CHARACTERIZATION OF $Y_2$-EQUIVALENCE FOR HOMOLOGY CYLINDERS

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ABSTRACT

For a compact connected oriented surface, we consider homology cylinders over \( \Sigma \); these are homology cobordisms with an extra homological triviality condition. When considered up to \( Y_2 \)-equivalence, which is a surgery equivalence relation arising from the Goussarov-Habiro theory, homology cylinders form an Abelian group.

In this paper, when \( \Sigma \) has one or zero boundary component, we define a surgery map from a certain space of graphs to this group. This map is shown to be an isomorphism, with inverse given by some extensions of the first Johnson homomorphism and Birman-Craggs homomorphisms.

Keywords: Homology cylinder; finite type invariant; clover; clasper.

1. Introduction

1.1. Homology cylinders

Homology cylinders are important objects in the theory of finite type invariants of Goussarov-Habiro: they have thus appeared in both [6] and [4]. Let us recall the definition of these objects.

Let \( \Sigma \) be a compact connected oriented surface. A homology cobordism over \( \Sigma \) is a triple \((M, i^+, i^-)\) where \( M \) is a compact oriented 3-manifold and \( i^\pm: \Sigma \to M \) are oriented embeddings with images \( \Sigma^\pm \), such that:

(i) \( i^\pm \) are homology isomorphisms;
(ii) \( \partial M = \Sigma^+ \cup (-\Sigma^-) \) and \( \Sigma^+ \cap (-\Sigma^-) = \pm \partial \Sigma^\pm \);
(iii) \( i^+|_{\partial \Sigma} = i^-|_{\partial \Sigma} \).

Homology cobordisms are considered up to orientation-preserving diffeomorphisms. When \( (i^-)^{-1} \circ (i^+) : H_1(\Sigma; \mathbb{Z}) \to H_1(\Sigma; \mathbb{Z}) \) is the identity, \( M \) is said to be a homology cylinder. The set of homology cobordisms is denoted here by \( \mathcal{C}(\Sigma) \), and \( \mathcal{HC}(\Sigma) \) denotes the subset of homology cylinders. If \( M = (M, i^+, i^-) \) and...
$N = (N, j^+, j^-)$ are homology cobordisms, we can define their *stacking product* by

$$M \cdot N := (M \cup_{i^{-} \circ (j^+)^{-}^{-}^{-}}, N, i^+, j^-).$$

This product induces a monoid structure on $C(\Sigma)$, with $\mathcal{HC}(\Sigma)$ a submonoid. The unit element is $1_\Sigma := (\Sigma \times I, Id, Id)$, where $I$ is the unit interval $[0,1]$ and where a collar of $\Sigma^\pm$ is stretched along $\partial \Sigma \times I$ so that the second defining condition for homology cobordisms is satisfied.

Habiro in [6, §8.5] outlined how homology cylinders can serve as a powerful tool in studying the mapping class groups of surfaces (see [3], [5], [12]). The connection lies on the homomorphism of monoids

$$\mathcal{T}(\Sigma) \xrightarrow{C} \mathcal{HC}(\Sigma)$$

sending each $h$ in the Torelli group of $\Sigma$ to the mapping cylinder $C_h = (\Sigma \times I, Id, h)$ (with, as above, a collar of $\Sigma^\pm$ stretched along $\partial \Sigma \times I$).

In the sequel, *we restrict ourselves* to the following two cases:

(i) $\Sigma = \Sigma_g$ is the standard closed oriented surface of genus $g \geq 0$, which here is referred to as the *closed case*;

(ii) $\Sigma = \Sigma_{g,1}$ is the standard compact oriented surface of genus $g \geq 0$ with one boundary component, which here is referred to as the *boundary case*.

The usual notations $T_{g,1} = T(\Sigma_{g,1})$ and $T_g = T(\Sigma_g)$ for the Torelli groups will be used. Also denote by $H$ the first homology group of $\Sigma$ with integer coefficients, by $\bullet$ the intersection form on $H$ and by $(x_i, y_i)_{i=1}^g$ a symplectic basis for $(H, \bullet)$.

**1.2. Y_k-equivalence**

The theory of finite type invariants of Goussarov-Habiro has come equipped with a topological calculus toolbox: this was called *calculus of claspers* in [6] or alternatively *clovers* in [2]. We will assume a certain familiarity of the reader with these techniques.

In particular, let us recall that, for $k \geq 1$ an integer, the *Y_k-equivalence* is the equivalence relation generated by surgery on connected clovers of degree $k$. Following Habiro in [6], we can then define a descending filtration of monoids

$$C(\Sigma) \supset C_1(\Sigma) \supset C_2(\Sigma) \supset \cdots \supset C_k(\Sigma) \supset \cdots$$

where $C_k(\Sigma)$ is the submonoid consisting of the homology cobordisms which are $Y_k$-equivalent to the trivial cobordism $1_\Sigma$. Note the following fact, a proof of which has been inserted in §4.

**Proposition 1.1.** If $\Sigma = \Sigma_g$ or $\Sigma_{g,1}$, then $\mathcal{HC}(\Sigma) = C_1(\Sigma)$.

*This equivalence relation is called $(k-1)$-equivalence in [4], and $A_k$-equivalence in [6].*
As mentioned by Habiro, we can show from the calculus of clovers that for every \( k \geq 1 \), the quotient monoid 
\[
\overline{C}_k(\Sigma) := \overline{C}_k(\Sigma) / Y_{k+1}
\]
is an Abelian group. In particular, \( \overline{C}_1(\Sigma) \) is the Abelian group of homology cylinders over \( \Sigma \) up to \( Y_2 \)-equivalence. This group is the subject of the present paper.

For \( k \geq 2 \), Habiro gives a combinatorial upper bound for the Abelian group \( \overline{C}_k(\Sigma) \). Precisely, he defines \( \mathcal{A}_k(H) \) to be the Abelian group (finitely) generated by unitrivalent graphs of internal degree \( k \), with cyclic orientation at each trivalent vertex and whose univalent vertices are labelled by elements of \( H \) and are totally ordered. These graphs are considered modulo the well-known AS, IHX, multilinearity relations, and up to some “STU-like relations” dealing with the order of the univalent vertices. In the closed case, some relations of a symplectic type can be added. Then, there is a surjective surgery map 
\[
\mathcal{A}_k(H) \xrightarrow{\psi_k} \overline{C}_k(\Sigma)
\]
sending each graph \( G \) to \((1_\Sigma)\tilde{G}\), where \( \tilde{G} \) is a clover in the manifold \( 1_\Sigma \) with \( G \) as associated abstract graph, whose leaves are stacked from the upper surface \( \Sigma \times 1 \) according to the total order, framed along this surface and embedded according to the labels of the corresponding univalent vertices. The fact that \( \psi_k \) is well-defined also follows from the calculus of clovers.

As for the case \( k = 1 \), Habiro does not define any space of graphs but announces the following isomorphisms
\[
\begin{align*}
\overline{C}_1(\Sigma_{g,1}) &\simeq \Lambda^3 H \oplus \Lambda^2 H_{(2)} \oplus H_{(2)} \oplus \mathbb{Z}_2 \\
\overline{C}_1(\Sigma_g) &\simeq \Lambda^3 H / (\omega \wedge H) \oplus \Lambda^2 H_{(2)} / \omega(2) \oplus H_{(2)} \oplus \mathbb{Z}_2
\end{align*}
\]
where \( H_{(2)} = H \otimes \mathbb{Z}_2 \) and where
\[
\omega = \sum_{i=1}^{g} x_i \wedge y_i \in \Lambda^2 H
\]
is the symplectic element. This fact has been used afterwards in [12].

The goal of this paper is to prove these isomorphisms, in a diagrammatic way, by again defining a surgery map 
\[
\mathcal{A}_1(P) \xrightarrow{\psi_1} \overline{C}_1(\Sigma).
\]

The space of graphs \( \mathcal{A}_1(P) \) and the map \( \psi_1 \) appear to be meaningfully different from \( \mathcal{A}_k(H) \) and \( \psi_k \) for \( k > 1 \), making thus the case \( k = 1 \) exceptional. Indeed, their definition will involve both the homology group \( H \) and \( Spin(\Sigma) \), the set of spin structures on \( \Sigma \).
1.3. The Abelianized Torelli group

We denote by $\Omega_g$ the set of quadratic forms with $\bullet : H(2) \times H(2) \rightarrow \mathbb{Z}_2$ as associated bilinear form, namely

$$\Omega_g = \left\{ q : H(2) \rightarrow \mathbb{Z}_2 \mid \forall x, y \in H(2), \; q(x + y) - q(x) - q(y) = x \bullet y \right\}.$$ 

Note that $\Omega_g$ is an affine space over $H(2)$, with action given by

$$\forall q \in \Omega_g, \forall x \in H(2), \; x \cdot q := q + x \bullet (-).$$

Thus, among the maps $\Omega_g \rightarrow \mathbb{Z}_2$, there are the affine functions, and more generally there are the Boolean polynomials which are defined to be sums of products of affine ones (see [8, §4]). These polynomials form a $\mathbb{Z}_2$-algebra denoted by $B_g$, which is filtered by the degree (defined in the obvious way):

$$B_g^{(0)} \subset B_g^{(1)} \subset \cdots \subset B_g.$$ 

For instance, $B_g^{(1)}$ is the space of affine functions on $\Omega_g$; the constant function $\mathbb{T} : \Omega_g \rightarrow \mathbb{Z}_2$ sending each $q$ to 1 and, for $h \in H$, the function $\mathbb{h}$ sending each $q$ to $q(h)$ are affine functions. Note the following identity:

$$\forall h_1, h_2 \in H, \; h_1 + h_2 = h_1 + \overline{h_2} + (h_1 \bullet h_2) \cdot \mathbb{T} \in B_g^{(1)}. \quad (1.2)$$

Another example of Boolean polynomial is the quadratic Boolean function

$$\alpha = \sum_{i=1}^{q} \overline{x_i} \cdot y_i,$$

which is known as the Arf invariant. For any basis $(e_i)_{i=1}^{2g}$ for $H$, there is an isomorphism of algebras:

$$B_g \simeq \mathbb{Z}_2[t_1, \ldots, t_{2g}] / t_i^2 = t_i \quad (1.3)$$

sending $\mathbb{T}$ to 1 and $\overline{t_i}$ to $t_i$.

