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FINITE TYPE INVARIANTS AND MILNOR INVARIANTS FOR BRUNNIAN LINKS

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A link L in the 3-sphere is called *Brunnian* if every proper sublink of L is trivial. In a previous paper, Habiro proved that the restriction to Brunnian links of any Goussarov–Vassiliev finite type invariant of $(n+1)$-component links of degree $< 2n$ is trivial. The purpose of this paper is to study the first nontrivial case. We show that the restriction of an invariant of degree $2n$ to $(n+1)$-component Brunnian links can be expressed as a quadratic form on the Milnor link-homotopy invariants of length $n+1$.

Keywords: Brunnian links; Goussarov–Vassiliev finite type invariants; Milnor link-homotopy invariants; claspers.

Mathematics Subject Classification 2000: 57M25, 57M27

1. Introduction

The notion of *Goussarov–Vassiliev finite type link invariants* [7, 8, 28] enables us to understand the various quantum invariants from a unifying viewpoint, see e.g. [1, 26]. The theory involves a descending filtration

$$\mathbb{Z}\mathcal{L}(m) = J_0(m) \supset J_1(m) \supset \cdots$$

of the free abelian group $\mathbb{Z}\mathcal{L}(m)$ generated by the set $\mathcal{L}(m)$ of the ambient isotopy classes of m-component, oriented, ordered links in a fixed 3-mainfold M. Here each $J_n(m)$ is generated by alternating sums of links over n independent crossing changes. A homomorphism from $\mathbb{Z}\mathcal{L}(m)$ to an abelian group A is said to be a Goussarov–Vassiliev invariant of degree n if it vanishes on $J_{n+1}(m)$. Thus, for $L, L' \in \mathcal{L}(m)$, we have $L - L' \in J_{n+1}(m)$ if and only if L and L' have the same values of Goussarov–Vassiliev invariants of degree $\leq n$ with values in any abelian group.

It is natural to ask what kind of informations a Goussarov–Vassiliev link invariants can contain and what the topological meaning of the unitrivalent diagrams

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is. *Calculus of claspers*, introduced by Goussarov and the first author [9, 10, 15], answers these questions. (We will recall the definition of claspers in Sec. 2.) A special type of claspers, called *graph claspers*, can be regarded as topological realizations of unitrivalent diagrams. For *knots*, claspers enables us to give a complete topological characterization of the informations that can be contained by Goussarov–Vassiliev invariants of degree $< n$ [10, 15]: the difference of two knots is in J_n if and only if these two knots are C_n*-equivalent*. Here C_n-equivalence is generated by a certain type of local moves, called C_n*-moves* (called $(n-1)$*-variations* by Goussarov), which is defined as surgeries along certain tree claspers.

For links with more than 1 components, the above-mentioned properties of Goussarov–Vassiliev invariants does not hold. It is true that if $L, L' \in \mathcal{L}(m)$ are C_n-equivalent, then we have $L - L' \in J_n(m)$, but the converse does not hold in general. A counterexample is Milnor's link L_{n+1} of $n+1$ components depicted in Fig. 1: if $n \geq 2$, L_n is (C_n-equivalent but) not C_{n+1}-equivalent to the $(n+1)$-component unlink U, while we have $L_{n+1} - U \in J_{2n}(n+1)$ (but $L_{n+1} - U \notin J_{2n+1}(n+1)$), see [15, Proposition 7.4]. (This fact is contrasting to the case of *string links*: conjecturally [15, Conjecture 6.13], two string links L, L' of the same number of components are C_n-equivalent if and only if $L - L' \in J_n$.)

Milnor's links are typical examples of *Brunnian links*. Recall that a link in a connected 3-manifold is said to be Brunnian if every proper sublink of it is an unlink. In some sense, an n-component Brunnian link is a "pure n-component linking". Thus studying the behavior of Goussarov–Vassiliev invariants on Brunnian links would be a first step in understanding the Goussarov–Vassiliev invariants for links.

The first author generalized a part of the above-mentioned properties of Milnor's links to Brunnian links:

Theorem 1.1 ([16]). *Let L be an $(n+1)$-component Brunnian link in a connected, oriented 3-manifold M $(n \geq 1)$, and let U be an $(n+1)$-component unlink in M. Then we have the following.*

(1) *L and U are C_n-equivalent.*
(2) *If $n \geq 2$, then we have $L - U \in J_{2n}(n+1)$. Hence L and U are not distinguished by any Goussarov–Vassiliev invariants of degree $< 2n$.*

The case $M = S^3$ of Theorem 1.1 was announced in [15], and was later proved also by Miyazawa and Yasuhara [24], independently to [16].

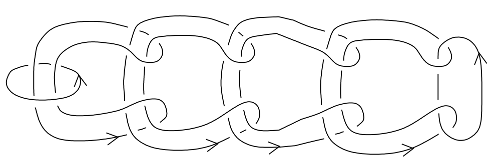

Fig. 1. Milnor's link L_6 of 6 components.

The purpose of the present paper is to study the restrictions of Goussarov–Vassiliev invariants of degree $2n$ to $(n+1)$-component Brunnian links in S^3, which is the first nontrivial case according to Theorem 1.1. The main result in the present paper expresses any such restriction as a *quadratic* form of Milnor $\bar{\mu}$ link-homotopy invariants of length $n+1$:

Theorem 1.2. *Let f be any Goussarov–Vassiliev link invariant of degree $2n$ valued in an abelian group A. Then there are (non-unique) elements $f_{\sigma,\sigma'} \in A$ for σ, σ' in the symmetric group S_{n-1} on the set $\{1, \ldots, n-1\}$ such that, for any $(n+1)$-component Brunnian link L, we have*

$$f(L) - f(U) = \sum_{\sigma,\sigma' \in S_{n-1}} f_{\sigma,\sigma'} \bar{\mu}_{\sigma}(L) \bar{\mu}_{\sigma'}(L). \tag{1.1}$$

Here, U is an $(n+1)$-component unlink, and we set, for $\sigma \in S_{n-1}$,

$$\bar{\mu}_{\sigma}(L) = \bar{\mu}_{\sigma(1),\sigma(2),\ldots,\sigma(n-1),n,n+1}(L) \in \mathbb{Z}.$$

A possible choice for the $f_{\sigma,\sigma'}$ is given in terms of tree claspers in Sec. 7.3.

Remark 1.3. The proof of Theorem 1.2 involves calculus of claspers. The first preprint version of the present paper (arxiv:math.GT/0510534v1) contained a one-page sketch of an alternative proof of Theorem 1.2 using the Kontsevich integral. This alternative proof has been separated from the present paper, and has been published in [17]. Though shorter than the clasper-based proof below, the proof in [17] relies heavily on the properties of unitrivalent diagrams, and the topological meaning of the steps in the proof are therefore not always very clear. The present proof gives a better understanding of Theorem 1.2 from that point of view. Also, the proof in [17] only works in the case of a $\mathbb{Z}$-valued Goussarov–Vassiliev invariant, whereas Theorem 1.2 is proved here for *any* abelian group.

