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ON C_n-MOVES FOR LINKS

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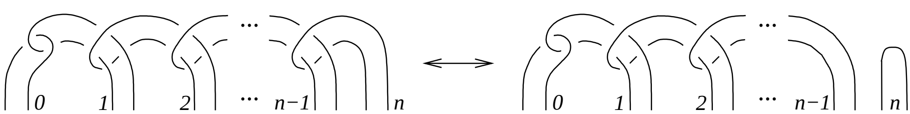
Dedicated to the memory of Xiao-Song Lin

A C_n-move is a local move on links defined by Habiro and Goussarov, which can be regarded as a 'higher order crossing change'. We use Milnor invariants with repeating indices to provide several classification results for links up to C_n-moves, under certain restrictions. Namely, we give a classification up to C_4-moves of 2-component links, 3-component Brunnian links and n-component C_3-trivial links. We also classify n-component link-homotopically trivial Brunnian links up to C_{n+1}-moves.

1. Introduction

A C_n-move is a local move on links as illustrated below. It involves $n+1$ strands, labeled here by integers between 0 and n, and can be regarded as a kind of 'higher order crossing change' (in particular, a C_1-move is a crossing change). These local moves were introduced by Habiro [1994] and independently by Goussarov [2000].



The C_n-move generates an equivalence relation on links, called C_n*-equivalence*. This notion can also be defined by using the theory of claspers (see Section 2). The C_n-equivalence relation becomes finer as n increases, that is, C_m-equivalence implies C_k-equivalence for $m > k$. It is well known that C_n-equivalence approximates the topological information carried by Goussarov–Vassiliev invariants. Namely, two links cannot be distinguished by any Goussarov–Vassiliev invariant of order less than n if they are C_n-equivalent. See [Gusarov 2000; Habiro 2000].

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Denote by $\mathcal{L}_k(n)$ the set of C_k*-trivial* n-component links, that is, links that are C_k-equivalent to the trivial link. We have a filtration

$$\mathcal{L}_1(n) \supset \mathcal{L}_2(n) \supset \mathcal{L}_3(n) \supset \cdots .$$

The quotient $\mathcal{L}_k(n)/C_{k+1}$ forms an abelian group under a certain geometric operation, with $\mathcal{L}_{k+1}(n)$ as unit element [Taniyama and Yasuhara 2003]. Note that $\mathcal{L}_1(n)$ is just the set of n-component links, and $\mathcal{L}_2(n)$ is the set of n-component algebraically split links [Murakami and Nakanishi 1989]. So the classifications of $\mathcal{L}_1(n)/C_2$ and $\mathcal{L}_2(n)/C_3$ are given by [Murakami and Nakanishi 1989] and [Taniyama and Yasuhara 2002], respectively. These classifications give us that the abelian group $\mathcal{L}_1(n)/C_2$ is free with rank $n(n-1)/2$, and $\mathcal{L}_2(n)/C_3$ is isomorphic to a direct sum of $n+n(n-1)(n-2)/6$ copies of $\mathbb{Z}$ and $n(n-1)/2$ copies of $\mathbb{Z}_2$. These classifications are given by using Milnor $\bar{\mu}$ invariants (of length ≤ 3) with distinct indices and the Conway polynomial. (For the definition of Milnor invariants, see Section 3.) In this paper, we use Milnor $\bar{\mu}$ invariants with (possibly) repeating indices to classify $\mathcal{L}_3(n)/C_4$. We obtain the following.

Theorem 1.1. *Let L and L' be n-component C_3-trivial links. Then L and L' are C_4-equivalent if and only if they satisfy the properties that*

(1) $\bar{\mu}_L(I) = \bar{\mu}_{L'}(I)$ *for all multiindices I with $|I| = 4$, and*

(2) *no Vassiliev knot invariant of order* 3 *can distinguish the i-th component of L from the i-th component of L', for all $1 \leq i \leq n$.*

Here, a multi-index I is a sequence of not necessarily distinct integers in $\{1, \ldots, n\}$, and $|I|$ denotes the number of entries in I.

Remark 1.2. The proof of Theorem 1.1 shows the following. The classification is given by $\mu(I)$ with $I = iijj$ for $1 \leq i < j \leq n$, $ijkk$ for $1 \leq i < j \leq n$ for $1 \leq k \leq n$, $ijkl$ for $1 \leq i \neq j < k < l \leq n$ and an order 3 Vassiliev invariant of each component. The abelian group $\mathcal{L}_3(n)/C_4$ is thus free with rank $n(n-1)/2 + n(n-1)(n-2)/2 + n(n-1)(n-2)(n-3)/12 + n$, which is the number of these invariants. Since these invariants are additive under the band sum, $\mathcal{L}_3(n)/C_4$ forms an abelian group under the band sum.

Note that Theorem 1.1, together with [Murakami and Nakanishi 1989] and [Taniyama and Yasuhara 2002], implies the following.

Corollary 1.3. *An n-component link L is C_4-trivial if and only if $\bar{\mu}_L(I) = 0$ for all multiindices I with $|I| \leq 4$, and any Vassiliev knot invariant of order ≤ 3 vanishes for each component.*

For 2-component links, we obtain a refinement of a result of H. A. Miyazawa [2003, Theorem 1.5].

Proposition 1.4. *Let L and L' be 2-component links. Then L and L' are C_4-equivalent if and only if they are not distinguished by any Vassiliev invariant of order ≤ 3.*

Remark 1.5. Two knots are C_k-equivalent if and only if they are not distinguished by any Vassiliev invariant of order $\leq k-1$ [Gusarov 2000; Habiro 2000]. For $k = 2$, this equivalence is true for all links [Murakami and Nakanishi 1989]. In general, as we mentioned before, the 'only if' part holds for links. But the 'if' part does not hold in general, even for 2-component links. For example, the Whitehead link, whose Vassiliev invariants of order ≤ 2 vanish, is not C_3-trivial. Hence, for 2-component links, the 'if' part holds when $k = 2$ and does not hold when $k = 3$. Proposition 1.4 means that, unexpectedly, it holds when $k = 4$.

On the other hand, we consider Brunnian links. Recall that a link L in the 3-sphere S^3 is *Brunnian* if every proper sublink of L is trivial. In particular, all trivial links are Brunnian. It is known that an n-component link is Brunnian if and only if it can be turned into the trivial link by a sequence of C_{n-1}-moves of a specific type, called C^a_{n-1}*-moves*, involving *all* the components [Habiro 2007; Miyazawa and Yasuhara 2006]. Denote by $BL(n)$ the set of n-component Brunnian links, and by $B_k(n)$ the set of n-component C_k-trivial Brunnian links. We have a descending filtration

$$BL(n) = B_{n-1}(n) \supset B_n(n) \supset B_{n+1}(n) \supset \cdots .$$

As in the case of arbitrary links, the quotient $B_k(n)/C_{k+1}$ forms an abelian group with the unit element $B_{k+1}(n)$ [Taniyama and Yasuhara 2003]. The abelian group $BL(n)/C_n$ is well understood and coincides with the abelian group of n-component Brunnian links up to link-homotopy [Habiro 2007; Miyazawa and Yasuhara 2006]. Recall that two links are *link-homotopic* if they are related by a sequence of isotopies and self-crossing changes, that is, crossing changes involving two strands of the same component. Habiro and Meilhan [2008] showed that n-component Brunnian links are link-homotopic if and only if their Milnor invariants

$$\bar{\mu}(\sigma(1), \ldots, \sigma(n-2), n-1, n)$$

coincide for all σ in the symmetric group S_{n-2}.

Here, we consider the next stage, namely the quotient $B_n(n)/C_{n+1}$. Given any $k \in \{1, \ldots, n\}$ and a bijection τ from $\{1, \ldots, n-1\}$ to $\{1, \ldots, n\} \setminus \{k\}$, set

$$\mu_\tau(L) := \bar{\mu}_L(\tau(1), \ldots, \tau(n-1), k, k).$$

We obtain the following.

Theorem 1.6. *Let $n \geq 3$. Let L and L' be n-component link-homotopically trivial Brunnian links. Then, the following assertions are equivalent*:

(1) *L and L' are C_{n+1}-equivalent.*

(2) $\bar{\mu}_L(I) = \bar{\mu}_{L'}(I)$ *for any multi-index I with $|I| = n+1$.*

(3) $\mu_\tau(L) = \mu_\tau(L')$ *for all $k \in \{1, \dots, n\}$ and $\tau \in \mathcal{B}(k)$, where $\mathcal{B}(k)$ denotes the set of all bijections τ from $\{1, \dots, n-1\}$ to $\{1, \dots, n\} \setminus \{k\}$ such that $\tau(1) < \tau(n-1)$.*

Remark 1.7. The abelian group $BL(n)/C_n$ is free with rank $|S_{n-1}|$; see [Habiro and Meilhan 2008]. In the proof of Theorem 1.6, it is shown that the abelian group $B_n(n)/C_{n+1}$ is free with rank $\left|\bigcup_{k=1}^{n} \mathcal{B}_k(k)\right|$. As in case of $\mathcal{L}_3(n)/C_4$, the quotient $B_n(n)/C_{n+1}$ forms an abelian group under the band sum.

Remark 1.8. Theorem 1.6 is not true for $n = 2$. The Whitehead link, for example, is not C_3-trivial (by [Taniyama and Yasuhara 2002]), but all its Milnor invariants $\bar{\mu}(I)$ with $|I| \le 3$ vanish. So the condition $n \ge 3$ is essential.