Recall now from [8], that the many Birman-Craggs homomorphisms can be summed up into a single homomorphism

$$\mathbb{T}_{g,1} \xrightarrow{\beta} B_g^{(3)} \quad \text{or} \quad \mathbb{T}_{g} \xrightarrow{\beta} \frac{B_g^{(3)}}{\alpha \cdot B_g^{(1)}},$$

according to whether one is considering the boundary case or the closed case. Recall also from [9] that the first Johnson homomorphism is a homomorphism

$$\mathbb{T}_{g,1} \xrightarrow{\eta_1} \Lambda^3 H \quad \text{or} \quad \mathbb{T}_g \xrightarrow{\eta_1} \frac{\Lambda^3 H}{\omega \wedge H},$$

where $\omega$ is the symplectic form on $H$.
Form the following pull-back:

\[
\begin{array}{cccc}
\Lambda^3 H \times_{\Lambda^3 H(2)} B^{(3)}_g & \to & B^{(3)}_g \\
\downarrow & & \downarrow q \\
\Lambda^3 H & \to & \Lambda^3 H(2),
\end{array}
\]

where the map \(q\) is the canonical projection \(B^{(3)}_g \to B^{(2)}_g\) followed by the isomorphism \(B^{(3)}_g / B^{(2)}_g \cong \Lambda^3 H(2)\) which identifies the cubic polynomial \(h_1 h_2 h_3\) with \(h_1 \wedge h_2 \wedge h_3\) (this is well-defined because of (1.2) and (1.3)).

We denote by \(S\) the subgroup of this pull-back corresponding to \(\Lambda^3 H \times_{\Lambda^3 H(2)} B^{(3)}_g\) and \(\Lambda^3 H(2)\). Johnson has shown in [10] that, under the assumption \(g \geq 3\), the homomorphisms \(\eta_1\) and \(\beta\) induce isomorphisms

\[
\frac{T_{g,1}}{T'_{g,1}} (\eta_1, \beta) \cong \Lambda^3 H \times_{\Lambda^3 H(2)} B^{(3)}_g \quad \text{and} \quad \frac{T_{g}}{T'_{g}} (\eta_1, \beta) \cong \Lambda^3 H \times_{\Lambda^3 H(2)} B^{(3)}_g / S.
\]

**Remark 1.2.** Note that, because of (1.3), the codomains of these maps are respectively non-canonically isomorphic to \(\Lambda^3 H \oplus \Lambda^2 H(2) \oplus H(2) \oplus \mathbb{Z}_2\) and \(\Lambda^3 H / (\omega \wedge H) \oplus \Lambda^2 H(2) / \omega(2) \oplus H(2) \oplus \mathbb{Z}_2\).

### 1.4. Statement of the results

In §2, we will construct the space of graphs \(A_1(P)\) and the surgery map \(\psi_1 : A_1(P) \to \overline{C}_1(\Sigma)\). Spin structures play a prominent role in their definitions.

Observe that, \(\overline{C}_1(\Sigma)\) being an Abelian group, the mapping cylinder construction induces a group homomorphism

\[
\frac{T(\Sigma)}{T(\Sigma)'} \xrightarrow{C} \overline{C}_1(\Sigma).
\]

As pointed out by Garoufalidis and Levine in [3] and [12], Johnson homomorphisms and Birman-Craggs homomorphisms factor through \(C : T(\Sigma) \to HC(\Sigma)\). These extensions will be detailed in §3.

Next, we will specify in §4 an isomorphism \(\rho : A_1(P) \to \Lambda^3 H \times_{\Lambda^3 H(2)} B^{(3)}_g\) and the following two theorems will be proved from the previous material.

**Theorem 1.3.** In the boundary case, the diagram

\[
\begin{array}{ccc}
A_1(P) & \xrightarrow{\psi_1} & \overline{C}_1(\Sigma_{g,1}) \\
\downarrow \rho & & \downarrow \text{C} \frac{T_{g,1}}{T'_{g,1}} \xrightarrow{\eta_1, \beta} \\
\Lambda^3 H \times_{\Lambda^3 H(2)} B^{(3)}_g
\end{array}
\]
commutes and all of its arrows are isomorphisms, except for the two maps starting from $T_{g,1}/T'_{g,1}$ when $g < 3$.

**Theorem 1.4.** In the closed case, the diagram

\[
\begin{array}{ccc}
\mathcal{A}_1(P) & \xrightarrow{\psi_1} & \mathcal{C}_1(\Sigma_g) \\
\rho^{-1}(S) & \xrightarrow{\rho} & \Lambda^3 H \times_{\Lambda^3 H(2)} B^{(3)}_{\Sigma_g} \\
\xrightarrow{(\eta_1, \beta)} & \xrightarrow{(\eta_1, \beta)} & S
\end{array}
\]

commutes and all of its arrows are isomorphisms, except for the two maps starting from $T_g/T'_g$ when $g < 3$.

Note that Theorems 1.3 and 1.4 together with Remark 1.2, give Habiro’s isomorphisms (1.1), which are non-canonical. Also, we will easily deduce the following.

**Corollary 1.5.** For $\Sigma = \Sigma_{g,1}$ or $\Sigma_g$, let $M$ and $M'$ be two homology cylinders over $\Sigma$. Then, the following assertions are equivalent:

(a) $M$ and $M'$ are $Y_2$-equivalent;
(b) $M$ and $M'$ are not distinguished by degree 1 Goussarov-Habiro finite type invariants;
(c) $M$ and $M'$ are not distinguished by the first Johnson homomorphism nor Birman-Craggs homomorphisms.

Finally, if an embedding $\Sigma_{g,1} \hookrightarrow \Sigma_g$ is fixed, there is an obvious “filling-up” map $\mathcal{C}_1(\Sigma_{g,1}) \rightarrow \mathcal{C}_1(\Sigma_g)$, through which the commutative diagrams of Theorems 1.3 and 1.4 are compatible. The reader is referred to §4 for a precise statement.

2. **Definition of the Surgery Map $\psi_1$**

In this section, we define the space of graphs $\mathcal{A}_1(P)$ and the surgery map $\psi_1$ announced in the introduction.

2.1. **Special Abelian groups and the $\mathcal{A}_1$ functor**

Let us denote by $\text{Ab}$ the category of Abelian groups. An **Abelian group with special element** is a pair $(G, s)$ where $G$ is an Abelian group and $s \in G$ is of order at most 2. We denote by $\text{Ab}_s$ the category of special Abelian groups whose morphisms are group homomorphisms preserving the special elements. We now define a functor $\text{Ab}_s \xrightarrow{\mathcal{A}_1} \text{Ab}$ in the following way. For $(G, s)$ an object in $\text{Ab}_s$, $\mathcal{A}_1(G, s)$ is the free Abelian group generated by Y-shaped unitrivalent graphs, whose trivalent vertex is equipped with
a cyclic order on the incident edges and whose univalent vertices are labelled by $G$, subject to some relations. The notation

$$Y[z_1, z_2, z_3]$$

will stand for the $Y$-shaped graph whose univalent vertices are colored by $z_1, z_2$ and $z_3 \in G$ in accordance with the cyclic order, so that our notation is invariant under cyclic permutation of the $z_i$'s. The relations are the following ones:

- **Antisymmetry (AS)**: $Y[z_1, z_2, z_3] = -Y[z_2, z_1, z_3]$,
- **Multilinearity of colors**: $Y[z_0 + z_1, z_2, z_3] = Y[z_0, z_2, z_3] + Y[z_1, z_2, z_3]$,
- **Slide**: $Y[z_1, z_1, z_2] = Y[s, z_1, z_2]$,

where $z_0, z_1, z_2, z_3 \in G$. For $(G, s) \mapsto (G', s')$ a morphism in $\mathcal{A}b_s$, $\mathcal{A}_1(f)$ maps each generator $Y[z_1, z_2, z_3]$ of $\mathcal{A}_1(G, s)$ to $Y[f(z_1), f(z_2), f(z_3)] \in \mathcal{A}_1(G', s')$.

**Example 2.6.** The map $[G \to (G, 0)]$ makes $\mathcal{A}b$ a (full) subcategory of $\mathcal{A}b_s$. It follows from the definitions that the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{A}b & \longrightarrow & \mathcal{A}b_s \\
\Lambda^3(-) & \downarrow & \mathcal{A}_1 \\
& \downarrow & \mathcal{A}b \\
\end{array}$$

Non-trivial examples will be given in the next paragraph. For future use, note that this category has an obvious pull-back construction extending that of $\mathcal{A}b$:

$$\begin{array}{ccc}
(G_1, s_1) \times_{(G, s)} (G_2, s_2) & \longrightarrow & (G_2, s_2) \\
\downarrow & & \downarrow f_2 \\
(G_1, s_1) & \longrightarrow & (G, s) \\
\downarrow f_1 & & \downarrow \\
\end{array}$$

where $(G_1, s_1) \times_{(G, s)} (G_2, s_2)$ is the subgroup of $G_1 \times G_2$ consisting of those $(z_1, z_2)$ such that $f_1(z_1) = f_2(z_2)$, and with special element $(s_1, s_2)$.

### 2.2. Spin structures and the special Abelian group $P$

In this paragraph, let $M$ be a compact oriented 3-manifold endowed with a Riemannian metric, and let $FM$ be its bundle of oriented orthonormal frames:

$$SO(3) \xrightarrow{i} E(FM) \xrightarrow{p} M.$$ 

Let $s \in H_1(E(FM); \mathbb{Z})$ be the image by $i_*$ of the generator of $H_1(SO(3); \mathbb{Z}) \cong \mathbb{Z}_2$. Recall that $M$ is spinnable and that $Spin(M)$ can be defined as

$$Spin(M) = \{ y \in H^1(E(FM); \mathbb{Z}_2), \langle y, s \rangle \neq 0 \}$$,
which is essentially independent of the metric. The manifold $M$ being spinnable, $s$ is not 0 (and so is of order 2).