Recall that Milnor invariants of length $n+1$ for string links are Goussarov–Vassiliev invariants of degree $\leq n$ [2, 19] (see also [14]). As is well-known, Milnor's invariants is not well-defined for all links, and hence it does not make sense to ask whether Milnor invariants of length $n+1$ are of degree $\leq n$ or not. However, as Theorem 1.2 indicates, a quadratic expression in such Milnor invariants, which is well-defined at least for $(n+1)$-component Brunnian links, may extend to a link invariant of degree $\leq 2n$.

In the study of Milnor's invariants, tree claspers seem at least as useful as Cochran's construction [3]. For the use of claspers in the study of the Milnor invariants, see also [6, 11, 22]. For other relationships between finite type invariants and the Milnor invariants, see [2, 19, 14, 13, 20].

We organize the rest of the paper as follows.

In Sec. 2, we recall some definitions from clasper calculus.

In Sec. 3, we recall the notion of C_k^a-equivalence for links, studied in [16]. If a link L is C_k^a-equivalent (for any k) to a Brunnian link, then L also is a Brunnian link.

In Sec. 4, we study the group $\overline{BSL}_{n+1}$ of C^a_{n+1}-equivalence classes of $(n+1)$-component Brunnian string links. We establish an isomorphism

$$\theta_n\colon \mathcal{T}_{n+1} \xrightarrow{\simeq} \overline{BSL}_{n+1}$$

from an abelian group $\mathcal{T}_{n+1}$ of certain tree diagrams. This map is essentially the inverse to the Milnor link-homotopy invariants of length $n+1$.

In Sec. 5, we apply the results in Sec. 4 to Brunnian links. The operation of closing string links induces a bijection

$$\bar{c}_{n+1}\colon \overline{BSL}_{n+1} \xrightarrow{\simeq} \overline{B}_{n+1},$$

where $\overline{B}_{n+1}$ is the set of C^a_{n+1}-equivalence classes of $(n+1)$-component Brunnian links. As a byproduct, we obtain another proof of a result of Miyazawa and Yasuhara [24].

In Sec. 6, we recall the definition of the Goussarov–Vassiliev filtration for links using claspers.

In Sec. 7, we study the behavior of Goussarov–Vassiliev invariants of degree $2n$ for $(n+1)$-component Brunnian links. We first show that two C^a_{n+1}-equivalent, $(n+1)$-component Brunnian links cannot be distinguished by Goussarov–Vassiliev invariants of degree $2n$. We have a quadratic map

$$\kappa_{n+1} : \overline{B}_{n+1} \to \bar{J}_{2n}(n+1)$$

defined by $\kappa_{n+1}([L]_{C^a_{n+1}}) = [L-U]_{J_{2n+1}}$ where $\bar{J}_{2n}(n+1)$ denotes the quotient space $\bar{J}_{2n}(n+1)/\bar{J}_{2n+1}(n+1)$. We prove Theorem 1.2, using κ_{n+1}.

2. Claspers

In this section, we recall some definitions from calculus of claspers. For the details, we refer the reader to [15].

A *clasper* in an oriented 3-manifold M is a compact, possibly unorientable, embedded surface G in $\operatorname{int} M$ equipped with a decomposition into connected subsurfaces called *leaves*, *disk-leaves*, *nodes*, *boxes*, and *edges*. Two distinct non-edge subsurfaces are disjoint. Edges are disjoint bands which connect two subsurfaces of the other types. A connected component of the intersection of one edge E and another subsurface F (of different type), which is an arc in $\partial E \cap \partial F$, is called an *attaching region* of F.

- A *leaf* is an annulus with one attaching region.
- A *disk-leaf* is a disk with one attaching region.
- A *node* is a disk with three attaching regions. (Usually, a node is incident to three edges, but it is allowed that the two ends of one edge are attached to the same node.)
- A *box* is a disk with three attaching regions. (The same remark as that for node applies here, too.) Moreover, one attaching region is distinguished from

the other two. (This distinction is done by drawing a box as a rectangle, see [15].)

A clasper G *for* a link L in M is a clasper in M such that the intersection $G \cap L$ consists of finitely many transverse double points and is contained in the interior of the union of disk-leaves.

We often use the drawing convention for claspers as described in [15].

Surgery along a clasper G is defined to be surgery along the *associated framed link* L_G to G. Here L_G is obtained from G by the rules described in Fig. 2.

A *tree clasper* is a connected clasper T without boxes, such that the union of edges and nodes of T is simply connected. A tree clasper T is called *strict* if each component of T has no leaves and at least one disk-leaf. Surgery along a strict tree clasper T is *tame* in the sense of [15, Sec. 2.3], i.e. the result of surgery along T preserves the 3-manifold and the surgery may be regarded as a move on a link.

A tree clasper T for a link L is *simple* (with respect to L) if each disk-leaf of T has exactly one intersection point with L.

The *degree* of a strict tree clasper G is defined to be the number of nodes of T plus 1. For $n \geq 1$, a C_n*-tree* is a strict tree clasper of degree n. A (*simple*) C_n*-move* is a local move on links defined as surgery along a (simple) C_n-tree. The C_n*-equivalence* is the equivalence relation on links generated by C_n-moves. This equivalence relation is also generated by simple C_n-moves. The C_n-equivalence becomes finer as n increases.

Note, for example, that a simple C_1-move is a crossing change, and a simple C_2-move is a delta move [21, 25]. Also, Milnor link's L_{n+1} discussed in the introduction and shown in Fig. 1 is obtained from the $(n+1)$-component unlink by a simple c_{n+1}-more. More precisely, L_{n+1} can be obtained from the $(n+1)$-component unlink by surgery along the C_n-tree represented in Fig. 5.