In the case of 3-component Brunnian links, we have the following improvement of Theorem 1.6.

Theorem 1.9. *Let L and L' be 3-component Brunnian links. Then the following assertions are equivalent*:

(1) *L and L' are C_4-equivalent.*

(2) $\bar{\mu}_L(I) = \bar{\mu}_{L'}(I)$ *for any multi-index I with $|I| \le 4$.*

(3) $\bar{\mu}_L(123) = \bar{\mu}_{L'}(123)$, $\quad \bar{\mu}_L(1233) = \bar{\mu}_{L'}(1233)$,
$\bar{\mu}_L(1322) = \bar{\mu}_{L'}(1322)$, $\quad \bar{\mu}_L(2311) = \bar{\mu}_{L'}(2311)$.

Note that $\bar{\mu}_L(ijkk)$ denotes here the *residue class* of the integer $\mu_L(ijkk)$ (defined in Section 3) modulo $\bar{\mu}_L(ijk)$.

Remark 1.10. One may wonder if the equivalence of (1) and (2) remains true for Brunnian links with $m \ne 3$ components. First, observe that all m-component Brunnian links are C_4-equivalent (namely, C_4-trivial) for $m > 4$ [Habiro 2007; Miyazawa and Yasuhara 2006]. For $m = 4$ the answer is positive and follows from [Habiro and Meilhan 2008] and [Habiro 2000, Theorem 7.2] (as the C_4-equivalence coincides here with link-homotopy). The case $m = 2$ seems to be still open.

Remark 1.11. Similarly, one may ask, for 3-component Brunnian links L and L' and $k \ne 4$, whether C_k-equivalence of L and L' is equivalent to the condition $\bar{\mu}_L(I) = \bar{\mu}_{L'}(I)$ for any $|I| \le k$. As we already saw, the case $k \ne 2$ is vacuous and the case $k = 3$ holds true. But this is not true in general for $k > 4$. Consider for example the Whitehead double L of the Borromean rings (see [Fleming and Yasuhara 2008, Figure 4] for a diagram of L). We have $\bar{\mu}_L(I) = 0$ for all $|I| \le 5$. However, L is not C_5-trivial. Indeed, L is distinguished from the trivial link by the fourth derivative of the Jones polynomial evaluated at 1, which is a C_5-equivalence invariant.

The rest of the paper is organized as follows. In Section 2, we recall elementary notions of the theory of claspers. In Section 3, we recall the definition of Milnor invariants for (string) links and give some lemmas. Section 4 considers Brunnian string links; its main result is Proposition 4.5, which gives a set of generators for the abelian group of C_{n+1}-equivalence classes of n-component Brunnian string links. In Section 5, we use results of Section 4 to prove Theorems 1.6 and 1.9. In Section 6, we prove Theorem 1.1 and Proposition 1.4. In Section 7 we give proofs of Propositions 2.12 and 2.14; these proofs are independent from the rest of the paper.

2. Claspers and local moves on links

A brief review of clasper theory. Let us briefly recall from [Habiro 2000] the basic notions of clasper theory for (string) links. In this paper, we essentially only need the notion of C_k-tree. See [Habiro 2000] for a general definition of claspers.

Definition 2.1. Let L be a link in S^3. An embedded disk F in S^3 is called a *tree clasper* for L if it satisfies these three properties:

(1) F is decomposed into disks and bands, called *edges*, each of which connects two distinct disks.

(2) The disks have either 1 or 3 incident edges, called *leaves* or *nodes*, respectively.

(3) L intersects F transversely, and the intersections are contained in the union of the interior of the leaves.

The *degree* of a tree clasper is one less than the number of leaves.

A degree k tree clasper is called a C_k-tree. A C_k-tree is *simple* if each leaf intersects L at one point.

We will make use of the drawing convention for claspers of [Habiro 2000, Figure 7], with the exception that a $\oplus$ (respectively $\ominus$) on an edge represents a positive (respectively negative) half-twist. (This replaces the convention of a circled S (respectively S^{-1}) used in [Habiro 2000].)

Given a C_k-tree G for a link L in S^3, there is a procedure to construct, in a regular neighborhood of G, a framed link $\gamma(G)$. There is thus a notion of *surgery along* G, which is defined as surgery along $\gamma(G)$. There exists a canonical diffeomorphism between S^3 and the manifold $S^3_{\gamma(G)}$: surgery along the C_k-tree G can thus be regarded as a local move on L in S^3. We say that the resulting link L_G in S^3 is obtained by surgery on L along G. In particular, surgery along a simple C_k-tree, as illustrated in Figure 2.2, is equivalent to band-summing a copy of the $(k+1)$-component Milnor's link L_{k+1} (see [Milnor 1954, Figure 7]), and is equivalent to a C_k-move as defined on page 119. In Figure 2.2, a C_k-tree G having the shape

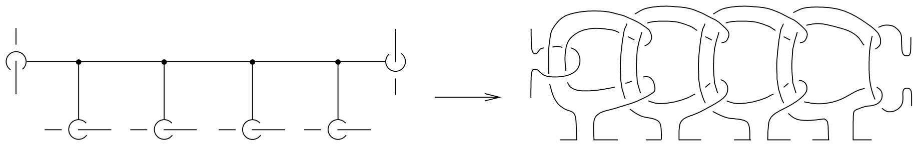

Figure 2.2. Surgery along a simple C_5-tree.

of the tree clasper is called *linear*, and the leftmost and rightmost leaves of G are called the *ends* of G.

The C_k-equivalence (as defined in the introduction) coincides with the equivalence relation on links generated by surgery along C_k-trees and isotopies. We use the notation $L \sim_{C_k} L'$ for C_k-equivalent links L and L'.

Some lemmas. This subsection gives some basic results of calculus of claspers, whose proofs can be found in [Habiro 2000] or [Meilhan 2003]. For convenience, we give the statements for string links. Recall that a string link is a pure tangle without closed components (see [Habegger and Lin 1990] for a precise definition). Denote by $SL(n)$ the set of n-component string links up to isotopy with respect to the boundary. The set $SL(n)$ has a monoid structure with composition given by the *stacking product*, denoted by $\cdot$, and with the trivial n-component string link $\mathbf{1}_n$ as unit element.

Lemma 2.3. *Let T be a union of C_k-trees for a string link L, and let T' be obtained from T by passing an edge across L or across another edge of T, or by sliding a leaf over a leaf of another component of T (see Figure 2.4). Then $L_T \sim_{C_{k+1}} L_{T'}$.*

Lemma 2.5. *Let T be a C_k-tree for $\mathbf{1}_n$, and let $\overline{T}$ be a C_k-tree obtained from T by adding a half-twist on an edge. Then $(\mathbf{1}_n)_T \cdot (\mathbf{1}_n)_{\overline{T}} \sim_{C_{k+1}} \mathbf{1}_n$.*

Lemma 2.6. *Consider some C_k-trees T and T' (respectively T_I, T_H and T_X) for $\mathbf{1}_n$ that differ only in a small ball as depicted in Figure 2.7. Then $(\mathbf{1}_n)_T \cdot (\mathbf{1}_n)_{T'} \sim_{C_{k+1}} \mathbf{1}_n$ (respectively $(\mathbf{1}_n)_{T_I} \sim_{C_{k+1}} (\mathbf{1}_n)_{T_H} \cdot (\mathbf{1}_n)_{T_X}$).*

Lemma 2.8. *Let G be a C_k-tree for $\mathbf{1}_n$. Let f_1 and f_2 be two disks obtained by splitting a leaf f of G along an arc α as shown in Figure 2.9 (that is, $f = f_1 \cup f_2$*

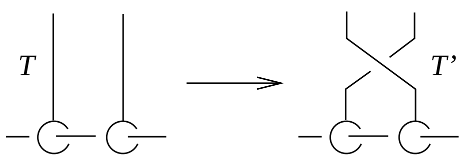

Figure 2.4. Sliding a leaf over another leaf.

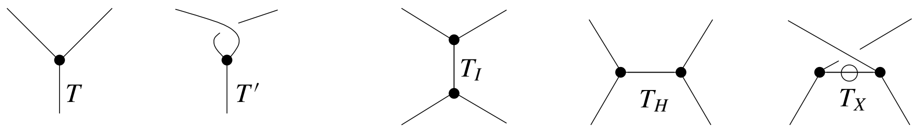

Figure 2.7. The AS and IHX relations for C_k-trees.

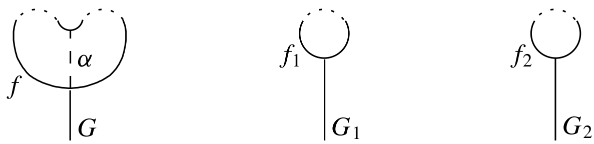

Figure 2.9. The C_k-trees G, G_1 and G_2 are identical outside a small ball, where they are as depicted.

and $f_1 \cap f_2 = \alpha$). Then, $(\mathbf{1}_n)_G \sim_{C_{k+1}} (\mathbf{1}_n)_{G_1} \cdot (\mathbf{1}_n)_{G_2}$, where G_i denotes the C_k-tree for $\mathbf{1}_n$ obtained from G by replacing f by f_i for $i = 1, 2$.

C_k^a-trees and C_k^a-equivalence.

Definition 2.10. Let L be an m-component link in a 3-manifold M. For $k \geq m-1$, a (simple) C_k-tree T for L in M is a (*simple*) *C_k^a-tree* if it satisfies the following:

(1) $f \cap L$ is contained in a single component of L for each leaf f of T.

(2) T intersects *all* the components of L.