Now, $\text{Spin}(M)$ being an affine space over $H^1(M; \mathbb{Z}_2)$ with action given by
\[
\forall x \in H^1(M; \mathbb{Z}_2), \forall \sigma \in \text{Spin}(M), \quad x \cdot \sigma := \sigma + p^*(x),
\]
we can consider the space
\[
A(\text{Spin}(M), \mathbb{Z}_2)
\]
of $\mathbb{Z}_2$-valued affine functions on $\text{Spin}(M)$. For instance, $\mathbf{T} \in A(\text{Spin}(M), \mathbb{Z}_2)$ will denote the constant map defined by $\sigma \mapsto 1$.

There is a canonical map
\[
A(\text{Spin}(M), \mathbb{Z}_2) \xrightarrow{\kappa} H_1(M; \mathbb{Z}_2).
\]
For $f \in A(\text{Spin}(M), \mathbb{Z}_2)$, the homology class $\kappa(f)$ is defined unambiguously by
\[
\forall \sigma, \sigma' \in \text{Spin}(M), \quad f(\sigma') - f(\sigma) = \langle \sigma' / \sigma, \kappa(f) \rangle \in \mathbb{Z}_2,
\]
where $\sigma' / \sigma \in H^1(M; \mathbb{Z}_2)$ is defined by the affine action of $H^1(M; \mathbb{Z}_2)$ on $\text{Spin}(M)$.

Another canonical map is
\[
H_1(E(FM); \mathbb{Z}) \xrightarrow{e} A(\text{Spin}(M), \mathbb{Z}_2)
\]
sending a $x$ to the map defined by $\sigma \mapsto < \sigma, x >$. Next lemma gives us a nice understanding of the special Abelian group $(H_1(E(FM); \mathbb{Z}), s)$.

**Lemma 2.7.** (a) The following diagram of special groups is a pull-back diagram:
\[
\begin{array}{ccc}
(H_1(E(FM); \mathbb{Z}), s) & \xrightarrow{e} & (A(\text{Spin}(M), \mathbb{Z}_2), \mathbf{T}) \\
p_* & & \kappa \\
(H_1(M; \mathbb{Z}), 0) & \xrightarrow{\oplus} & (H_1(M; \mathbb{Z}_2), 0).
\end{array}
\]

(b) Let $t$ be the map
\[
\{ \text{Oriented framed knots in } M \} \xrightarrow{t} H_1(E(FM); \mathbb{Z})
\]
which adds to any oriented framed knot $K$ an extra $(+1)$-twist, and then sends it to the homology class of its lift in $FM$. Then,

(i) $t$ is surjective;
(ii) $t_{K_1} = t_{K_2}$ if and only if $K_1$ and $K_2$ are cobordant as oriented knots in $M$ and if their framings with respect to a surface with boundary $(K_1) \cup (-K_2)$ then differ from each other by an even integer;
(iii) if $K_1 \sharp K_2$ denotes the band connected sum of $K_1$ and $K_2$, then $t_{K_1 \sharp K_2} = t_{K_1} + t_{K_2}$;
(iv) the $k$-framed trivial oriented knot ($k \in \mathbb{Z}$) is sent by $t$ to $k \cdot s$. 
Proof. We begin by proving (a). The commutativity of the diagram of special groups is easy to verify. By functoriality, we get a map

\[(H_1 (E(FM); \mathbb{Z}), e) \xrightarrow{(p, e)} (H_1 (M; \mathbb{Z}), 0) \times_{(H_1 (M; \mathbb{Z}), 0)} (A(Spin(M), \mathbb{Z}_2), \mathbb{T}).\]

The Serre sequence associated to the fibration FM gives for homology with integer coefficients:

\[0 \to H_1(SO(3); \mathbb{Z}) \xrightarrow{i^*} H_1(E(FM); \mathbb{Z}) \xrightarrow{p_*} H_1(M; \mathbb{Z}) \to 0.\]

The bijectivity of \((p_*, e)\) follows from the exactness of this sequence.

We now prove (b) and we begin with assertion (iv). Let \(K\) be a trivial \(k\)-framed oriented knot, let \(* \in K\) and let \(e = (e_1, e_2, e_3) \in p^{-1}(*)\) be the framing of \(K\) at \(*\). We denote by \(\tilde{K}\) the lift of \(K\) to \(FM\). Then, as a loop in \(E(FM)\), \(K\) is homotopic to the loop in the fiber \(p^{-1}(*)\) defined by

\[0, 1) \ni t \mapsto R_{\theta}(t)\],

where \(R_{\theta}\) (with \(\theta \in \mathbb{R}\)) denotes the rotation of oriented axis directed by \(e_3\) and angle \(\theta\). From an appropriate description of the generator of \(\pi_1(SO(3)) \simeq \mathbb{Z}_2\), it follows that \([\tilde{K}] = (k + 1) \cdot s \in H_1(E(FM); \mathbb{Z})\), and assertion (iv) then follows.

Let us make an observation. Let \(K\) be any oriented framed knot in \(M\); since the framing of \(K\) determines a trivialization of its normal bundle in \(M\), it allows us to restrict any spin structure on \(M\) to \(K\). Recall now that the cobordism group \(\Omega_1^{Spin}\) is isomorphic to \(\mathbb{Z}_2\) (with generator given by \(S^1\) endowed with the spin structure induced by its Lie group structure: see [11, pp. 35 and 36]). The following observation then makes sense:

\[\forall \sigma \in Spin(M), \quad e(t_K)|\sigma_K = (K, \sigma)|_K \in \Omega_1^{Spin} \simeq \mathbb{Z}_2,\]

and can be derived from an appropriate characterization of the spin structures on the circle (see [11, pp. 35, 36]).

Let now \(K_1\) and \(K_2\) be some disjoint oriented framed knots in \(M\). There is an obvious genus 0 surface with boundary \(K_1 \cup K_2 \cup (-K_1) \cup (-K_2)\). Then, according to (2.4), we have \(e(t_{K_1 \cup K_2}) = e(t_{K_1}) + e(t_{K_2})\). Also, \(p_*(t_{K_1 \cup K_2}) = [K_1] + [K_2] = p_*(t_{K_1}) + p_*(t_{K_2})\), and so by (a), we obtain that assertion (iii) holds for \(K_1\) and \(K_2\).

We now justify assertion (ii). According to (a), \(t_{K_1} = t_{K_2}\) if and only if \(p_*(t_{K_1}) = p_*(t_{K_2})\) and \(e(t_{K_1}) = e(t_{K_2})\). Also, the condition \(p_*(t_{K_1}) = p_*(t_{K_2})\) holds if and only if \(K_1\) and \(K_2\) are homologous in \(M\). In this case, let \(S\) be an embedded oriented surface in \(M\) such that \(\partial S = K_1 \cup (-K_2)\). Let \(k_i\) be the framing of \(K_i\) with respect to \(S\) and let \(K'_i\) be the oriented framed knot obtained from \(K_i\) by adding an extra \((-k_i)\)-twist, so that the framing of \(K'_i\) is given by \(S\). Then, according to (2.4), we have \(e(t_{K'_1}) = e(t_{K'_2})\). Moreover, applying assertions (iii) and (iv), we obtain: \(e(t_{K'_1}) = e(t_{K_1}) + k_i \cdot s\). We conclude that \(e(t_{K_1}) = e(t_{K_2})\) if and only if \(k_1\) and \(k_2\) are equal modulo 2, proving thus assertion (ii).
Let \( x \in H_1(E(FM); \mathbb{Z}) \), then \( p_*(x) \in H_1(M; \mathbb{Z}) \) can be realized by an oriented knot \( K \) in \( M \): we give it an arbitrary framing. By construction, \( p_*(t_K - x) = 0 \in H_1(M; \mathbb{Z}) \), and so by exactness of the Serre sequence, \( t_K - x = \varepsilon \cdot s \) with \( \varepsilon \in \{0, 1\} \). By possibly band-summing \( K \) with a trivial (+1)-framed knot when \( \varepsilon = 1 \), and according to assertion (iii) and (iv), the framed knot \( K \) can be supposed to be such that \( t_K = x \); this proves assertion (i).

We now restrict ourselves to the 3-manifold \( M = 1_\Sigma = \Sigma \times I \) where \( \Sigma \) can be \( \Sigma_g \) or \( \Sigma_{g,1} \). The inclusion \( i^+: \Sigma \longrightarrow 1_\Sigma \), with image \( \Sigma^+ \), induces an isomorphism between \( H \) and \( H_1(M; \mathbb{Z}) \) and a bijection between \( \text{Spin}(\Sigma) \) and \( \text{Spin}(M) \). As shown by Johnson in [7], there is an algebraic way to think of \( \text{Spin}(\Sigma) \). Indeed, there exists a canonical affine isomorphism

\[
\text{Spin}(\Sigma) \xrightarrow{\simeq} \Omega_g,
\]

sending any spin structure \( \sigma \) to a quadratic form \( q_\sigma \) which can be defined as follows. Let \( x \in H_{(2)} = H_1(\Sigma; \mathbb{Z}_2) \) be represented by an oriented simple closed curve on \( \Sigma^+ \); by framing it along \( \Sigma^+ \) and pushing it into the interior of \( 1_\Sigma \), we get a framed oriented knot \( K \) in \( 1_\Sigma \). Then,

\[
q_\sigma(x) = e(t_K)(\sigma \times I) \in \mathbb{Z}_2.
\]

Therefore, according to Lemma 2.7(a), \((H_1(E(F1_\Sigma); \mathbb{Z}), s)\) is canonically isomorphic to the special Abelian group defined by the pull-back construction

\[
\begin{array}{ccc}
(H,0) \times (H_{(2)},0) & \xrightarrow{p} & B_{g}^{(1)} \times (\mathbb{Z}_2,0) \\
v & & \kappa \\
(H,0) & \xrightarrow{- \otimes \mathbb{Z}_2} & (H_{(2)},0)
\end{array}
\]

whose projections are denoted by \( p \) and \( e \), and where \( \kappa \) is the composite

\[
B_{g}^{(1)} \xrightarrow{\text{id}} B_{g}^{(1)} / B_{g}^{(0)} \xrightarrow{\simeq} H_{(2)}.
\]

The last isomorphism here identifies \( \overline{h} \) with \( h_{(2)} \) for all \( h \in H \) (this is well-defined by (1.2) and (1.3)). We define the special Abelian group \( P \) to be

\[
P = (H,0) \times (H_{(2)},0) \left( B_{g}^{(1)} \times (\mathbb{Z}_2,0) \right),
\]

and \( A_1(P) \) is the space of graphs announced in the introduction.