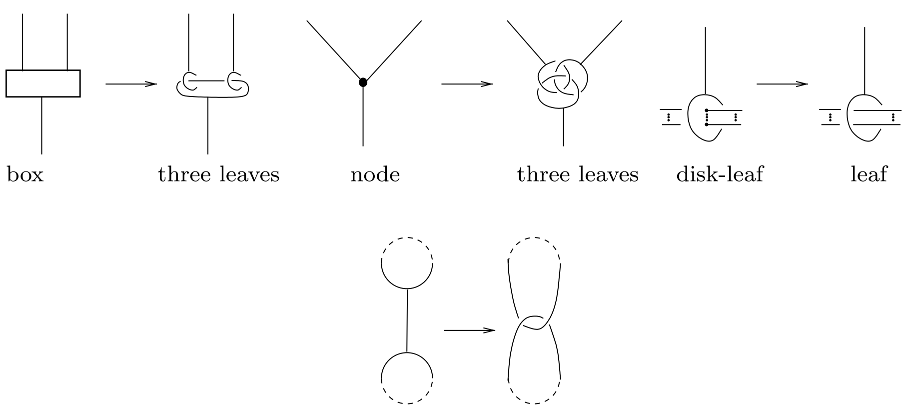

Fig. 2. How to obtain the associated framed link L_G from G. First one replaces boxes, nodes and disk-leaves by leaves. Then replace each "I-shaped" clasper by a 2-component framed link as depicted.

3. C_k^a-Equivalence

We recall from [16] the definition of the C_k^a-equivalence.

Definition 3.1. Let L be an m-component link in a 3-manifold M. For $k \geq m-1$, a C_k^a*-tree* for L in M is a C_k-tree T for L in M, such that

(1) for each disk-leaf A of T, all the strands intersecting A are contained in the same component of L, and
(2) each component of L intersects at least one disk-leaf of T, i.e. T intersects all the components of L.

Note that the condition (1) is vacuous if T is simple.

A C_k^a*-move* on a link is the surgery along a C_k^a-tree. The C_k^a*-equivalence* is the equivalence relation on links generated by C_k^a-moves. A C_k^a*-forest* is a clasper consisting only of C_k^a-trees.

Clearly, the above notions are defined also for tangles, particularly for string links.

What makes the notion of C_k^a-equivalence useful in the study of Brunnian links is the fact that a link which is C_k^a-equivalent (for any k) to a Brunnian link is again a Brunnian link ([16, Proposition 5]).

Note that the C_k^a-equivalence is generated by *simple* C_k^a-moves, i.e. surgeries along simple C_k^a-trees [16]. In the following, we use technical lemmas from [16].

Lemma 3.2 ([16, Lemma 7], C^a-version of [15, Theorem 3.17]). *For two tangles β and β' in a 3-manifold M, and an integer $k \geq 1$, the following conditions are equivalent.*

(1) *β and β' are C_k^a-equivalent.*
(2) *There is a simple C_k^a-forest F for β in M such that $\beta_F \cong \beta'$.*

Lemma 3.3 ([16, Lemma 8], C^a-version of [15, Proposition 4.5]). *Let β be a tangle in a 3-manifold M, and let β_0 be a component of β. Let T_1 and T_2 be C_k-trees for the tangle β, differing from each other by a crossing change of an edge with the component β_0. Suppose that T_1 and T_2 are C_k^a-trees for either β or $\beta \setminus \beta_0$. Then β_{T_1} and β_{T_2} are related by one C_{k+1}^a-move.*

4. The Group $\overline{BSL}_{n+1}$

4.1. *The monoids BSL_{n+1} and $\overline{BSL}_{n+1}$*

Let us recall the definition of string links. (For the details, see e.g. [12, 15]). Let $x_1, \ldots, x_{n+1} \in \operatorname{int} D^2$ be distinct points. An $(n+1)$-component *string link* $\beta = \beta_1 \cup \cdots \cup \beta_{n+1}$ is a tangle in the cylinder $D^2 \times [0,1]$, consisting of arc components $\beta_1, \ldots, \beta_{n+1}$ such that $\partial\beta_i = \{x_i\} \times \{0,1\}$ for each i. Let SL_{n+1} denote the set of $(n+1)$-component string links up to ambient isotopy fixing $\partial(D^2 \times [0,1])$

pointwisely. There is a natural, well-known monoid structure for SL_{n+1} with multiplication given by "stacking" of string links. The identity string link is denoted by $\mathbf{1} = \mathbf{1}_{n+1}$.

Let BSL_{n+1} denote the submonoid of SL_{n+1} consisting of *Brunnian string links*. Here a string link β is said to be Brunnian if every proper subtangle of β is the identity string link.

We have the following characterization of Brunnian string links.

Theorem 4.1 ([16, Theorem 9] [24, Proposition 4.1]). *An $(n+1)$-component link (respectively string link) is Brunnian if and only if it is C_n^a-trivial, i.e. it is C_n^a-equivalent to the unlink (respectively the identity string link).*

Set

$$\overline{BSL}_{n+1} = BSL_{n+1}/(C_{n+1}^a\text{-equivalence}).$$

By Theorem 4.1, $\overline{BSL}_{n+1}$ can be regarded as the monoid of C_{n+1}^a-equivalence classes of C_n^a-trivial, $(n+1)$-component string links (in $D^2 \times [0,1]$).

In the rest of this section, we will describe the structure of $\overline{BSL}_{n+1}$.

4.2. *The group $\overline{BSL}_{n+1}$ and the surgery map $\theta_n\colon \mathcal{T}_{n+1} \to \overline{BSL}_{n+1}$*

Proposition 4.2. *$\overline{BSL}_{n+1}$ is a finitely generated abelian group.*

Proof. The assertion is obtained by adapting the proof of [15, Lemma 5.5, Corollary 5.6] into the C^a setting. □

Let $n \geq 1$. By a (*labeled*) *unitrivalent tree* of degree n, we mean a vertex-oriented, unitrivalent graph t such that the $n+1$ univalent vertices of t are labeled by distinct elements from $\{1, 2, \ldots, n+1\}$. In figures, the counterclockwise vertex-orientation is assumed at each trivalent vertex.

Let $\mathcal{T}_{n+1}$ denote the free abelian group generated by unitrivalent trees of degree n, modulo the well-known IHX and AS relations.

Let t be a unitrivalent tree in $\mathcal{T}_{n+1}$, Consider a disjoint union D of $(n+1)$ oriented disks in $D^2 \times [0,1]$ such that D intersects every components of $\mathbf{1}$ and such that each component of $\mathbf{1}$ intersects exactly one disk in D, transversely, at its interior, in the positive normal direction. Consider also a disjoint union D' of $(n-1)$ oriented disks in $D^2 \times [0,1]$, such that D' is disjoint from $1 \cup D$. Connect the $2n$ disks in $D \cup D'$ by $2n-1$ disjoint bands in $(D^2 \times [0,1]) \backslash 1$ such that the resulting surface is a C_{n+1}^a-tree T_t for $\mathbf{1}$ whose shape and labelling are induced by those of t. These band sums are required to be compatible with the orientations of the disks, and to be coherent with the vertex-orientation of t. See, for example, Fig. 3. Note that there are many possibilities for such a C_{n+1}^a-tree T_t: here a choice is done.