The *C_k^a-equivalence* is an equivalence relation on links generated by surgeries along C_k^a-trees and isotopies. The next result shows the relevance of this notion in the study of Brunnian (string) links.

Theorem 2.11 [Habiro 2007; Miyazawa and Yasuhara 2006]. *Suppose L is an n-component link in S^3. Then L is Brunnian if and only if it is C_{n-1}^a-equivalent to the n-component trivial link.*

Further, it is known from [Miyazawa and Yasuhara 2006] that for n-component Brunnian links, C_n-equivalence coincides with C_n^a-equivalence (and with link-homotopy). See also [Habiro and Meilhan 2008]. We observe the following.

Proposition 2.12. *Let $k \geq n-1$. An n-component Brunnian (string) link is C_k-trivial if and only if it is C_k^a-equivalent to the trivial (string) link.*

Remark 2.13. It seems that Proposition 2.12 can be generalized: for $k \geq n-1$, n-component Brunnian (string) links are C_k-equivalent if and only if they are C_k^a-equivalent. The string link case holds (see the proposition below), but the link case is still open.

Proposition 2.14. *Let $k \geq n-1$. Then two n-component Brunnian string links are C_k-equivalent if and only if they are C_k^a-equivalent.*

We prove Propositions 2.12 and 2.14 in Section 7.

3. On Milnor invariants

A short definition. J. Milnor [1954] defined a family of invariants of oriented, ordered links in S^3, known as Milnor's $\bar{\mu}$-invariants.

Given an n-component link L in S^3, denote by π the fundamental group of $S^3 \setminus L$, and by π_q the q-th subgroup of the lower central series of π. We have a presentation of π/π_q with n generators, given by a meridian m_i of the i-th component of L. So for $1 \leq i \leq n$, the longitude l_i of the i-th component of L is expressed modulo π_q as a word in the m_i. (Abusing notation, we still denote this word by l_i.)

The *Magnus expansion* $E(l_i)$ of l_i is the formal power series in noncommuting variables $X_1, \dots, X_n$ obtained by replacing m_j by $1+X_j$ and replacing m_j^{-1} by $1-X_j+X_j^2-X_j^3+\cdots$ for $1 \leq j \leq n$. We use the notation $E_k(l_i)$ to denote the degree k part of $E(l_i)$, where the degree of a monomial in the X_j is simply defined by the sum of the powers.

Let $I = i_1 i_2 \dots i_{k-1} j$ be a multi-index (that is, a sequence of possibly repeating indices) among $\{1, \dots, n\}$. Denote by $\mu_L(I)$ the coefficient of $X_{i_1} \dots X_{i_{k-1}}$ in the Magnus expansion $E(l_j)$. The *Milnor invariant* $\bar{\mu}_L(I)$ is the residue class of $\mu_L(I)$ modulo the greatest common divisor of all Milnor invariants $\mu_L(J)$ such that J is obtained from I by removing at least one index and permuting the remaining indices cyclically. We call $|I| = k$ the *length* of Milnor invariant $\bar{\mu}_L(I)$.

The indeterminacy comes from the choice of the meridians m_i. Equivalently, it comes from the indeterminacy of representing the link as the closure of a string link [Habegger and Lin 1990]. Indeed, $\mu(I)$ is a well-defined invariant for string links. Furthermore, $\mu(I)$ is known to be a Goussarov–Vassiliev invariant of degree $|I|-1$ for string links [Bar-Natan 1995; Lin 1997].

Some lemmas. Let us first recall a result due to Habiro.

Lemma 3.1 [Habiro 2000]. *Milnor invariants of length k for* (*string*) *links are invariants of C_k-equivalence.*

Next we state a simple lemma, which will be used in the following.

Lemma 3.2. *Let L be an n-component string link obtained from $\mathbf{1}_n$ by surgery along a union F of C_k-trees that is disjoint from the j-th component of $\mathbf{1}_n$. Then $\mu_L(I) = 0$ for all multiindices I containing j and satisfying $|I| \leq k+1$.*

Proof. Consider a diagram of $\mathbf{1}_n$ together with F. The diagram contains several crossings between an edge of F and the j-th component of $\mathbf{1}_n$. Denote by F_o

(respectively F_u) the union of C_k-trees obtained from F by performing crossing changes so that the j-th component of $\mathbf{1}_n$ overpasses (respectively underpasses) all edges. By Lemma 2.3, we have $L \sim_{C_{k+1}} U_{F_o} \sim_{C_{k+1}} U_{F_u}$. The result then follows from Lemma 3.1 and the following observation.

Consider the diagram D of a string link K. If the i-th component of K overpasses all the other components in D, it follows from the definition of Milnor invariants that $\mu_K(I) = 0$ for any multi-index I with last index i. Similarly, if the i-th component of K underpasses all the other components in D, then $\mu_K(I) = 0$ for any multi-index I containing i and with last index not equal to i. □

We have the following simple additivity property.

Lemma 3.3. *Let L and L' be n-component string links such that all Milnor invariants of L (respectively L') of length $\leq m$ (respectively $\leq m'$) vanish. Then $\mu_{L\cdot L'}(I) = \mu_L(I) + \mu_{L'}(I)$ for all I of length $\leq m + m'$.*

Proof. The Milnor invariant of $L \cdot L'$ is computed by taking the Magnus expansion of the k-th longitude L_k of $L \cdot L'$. Denote respectively by l_i and m_i (respectively l'_i and m'_i) the i-th meridian and longitude of L (respectively L'), where $1 \leq i \leq n$. We have $L_k = l_k \cdot \tilde{l}'_k$, where $\tilde{l}'_k$ is obtained from l'_k by replacing m'_i with $M_i = l_i^{-1} m_i l_i$ for each i. So $E(L_k) = E(l_k) \cdot E(\tilde{l}'_k)$, where $E(\tilde{l}'_k)$ is obtained from $E(l'_k)$ by substituting $\widetilde{X}_i$ for X_i in $E(l'_k)$, where $\widetilde{X}_i := E(M_i) - 1$.

The Magnus expansion of l_i is the form $E(l_i) = 1 + (\text{terms of degree} \geq m)$, so

$$\begin{aligned} E(M_i) &= E(l_i^{-1}) E(m_i) E(l_i) \\ &= E(l_i^{-1}) E(l_i) + E(l_i^{-1}) X_i E(l_i) \\ &= 1 + X_i + (\text{terms of degree} \geq m + 1). \end{aligned}$$

So $E(\tilde{l}'_k)$ is obtained from $E(l'_k) = 1 + \sum_{j \geq m'} E_j(l'_k)$ by replacing each X_i by $X_i + (\text{terms of degree} \geq m + 1)$ for all i. It follows that

$$E(\tilde{l}'_k) = 1 + \sum_{m+m'-1 \geq j \geq m'} E_j(l'_k) + (\text{terms of degree} \geq (m + m')).$$

It follows that $E(L_k) = E(l_k) E(\tilde{l}'_k)$ has the form

$$1 + \sum_{m+m'-1 \geq j \geq m} E_j(l_k) + \sum_{m+m'-1 \geq j \geq m'} E_j(l'_k) + (\text{terms of degree} \geq (m + m')),$$

which implies that all Milnor invariants of length $\leq m + m'$ of $L \cdot L'$ are additive. □

4. C_{n+1}-moves for n-component Brunnian string links

An n-component string link L is Brunnian if every proper substring link of L is the trivial string link. In particular, any trivial string link is Brunnian. The set of n-component Brunnian string links form a submonoid of $SL(n)$, denoted by $BSL(n)$.

Recall that, given $L \in SL(n)$, the *closure* $\mathrm{cl}(L)$ of L is an n-component link in S^3 [Habegger and Lin 1990]. By [Habiro 2007], an n-component link is Brunnian if and only if it is the closure of a certain Brunnian string link.

n-component Brunnian string links up to C_n-equivalence. Let $BSL(n)/C_n$ denote the abelian group of C_n-equivalence classes of n-component Brunnian string links. Habiro and Meilhan [2008] gave a basis for $BSL(n)/C_n$ as follows.

Let σ be an element in the symmetric group S_{n-2}. Let L_σ be the n-component string link obtained from $\mathbf{1}_n$ by surgery along the C^a_{n-1}-tree T_σ shown in Figure 4.1. Likewise, denote by $(L_\sigma)^{-1}$ the n-component string link obtained from the C^a_{n-1}-tree $\overline{T}_\sigma$, which is obtained from T_σ by adding a positive half-twist in the edge e (see Figure 4.1).

Let $\mu_\sigma(L)$ denote the Milnor invariant $\mu_L(\sigma(1), \dots, \sigma(n-2), n-1, n)$ for any element $\sigma \in S_{n-2}$.

Proposition 4.2 [Habiro and Meilhan 2008]. *Let L be an n-component Brunnian string link. Then*

$$L \sim_{C_n} \prod_{\sigma \in S_{n-2}} (L_\sigma)^{\mu_\sigma(L)}.$$

Remark 4.3. Recall from [Habiro and Meilhan 2008; Miyazawa and Yasuhara 2006] that C_n-equivalence, link-homotopy, and C^a_n-equivalence all coincide on $BSL(n)$.

n-component Brunnian string links up to C_{n+1}-equivalence. In this section, we study the quotient $BSL(n)/C_{n+1}$. Note that $BSL(n)/C_{n+1}$ is a finitely generated abelian group (this is shown by using the same arguments as in the proof of [Habiro 2000, Lemma 5.5]).