**Remark 2.8.** Thus, any element \( z \) of \( P \) can be written as

\[
z = (h, \overline{h} + \varepsilon \cdot 1) \in P,
\]
with \( h \in H \) and \( \varepsilon \in \{0, 1\} \). Observe also the following. Suppose that there exists a simple oriented closed curve in \( \Sigma^+ \) with homology class \( h \). Let \( K \) be the push-off of this curve, framed along \( \Sigma^+ \), with an extra \( \varepsilon \)-twist. Then, it follows from (2.5) that \( t_K = z \in P \simeq H_1(E(F1_\Sigma); \mathbb{Z}) \).

**Remark 2.9.** According to the proof of Lemma 2.7, the Serre sequence for homology associated to the bundle \( F1_\Sigma \) gives the following short exact sequence:

\[
0 \rightarrow \mathbb{Z}_2 \rightarrow P \xrightarrow{p} H \rightarrow 0,
\]

where \( \mathbb{Z}_2 \) is injected into \( P \) by sending 1 to \((0, 1)\). The map \( s : H \rightarrow P \) defined by \( s(h) = (h, h) \) is a set-theoretic section. According to (1.2), the associated 2-cocycle \( H \times H \rightarrow \mathbb{Z}_2 \) is the mod 2 reduced intersection form of \( \Sigma \). Thus, \( P \) is isomorphic to \( H \otimes \mathbb{Z}_2 \) with crossed product defined by

\[
(h_1, \varepsilon_1) \cdot (h_2, \varepsilon_2) = (h_1 + h_2, \varepsilon_1 + \varepsilon_2 + h_1 \cdot h_2).
\]

The element \((h, \varepsilon + \varepsilon \cdot 1) \in P \) corresponds to \((h, \varepsilon) \in H \otimes \mathbb{Z}_2 \).

### 2.3. The surgery map \( \psi_1 \)

In this paragraph, \( \Sigma \) is allowed to be \( \Sigma_g \) or \( \Sigma_{g,1} \) and the surgery map \( \psi_1 : \tilde{A}_1(P) \rightarrow \tilde{C}_1(\Sigma) \) is constructed by means of calculi of clovers.

**Convention 2.10.** Here, we adopt Goussarov’s convention for the surgery meaning of \( Y \)-graphs and clovers [4, 2].

Denote by \( \tilde{A}_1(P) \) the free Abelian group generated by abstract \( Y \)-shaped graphs whose univalent vertices are labelled by \( P \), and which are equipped with an orientation at their trivalent vertex: \( \tilde{A}_1(P) \) is a quotient of \( \tilde{A}_1(P) \). For each generator \( \mathcal{Y}[z_1, z_2, z_3] \) of \( \tilde{A}_1(P) \), where \( z_i \in P \), pick some disjoint oriented framed knots \( K_i \) in the interior of \( 1_\Sigma \) such that \( t_{K_i} = z_i \in P \simeq H_1(E(F1_\Sigma); \mathbb{Z}) \); this is possible according to Lemma 2.7(b) (i). Next, pick an embedded 2-disk \( D \) in the interior of \( 1_\Sigma \) and disjoint from the \( K_i \)'s, orient it in an arbitrary way, and connect it to the \( K_i \)'s with some bands \( \varepsilon_i \). These band sums are required to be compatible with the orientations, and to be coherent with the cyclic ordering \((1, 2, 3)\). See Fig. 1 as an illustration. What we obtain in \( 1_\Sigma \) is precisely a \( Y \)-graph, as defined by Goussarov in [4]. We denote it by \( \phi(\mathcal{Y}[z_1, z_2, z_3]) \). For example, as follows from Lemma 2.7(b) (iv), if \( z_1 \) is the special element \( s \) of \( P \), the corresponding leaf \( K_1 \) of \( \phi(\mathcal{Y}[s, z_2, z_3]) \) can be chosen to be unknotted and \((+1)\)-framed; such a leaf is called special in [2].

We now put \( \tilde{\psi}_1(\mathcal{Y}[z_1, z_2, z_3]) \) to be the \( Y_2 \)-equivalence class of the surgered manifold \((1_\Sigma) \phi(\mathcal{Y}[z_1, z_2, z_3]) \), so that we get a map

\[
\tilde{A}_1(P) \xrightarrow{\tilde{\psi}_1} \tilde{C}_1(\Sigma).
\]
Theorem 2.11. The map $\tilde{\psi}_1$ does not depend on the choice of $\phi$, and induces a surjective quotient map
$$\mathcal{A}_1(P) \overset{\psi_1}{\longrightarrow} C_1(\Sigma).$$

Proof. The proof might be read with a copy of [2] in hand. Using the above notation, we begin with showing that $\tilde{\psi}_1(\gamma[z_1, z_2, z_3])$ does not depend on the choice of $\phi(\gamma[z_1, z_2, z_3])$. For this, we recall two facts concerning any Y-graph $G$ in a homology cylinder $M$ (see Remark 2.12 below):

**Fact 1:** the $Y_2$-equivalence class of $M_G$ is not modified when an edge of $G$ is band-summed with a (disjoint) oriented framed knot of $M$;

**Fact 2:** the $Y_2$-equivalence class of $M_G$ is inverted when an edge of $G$ is half-twisted.

Using these, the independance on the choice of the disk $D$, its orientation and the edges $e_i$ is easily shown.

We now show the independance on the choice of the leaves $K_i$. Suppose for example that $K'_1$ is another choice of $K_1$. Then, according to Lemma 2.7(b) (ii), there exists an embedded oriented surface $F$ in $1_S$ such that $\partial F = K_1 \cup (-K'_1)$ and such that, if $k$ (resp. $k'$) is the framing of $K_1$ (resp. $K'_1$) with respect to $F$, $(k - k')$ is even. We also assume transversality of $F$ with the edges of the Y-graph, and with the two other leaves $K_2$ and $K_3$. Let $g(F)$ denote the genus of $F$, let $m$ be the number of intersection points of $F$ with the edges, and for $i = 2, 3$, let $n_i$ be the number of intersection points of $F$ with $K_i$. If all of the integers $g(F), (k - k'), m, n_2$ and $n_3$ are zero, the two Y-graphs are isotopic and we are done. In the general case, recall from [2, §4.3] that there is a procedure for simplifying the leaves. The main tool for this is the following:
Fact 3: if $G_1$ and $G_2$ are two Y-graphs in $1_\Sigma$ obtained from a Y-graph $G$ by splitting a leaf, then $(1\Sigma)_G = (1\Sigma)_{G_1} \cdot (1\Sigma)_{G_2} \in \overline{\mathcal{C}}_1(\Sigma)$ (see Remark 2.12).

Splitting $(g(F) + |k-k'|/2 + m + n_2 + n_3)$ times the leaf $K_1$, splitting $n_2$ times the leaf $K_2$ and splitting $n_3$ times the leaf $K_3$, we see that the result $\hat{\psi}_1(Y[z_1, z_2, z_3])$ in $\overline{\mathcal{C}}_1(\Sigma)$ defined by the choice of $K_1$ differs from the one defined by $K'_1$ by some elements of the form $(1\Sigma)_G$, where $G$ satisfies one of the following conditions:

(i) $G$ has a leaf which bounds a genus 1 surface disjoint from $G$ and with respect to which the leaf is 0-framed;
(ii) $G$ has a leaf which bounds a disk disjoint from $G$, and with respect to which the leaf is $(\pm 2)$-framed;
(iii) $G$ has a leaf which bounds a disk with respect to which it is 0-framed, and this disk intersects $G$ in exactly one point belonging to an edge;
(iv) $G$ has two leaves which are linked as the Hopf link.

Let us now verify that all of these elements vanish in $\overline{\mathcal{C}}_1(\Sigma)$. If $G$ is of type (i), the surgery effect of $G$ is the same as a clover of degree 2 (apply [2, Lemma 5.1] and [2, Theorem 2.4]). If $G$ is of type (ii), by again cutting its leaf we get $(1\Sigma)_G = 2 \cdot (1\Sigma)_{G'}$ where $G'$ has a special leaf; but $(1\Sigma)_{G'} = -(1\Sigma)_{G'}$ by Fact 2. If $G$ is of type (iii), by applying Fact 1 the edge can be slid away from the leaf, we then get a Y-graph with a trivial leaf which has no surgery effect by the “blow-up move” of [2, Fig. 6]. If $G$ is of type (iv), by applying [2, Theorem 2.4], we obtain a Y-graph with a looped edge, but this is stated to be 0 in $\overline{\mathcal{C}}_1(\Sigma)$ by the so-called LOOP relation. This relation is easily shown from [2, Lemma 2.3] and from Facts 1 and 2. This completes the proof of the independency of $\hat{\psi}_1$ on $\phi$.

The fact that $\hat{\psi}_1$ is surjective follows immediately from the fact that the Abelian group $\overline{\mathcal{C}}_1(\Sigma)$ is generated by the homology cylinders $(1\Sigma)_G$ where $G$ is a single Y-graph (this is also proved by standard calculi of clovers).

We now show that the map $\hat{\psi}_1$ factors through $\tilde{A}_1(P) \longrightarrow A_1(P)$. The AS relation is proved in $\overline{\mathcal{C}}_1(\Sigma)$ from Fact 2 and an isotopy of the Y-graph – see [2, Corollary 4.6].