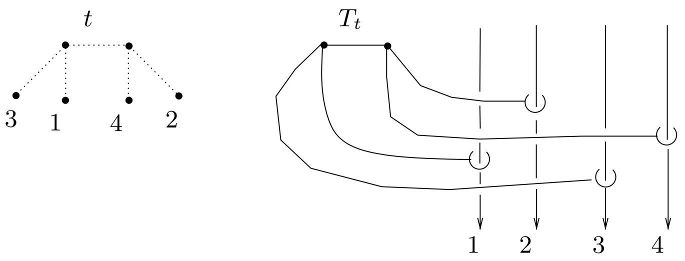

Fig. 3.

Proposition 4.3. *There is a unique isomorphism*

$$\theta_{n+1}\colon \mathcal{T}_{n+1} \xrightarrow{\simeq} \overline{BSL}_{n+1}$$

such that $\theta_{n+1}(t) = [\mathbf{1}_{T_t}]_{C^a_{n+1}}$ *for each unitrivalent tree* t*, where* T_t *is as above. This isomorphism does not depend on the choice of the* T_t*.*

Proof. Let $\mathcal{T}'_{n+1}$ be the free abelian group generated by unitrivalent trees of degree n, modulo the AS relations. By adapting the proof of [15, Theorem 4.7] into the C^a setting, we see that there is a unique surjective homomorphism

$$\theta'_{n+1}\colon \mathcal{T}'_{n+1} \to \overline{BSL}_{n+1}.$$

which does not depend on the choice of the T_t.

To see that θ'_{n+1} factors through the projection $\mathcal{T}'_{n+1} \to \mathcal{T}_n$, it suffices to see that the IHX relation is valid in $\overline{BSL}_{n+1}$, i.e. $t_I - t_H + t_X \in \mathcal{T}'_{n+1}$ is mapped to 0, where t_I, t_H, t_X locally differs as in the definition of the IHX relation. This can be checked by adapting the IHX relation for tree claspers (see e.g. [10, 5, 4]) into the C^a setting.

Let

$$\theta_{n+1}\colon \mathcal{T}_{n+1} \to \overline{BSL}_{n+1}$$

be the surjective homomorphism induced by θ'_{n+1}. As in the statement of Theorem 1.2, for $\sigma \in S_{n-1}$ and $T \in \overline{BSL}_{n+1}$, we set

$$\mu_\sigma(T) = \mu_{\sigma(1),\sigma(2),\ldots,\sigma(n-1),n,n+1}(T),$$

where $\mu_{\sigma(1),\sigma(2),\ldots,\sigma(n-1),n,n+1}(T) \in \mathbb{Z}$ is the Milnor string link invariant of T. Let t_σ denote the unitrivalent tree as depicted in Fig. 4. Using the IHX and AS relations,

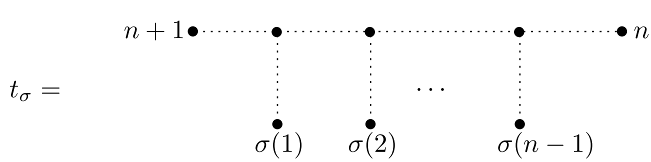

Fig. 4.

any element of $\mathcal{T}_{n+1}$ can be written as a linear combination of the t_σ for $\sigma \in S_{n-1}$. Define a homomorphism

$$\mu_{n+1}\colon \overline{BSL}_{n+1} \to \mathcal{T}_{n+1}$$

by

$$\mu_{n+1}(L) = \sum_{\sigma \in S_{n-1}} \mu_\sigma(L) t_\sigma.$$

By [15, Theorem 7.2], μ_{n+1} is well defined.

To show that μ_{n+1} is left inverse to θ_{n+1}, it suffices to prove that $\mu_{n+1}\theta_{n+1}(t_\sigma) = t_\sigma$ for $\sigma \in S_{n-1}$. Let L_σ denote the closure of the string link obtained from 1 by surgery along $T_{t\sigma}$. L_σ is Milnor's link as depicted in Fig. 1. Milnor [23] proved that for $\tau \in S_{n-1}$

$$\mu_\tau(L_\sigma) = \begin{cases} 1 & \text{if } \tau = \sigma, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

Hence we have

$$\mu_{n+1}\theta_{n+1}(t_\sigma) = \sum_{\tau \in S_{n-1}} \bar{\mu}_\tau(L_\sigma) t_\tau = t_\sigma.$$

This completes the proof. □

Corollary 4.4. *For two Brunnian $(n+1)$-component string links T, $T' \in BSL_{n+1}$, the following conditions are equivalent.*

(1) *T and T' are C^a_{n+1}-equivalent.*
(2) *T and T' have the same Milnor invariants of length $n+1$.*
(3) *T and T' are link-homotopic.*

Proof. The equivalence (2) ⇔ (3) is due to Milnor [23]. The equivalence (1) ⇔ (2) follows from the proof of Proposition 4.3. □

Remark 4.5. Miyazawa and Yasuhara [24] prove a similar result for Brunnian links. It seems that their proof can be applied to the case of string links. See also the Remark 5.4 below.

Corollary 4.6. *The abelian group $\overline{BSL}_{n+1}$ is free with rank $(n-1)!$.*

Proof. As seen in the proof of Proposition 4.3, the classification of $\overline{BSL}_{n+1}$ is actually given the Milnor invariants $\mu_\sigma = \mu_{\sigma(1),\sigma(2),\ldots,\sigma(n-1),n,n+1}$ for $\sigma \in S_{n-1}$, and there are $(n-1)!$ of these invariants. □

Remark 4.7. Observe that the string links $\mathbf{1}_{T_{t_\sigma}}$ for $\sigma \in S_{n-1}$ form basis of BSL_{n+1}, which is carried over to the system of generator to $\mathcal{T}_{n+1}$ by the isomorphism μ_{n+1}. It follows that the t_σ, for $\sigma \in S_{n-1}$, form a basis of $\mathcal{T}_{n+1}$.

5. The Group $\overline{B}_{n+1}$

5.1. *The set* B_{n+1}

Let B_{n+1} denote the set of the ambient isotopy classes of $(n+1)$-component Brunnian links in S^3. Let

$$c_{n+1}\colon BSL_{n+1} \to B_{n+1} \tag{5.1}$$

denote the map such that $c_{n+1}(\beta)$ is obtained from $\beta \in BSL_{n+1}$ by closing each component in the well-known manner.

Proposition 5.1. *The map c_{n+1} is onto.*

Proof. This is an immediate consequence of [16, Proposition 12]. □

5.2. *The isomorphism* $\bar{c}_{n+1}\colon \overline{BSL}_{n+1} \to \overline{B}_{n+1}$

Set

$$\overline{B}_{n+1} = B_{n+1}/(C^a_{n+1}\text{-equivalence}),$$

and let

$$\bar{c}_{n+1}\colon \overline{BSL}_{n+1} \to \overline{B}_{n+1}$$

denote the map induced by c_{n+1}, which is onto by Proposition 5.1.