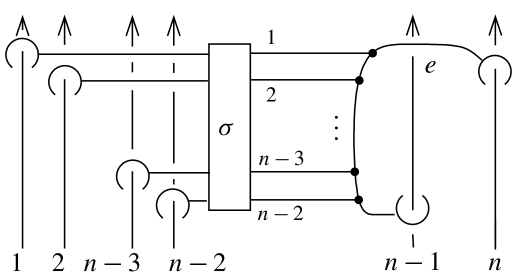

Figure 4.1. The simple C^a_n-tree T_σ. Here, the numbering of the edges just indicates how $\sigma \in S_{n-1}$ acts on the edges of T_σ (a similar notation is used in Figure 4.4).

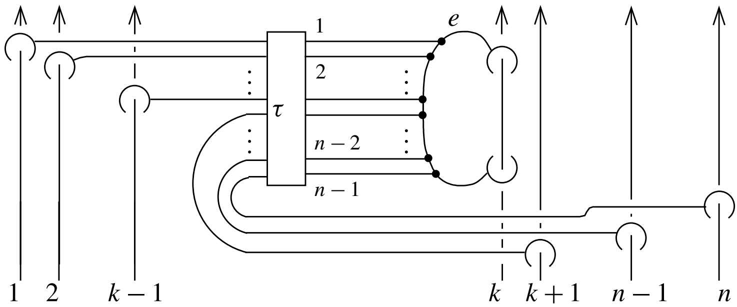

Figure 4.4. The simple C_n^a-tree G_τ.

For $k \in \{1, \dots, n\}$, consider a bijection τ from $\{1, \dots, n-1\}$ to $\{1, \dots, n\} \setminus \{k\}$. Denote by V_τ the n-component string link obtained from $\mathbf{1}_n$ by surgery along the C_n^a-tree G_τ shown in Figure 4.4. Denote by $\overline{G}_\tau$ the C_n^a-tree for $\mathbf{1}_n$ obtained from G_τ by adding a positive half-twist in the edge e (see Figure 4.1). Let $(V_\tau)^{-1}$ be the n-component string link obtained from $\mathbf{1}_n$ by surgery along $\overline{G}_\tau$.

Set $\mu_\tau(L) := \mu_L(\tau(1), \dots, \tau(n-1), k, k)$. Denote by $\mathcal{B}(k)$ the set of all bijections τ from $\{1, \dots, n-1\}$ to $\{1, \dots, n\} \setminus \{k\}$ such that $\tau(1) < \tau(n-1)$, and denote by ρ a bijection from $\{1, \dots, n-1\}$ to itself defined by $\rho(i) = n - i$. We have the following.

Proposition 4.5. *Let L be an n-component Brunnian string link. Then*

$$L \sim_{C_{n+1}} \Big(\prod_{\sigma \in S_{n-2}} (L_\sigma)^{\mu_\sigma(L)} \Big) \cdot L_1 \cdots L_n, \tag{4-1}$$

where, for each k in $1 \le k \le n$, the factor L_k is the n-component Brunnian string link

$$\prod_{\tau \in \mathcal{B}(k)} (V_\tau)^{n_\tau(L)} \cdot (V_{\tau\rho})^{n'_\tau(L)}$$

such that, for any $\tau \in \mathcal{B}(k)$ for $k = 1, \dots, n$, the exponents $n_\tau(L)$ and $n'_\tau(L)$ are two integers satisfying

$$n_\tau(L) + (-1)^{n-1} n'_\tau(L) = \mu_\tau(L_1 \cdots L_n). \tag{4-2}$$

Proof. By Proposition 4.2 and Remark 4.3, L is obtained from the n-component string link

$$L_0 := \prod_{\sigma \in S_{n-2}} (L_\sigma)^{\mu_\sigma(L)}$$

by surgery along a disjoint union F of simple C_n^a-trees. By Lemma 2.3, we have $L \sim_{C_{n+1}} L_0 \cdot (\mathbf{1}_n)_{G_1} \cdot \dots \cdot (\mathbf{1}_n)_{G_p}$, where G_j for $1 \le j \le p$ are simple C_n^a-trees for $\mathbf{1}_n$. Denote by k_j the (unique) element of $\{1, \dots, n\}$ such that G_j intersects twice the

k_j-th component of $\mathbf{1}_n$ for $1 \le j \le p$. We can use the AS and IHX relations for tree claspers to replace, up to C_{n+1}-equivalence, each of these C_n^a-trees with a union of linear C_n^a-trees whose ends intersect the k_j-th component. More precisely, by Lemmas 2.6, 2.5 and 2.3 we have for each $1 \le j \le p$ that

$$(\mathbf{1}_n)_{G_j} \sim_{C_{n+1}} \prod_{i=1}^{m_j} (V_{\nu_{ij}})^{\varepsilon_{ij}},$$

where $\varepsilon_{ij} \in \mathbf{Z}$ and where ν_{ij} is a bijection from $\{1, \dots, n-1\}$ to $\{1, \dots, n\} \setminus \{k_j\}$. Since there exists, for each such ν_{ij}, a unique element τ of $\mathcal{B}(k_j)$ such that ν_{ij} is equal to either τ or $\tau\rho$, it follows that L is C_{n+1}-equivalent to an n-component string link of the form given in (4-1). It remains to prove (4-2).

First, let us compute $\mu_\tau(V_\eta)$ for all $\tau \in \mathcal{B}(k)$ and $\eta \in \mathcal{B}(l)$, where $k, l = 1, \dots, n$. By [Milnor 1957, Theorem 7], we have $\mu_\tau(V_\eta) = \mu_{\tau,n+1}(W_\eta)$, where $\mu_{\tau,n+1}$ is Milnor invariant $\mu(\tau(1), \dots, \tau(n-1), k, n+1)$ and where W_η denotes the $(n+1)$-component string link obtained from V_η by taking, as the $(n+1)$-st component, a parallel copy of the k-th component (so that the k-th and the $(n+1)$-st components of W_η have linking number zero). Now recall that $V_\eta \cong (\mathbf{1}_n)_{G_\eta}$, where G_η is a C_n^a-tree as shown in Figure 4.4. So $W_\eta \cong (\mathbf{1}_{n+1})_{\widetilde{G}_\eta}$, where $\widetilde{G}_\eta$ is a C_n^a-tree obtained from G_η by replacing each leaf intersecting the k-th component of $\mathbf{1}_n$ with a leaf intersecting components k and $n+1$, as depicted in Figures 4.6 and 4.7.

If $k \neq l$, then $\widetilde{G}_\eta$ contains exactly one leaf f intersecting both the k-th and the $(n+1)$-st components of $\mathbf{1}_{n+1}$. By Lemma 2.8, we have

$$(\mathbf{1}_{n+1})_{\widetilde{G}_\eta} \sim_{C_{n+1}} (\mathbf{1}_{n+1})_{G_\eta^1} \cdot (\mathbf{1}_{n+1})_{G_\eta^2},$$

where G_η^i denotes the simple C_n-tree for $\mathbf{1}_{n+1}$ obtained from $\widetilde{G}_\eta$ by replacing f by f_i for $i = 1, 2$ as shown in Figure 4.6. By Lemmas 3.1 and 3.3, $\mu_\tau(V_\eta)$ is thus equal to $\mu_{\tau,n+1}((\mathbf{1}_{n+1})_{G_\eta^1}) + \mu_{\tau,n+1}((\mathbf{1}_{n+1})_{G_\eta^2})$. It follows from Lemma 3.2 that $\mu_\tau(V_\eta) = 0$.

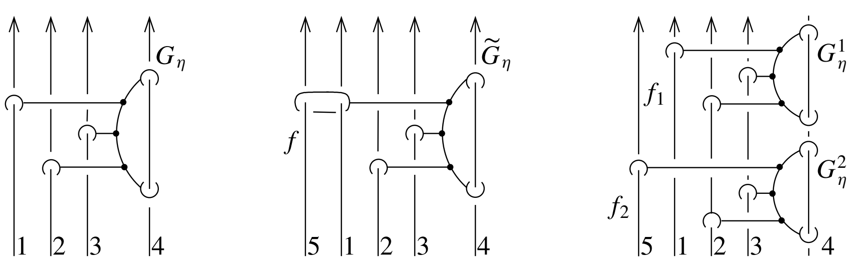

Figure 4.6. Here and subsequently we fix, for simplicity, $n = 4$, $k = 1$, and $l = 4$. We let η be the permutation $(23) \in S_3$.