The multilinearity relation follows from Fact 3. Indeed, let $G$ be a Y-graph in $1_\Sigma$ with $K$ as a leaf. Split the leaf $K$ to $K_1$ and $K_2$, and let $G_1$ and $G_2$ be the corresponding new Y-graphs. Then, $\hat{\psi}_1(Y[z_1, z_2, z_3]) = Y[z_1, z_2, z_3] (z_1, z_3 \in P)$ is satisfied in $\overline{\mathcal{C}}_1(\Sigma)$. The slide relation, as stated in §2.1, follows then from the AS and multilinearity relations.
Remark 2.12. The proofs of Facts 1, 2 and 3 use calculus of clovers and can respectively be obtained from the proof of Corollary 4.2, Lemma 4.4 and Corollary 4.3 in [2]. Alternatively, those facts can be considered as corollaries of those results in the following way. Denote by $\mathcal{ZC}_1(\Sigma)$ the free Abelian group generated by the set $\mathcal{C}_1(\Sigma)$, and let

$$\mathcal{ZC}_1(\Sigma) = \mathcal{F}^Y_0(1_\Sigma) \supset \mathcal{F}^Y_1(1_\Sigma) \supset \mathcal{F}^Y_2(1_\Sigma) \supset \cdots$$

be its Goussarov-Habiro filtration [2, §1.4]. Results are stated in [2] to hold in the graded space $G_k(1_\Sigma) = \mathcal{F}^Y_k(1_\Sigma)/\mathcal{F}^Y_{k+1}(1_\Sigma)$. Consider also the homomorphism of Abelian groups

$$\mathcal{ZC}_1(\Sigma) \longrightarrow \mathcal{C}_1(\Sigma)$$

which assigns to any homology cylinder its $Y_2$-equivalence class. The invariant $v$ is primitive, in the sense that it restricts to $\mathcal{C}_1(\Sigma)$ to a monoid homomorphism, and is a degree 1 invariant\(^b\) as follows from calculus of clovers. In particular, $v$ induces a homomorphism $G_1(1_\Sigma) \longrightarrow \mathcal{C}_1(\Sigma)$, by which Facts 1, 2 and 3 are respectively the images of Corollary 4.2, Lemma 4.4 and Corollary 4.3.

3. Johnson Homomorphism and Birman-Craggs Homomorphisms for Homology Cylinders

In this section, the first Johnson homomorphism and the Birman-Craggs homomorphisms are extended to the monoid of homology cylinders.

3.1. The first Johnson homomorphism for homology cylinders

In [3] the notion of Johnson homomorphisms for homology cobordisms over $\Sigma_{g,1}$ was introduced. In this paragraph, we allow $\Sigma$ to be $\Sigma_g$ or $\Sigma_{g,1}$, and give the definition of the first Johnson homomorphism in both cases.

The fundamental group of $\Sigma$ with base point $* \in \Sigma$ will be denoted by $\pi(\cdot)$, and $\pi_k(\cdot)$ will denote the $k^{th}$ term of its lower central series, beginning at $\pi_1(\cdot) = \pi(\cdot)$. We denote by $(x_i, y_i)_{i=1}^g$ the based loops depicted in Fig. 2 or their corresponding images under an inclusion $\Sigma_{g,1} \subset \Sigma_g$. Then,

in the boundary case, $\pi(\cdot) = F(x_1, \ldots, x_g, y_1, \ldots, y_g),$

and in the closed case, $\pi(\cdot) = (x_1, \ldots, x_g, y_1, \ldots, y_g | \prod_{i=1}^g [x_i, y_i] = 1).$

Given a homology cobordism $(M, i^+, i^-) \in \mathcal{C}(\Sigma)$, the map $i^\pm$ induces an isomorphism at the level of each nilpotent quotient (by Stallings [14]). We choose a

\(^b\)In fact, $v$ is a universal degree 1 primitive invariant for homology cylinders. See [6, §6.4] for a similar invariant, of any degree, for knots in the 3-sphere.
path $\gamma \subset M$ going from $i^+(*')$ to $i^-(*')$, and then consider the following composite:

$$\pi_3^{(*)} \xrightarrow{i_3^+} \pi_1(M, i^+(*)) \xrightarrow{c_{\gamma}} \pi_1(M, i^-(*)) \xrightarrow{(i_3^-)^{-1}} \pi_3^{(*)}.$$ 

Up to inner automorphisms, this is independent on the choice of $\gamma$, so that there is a well-defined map

$$\mathcal{C}(\Sigma) \xrightarrow{\eta_{1}^{(*)}} Out \left( \frac{\pi_1}{\pi_3^{(*)}} \right),$$

satisfying $\eta_{1}^{(*)}(M \cdot N) = \eta_{1}^{(*)}(N) \cdot \eta_{1}^{(*)}(M)$. Let $\ast$ be another base point in $\Sigma$, and $\gamma$ an arbitrary path between $\ast$ and $\ast$. Conjugation by $\gamma$ induces an isomorphism $Out \left( \frac{\pi_1}{\pi_3^{(*)}} \right) \simeq Out \left( \frac{\pi_1}{\pi_3^{(*)}} \right)$. This isomorphism is independent on the choice of the path $\gamma$, and the maps $\eta_{1}^{(*)}$ and $\eta_{1}^{(*)}$ are compatible through it. Therefore, we get a well-defined group denoted by $Out(\pi/\pi_3)$ and an anti-homomorphism of monoids

$$\mathcal{C}(\Sigma) \xrightarrow{\eta_{1}} Out \left( \frac{\pi}{\pi_3} \right).$$  \hspace{1cm} (3.6)

If we restrict ourselves to homology cylinders, we are led to a map

$$\mathcal{C}_1(\Sigma) \xrightarrow{\eta_{1}} Ker \left( Out \left( \frac{\pi}{\pi_3} \right) \rightarrow Out \left( \frac{\pi}{\pi_2} \right) \right).$$

Observe the following exact sequence:

$$1 \rightarrow \text{Hom} \left( H, \frac{\pi_2}{\pi_3} \right) \rightarrow \text{Aut} \left( \frac{\pi_3^{(*)}}{\pi_3^{(*)}} \right) \rightarrow \text{Aut} \left( \frac{\pi_1^{(*)}}{\pi_3^{(*)}} \right) \rightarrow \text{Out} \left( \frac{\pi}{\pi_3} \right) \rightarrow \text{Out} \left( \frac{\pi}{\pi_2} \right).$$

where any $f \in \text{Hom} \left( H, \frac{\pi_2}{\pi_3} \right)$ is sent to the automorphism of $\pi_3^{(*)}$ which sends $\pi$ to $\pi f(\pi)$ (with $x \in \pi^{(*)}$). Hence we have the following exact sequence:

$$1 \rightarrow \text{Hom} \left( H, \frac{\pi_2^{(*)}}{\pi_3^{(*)}} \right) \rightarrow \text{Out} \left( \frac{\pi}{\pi_3} \right) \rightarrow \text{Out} \left( \frac{\pi}{\pi_2} \right).$$
Here, \([H, -]\) stands for the subgroup of \(\text{Hom}\left(H, \pi_2^{(s)} / \pi_3^{(s)}\right)\) consisting of those homomorphisms \([h, -]\) defined for any \(h \in H\) by \(x \mapsto [h, x]\), where \(H\) is identified with \(\pi_1^{(s)} / \pi_2^{(s)}\). Consequently, we have defined an anti-homomorphism of monoids

\[ C_1(\Sigma) \xrightarrow{\eta} \text{Hom}\left(H, \pi_2^{(s)} / \pi_3^{(s)}\right) / \langle [H, -] \rangle. \]

In the sequel, we denote by \(L(H) = \oplus_n L_n(H)\), the free Lie \(\mathbb{Z}\)-algebra on the \(\mathbb{Z}\)-module \(H\), and distinguish the boundary case from the closed case.

In the boundary case, as \(\pi^{(s)}\) is free and \(H\) is the Abelianized of \(L^2(H)\) is canonically isomorphic to \(\pi_2^{(s)} / \pi_3^{(s)}\). Also, there is a sequence of isomorphisms \(\text{Hom}(H, L_2(H)) \cong H^* \otimes L_2(H) \cong H \otimes L_2(H)\), with last one induced by \(\bullet\)-duality. Through these, \([H, -] \subset \text{Hom}(H, L_2(H))\) becomes \(A_{g,1} \subset H \otimes L_2(H)\) defined by

\[ A_{g,1} = \left\{ \sum_{i=1}^{g} (x_i \otimes [h, y_i] - y_i \otimes [h, x_i]) \mid h \in H \right\}. \]

Thus, \(\eta_1\) takes values in

\[ \text{Hom}\left(H, \pi_2^{(s)} / \pi_3^{(s)}\right) / \langle [H, -] \rangle \cong H \otimes L_2(H) / A_{g,1}. \]

The group \(\Lambda^3 H\) can be seen as a subgroup of \(H \otimes L_2(H)\) in the following manner:

\[ 0 \longrightarrow \Lambda^3 H \xrightarrow{\nu} H \otimes L_2(H) \xrightarrow{[\cdot, -]} L_3(H), \]

where \(\nu\) is defined by \(\nu(x \wedge y \wedge z) = x \otimes [y, z] + y \otimes [z, x] + z \otimes [x, y]\). Composing \(\nu\) with the projection \(H \otimes L_2(H) \longrightarrow H \otimes L_2(H) / A_{g,1}\) still gives an injection

\[ \Lambda^3 H \xrightarrow{\nu} H \otimes L_2(H) / A_{g,1}. \]

This follows from the fact that

\[ \forall h \in H, \quad [h, \omega] = 0 \in L_3(H) \implies h = 0, \quad (3.7) \]

where \(\omega = \sum_i [x_i, y_i] \in L_2(H)\) corresponds via the canonical isomorphism \(L_2(H) \cong \Lambda^2 H\) to the symplectic element \(\omega\), defined in the introduction.

We now prove that \(\eta_1\) takes values in the subgroup \(\Lambda^3 H\). Suppose for this that \(f \in \text{Hom}\left(H, \pi_2^{(s)} / \pi_3^{(s)}\right) \subset \text{Aut}\left(\pi_2^{(s)} / \pi_3^{(s)}\right)\) is such that there exists a lift \(\tilde{f} \in \text{End}(\pi^{(s)})\) of \(f\) fixing the boundary element \(\hat{\partial} := \prod_{i=1}^{g} [x_i, y_i]\) modulo \(\pi_4^{(s)}\).