Proposition 5.2. *$\bar{c}_{n+1}$ is one-to-one.*

Proof. It suffices to prove that there is a map $\overline{B}_{n+1} \to \mathcal{T}_{n+1}$ which is inverse to $\bar{c}_{n+1}\theta_n\colon \mathcal{T}_{n+1} \to \overline{B}_{n+1}$. This is proved similarly as in the proof of Proposition 4.3. □

Proposition 5.2 provides the set $\overline{B}_{n+1}$ the well-known abelian group structure, with multiplication induced by band sums of Brunnian links.

As a corollary, we obtain another proof of a result of Miyazawa and Yasuhara [24].

Corollary 5.3 ([24, Theorem 1.2]). *Let L and L' be two $(n+1)$-component Brunnian links in S^3. Then the following conditions are equivalent.*

(1) *L and L' are C^a_{n+1}-equivalent.*
(2) *L and L' are C_{n+1}-equivalent.*
(3) *L and L' are link-homotopic.*

Proof. The result follows immediately from Propositions 4.4 and 5.2. □

Remark 5.4. Miyazawa and Yasuhara [24] do not explicitly state the equivalence of (1) and others, but this equivalence follows from their proof.

Note that, unlike the C^a_{n+1}-equivalence, neither the C_{n+1}-equivalence nor the link-homotopy are closed for Brunnian links.

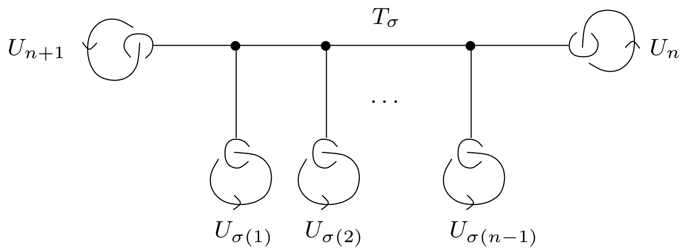

Fig. 5.

Remark 5.5. It is possible to show directly that $\mathcal{T}_{n+1}$ is isomorphic to $\overline{B}_{n+1}$, without using string links and the closure map $\bar{c}_{n+1}$. The proof uses Milnor's $\overline{\mu}$-invariants and the above result of Miyazawa and Yasuhara. Our approach provides an alternative proof of the latter (instead of using it).

5.3. *Trees and the Milnor invariants*

In this subsection, we fix some notations which are used in later sections. (Some have appeared in the proof of Proposition 4.3.)

For $\sigma \in S_{n-1}$, let t_σ denote the unitrivalent tree as depicted in Fig. 4. The t_σ for $\sigma \in S_{n-1}$ form a basis of $\mathcal{T}_{n+1}$. Let T_σ denote the corresponding C^a_n-tree for the $(n+1)$-component unlink $U = U_1 \cup \cdots \cup U_{n+1}$, see Fig. 5.

For $i_1, \ldots, i_{n+1}$ with $\{i_1, \ldots, i_{n+1}\} = \{1, \ldots, n+1\}$, let

$$\bar{\mu}_{i_1,\ldots,i_{n+1}} \colon B_{n+1} \to \mathbb{Z}$$

denote the Milnor invariant, which is additive under connected sum [23] (see also [3, 27, 18]). For $\sigma \in S_{n-1}$, we set

$$\bar{\mu}_\sigma = \bar{\mu}_{\sigma(1),\sigma(2),\ldots,\sigma(n-1),n,n+1} \colon B_{n+1} \to \mathbb{Z}.$$

It is well known [23] that for $\rho \in S_{n-1}$

$$\bar{\mu}_\rho(U_{T_\sigma}) = \begin{cases} 1 & \text{if } \rho = \sigma, \\ 0 & \text{otherwise.} \end{cases}$$

6. The Goussarov–Vassiliev Filtration for Links

In this section, we briefly recall the formulation using claspers of the Goussarov–Vassiliev filtrations for links. See [15] for details.

6.1. *Forest schemes and Goussarov–Vassiliev filtration*

A *forest scheme* of degree k for a link L in a 3-manifold M will mean a collection $S = \{G_1, \ldots, G_l\}$ of disjoint (strict) tree claspers $G_1, \ldots, G_l$ for L such that $\sum_{i=1}^k \deg G_i = k$. A forest scheme S is said to be *simple* if every element of S is simple.

For $n \geq 0$, let $\mathcal{L}(M, n)$ denote the set of ambient isotopy classes of oriented, n-component, ordered links in M.

For a forest scheme $S = \{G_1, \ldots, G_l\}$ for a link L in M, we set

$$[L, S] = [L; G_1, \ldots, G_l] = \sum_{S' \subset S} (-1)^{|S'|} L_{\bigcup S'} \in \mathbb{Z}\mathcal{L}(M, n),$$

where the sum is over all subsets S' of S, and $|S'|$ denote the number of elements of S'.

For $k \geq 0$, let $J_k(M, n)$ (sometimes denoted simply by J_k) denote the $\mathbb{Z}$-submodule of $\mathbb{Z}\mathcal{L}(M, n)$ generated by the elements of the form $[L, S]$, where $L \in \mathcal{L}(M, n)$ and S is a forest scheme for L of degree k. We have

$$\mathbb{Z}\mathcal{L}(M, n) = J_0(M, n) \subset J_1(M, n) \subset \cdots,$$

which coincides with the Goussarov–Vassiliev filtration using alternating sums of links determined by singular links, see [15, Sec. 6].

6.2. *Crossed edge notation*

It is useful to introduce a notation for depicting certain linear combinations of surgery along claspers, which we call *crossed edge notation*.

Let G be a clasper for a link L in a 3-manifold M. Let E be an edge of G. By putting a cross on the edge E in a figure, we mean the difference $L_G - L_{G_0}$, where G_0 is obtained from G by inserting two trivial leaves into E. See Fig. 6. If we put several crosses on the edges of G, then we understand it in a multilinear way, i.e. a clasper with several crosses is an alternating sum of the result of surgery along claspers obtained from G by inserting pairs of trivial, unlinked leaves into the crossed edges. We will freely use the identities depicted in Fig. 7, which can be easily verified. The second identity implies that if G' is a connected graph clasper contained in G and there are several crosses on G', then one can safely replace these crosses by just one cross on one edge in G'. This properties can be generalized to the case where G' is a connected subsurface of G consisting only of nodes, edges, leaves and disk-leaves. Note also that if $S = \{G_1, \ldots, G_l\}$, is a forest scheme for L,

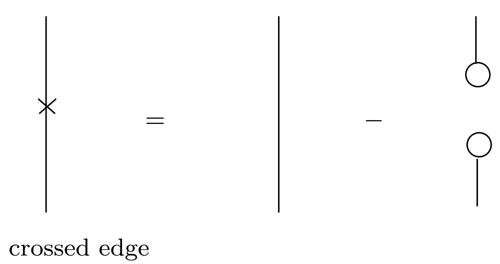

Fig. 6. The crossed edge notation.