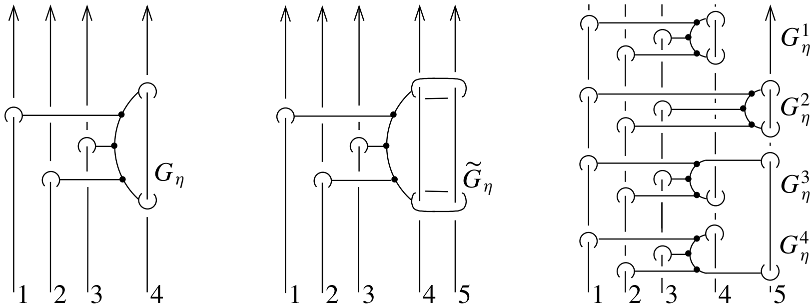

Figure 4.7

Now suppose that $k = l$. Then $\widetilde{G}_\eta$ contains two leaves intersecting both the k-th and the $(n+1)$-st components of $\mathbf{1}_{n+1}$. By Lemma 2.8, we obtain

$$(\mathbf{1}_{n+1})_{\widetilde{G}_\eta} \sim_{C_{n+1}} (\mathbf{1}_{n+1})_{G^1_\eta} \cdot (\mathbf{1}_{n+1})_{G^2_\eta} \cdot (\mathbf{1}_{n+1})_{G^3_\eta} \cdot (\mathbf{1}_{n+1})_{G^4_\eta},$$

where, for $1 \le i \le 4$, G^i_η is a simple C_n-tree for $\mathbf{1}_{n+1}$ as depicted in Figure 4.7. By Lemmas 3.1, 3.2 and 3.3, it follows that

$$\mu_\tau(V_\eta) = \mu_{\tau,n+1}((\mathbf{1}_{n+1})_{G^3_\eta}) + \mu_{\tau,n+1}((\mathbf{1}_{n+1})_{G^4_\eta}).$$

Observe that the closure of each of these two string links is a copy of Milnor's link [Milnor 1954, Figure 7]. By a formula of Milnor [1954, page 190], we obtain $\mu_{\tau,n+1}((\mathbf{1}_{n+1})_{G^3_\eta}) = \delta_{\tau,\eta}$ and $\mu_{\tau,n+1}((\mathbf{1}_{n+1})_{G^4_\eta}) = 0$, where δ denotes Kronecker's symbol. So we obtain that $\mu_\tau(V_\eta) = \delta_{\tau,\eta}$. Moreover, it follows from Lemmas 3.3 and 2.5 that $\mu_\tau((V_\eta)^{-1}) = -\delta_{\tau,\eta}$.

Now consider the string link $V_{\eta\rho}$. By the same arguments as above, we have $\mu_\tau(V_{\eta\rho}) = \mu_\tau((V_{\eta\rho})^{-1}) = 0$ if $k \ne l$. If $k = l$, it follows from the same arguments as above that

$$\mu_\tau(V_{\eta\rho}) = \mu_{\tau,n+1}((\mathbf{1}_{n+1})_{G^1_{\eta\rho}}) + \mu_{\tau,n+1}((\mathbf{1}_{n+1})_{G^2_{\eta\rho}}),$$

where $G^1_{\eta\rho}$ and $G^2_{\eta\rho}$ are two simple C^a_n-trees for $\mathbf{1}_{n+1}$ as depicted in Figure 4.8. By Lemma 2.3 and isotopy, $(\mathbf{1}_{n+1})_{G^i_{\eta\rho}}$ is C_{k+1}-equivalent to $(\mathbf{1}_{n+1})_{T^i_\eta}$, where T^i_η is as shown in Figure 4.8 for $i = 1, 2$. By Lemma 2.5, we thus obtain

$$\mu_\tau(V_{\eta\rho}) = (-1)^{n-1}\delta_{\tau,\eta}.$$

We conclude that

$$\mu_\tau(L_1 \cdots L_n) = \sum_{1\le i\le n} \mu_\tau(L_i) = n_\tau(L) + (-1)^{n-1} n'_\tau(L). \qquad \square$$

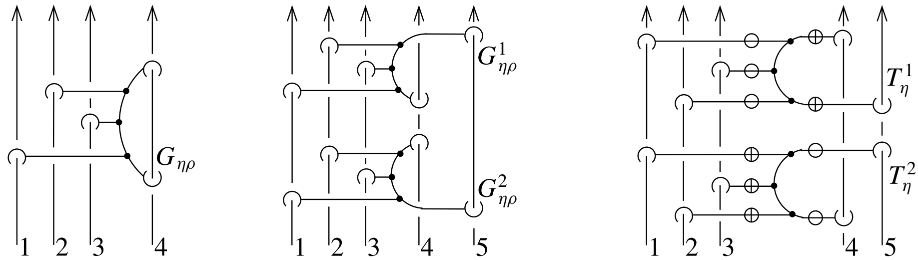

Figure 4.8

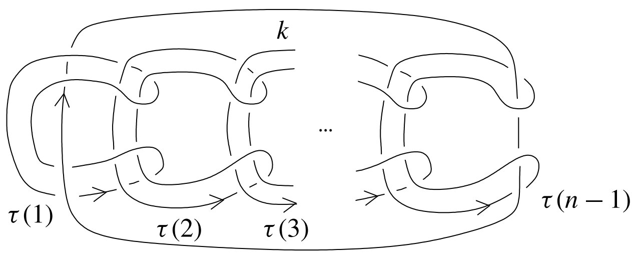

Figure 4.10. The link B_τ.

Remark 4.9. Observe that we obtain the following as a byproduct of the proof of Proposition 4.5. Consider the n-component Brunnian link B_τ represented in Figure 4.10, for some $\tau \in \mathcal{B}(k)$. B_τ is the closure of the n-component string link V_τ considered above. We showed that, for $1 \le l \le n$ and $\eta \in \mathcal{B}(l)$,

$$\bar{\mu}_\eta(B_\tau) = \mu_\eta(B_\tau) = \delta_{\eta,\tau}.$$

We conclude this section by showing that the string links V_τ and $V_{\tau\rho}$ are linearly independent in $BSL(n)/C_{n+1}$.

Proposition 4.11. *For any integer k in $\{1, \dots, n\}$ with $n \ge 3$ and any $\tau \in \mathcal{B}(k)$, we have $V_\tau \not\sim_{C_{n+1}} V_{\tau\rho}$ and $V_\tau \not\sim_{C_{n+1}} (V_{\tau\rho})^{-1}$.*

Remark 4.12. In contrast to the lemma above, we will see while proving Proposition 5.1 that either $\mathrm{cl}(V_\tau) \sim_{C_{n+1}} \mathrm{cl}(V_{\tau\rho})$ or $\mathrm{cl}(V_\tau) \sim_{C_{n+1}} \mathrm{cl}((V_{\tau\rho})^{-1})$.

Proof. Consider a diagram of an n-component string link L. The string link L lives in a copy of $D^2 \times I$ standardly embedded in S^3. The *origin* (respectively *terminal*) of the i-th component of L is the starting point (respectively ending point) of the component, according to the orientation of L. We can construct a knot $K_\tau(L)$ in S^3 as follows.

Connect the terminals of the k-th and the $\tau(1)$-st components by an arc a_1 in $S^3 \setminus (D^2 \times I)$. Next, connect the origins of the $\tau(1)$-st and the $\tau(2)$-nd components

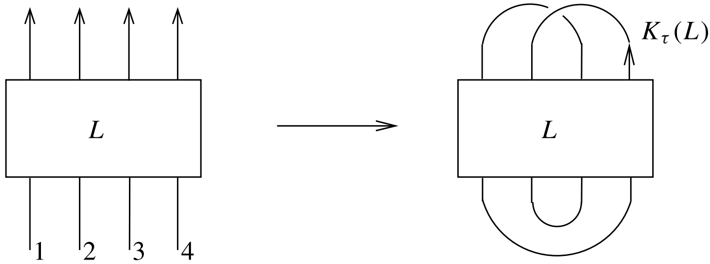

Figure 4.13. The knot $K_\tau(L)$.

by an arc a_2 in $S^3 \setminus (D^2 \times I)$ disjoint from a_1, then the terminals of the $\tau(2)$-nd and the $\tau(3)$-rd components by an arc a_3 in $S^3 \setminus (D^2 \times I)$ disjoint from $a_1 \cup a_2$. Repeat this construction until reaching the last component, the $\tau(n-1)$-st component, and connect the terminal or the origin (depending on whether n is even or odd) to the origin of the k-th component by an arc a_n in $S^3 \setminus (D^2 \times I)$ disjoint from $\bigcup_{1 \le i \le n-1} a_i$. The arcs are chosen so that, if a_i and a_j (with $i < j$) meet in the diagram of L, then a_i overpasses a_j. The orientation of K_τ is the one induced from the k-th component. An example is given in Figure 4.13 for the case $n = 4$, $k = 4$ and $\tau = (231) \in S_3$.

It follows immediately from the above construction and [Horiuchi 2007, Theorem 1.4] that

$$P_0^{(n)}(K_\tau(V_\tau); 1) = \pm n!\, 2^n \quad \text{and} \quad P_0^{(n)}(K_\tau(V_{\tau\rho}); 1) = P_0^{(n)}(K_\tau((V_{\tau\rho})^{-1}); 1) = 0,$$

where $P_l^{(k)}(K; 1)$ denotes the k-th derivative of the coefficient polynomial $P_k(K; t)$ of z^k in the HOMFLY polynomial $P(K; t, z)$ of a link K, evaluated in 1. The result then follows from [Habiro 2000, Corollary 6.8] and the fact that $P_0^{(n)}(K; 1)$ is a Goussarov–Vassiliev invariant of degree $\le n$ [Kanenobu and Miyazawa 1998]. □

5. C_{n+1}-moves for n-component Brunnian links

In this section, we prove Theorems 1.6 and 1.9. Let us begin with stating the following link version of Proposition 4.5.