Note that this property is verified by a representative for \(\eta_1(M)\) if \(M\) is a homology cylinder, so that proving that \(f \in \text{Ker}([\cdot, -])\) will prove that \(\text{Im}(\eta_1) \subset \Lambda^3 H\). Let \(X_i = x_i^{-1} \tilde{f}(x_i) \in \pi_2^{(s)}\) and \(Y_i = y_i^{-1} \tilde{f}(y_i) \in \pi_2^{(s)}\). We have

\[ \tilde{f}(\hat{\partial}) = \prod_i [\tilde{f}(x_i), \tilde{f}(y_i)] = \prod_i [x_i X_i, y_i Y_i] = \prod_i [x_i, y_i] [X_i, y_i] [x_i, Y_i] \mod \pi_4^{(s)}, \]
which implies that \( \prod_i [X_i, y_i][x_i, Y_i] \equiv 1 \mod \pi_2^*(\Sigma) \). Consequently,
\[
\sum_i (x_i \otimes Y_i - y_i \otimes X_i) \in H \otimes L_2(H),
\]
which essentially corresponds to \( f \), goes to 0 by the bracketting map.

Let us now focus on the closed case. The canonical map \( L_2(H) \to \pi_2^*(\Sigma) / \pi_3^*(\Sigma) \) induces an isomorphism between \( \pi_2^*(\Sigma) / \pi_3^*(\Sigma) \) and \( L_2(H) / \omega \). Thus, in this case, \( \eta_1 \) takes values in
\[
\text{Hom}
\left(
\frac{H_2}{[H,-]}
\right)
\cong \frac{H \otimes L_2(H)}{A_g}
\]
where \( A_g = A_{g,1} + H \otimes \omega \). Since \( \nu(\omega \wedge H) \subset A_g \), \( \nu \) factors to give
\[
\frac{\Lambda^3 H}{\omega \wedge H} \to \frac{H \otimes L_2(H)}{A_g}.
\]
It also follows from (3.7) that this new \( \nu \) is still injective. Then, \( \Lambda^3 H / \omega \wedge H \) can be seen as a subgroup of \( H_2 / [H, \pi_2^*(\Sigma) / \pi_3^*(\Sigma)] / [H, -] \). Similarly to the boundary case, one shows that \( \eta_1 \) takes values in \( \Lambda^3 H / \omega \wedge H \).

So far, we have defined some anti-homomorphisms of monoids
\[
\mathcal{C}_1(\Sigma_{g,1}) \xrightarrow{\eta_1} \Lambda^3 H \quad \text{and} \quad \mathcal{C}_1(\Sigma_g) \xrightarrow{\eta_1} \frac{\Lambda^3 H}{\omega \wedge H},
\]
but next lemma allows us to go a bit further.

**Lemma 3.13.** Let \( (M, K) \) be a homology cylinder over \( \Sigma \) together with a loop \( K \) based on \( * \in M \). Let also \( G \) be a degree 2 clover in \( M \) disjoint from \( K \) and let \( (M_G, K_G) \) be the result of the surgery along \( G \). Then, there exists an isomorphism
\[
\frac{\pi_1(M, *)}{\pi_1(M, *)_3} \cong \frac{\pi_1(M_G, *)}{\pi_1(M_G, *)_3}
\]
sending \([K] \) to \([K_G] \).

This lemma allows us to conclude with the following proposition-definition.

**Proposition 3.14.** For homology cylinders over \( \Sigma = \Sigma_{g,1} \) or \( \Sigma_g \), there are some well-defined homomorphisms
\[
\mathcal{C}_1(\Sigma_{g,1}) \xrightarrow{\eta_1} \Lambda^3 H \quad \text{and} \quad \mathcal{C}_1(\Sigma_g) \xrightarrow{\eta_1} \frac{\Lambda^3 H}{\omega \wedge H}.
\]
Induced by the map (3.6), they are called the first Johnson homomorphisms.

**Remark 3.15.** The composition of \( \eta_1 \) with the map \( C : T(\Sigma) \to \mathcal{C}_1(\Sigma) \) is the classical homomorphism defined in [9].
Proof of Lemma 3.13. Using [2, Lemma 5.1], one shows that
\[ M_G \cong_+ M \setminus \text{int}(N(G)) \cup_{j|_{\partial}} (H_4)_L \]
where \( H_4 \hookrightarrow M \) is an oriented embedding of the standard genus 4 handlebody onto \( N(G) \), which is a regular neighborhood of \( G \) in \( M \), and where \( L \) is the 2-component framed link shown\(^*\) on Fig. 3. Through this diffeomorphism \( K_G \) goes to \( K \subset M \setminus \text{int}(N(G)) \).

Moreover, \( L \) is Kirby-equivalent to the 3-component link \( N \) drawn on the right part of Fig. 3. It turns out that \( N \) is a boundary link. More precisely, up to a \((\pm 1)\)-framing correction, one can push disjointly \( N_3, N_1 \) and then \( N_2 \) to the boundary of \( H_4 \). We obtain some simple closed curves on \( \Sigma_4 = \partial H_4 \), which are bounding curves. Therefore, twist along each of these curves induces the identity at the level of \( \pi_1(\Sigma_4,*)/\pi_1(\Sigma_4,*)_3 \). We then obtain the lemma by a Van-Kampen type argument. \( \square \)

3.2. Birman-Craggs homomorphisms for homology cylinders

Birman-Craggs homomorphisms were defined in [1] and they were enumerated in [8]. Levine also outlined in [12] how they can be extended to homology cylinders. In this paragraph, we review Birman-Craggs homomorphisms in a self-contained way. For this, we use the spin refinement of the Goussarov-Habiro theory of finite type invariants, introduced by the first author in [13].

We first fix a few notation. If \((M, \sigma)\) is a closed spin 3-manifold, let \( R(M, \sigma) \in \mathbb{Z}_{16} \) denote its Rochlin invariant. If \( M \) is a homology sphere, we will denote its (unique) spin structure by \( \sigma_0 \). Recall from [13] that surgery along a \( Y \)-graph makes also sense among spin 3-manifolds:

\[
\left( \text{Data: } (i) \ (M, \sigma), \ a \ \text{closed spin 3-manifold} \right. \\
\left. (ii) \ G, \ a \ Y \text{-graph in } M \right) \rightarrow \text{Result: } (M_G, \sigma_G).
\]

\(^*\)Blackboard framing convention is used.
The following lemma describes precisely how the Rochlin invariant is modified during surgery along a Y-graph.

**Lemma 3.14.** Let \((M, \sigma)\) be a closed spin 3-manifold, and let \(G\) be a \(Y\)-graph in \(M\) whose leaves are ordered, oriented and denoted by \(K_1, K_2\) and \(K_3\). Then,

\[
R(M_G; \sigma_G) - R(M, \sigma) = 8 \cdot \prod_{k=1}^{3} e(t_{K_k})(\sigma) \in \mathbb{Z}_{16},
\]

(3.8)

where \(8: \mathbb{Z}_2 \rightarrow \mathbb{Z}_{16}\) denotes the usual injection, and where \(t_{K_k} \in H_1(E(FM); \mathbb{Z})\) and \(e(t_{K_k}) \in A(\text{Spin}(M), \mathbb{Z}_2)\) have been defined in \S 2.2.

**Proof.** Let \(j : H_3 \rightarrow M\) be the embedding of the genus 3 handlebody, determined (up to isotopy) by the \(Y\)-graph \(G\) in \(M\). Then, it follows from [13, Proposition 1], that the variation \(R(M_G, \sigma_G) - R(M, \sigma)\) only depends on \(j^*(\sigma) \in \text{Spin}(H_3)\). Also, according to Eq. (2.4) from the proof of Lemma 2.7, the rhs of (3.8) is determined by \(j^*(\sigma) \in \text{Spin}(H_3)\).

For \(i_1, i_2, i_3 \in \{0, 1\}\), we denote by \(G_{i_1, i_2, i_3}\) the trivial \(Y\)-graph in \(S^3\) (with ordered and oriented leaves) and whose leaf number \(k\) is trivial and \(i_k\)-framed; we also denote by \(j_{i_1, i_2, i_3} : H_3 \rightarrow S^3\) the corresponding embedding. Then,

\[
\text{Spin}(H_3) = \{j_{i_1, i_2, i_3}^*(\sigma_0) | i_1, i_2, i_3 \in \{0, 1\}\}.
\]

Thus, it is enough to prove (3.8) when \((M, \sigma)\) is \((S^3, \sigma_0)\) and when \(G\) is a \(G_{i_1, i_2, i_3}\), so that we now restrict ourselves to this case. By Lemma 2.7 (b) (iv), the rhs of Eq. (3.8) is 8 if \(i_1 = i_2 = i_3 = 1\) and is 0 otherwise. The same holds for the lhs of Eq. (3.8). Indeed, surgery along a \(Y\)-graph with a trivial leaf has no effect (by the “blow-up move” of [2, Fig. 6]), and surgery on \(S^3\) along \(G_{111}\) gives the Poincaré sphere whose Rochlin invariant is \(8 \in \mathbb{Z}_{16}\). It follows that Eq. (3.8) holds in these eight particular cases.