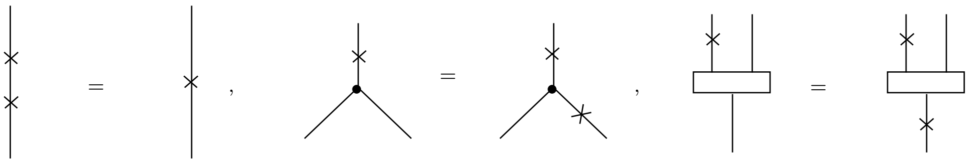

Fig. 7. Identities for the crossed edge notation.

then $[L, S]$ can be expressed by the clasper $G_1 \cup \cdots \cup G_l$ with one cross on each component G_i.

7. Goussarov–Vassiliev Invariants of Brunnian Links

Throughout this section, let $U = U_1 \cup U_2 \cup \cdots \cup U_{n+1}$ be the $(n+1)$-component unlink.

7.1. *The map* $\kappa_{n+1} : \overline{B}_{n+1} \to \bar{J}_{2n}(n+1)$

Proposition 7.1. *Let $n \geq 2$. Let L and L' be two $(n+1)$-component Brunnian links in an oriented, connected 3-manifold M. If L and L' are C^a_{n+1}-equivalent (or link-homotopic), then we have $L' - L \in J_{2n+1}$.*

Proposition 7.1 implies the following.

Corollary 7.2. *The restriction of any Goussarov–Vassiliev invariant of degree $2n$ to $(n+1)$-component Brunnian links is a link-homotopy invariant.*

Proof of Proposition 7.1. First, we consider the case $L = U$. By using the same arguments as in the proof of [16, Lemma 14], we see that there is a clasper G for U consisting of C^a_l-claspers with $n+1 \leq l < 2n+1$, such that U bounds $n+1$ disjoint disks which are disjoint from the edges and the nodes of G, and such that $U_G \sim_{C^a_{2n+1}} L'$. The latter implies that $U_G - L' \in J_{2n+1}$. We use the equality $U_G = \sum_{G' \subset G}(-1)^{|G'|}[U, G']$. Clearly $[U, G'] \in J_{2n+1}$ for $|G'| > 1$, so we may safely assume that G has only one component. We then have $U_G - U \in J_{2n+1}$ as a direct application of [16, Lemma 16]. This completes the proof of the case $L = U$.

Now consider the general case. We may assume that L' is obtained from L by one simple C^a_{n+1}-move. Since L is an $(n+1)$-component Brunnian link, it follows from Theorem 4.1 and Lemma 3.2 that there exists a simple C^a_n-forest F for U such that $L = U_F$. Also, there exists a simple C^a_{n+1}-tree T for $L = U_F$ such that $L' = L_T$. We may assume that T is a simple C^a_{n+1}-tree for U disjoint from F such that $L' = U_{F \cup T}$. Let S be the forest scheme consisting of the trees $T_1, \ldots, T_l$ of F. We have $L = \sum_{S' \subset S}(-1)^{|S'|}[U, S']$ and $L' = (-1)^{|S'|}\sum_{S' \subset S}[U_T, S']$. Hence we have

$$L' - L = \sum_{S' \subset S}(-1)^{|S'|}[U, S' \cup \{T\}].$$

Since $\deg T = n+1$ and $\deg T_i = n$ for all i, the term in the above sum is contained in J_{2n+1} unless $S' = \emptyset$. Hence we have

$$L' - L \equiv [U, T] \equiv 0 \quad (\mathrm{mod}\ J_{2n+1}),$$

where the second congruence follows from the first case. □

Let us denote by $\bar{J}_{2n}(n+1)$ the quotient space $J_{2n}(n+1)/J_{2n+1}(n+1)$. By Proposition 7.1, we have a map

$$\kappa_{n+1} : \overline{B}_{n+1} \to \bar{J}_{2n}(n+1)$$

defined by $\kappa_{n+1}(L) = [L - U]_{J_{2n+1}}$.

7.2. *Quadraticity of* $\boldsymbol{\kappa_{n+1}}$

Let $n \geq 2$. In this subsection, we establish the following commutative diagram.

$$\begin{array}{ccc} \mathcal{T}_{n+1} & \xrightarrow[\simeq]{\psi_{n+1}} & \overline{B}_{n+1} \\ {\scriptstyle q_{n+1}}\downarrow & & \downarrow{\scriptstyle \kappa_{n+1}} \\ \tilde{\mathrm{Sym}}^2\mathcal{T}_{n+1} & \xrightarrow[\delta_{n+1}]{} & \bar{J}_{2n}(n+1) \end{array} \tag{7.1}$$

Definitions of ψ_{n+1}, $\tilde{\mathrm{Sym}}^2\mathcal{T}_{n+1}$, q_{n+1} and δ_{n+1} are in order.

The isomorphism ψ_{n+1} is the composition of

$$\mathcal{T}_{n+1} \xrightarrow[\simeq]{\theta_{n+1}} \overline{BSL}_{n+1} \xrightarrow[\simeq]{\bar{c}_{n+1}} \overline{B}_{n+1}.$$

Let $\mathrm{Sym}^2\mathcal{T}^{\mathbb{Q}}_{n+1}$ denote the symmetric product of two copies of $\mathcal{T}^{\mathbb{Q}}_{n+1} := \mathcal{T}_{n+1} \otimes \mathbb{Q}$, and let $\tilde{\mathrm{Sym}}^2\mathcal{T}_{n+1}$ denote the $\mathbb{Z}$-submodule of $\mathrm{Sym}^2\mathcal{T}^{\mathbb{Q}}_{n+1}$ generated by $\frac{1}{2}x^2$, $x \in \mathcal{T}_{n+1}$. One can easily verify that $\tilde{\mathrm{Sym}}^2\mathcal{T}_{n+1}$ is $\mathbb{Z}$-spanned by the elements $\frac{1}{2}t_\sigma^2$ for $\sigma \in S_{n-1}$ and $t_\sigma t_{\sigma'}$ for $\sigma, \sigma' \in S_{n-1}$. (Of course we have $t_\sigma t_{\sigma'} = t_{\sigma'} t_\sigma$. Thus $\tilde{\mathrm{Sym}}^2\mathcal{T}_{n+1}$ is a free abelian group of rank $\frac{1}{2}(n-1)!((n-1)!+1)$.)