Proposition 5.1. *Let L be an n-component Brunnian link. Then*

$$L \sim_{C_{n+1}} \mathrm{cl}\Big(\prod_{\sigma \in S_{n-2}} (L_\sigma)^{\mu_\sigma(L)} \cdot \prod_{1 \le k \le n} L'_k \Big),$$

where, for each k with $1 \le k \le n$,

$$L'_k := \prod_{\tau \in \mathcal{B}(k)} (V_\tau)^{\mu_\tau(L'_1 \cdot \cdots \cdot L'_n)}.$$

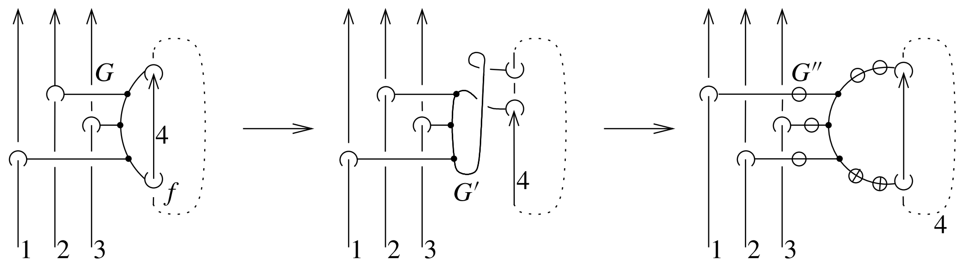

Figure 5.2

Proof. By Proposition 4.5, L is C_{n+1}-equivalent to the closure of the string link

$$l = \prod_{\sigma\in S_{n-2}} ((\mathbf{1}_n)_{T_\sigma})^{\mu_\sigma(L)} \cdot \prod_{1\le k\le n} \prod_{\tau\in\mathcal{B}(k)} ((\mathbf{1}_n)_{G_\tau})^{n_\tau(L)} \cdot ((\mathbf{1}_n)_{G_{\tau\rho}})^{n'_\tau(L)}, \tag{5-1}$$

where $n_\tau(L)$ and $n'_\tau(L)$ are two integers satisfying (4-2). Denote by F the union of all the tree claspers involved in (5-1), that is, $l = (\mathbf{1}_n)_F$.

For some $k \in \{1, \dots, n\}$ and $\tau \in \mathcal{B}(k)$, let G be a copy of the simple C_n-tree $G_{\tau\rho}$ in F. Let f be a leaf of G that intersects the k-th component of $\mathbf{1}_n$ (see Figure 5.2). When we close the k-th component of $\mathbf{1}_n$, we can slide f over leaves of the components of $F\setminus G$ until we obtain the C_n-tree G' of Figure 5.2. Denote by F' the union of tree claspers obtained from F by this operation. By Lemma 2.3, we have $\mathrm{cl}((\mathbf{1}_n)_F) \sim_{C_{n+1}} \mathrm{cl}((\mathbf{1}_n)_{F'})$. By Lemma 2.3 and isotopy, $(\mathbf{1}_n)_{G'}$ is C_{n+1}-equivalent to $(\mathbf{1}_n)_{G''}$, where G'' is the C_n-tree depicted in Figure 5.2. G'' differs from a copy of G_τ by $(n+3)$ half-twists on its edges. It thus follows from Lemma 2.5 that

$$\mathrm{cl}((\mathbf{1}_n)_{G_\tau} \cdot (\mathbf{1}_n)_{G_{\tau\rho}}) \sim_{C_{n+1}} \begin{cases} \mathrm{cl}(\mathbf{1}_n) & \text{if } n \text{ is even,} \\ \mathrm{cl}(((\mathbf{1}_n)_{G_\tau})^2) & \text{if } n \text{ is odd.} \end{cases}$$

L is thus C_{n+1}-equivalent to the closure of the string link

$$\prod_{\sigma\in S_{n-2}} ((\mathbf{1}_n)_{T_\sigma})^{\mu_\sigma(L)} \cdot \prod_{1\le k\le n} \prod_{\tau\in\mathcal{B}(k)} ((\mathbf{1}_n)_{G_\tau})^{n_\tau(L)+(-1)^{n-1}n'_\tau(L)}.$$

The result follows from (4-2). □

The link-homotopically trivial links case: Proof of Theorem 1.6.

Proof of Theorem 1.6. That (1) implies (2) follows immediately from Lemma 3.1, and (2) implies (3) is clear. So it remains to show that (3) implies (1).

By Proposition 4.2, if an n-component Brunnian link B is link-homotopically trivial, then $\mu_\sigma(B) = 0$ for all $\sigma \in S_{n-2}$. For all $\tau \in \mathcal{B}(k)$ with $k = 1, \dots, n$, $\mu_\tau(B)$ is thus a well-defined integer, which satisfies $\mu_\tau(B) = \mu_\tau(L(B))$ for any

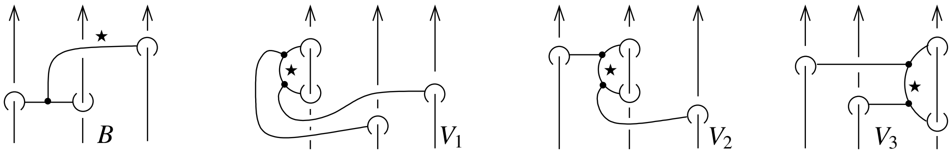

Figure 5.3. Here B^{-1} (respectively V_p^{-1} for $1 \le p \le 3$) is defined as obtained from B (respectively V_p for $1 \le p \le 3$) by a positive half-twist on the edge marked by a $\star$.

string link $L(B)$ whose closure is B. By Proposition 5.1, we have

$$B \sim_{C_{n+1}} \mathrm{cl}\Big(\prod_{1 \le k \le n} \prod_{\tau \in \mathcal{B}(k)} (V_\tau)^{\mu_\tau(B)} \Big).$$

The result follows immediately. □

5.1. *The 3-component links case: Proof of Theorem 1.9.*

Proof of Theorem 1.9. As in the proof of Theorem 1.6, we only have to show (3) implies (1). Let L be a 3-component Brunnian link. By Proposition 5.1, we have

$$L \sim_{C_4} \mathrm{cl}(L_0 \cdot L_1 \cdot L_2 \cdot L_3), \quad \text{with } L_p = \begin{cases} B^{\mu_L(123)} & \text{if } p = 0, \\ V_p^{n_p} & \text{if } p = 1, 2, 3, \end{cases} \tag{5-2}$$

where B and V_p for $p = 1, 2, 3$ are 3-component string links obtained from $\mathbf{1}_3$ by surgery along a C_2-tree and along C_3-trees, respectively, as shown in Figure 5.3, and where $n_k = \mu_{L_1 \cdot L_2 \cdot L_3}(ijkk)$ with $\{i, j, k\} = \{1, 2, 3\}$ and $i < j$. Note that $\mu_L(123) = \bar{\mu}_L(123)$ since L is Brunnian.

We now make an observation. Consider a union Y of u parallel copies of a simple C_2^a-tree for the 3-component trivial link $U = U_1 \cup U_2 \cup U_3$, and perform an isotopy as illustrated in Figure 5.4. Denote by Y' the resulting union of C_2-trees. Then by [Habiro 2000, Proposition 4.5], Y' can be deformed into Y by a sequence of u C_3-moves, corresponding to u parallel copies of a simple C_3-tree intersecting twice U_i and once U_j and U_k. So by Lemma 2.5, U_Y is C_4-equivalent to $\mathrm{cl}((\mathbf{1}_n)_Y \cdot (\mathbf{1}_n)_{V_i}^{\pm u})$. (Here, abusing notations, we still denote by Y a union of u simple C_2-trees for $\mathbf{1}_3$ such that $\mathrm{cl}((\mathbf{1}_3)_Y) \cong U_Y$.) Note that for any union F of C_3-trees, $U_{Y \cup F} \sim_{C_4} \mathrm{cl}((\mathbf{1}_n)_{Y \cup F} \cdot (\mathbf{1}_n)_{V_i}^{\pm u})$.

This observation implies that the n_p for $p = 1, 2, 3$ in (5-2) are changeable up to $|\mu_L(123)|$. So we can suppose that n_p for all $p = 1, 2, 3$ satisfies

$$0 \le n_p < |\mu_L(123)|. \tag{5-3}$$

Now by [Krushkal 1998] we have, for all $\{i, j, k\} = \{1, 2, 3\}$,

$$\mu_L(ijkk) \equiv \mu_{\mathrm{cl}(L_0)}(ijkk) + \mu_{\mathrm{cl}(L_1 \cdot L_2 \cdot L_3)}(ijkk) \mod \mu_L(123).$$

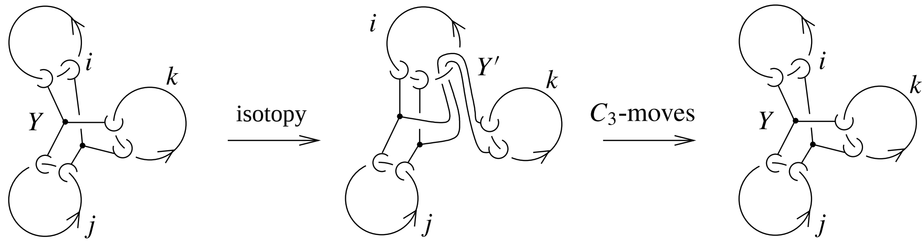

Figure 5.4

By Lemma 3.3, we have $\mu_{\mathrm{cl}(L_0)}(ijkk) \equiv 0 \bmod \mu_L(123)$ and

$$\mu_{\mathrm{cl}(L_1\cdot L_2\cdot L_3)}(ijkk) \equiv \sum_{1\le p\le 3} n_p \mu_{\mathrm{cl}(V_p)}(ijkk) \bmod \mu_L(123).$$

As seen in Remark 4.9, we have $\mu_{\mathrm{cl}(V_p)}(ijkk) = \delta_{p,k}$. It follows that

$$\mu_L(ijkk) \equiv n_k \bmod \mu_L(123). \tag{5-4}$$

Consider 3-component Brunnian links L and L' such that $\bar\mu_L(123) = \bar\mu_{L'}(123)$ and $\bar\mu_L(ijkk) = \bar\mu_{L'}(ijkk)$ for $(i, j, k) = (1, 2, 3)$, $(1, 3, 2)$ and $(2, 3, 1)$. It follows from (5-2), (5-4) and (5-3) that $L \sim_{C_4} L'$. This completes the proof. □

Minimal string link. Let L be an n-component Brunnian link in S^3. Denote by $\mathcal{L}(L)$ the set of all n-component string links l such that $\mathrm{cl}(l) = L$.