Let \(\Sigma\) be \(\Sigma_g\) or \(\Sigma_g.1\). Let \(j\) be an oriented embedding of \(\Sigma\) in \(S^3\), and let \(M = (M, i^+, i^-)\) be a homology cylinder over \(\Sigma\). We can then cut \(S^3\) along \(\text{Im}(j)\), and glue back \(M\) (using the identifications \(j, i^+\) and \(i^-\)). We get a new homology sphere which is denoted by

\[
S^3(M, j).
\]

It is shown in [13, Corollary 1] that the Rochlin invariant is a degree 1 invariant: in particular, it is preserved under a \(Y_2\)-surgery. Therefore, \(R(S^3(M, j), \sigma_0)\) only depends on the \(Y_2\)-equivalence class of \(M\) (and \(j\)). Suppose now we are given a surgery presentation of the \(Y_2\)-equivalence class of \(M\) on \(\Sigma\):

\[
\psi_1 \left( \sum_{i=1}^{n} Y \left[ z_1^{(i)}, z_2^{(i)}, z_3^{(i)} \right] \right) = M \in \mathbb{C}_1(\Sigma).
\]
Recall that the labels $z_k^{(i)}$ belong to $P$ and thus give some $e \left( z_k^{(i)} \right) \in B_g^{(1)}$. We also put $\sigma = j^*(\sigma_0) \in \text{Spin}(\Sigma)$, which can be identified with the quadratic form $q_\sigma \in \Omega_g$ according to the Johnson construction (see §2.2). We then deduce from (3.8) the following cubic formula:

$$R \left( \mathbf{S}^3(M, j), \sigma_0 \right) = \frac{\sum_{i=1}^{n} \prod_{k=1}^{3} e \left( z_k^{(i)} \right) (q_\sigma) \in \mathbf{Z}_2 \right).$$

In particular, this shows that:

(i) $R \left( \mathbf{S}^3(M, j), \sigma_0 \right)$ only depends on $\sigma = j^*(\sigma_0) \in \text{Spin}(\Sigma)$ (and the $Y_2$-equivalence class of $M$);
(ii) if $N$ is another homology cylinder over $\Sigma$, then:

$$R \left( \mathbf{S}^3(M \cdot N, j), \sigma_0 \right) = R \left( \mathbf{S}^3(M, j), \sigma_0 \right) + R \left( \mathbf{S}^3(N, j), \sigma_0 \right) \in \mathbf{Z}_2.$$

We now distinguish the case $\Sigma = \Sigma_g$ from the case $\Sigma = \Sigma_{g,1}$.

In the boundary case, any spin structure $\sigma$ on $\Sigma_{g,1}$ can be realized as a $j^*(\sigma_0)$ for a certain embedding $j : \Sigma_{g,1} \hookrightarrow \mathbf{S}^3$. In fact, the specific embeddings of $\Sigma_{g,1}$ whose images are depicted in Fig. 4 do suffice.

As for the closed case, observe that any embedding $j : \Sigma_g \hookrightarrow \mathbf{S}^3$ is splitting, so that $\sigma = j^*(\sigma_0)$ is spin-bounding. Conversely, any spin structure on $\Sigma_g$ which spin-bounds can be so realized: choose an appropriate embedding of $\Sigma_g$ among the particular ones whose images are shown in Fig. 5.

Two other facts about these structures still have to be mentioned. First, $\sigma \in \text{Spin}(\Sigma_g)$ spin-bounds if and only if the Arf invariant $\alpha(q_\sigma)$ vanishes (see [11, p. 36]). Second, if $f$ and $f'$ are two cubic polynomials on $\Omega_g$ (namely $f, f' \in B_g^{(3)}$), then they are identical on the quadratic forms with trivial Arf invariant if and only if $f - f'$ is a multiple of $\alpha$ (see [8, Lemma 14] for a proof\footnote{There, the proof is given for a genus $g \geq 3$, but the same arguments allow us to prove that this fact also holds for a genus $g = 0, 1$ or $2$.} of this algebraic fact).
Characterization of $Y_2$-Equivalence for Homology Cylinders

All of our present discussion leads to the following proposition-definition.

**Proposition 3.17.** There exist some well-defined homomorphisms

\[
\bar{\mathcal{C}}_1(\Sigma_{g,1}) \xrightarrow{\beta} B_g^{(3)} \quad \text{and} \quad \bar{\mathcal{C}}_1(\Sigma_g) \xrightarrow{\beta} \frac{B_g^{(3)}}{\alpha \cdot B_g^{(1)}},
\]

such that, for $M$ a homology cylinder over $\Sigma$ and for $j : \Sigma \hookrightarrow S^3$ an oriented embedding, we have

\[
\beta(M) \left( g_{j^+(\sigma_0)} \right) = \frac{R(S^3(M,j),\sigma_0)}{8} \in \mathbb{Z}_2.
\]

**Remark 3.18.** By composing $\beta$ with the map $C : T(\Sigma) \longrightarrow \bar{\mathcal{C}}_1(\Sigma)$, we obtain the classical Birman-Craggs homomorphisms, as presented by Johnson in [8].

### 4. Proof of the Results

In this section, we prove the results announced in the introduction.

**Convention 4.19.** In the proofs, we will use some specific techniques of Habiro. Recall that its calculus of claspers developed in [6] is based on the definition of surgery along a basic clasper. So as to be consistent with our Convention 2.10, we define here a basic clover $C$ in a 3-manifold $M$ to be the embedding into $M$ of the surface depicted on the left part of Fig. 6. **Surgery along** $C$ is defined as the surgery along the 2-component framed link shown in the right part of Fig. 6. Then, a basic clover is a basic clasper but with opposite surgery meaning. Consequently, before using one of the thirteen Habiro’s moves, we will have to take its mirror image.

#### 4.1. $Y$-equivalence: proof of Proposition 1.1

Since surgeries along clovers preserve homology, the inclusions $\mathcal{C}_1(\Sigma_g) \subset \mathcal{HC}(\Sigma_g)$ and $\mathcal{C}_1(\Sigma_{g,1}) \subset \mathcal{HC}(\Sigma_{g,1})$ are clear.

*Blackboard framing convention is used.*
We now prove the inclusion $\mathcal{HC}(\Sigma_{g,1}) \subset C_1(\Sigma_{g,1})$ using a result of Habegger. For this, we need the following definition. Let $k \geq 0$ be an integer, a homology handlebody of genus $k$ is a pair $(M, i)$ where

(i) $M$ is a compact oriented 3-manifold whose integral homology groups are isomorphic to those of $H_k$, the standard genus $k$ handlebody;

(ii) $i : \Sigma_k = \partial H_k \longrightarrow M$ is an oriented embedding with image $\partial M$.

**Theorem 4.20.** (Habegger, [5]) Let $(M_1, i_1)$ and $(M_2, i_2)$ be genus $k$ homology handlebodies such that

$$\text{Ker} \left( H_1(\Sigma_k; \mathbb{Z}) \xrightarrow{i_1*} H_1(M_1; \mathbb{Z}) \right) = \text{Ker} \left( H_1(\Sigma_k; \mathbb{Z}) \xrightarrow{i_2*} H_1(M_2; \mathbb{Z}) \right).$$

Then, $(M_1, i_1)$ and $(M_2, i_2)$ are $Y$-equivalent.

In the sequel we identify $H_2 g$ with $\Sigma_{g,1} \times I$, and so $\Sigma_{2g}$ with $\partial (\Sigma_{g,1} \times I)$. We also denote by $(H_{2g}, j)$ the standard genus $2g$ handlebody, with inclusion $j : \Sigma_{2g} \longrightarrow H_{2g}$. Any homology cobordism $M = (M, i^+, i^-)$ over $\Sigma_{g,1}$ produces a genus $2g$ homology handlebody $(M, i)$, by defining $i : \Sigma_{2g} \longrightarrow M$ to be the diffeomorphism obtained from the gluing of $i^+$ with $i^-$. Suppose now that $M$ is a homology cylinder. Proving that the homology handlebody $(M, i)$ is $Y$-equivalent to $(H_{2g}, j)$ will imply that the homology cylinder $M$ is $Y$-equivalent to $(\Sigma_{g,1} \times I, \text{Id, Id})$.

For this, let $x_1^*, \ldots, x_g^*, y_1^*, \ldots, y_g^*$ be some disjoint proper arcs in $\Sigma_{g,1}$, which are “dual” to the loops $x_1, \ldots, x_g, y_1, \ldots, y_g$ of Fig. 2, in the sense that $x_k^*$ (resp. $y_k^*$) transversely intersects $x_k$ (resp. $y_k$) once but does not intersect the other loops. For example, choose the first attaching region of each 1-handle. For each $k$, $X_k = x_k^* \times I$ and $Y_k = y_k^* \times I$ are discs in $\Sigma_{g,1} \times I$. The kernel of $i_* : H_1(\Sigma_{2g}) \longrightarrow H_1(\Sigma_{g,1} \times I)$ is spanned by $\partial X_1, \ldots, \partial X_g, \partial Y_1, \ldots, \partial Y_g$. On the other hand, observe that $\pm \partial Y_k$ (resp. $\pm \partial X_k$) is homologous to $x_k \times 0 - x_k \times 1$ (resp. to $y_k \times 0 - y_k \times 1$) in $\Sigma_{2g}$. Therefore, since $M$ is a homology cylinder, $i(\partial X_k)$ and $i(\partial Y_k)$ are null-homologous in $M$. As the kernel of $i_* : H_1(\Sigma_{2g}) \longrightarrow H_1(M)$ has to be of dimension $2g$, it is spanned by $\partial X_1, \ldots, \partial X_g, \partial Y_1, \ldots, \partial Y_g$. It follows from Theorem 4.20 that $(M, i)$ is $Y$-equivalent to $(H_{2g}, j)$, which proves the inclusion $\mathcal{HC}(\Sigma_{g,1}) \subset C_1(\Sigma_{g,1})$.

Let us now justify the inclusion $\mathcal{HC}(\Sigma_g) \subset C_1(\Sigma_g)$. Let $j : \Sigma_{g,1} \hookrightarrow \Sigma_g$ be an embedding and let $D \subset \Sigma_g$ be its complementary disk. Take a homology cobordism $M = (M, i^+, i^-)$ over $\Sigma_{g,1}$. Then, the embedding $(i^+) |_{\partial D} \circ (j |_{\partial D})^{-1} = (i^-) |_{\partial D} \circ (j |_{\partial D})^{-1} : \partial D \hookrightarrow \partial M$ can be stretched to an embedding $\partial D \times I \hookrightarrow \partial M$. The latter allows...
us to attach the 2-handle $D \times I$ to $M$. This results in a homology cylinder over $\Sigma_g$. We have thus defined a filling-up map

$$C(\Sigma_{g,1}) \to C(\Sigma_g),$$

which is obviously surjective. Let $M \in HC(\Sigma_g)$, and pick a $N \in C(\Sigma_{g,1})$ such that $M$ is a filling-up of $N$. Then, $N$ has to be a homology cylinder and so is $Y$-equivalent to $1_{\Sigma_{g,1}}$. We conclude that $M \in C_1(\Sigma_g)$, which completes the proof of Proposition 1.1.