The arrow q_{n+1} is the quadratic map defined by $q_{n+1}(x) = \frac{1}{2}x^2$ for $x \in \mathcal{T}_{n+1}$.

The arrow δ_{n+1} is the homomorphism defined as follows. For $\sigma, \sigma' \in S_{n-1}$, let T_σ and $T_{\sigma'}$ be the corresponding simple C^a_n-trees for U as in Sec. 5.3. Let $\tilde{T}_{\sigma'}$ denote a simple C^a_n-trees obtained from $T_{\sigma'}$ by a small isotopy if necessary so that $\tilde{T}_{\sigma'}$ is disjoint from T_σ. Set

$$\delta_{n+1}(t_\sigma t_{\sigma'}) = [U; T_\sigma, \tilde{T}_{\sigma'}]_{J_{2n+1}} \in \bar{J}_{2n}(n+1),$$

which does not depend on how we obtained $\tilde{T}_{\sigma'}$ from T_σ, since crossing changes between an edge of T_σ and an edge of $\tilde{T}_{\sigma'}$ preserves the right-hand side. (This can be verified by using a "C^a-version" of [15, Proposition 4.6].) For the case of $\frac{1}{2}t_\sigma^2$, we modify the above definition with $\sigma' = \sigma$ as follows. Let T_σ and $\tilde{T}_\sigma$ be as above. See Fig. 8. Let T' be the C_{n-1}-tree obtained from $\tilde{T}_\sigma$ by first removing the disk-leaf D intersecting U_{n+1}, the edge E incident to D, and the node N incident to E, and then gluing the ends of the two edges which were attached to N. Moreover, let C

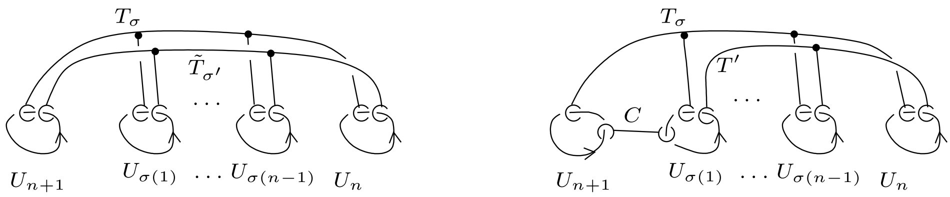

Fig. 8.

be a C_1-tree which intersects U_{n+1} and $U_{\sigma(1)}$ as depicted. Set

$$\delta_{n+1}\left(\frac{1}{2}t_\sigma^2\right) = [U; T_\sigma, T', C].$$

Lemma 7.3. *We have*

$$[U; T_\sigma, \tilde{T}_\sigma] \equiv 2[U; T_\sigma, T', C] \pmod{J_{2n+1}}. \tag{7.2}$$

Proof. By [15, Sec. 8.2], it suffices to prove the identity in the space of unitrivalent diagram depicted in Fig. 9, which can be easily verified using the STU relation several times. □

It follows from Lemma 7.3 that δ_{n+1} is a well-defined homomorphism. Set

$$\frac{1}{2}[U; T_\sigma, \tilde{T}_\sigma]_{J_{2n+1}} = [U; T_\sigma, T', C]_{J_{2n+1}}.$$

We have

$$\delta_{n+1}\left(\frac{1}{2}t_\sigma^2\right) = \frac{1}{2}[U; T_\sigma, \tilde{T}_\sigma]_{J_{2n+1}}.$$

Theorem 7.4. *The diagram* (7.1) *commutes. In particular*, κ_{n+1} *is a quadratic map.*

We need the following lemma before proving Theorem 7.4.

Lemma 7.5. *Let T be a clasper for a link L such that there is a disk-leaf D of T which "monopolizes" a component K of L in the sense of* [16, *Definition* 15], *and such that D is adjacent to a node. That is*, T *and L looks as depicted in the left-hand side of Fig.* 10. *Then we have the identity as depicted in the figure.*

Proof. The identity is easily verified and left to the reader. (Note that Lemma 7.5 is essentially the same as [16, (4.4)].) □

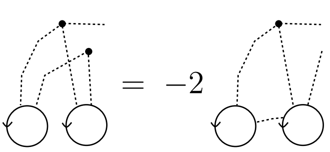

Fig. 9.

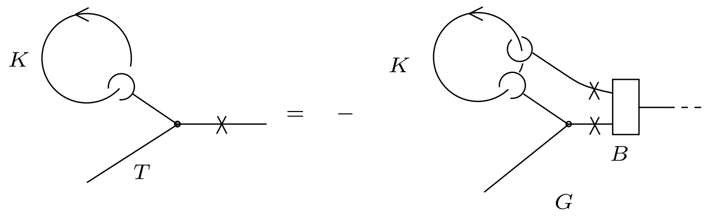

Fig. 10.

Proof of Theorem 7.4. Let $\sigma \in S_{n-1}$. We must show that

$$[U; T_\sigma]_{J_{2n+1}} = \frac{1}{2}[U; T_\sigma, \tilde{T}_\sigma]_{J_{2n+1}}.$$

For $i = 1, \ldots, n+1$, let D_i denote the disk-leaf of T_σ intersecting U_i, and let E_i denote the incident edge. For $i = 1, \ldots, n-1$, let N_i denote the node incident to E_i.

By applying Lemma 7.5 to the edge of T_σ which is incident to $N_{\sigma(1)}$ but not to D_{n+1} or $D_{\sigma(1)}$, we obtain the identity depicted in Fig. 11. Let B be the box and E be the edge as depicted. Let G be the clasper in the right-hand side. By zip construction [15, Sec. 3.3] at E, we obtain a crossed clasper depicted in Fig. 12, which consists of two components T_σ and P. The component P has $n-2$ (non-disk) leaves.

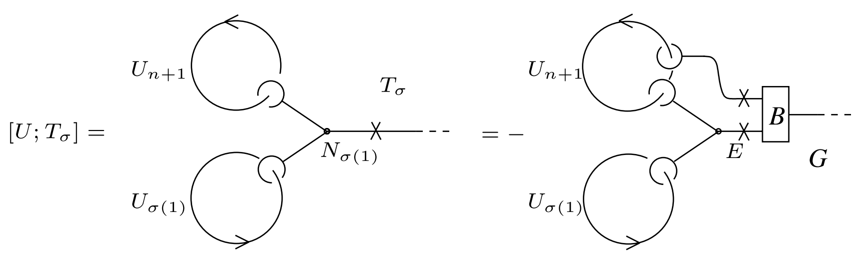

Fig. 11.