By Proposition 4.5, for each $l \in \mathcal{L}(L)$ there exists an $l' \in SL(n)$ such that l is C_{n+1}-equivalent to a string link of the form $\prod_{\sigma\in S_{n-2}} (L_\sigma)^{\mu_\sigma(l)} \cdot l'$.

Put any total order on the set $\mathcal{B} := \bigcup_{1\le k\le n} \mathcal{B}(k)$ and fix it. We denote by τ_i for $i = 1, \dots, m$ the elements of $\mathcal{B}$ according to this total order. For all $l \in \mathcal{L}(L)$, $\tau \in \mathcal{B}$, set $\alpha_\tau(l) := \mu_\tau(l')$. For each element $l \in \mathcal{L}(L)$, we can thus define a vector

$$v_l := (|\alpha_{\tau_1}(l)|, \dots, |\alpha_{\tau_k}(l)|, \dots, |\alpha_{\tau_m}(l)|, -\alpha_{\tau_1}(l), \dots, -\alpha_{\tau_k}(l), \dots, -\alpha_{\tau_m}(l)).$$

Set $\mathcal{V}_L = \{v_l \mid l \in \mathcal{L}(L)\}$. We have the following.

Proposition 5.5. *Two n-component Brunnian links L and L' are C_{n+1}-equivalent if and only if $\bar\mu_\sigma(L) = \bar\mu_\sigma(L')$ for all $\sigma \in S_{n-1}$ and* $\min \mathcal{V}_L = \min \mathcal{V}_{L'}$.

In Section 5.1, if we take $-|\mu_L(123)|/2 < n_k < (|\mu_L(123)| - 1)/2$ instead of inequality (5-3), then we have an explicit form of $\min \mathcal{V}_L$ for a 3-component Brunnian link L. In general, it is a problem to determine $\min \mathcal{V}_L$ from L.

6. C_4-equivalence for links

In this section we prove Theorem 1.1 and Proposition 1.4. The first subsection provides a lemma, which is the main new ingredient for the proofs of these results.

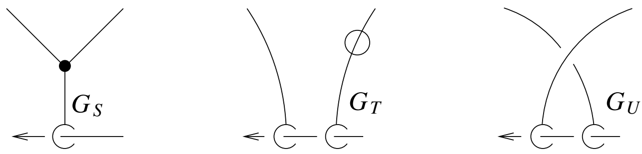

Figure 6.3. The STU relation for C_k-graphs.

6.1. *The index lemma.* Let T be a simple C_k-tree for an n-component link L. The *index* of T is the collection of all integers i such that T intersects the i-th component of L, counted with multiplicities. For example, a simple C_3-tree of index $\{2, 3^{(2)}, 5\}$ for L intersects component 3 twice and components 2 and 5 once (and is disjoint from all other components of L).

Lemma 6.1. *Suppose T is a simple C_k-tree with $k \geq 3$ of index $\{i, j^{(k)}\}$ for an n-component link L with $1 \leq i \neq j \leq n$. Then $L_T \sim_{C_{k+1}} L$.*

In order to prove this lemma, we need the notion of graph clasper introduced in [Habiro 2000, Section 8.2]. A *graph clasper* is defined as an embedded connected surface that is decomposed into leaves, nodes and bands as in Definition 2.1, but that is not necessarily a disk. A graph clasper may contain loops. The degree of a graph clasper G is defined as half of the number of nodes and leaves (which coincides with the usual degree if G is a tree clasper). We call a degree k graph clasper a C_k*-graph*. Two links related by surgery along a C_k-graph are C_k-equivalent; see [Habiro 2005]. A C_k-graph for a link L is *simple* if each of its leaves intersects L at one point.

Recall from [Habiro 2000, Section 8.2] that the STU relation holds for graph claspers.

Lemma 6.2. *Let G_S, G_T and G_U be three C_k-graphs for $\mathbf{1}_n$ that differ only in a small ball as depicted in Figure 6.3. Then $(\mathbf{1}_n)_{G_S} \sim_{C_{k+1}} (\mathbf{1}_n)_{G_T} \cdot (\mathbf{1}_n)_{G_U}$.*

It should be noted that, in contrast to the diagram case, this STU relation only holds among *connected* claspers. Note also that it differs by a sign from the STU relation for unitrivalent diagrams.

Lemma 6.4. *Let C be a simple C_k-graph for an n-component link L in S^3, which intersects a certain component of L exactly once. If C contains a loop (that is, if C is not a C_k-tree), then $L_C \sim_{C_{k+1}} L$.*

Proof. Suppose that C intersect the i-th component of L exactly once. By [Habiro 2000] and Lemma 2.3, there exists a union F of tree claspers for $\mathbf{1}_n$ and a simple C_k-tree G for $\mathbf{1}_n$ containing a loop and intersecting the i-th component once, such that $L \cong \mathrm{cl}((\mathbf{1}_n)_F)$ and $L_C \sim_{C_{k+1}} \mathrm{cl}((\mathbf{1}_n)_F \cdot (\mathbf{1}_n)_G)$.

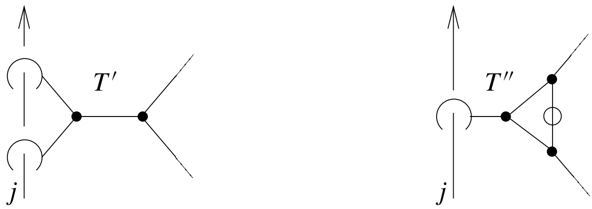

Figure 6.5

Consider the unique leaf f of G intersecting the i-th component. This leaf f is connected to a loop γ of G by a path P of edges and nodes. We proceed by induction on the number n of nodes in P.

If $n = 0$, that is, if f is connected to γ by a single edge, apply Lemma 6.2 at this edge. The result then follows from Lemmas 2.3 and 2.5 by arguments similar to those in the proof of Proposition 5.1.

For an arbitrary $n \geq 1$, apply the IHX relation at the edge of P incident to γ. By Lemma 2.6,[1] we obtain $(\mathbf{1}_n)_G \sim_{C_{k+1}} (\mathbf{1}_n)_{G'} \cdot (\mathbf{1}_n)_{G''}$, where G' and G'' are C_k-graphs, each of which has a unique leaf intersecting the i-th component connected to a loop by a path with $(n-1)$ nodes. By the induction hypothesis, we thus have $(\mathbf{1}_n)_{G'} \sim_{C_{k+1}} \mathbf{1}_n \sim_{C_{k+1}} (\mathbf{1}_n)_{G''}$. □

Proof of Lemma 6.1. Let T be a simple C_k-tree of index $\{i, j^{(k)}\}$ for an n-component link L with $1 \leq i \neq j \leq n$. By several applications of Lemmas 6.2, 6.4, 2.3 and 2.5, one can easily verify that $L_T \sim_{C_{k+1}} L_{T'}$, where T' is a simple C_k-tree of index $\{i, j^{(k)}\}$ for L that contains two leaves as depicted in Figure 6.5. By applying the IHX and STU relations, we have $L_{T'} \sim_{C_{k+1}} L_{T''}$, where T'' is a C_k-graph for L as illustrated in Figure 6.5. T'' clearly satisfies the hypothesis of Lemma 6.4. We thus have $L_T \sim_{C_{k+1}} L_{T''} \sim_{C_{k+1}} L$. □

Proof of Theorem 1.1. We only need to prove the 'if' part of the statement. Let L be a C_3-trivial n-component link. Consider an n-component string link l such that its closure is L and such that $l \sim_{C_3} \mathbf{1}_n$. By Lemmas 2.3, 2.5 and 2.6 and the arguments used in the proof of Proposition 5.1, we have

$$l \sim_{C_4} l_0 \cdot l_1 \cdot l_2 \cdot l_3 \cdot l_4,$$

where the l_i are defined as follows:

- $l_0 = \prod_i (\mathbf{1}_n)_{U_i}$, where U_i is union of simple C_3-trees of index $\{i^{(4)}\}$ contained in a regular neighborhood of the i-th component of $\mathbf{1}_n$, and $1 \leq i \leq n$.

[1] Strictly speaking, we cannot apply Lemma 2.6 here, as G is not a C_k-tree. However, similar relations hold among C_k-graphs [Habiro 2000, Section 8.2].

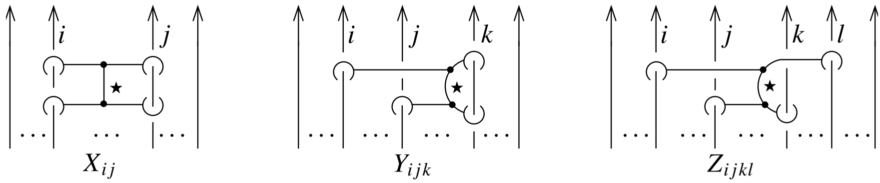

Figure 6.6. Here X_{ij}^{-1} (respectively Y_{ijk}^{-1}, Z_{ijkl}^{-1}) is defined as obtained from X_{ij} (respectively Y_{ijk}, Z_{ijkl}) by a positive half-twist on the edge marked by a $\star$.

