2$.)

So applying ε_2 in this way yields precisely the FK diagram for $f(T)$, where the box corresponding to the i-labeled vertex represents the ith copy of V_2. This is illustrated on an example in Fig. 13.

Now, it remains to observe that the elements $\tilde{c}$ and $\tilde{b}$, defined in Example 4.2 and represented by the diagrams D_c and D_b respectively, correspond to the elements c and b of Equations (2) and (3) via the map

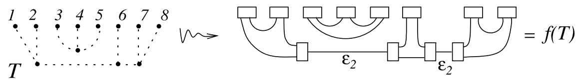


Fig. 13. Reformulating $f(T)$, for the Riordan partition $\{\{1,2,6,7,8\};\{3,4,5\}\}$.

ρ as follows:

$$\rho(\tilde{c}) = -c \tag{16}$$

and

$$\rho(\tilde{b}) = \frac{1}{2}b, \tag{17}$$

and that the contraction maps κ and ε_2, used in the definitions of $W(T)$ and $f(T)$ respectively, are related by

$$(\varepsilon_2)_{|q=1} = -\kappa \circ \rho. \tag{18}$$

Notice in particular that the $\frac{1}{2^{\mathrm{tri}(T)}}$ coefficient in the statement comes from the application of (17) at each trivalent vertex, while the sign $(-1)^{\deg(T)}$ is given by applying (16) at each strut component (which has degree 1), and (18) at each internal edge (since the degree of a linear tree is the number of internal edges plus 2). This concludes the proof. □

Acknowledgements

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