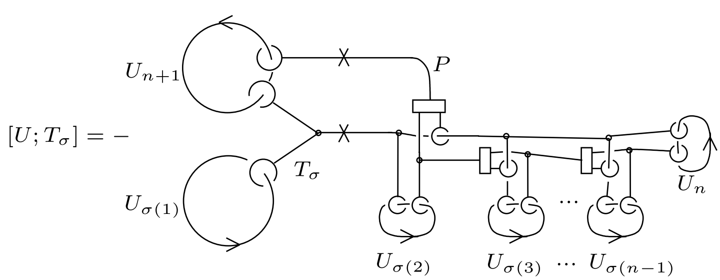

Fig. 12.

We claim that we can unlink the leaves of P from T_σ without changing the class in $\bar{J}_{2n}(n+1)$. To see this, it suffices to show that

$$U_{T_\sigma \cup P} - U_{T_\sigma \cup P'} \in J_{2n+1}, \tag{7.3}$$

where P' is obtained from P by the unlinking operation. Note that each unlinking is performed by a sequence of crossing changes between an edge of the C_n^a-tree T_σ and a link component (after performing surgery along P in the regular neighborhood of P), and thus can be performed by C_{n+1}^a-moves. Since all the links appearing in this sequence is Brunnian, we have (7.3) by Proposition 7.1. This completes the proof of the claim.

By the above claim, it follows that

$$[U; T_\sigma, P] \equiv [U; T_\sigma, T'] \quad (\text{mod } J_{2n+1}),$$

where T' is obtained from P by removing the leaves, the incident edges, and the boxes, and then smoothing the open edges, see the left-hand side of Fig. 13, which is equal to the right-hand side by Lemma 7.5. The result is related to the desired clasper defining $[U; T_\sigma, T', C]$ by half twists of two edges and homotopy with respect to U, and hence equivalent modulo J_{2n+1} to $[U; T_\sigma, T', C]$. (Recall that C is the simple C_1-tree represented in Fig. 8) This completes the proof. □

7.3. *Proof of Theorem 1.2*

In this subsection, we prove Theorem 1.2.

Let $L \in B_{n+1}$. We have

$$[L]_{C_{n+1}^a} = \sum_{\sigma \in S_{n-1}} \bar{\mu}_\sigma(L)[U_{T_\sigma}]_{C_{n+1}^a}$$

in $\overline{B}_{n+1}$. (Recall that the sum is induced by band-sum in $\overline{B}_{n+1}$.) Hence we have by the commutativity of (7.1)

$$\begin{aligned}[L - U]_{J_{2n+1}} &= \kappa_{n+1}([L]_{C_{n+1}^a}) \\ &= \delta_{n+1} q_{n+1} \psi_{n+1}^{-1} \left(\sum_{\sigma \in S_{n-1}} \bar{\mu}_\sigma(L)[U_{T_\sigma}]_{C_{n+1}^a} \right)\end{aligned}$$

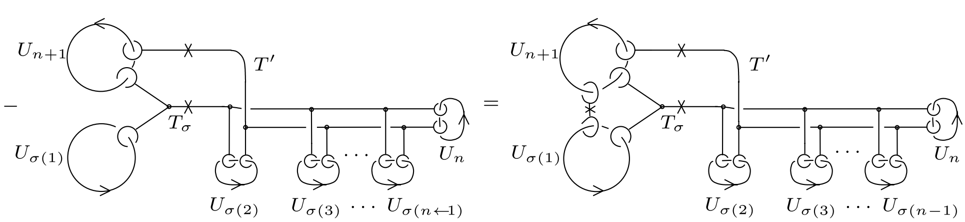

Fig. 13.

$$= \delta_{n+1} q_{n+1} \left(\sum_{\sigma \in S_{n-1}} \bar{\mu}_\sigma(L) t_\sigma \right)$$

$$= \delta_{n+1} \left(\frac{1}{2} \left(\sum_{\sigma \in S_{n-1}} \bar{\mu}_\sigma(L) t_\sigma \right)^2 \right)$$

$$= \delta_{n+1} \left(\frac{1}{2} \sum_{\sigma,\sigma' \in S_{n-1}} \bar{\mu}_\sigma(L) \bar{\mu}_{\sigma'}(L) t_\sigma t_{\sigma'} \right)$$

$$= \frac{1}{2} \sum_{\sigma,\sigma' \in S_{n-1}} \bar{\mu}_\sigma(L) \bar{\mu}_{\sigma'}(L) [U; T_\sigma, \tilde{T}_{\sigma'}]_{J_{2n+1}}.$$

We give any total order on the set S_{n-1}. Then we have

$$[L - U]_{J_{2n+1}} = \sum_{\sigma \in S_{n-1}} \bar{\mu}_\sigma(L) \bar{\mu}_\sigma(L) \left(\frac{1}{2} [U; T_\sigma, \tilde{T}_\sigma]) \right) + \sum_{\sigma < \sigma'} \bar{\mu}_\sigma(L) \bar{\mu}_{\sigma'}(L) [U; T_\sigma, \tilde{T}_{\sigma'}] \tag{7.4}$$

Recall that $\frac{1}{2}[U; T_\sigma, \tilde{T}_\sigma]$ denotes the element $S_\sigma = [U; T_\sigma, \tilde{T}', C]$ of $\tilde{J}_{2n}(n+1)$, described in Sec. 7.2, satisfying $2S_\sigma \equiv [U; T_\sigma, \tilde{T}_\sigma] \pmod{J_{2n+1}}$. See Lemma 7.3 and Fig. 8.

Hence we have

$$f(L) - f(U) = \sum_{\sigma \in S_{n-1}} f\left(\frac{1}{2} [U; T_\sigma, \tilde{T}_\sigma] \right) \bar{\mu}_\sigma(L) \bar{\mu}_\sigma(L) + \sum_{\sigma < \sigma'} f([U; T_\sigma, \tilde{T}_{\sigma'}]) \bar{\mu}_\sigma(L) \bar{\mu}_{\sigma'}(L).$$

Note that $f(1/2[U; T_\sigma, \tilde{T}_\sigma]) \in A$ and $f([U; T_\sigma, \tilde{T}_{\sigma'}]) \in A$. Hence we have (1.1) by setting

$$f_{\sigma,\sigma'} = \begin{cases} f(\frac{1}{2}[U; T_\sigma, \tilde{T}_\sigma]) & \text{if } \sigma = \sigma', \\ f([U; T_\sigma, \tilde{T}_{\sigma'}]) & \text{if } \sigma < \sigma', \\ 0 & \text{if } \sigma > \sigma'. \end{cases}$$

This completes the proof of Theorem 1.2.

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