- $l_1 = \prod_{i<j}((\mathbf{1}_n)_{X_{ij}})^{x_{ij}}$, where X_{ij} is the simple C_3-tree of index $\{i^{(2)}, j^{(2)}\}$ represented in Figure 6.6, and where $x_{ij} \in \mathbb{Z}$.
- $l_2 = \prod_{i<j;k}((\mathbf{1}_n)_{Y_{ijk}})^{y_{ijk}}$, where Y_{ijk} is the simple C_3-tree of index $\{i, j, k^{(2)}\}$ represented in Figure 6.6.
- $l_3 = \prod_{i\neq j<k<l}((\mathbf{1}_n)_{Z_{ijkl}})^{z_{ijkl}}$, where Z_{ijkl} is the simple C_3-tree whose index is $\{i, j, k, l\}$ and which is represented in Figure 6.6.
- l_4 is obtained from $\mathbf{1}_n$ by surgery along simple C_3-trees with index of the form $\{i, j^{(3)}\}$ for $1 \leq i \neq j \leq n$.

As an immediate consequence of Lemma 6.1, we thus have

$$L = \mathrm{cl}(l) \sim_{C_4} \mathrm{cl}(l_0 \cdot l_1 \cdot l_2 \cdot l_3).$$

It follows from standard computations (see preceding sections) that

$$\begin{aligned}
\bar{\mu}_L(iijj) &= \mu_{l_1}(iijj) = 2x_{ij} && \text{for all } 1 \leq i < j \leq n,\\
\bar{\mu}_L(ijkk) &= \mu_{l_2}(ijkk) = y_{ijk} && \text{for all } 1 \leq i < j \leq n \text{ and } 1 \leq k \leq n,\\
\bar{\mu}_L(ijkl) &= \mu_{l_3}(ijkl) = z_{ijkl} && \text{for all } 1 \leq i \neq j < k < l \leq n.
\end{aligned}$$

Now, consider another C_3-trivial n-component link L', such that L and L' satisfy assertions (1) and (2) of Theorem 1.1. By the same construction as above and Theorem 1.1(1), we have

$$L' \sim_{C_4} \mathrm{cl}(l'_0 \cdot l_1 \cdot l_2 \cdot l_3).$$

Here $l'_0 = \prod_i (\mathbf{1}_n)_{U'_i}$, where U'_i is union of simple C_3-trees of index $\{i^{(4)}\}$ contained in a regular neighborhood of the i-th component of $\mathbf{1}_n$ for $1 \leq i \leq n$. Denote respectively by $(l_0)_i$ and $(l'_0)_i$ the i-th components of l_0 and l'_0. By Theorem 1.1(2) and [Habiro 2000, Theorem 6.18], we have $(l_0)_i \sim_{C_4} (l'_0)_i$ for all i in $\{1, \ldots, n\}$. We thus have $l_0 \sim_{C_4} l'_0$, which implies the result. □

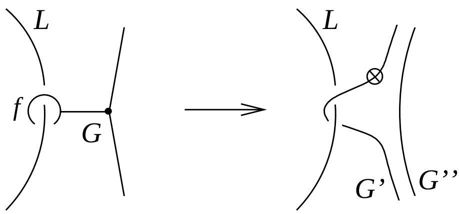

Figure 7.2

Proof of Proposition 1.4. It suffices to show that 2-component links L and L' that are not distinguished by Vassiliev invariants of order ≤ 3 are C_4-equivalent (the converse is well known).

By [Miyazawa 2003, Theorem 1.5], L' can be obtained from L by a sequence of surgeries along

(1) C_4-trees and

(2) simple C_3-trees with index $\{i, j^{(3)}\}$, $\{i, j\} = \{1, 2\}$.

By Lemma 6.1, each surgery of type (2) can be achieved by surgery along C_4-trees. It follows that $L \sim_{C_4} L'$. □

7. C_k and C_k^a-triviality for Brunnian links

In this section we prove Propositions 2.12 and 2.14. We will need the following 'C_k^a-version' of [Habiro 2000, Proposition 3.7].

Lemma 7.1. *If $n - 1 \leq k \leq l$, the C_l^a-equivalence implies the C_k^a-equivalence for n-component* (*string*) *links.*

Proof. It suffices to show the case $l = k + 1$. Let G be a C_{k+1}^a-tree for an n-component (string) link L. By [Habiro 2007, Lemma 6], we may assume that G is simple. There exists $j \in \{1, \ldots, n\}$ such that at least two leaves of G intersect the j-th component of L. Denote by f one of these leaves, and consider the node of G connected to f by an edge (see Figure 7.2). By applying [Meilhan 2006, Lemma 2.4] at this node, followed by [Habiro 2000, Proposition 2.7] and a zip construction, G is equivalent to the union $G' \cup G''$ of two C_k^a-trees as represented in Figure 7.2, where G'' lives in a regular neighborhood of G' (here, we use the zip construction from the point of view of [Conant and Teichner 2004]). This proves $L_G \sim_{C_k^a} L$.

Note that similar arguments appear in the proof of [Fleming and Yasuhara 2008, Proposition 3.1]. □

Proof of Proposition 2.12. First, observe that it suffices to show the result for links. For string links, the lemma can be shown by similar arguments.

Denote by $O_n = U_1 \cup \cdots \cup U_n$ the n-component trivial link. The 'if' part of the statement is obvious. Here we consider a link L that is C_k-equivalent to O_n, and we prove that $L \sim_{C_k^a} O_n$.

For any tree clasper T for O_n, set

$$\mathrm{D}(T) := \{i \in \{1, \ldots, n\} \mid T \cap U_i \neq \varnothing\}.$$

Note that $D(T)$ differs from the index of T introduced in Section 6.1 (here we consider elements of $\{1, \ldots, n\}$ without multiplicity). By assumption, $L \cong (O_n)_G$, where $G = G_1 \cup \cdots \cup G_p$ is a union of simple tree claspers of degree $\geq k$. Set

$$\mathrm{D}(G) := \bigcap_{i=1}^{p} \mathrm{D}(G_i).$$

Consider $j \in \{1, \ldots, n\} \setminus \mathrm{D}(G)$. Denote by $G(j)$ the union of all tree claspers of G that are disjoint from U_j. As L is Brunnian, we have $(O_n \setminus U_j)_{G(j)} \cong O_{n-1}$. By a sequence of crossing changes between edges of $G(j)$ and U_j, we can move U_j into the exterior of a 3-ball containing $(O_n \setminus U_j) \cup G(j)$. By the proof of [Habiro 2000, Proposition 4.5], each such crossing change is realized by surgery along one $C_{\deg(G_i)+1}$-tree T such that $\mathrm{D}(T) = \mathrm{D}(G_i) \cup \{j\}$, where $G_i \subset G(j)$ contains the edge involved in the crossing change. So there exists a union $F(j)$ of tree claspers $T_1 \cup \cdots \cup T_m$ of degree $> k$ with $\mathrm{D}(F(j)) \supset \mathrm{D}(G(j)) \cup \{j\}$ such that $L \cong (O_{n-1} \sqcup U_j)_{G \cup F(j)}$, where $\sqcup$ denotes the split union. So we have

$$L \cong ((O_{n-1})_{G(j)} \sqcup U_j)_{(G \setminus G(j)) \cup F(j)} \cong (O_{n-1} \sqcup U_j)_{(G \setminus G(j)) \cup F(j)}.$$

Set $G' := (G \setminus G(j)) \cup F(j)$. We have $L \cong (O_n)_{G'}$, and clearly $\mathrm{D}(G') \supset \mathrm{D}(G) \cup \{j\}$.

So by repeating this procedure, we obtain a union G'' of tree claspers for O_n such that $L \cong (O_n)_{G''}$. This union satisfies $\mathrm{D}(G'') = \{1, \ldots, n\}$, that is, each component of G'' is a C_p^a-tree for some $p \geq k$. The result then follows from Lemma 7.1. □

Proof of Proposition 2.14. Consider n-component Brunnian string links L and L' such that $L \sim_{C_k} L'$ for some $k \geq n-1$. Then $L \cong (\mathbf{1}_n)_{F \cup G}$, where F is a union of C_{n-1}^a-trees such that $(\mathbf{1}_n)_F \cong L'$, and G is a union of tree claspers of degree $\geq k$. Let $F' \cup G'$ be obtained from $F \cup G$ by passing an edge of G across an edge of F or sliding a leaf of G over a leaf of F (see Figure 2.4). By examining the proofs of [Habiro 2000, Propositions 4.6 and 4.4], one easily sees that $(\mathbf{1}_n)_{F \cup G} \sim_{C_p^a} (\mathbf{1}_n)_{F' \cup G'}$ for $p \geq n+k-1$. So by Lemma 7.1 we obtain

$$L \sim_{C_k^a} (\mathbf{1}_n)_F \cdot (\mathbf{1}_n)_G,$$

where G is a union of tree clasper of degree $\geq k$. Since L is Brunnian, $(\mathbf{1}_n)_F \cdot (\mathbf{1}_n)_G$ is also Brunnian. This and the fact that F is a union of C_{n-1}^a-trees imply that $(\mathbf{1}_n)_G \cong L''$ is Brunnian. Now, $(\mathbf{1}_n)_F \cong L'$, and $(\mathbf{1}_n)_G \cong L''$ is a Brunnian string link that is C_k-equivalent to the trivial string link. So by Proposition 2.12, $L'' \sim_{C_k^a} \mathbf{1}_n$. It follows that $L \sim_{C_k^a} L'$. □

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