4.2. The boundary case: proof of Theorem 1.3

Recall from Example 2.6 that the Abelian group $A_1(H,0)$ can be identified with $\Lambda^3H$, and likewise $A_1(H(2),0)$ with $\Lambda^3H(2)$. The following lemma will allow us to identify $A_1\left(B^{(1)}_g,\bar{T}\right)$ with $B^{(3)}_g$.

**Lemma 4.21.** Let $\gamma : A_1\left(B^{(1)}_g,\bar{T}\right) \to B^{(3)}_g$ be the map given by multiplying the labels of the abstract $Y$-graphs: $\gamma(\bar{Y}[z_1,z_2,z_3]) = z_1z_2z_3$. Then, $\gamma$ is a well-defined isomorphism.

**Proof.** The fact that $\gamma$ is well-defined is clear. In order to show that $\gamma$ is an isomorphism, it suffices to construct an epimorphism $B^{(3)}_g \to A_1\left(B^{(1)}_g,\bar{T}\right)$ such that $\gamma \circ \epsilon$ is the identity.

By choosing a basis $(e_j)_{j=1}^g$ for $H$, one determines an isomorphism between $B^{(3)}_g$ and $\mathbb{Z}_2 \oplus H(2) \oplus \Lambda^2H(2) \oplus \Lambda^3H(2)$: for $k = 1, 2, 3$ and $j_1, \ldots, j_k \in \{1, \ldots, 2g\}$ pairwise distinct, the monomial $\prod_{i=1}^k e_{j_i}$ is identified with the wedge product $\Lambda_{j_1}^{k-1}e_{j_1}$, and $\bar{T}$ with $1 \in \mathbb{Z}_2$. Since $B^{(1)}_g$ is a period 2 group, so is $A_1\left(B^{(1)}_g,\bar{T}\right)$ by the multilinearity relation. Then, it suffices to define $\epsilon$ on the above mentioned $\mathbb{Z}_2$-basis of $\mathbb{Z}_2 \oplus H(2) \oplus \Lambda^2H(2) \oplus \Lambda^3H(2) \simeq B^{(3)}_g$. We put $\epsilon(1) = \bar{Y} \left[\bar{T},\bar{T},\bar{T}\right]$, $\epsilon(e_j) = \bar{Y} \left[\bar{e}_j,\bar{T},\bar{T}\right]$, $\epsilon(e_{j_1} \land e_{j_2}) = \bar{Y} \left[\bar{e}_{j_1},\bar{e}_{j_2},\bar{T}\right]$ (with $j_1 \neq j_2$) and $\epsilon(e_{j_1} \land e_{j_2} \land e_{j_3}) = \bar{Y} \left[\bar{e}_{j_1},\bar{e}_{j_2},\bar{e}_{j_3}\right]$ (with $j_1, j_2, j_3$ pairwise distinct). The map $\epsilon$ is surjective by the multilinearity and slide relations, and obviously satisfies $\gamma \circ \epsilon = Id$.

Recall from §2.2 that the maps

$$P \to (H,0) \text{ and } P \to \left(B^{(1)}_g,\bar{T}\right)$$

are the canonical projections of the pullback of special Abelian groups

$$P = (H,0) \times_{(H(2),0)} \left(B^{(1)}_g,\bar{T}\right).$$

They happen to be surjective.
Lemma 4.22. The following diagram commutes:

\[
\begin{array}{ccc}
A_1(P) & \xrightarrow{\psi_1} & \mathcal{C}_1(\Sigma_{g,1}) \\
\downarrow{\eta_1} & & \downarrow{\eta_1} \\
A_1(p) & \xrightarrow{} & A_1(H,0).
\end{array}
\]

Proof. Let us verify that \( \eta_1(\psi_1(Y)) = A_1(p)(Y) \) for a generator \( Y = Y[z_1, z_2, z_3] \) of \( A_1(P) \). We put \( M = \psi_1(Y) \), so that \( M = (1_{\Sigma_{g,1}})_G \) where \( G \) is an appropriate \( Y \)-graph as described in \S 2.3. Its leaves are in particular ordered and oriented, they are denoted by \( K_1, K_2 \) and \( K_3; [K_i] = p(z_i) \in H \). Set \( \pi = \pi_1(\Sigma_{g,1}, *) \) and let \( \overline{\eta} \in \pi/\pi_3 \) be represented by \( \eta \in \pi \): we want to compute \( \eta_1(M) \) on \( \overline{\eta} \). This goes as follows: choose an immersed based curve \( k \) in \( \Sigma_{g,1}^+ \) representing \( Y \) (via the identification of \( \Sigma_{g,1} \) with \( \Sigma_{g,1}^+ \)), pick an oriented based knot \( K \subset M \) in a collar of \( \Sigma_{g,1}^+ \) which is a push-off of \( k \), and find another based knot \( K' \subset M \) in a collar of \( \Sigma_{g,1}^- \) such that the pairs \( (M,K) \) and \( (M,K') \) are \( Y_2 \)-equivalent. Then (via the identification of \( \Sigma_{g,1}^- \) with \( \Sigma_{g,1}^+ \)), this knot \( K' \) determines a \( \eta' \in \pi \), and by Lemma 3.13, the result \( \eta_1(M)(\overline{\eta}) \) is then \( \overline{\eta'} \in \pi/\pi_3 \). We now explain the procedure how to construct \( K' \) from \( K \).

In \( 1_{\Sigma_{g,1}} \setminus G \), \( K \) can be pushed down in a collar of \( \Sigma_{g,1}^- \) up to some “fingers” which are of two types (see Fig. 7):

(i) the finger is pointing on an edge of \( G \),
(ii) the finger is pointing on an leaf \( K_i \) of \( G \).

But, each finger of type (i) can be isotoped along the corresponding edge towards its leaf and so can be replaced by two fingers of type (ii), so that up to some isotopy of the immersed curve \( k \) in \( \Sigma_{g,1}^+ \), we can suppose each finger to be of type (ii). Since \( K_i \) has been oriented, each finger comes with a sign. Let \( k_i \) be an immersed curve
on $\Sigma^+_{g,1} \subset 1\Sigma_{g,1}$ such that $[k_i] = p(z_i) \in H$. We can suppose that $K_i$ is a push-off of $k_i$ (with possibly an additional twist): there are then as many fingers as intersection points of $k_i$ with $k$ in $\Sigma^+_{g,1}$; the sign of the finger corresponds with the sign of the intersection point contributing to $[k] \bullet [k_i] \in \mathbb{Z}$.

A finger move can be realized by surgery on a basic clover. Let $K'$ be a copy of $K$ in a collar of $\Sigma_{g,1}^- \subset 1\Sigma_{g,1} \setminus G$. There is then a family of basic clovers $(C_{i,j})_{j=1,\ldots,n_i}$ in $1\Sigma_{g,1} \setminus G$, such that each $C_{i,j}$ has a simple leaf which laces $K'$ and another simple leaf which laces the leaf $K_i$, and such that:

$$(M, K) \text{ is diffeomorphic to } (1\Sigma_{g,1}^-, K')_{\cup_{i,j} G^{(i)}(j)} \cup G.$$  

According to the sign of the corresponding finger, each basic clover comes with a sign denoted by $\varepsilon(i,j)$. Cutting the leaf $K_1$ (see [2, Corollary 4.3]) $n_1$ times, we obtain $n_1$ new $Y$-graphs $G^{(i)}_{j}$ ($j \in \{1, \ldots, n_1\}$): two leaves of $G^{(i)}_{j}$ are copies of $K_2$ and $K_3$, and the third leaf forms with a leaf of $C^{(i)}_{j}$ the Hopf-link. Hence, by applying Habiro’s move 2 (or [2, Theorem 2.4]) to $C^{(i)}_{j} \cup G^{(i)}_{j}$ we obtain a new $Y$-graph still denoted by $G^{(i)}_{j}$. We do the same for $i = 2$ and $i = 3$, therefore:

$$(M, K) \text{ is } Y_2\text{-equivalent to } (1\Sigma_{g,1}^-, K')_{\cup_{i,j} G^{(i)}(j)} \cup G.$$  

Up to $Y_2$-equivalence of the pair $(1\Sigma_{g,1}^-, K')_{\cup_{i,j} G^{(i)}(j)}$, one can suppose that, for each $(i, j)$, the whole of $G^{(i)}_{j}$ lies in a collar neighborhood of $\Sigma_{g,1}^- \subset 1\Sigma_{g,1}$. We now do the surgery along $G$, and then along each of the $G^{(i)}_{j}$; the latter does not modify the 3-manifold $M$ but changes the knot. The new knot we obtain is still denoted by $K'$ and satisfies the announced required properties.

We now calculate the $y' \in \pi$ defined by $K'$. In view of Habiro’s move 10, the contribution of each $Y$-graph $G^{(i)}_{j}$ to the modification of $K'$ is in $\pi$ the commutator $[k_2, k_3^{-1}]^{\varepsilon(1,j)}$. Therefore, we obtain:

$$y' \cdot y^{-1} = \prod_{i \in \mathbb{Z}_3} [k_{i+1}, k_{i+2}^{-1}]^{[k_i]} \in \frac{\pi_2}{\pi_3}, \quad (4.10)$$  

Then, as a homomorphism $H \longrightarrow \pi_2/\pi_3 = L_2(H), \eta_1(M)$ sends any $h \in H$ to

$$- \sum_{i \in \mathbb{Z}_3} (h \cdot p(z_i)) \cdot [p(z_{i+1}), p(z_{i+2})] \in L_2(H).$$

which corresponds to $\sum_{i \in \mathbb{Z}_3} p(z_i) \otimes [p(z_{i+1}), p(z_{i+2})]$ in $H \otimes L_2(H)$, to $p(z_1) \wedge p(z_2) \wedge p(z_3)$ in $\Lambda^3 H$, and so to $\mathcal{A}_1(p)(Y)$.
Lemma 4.23. The following diagram commutes:

\[
\begin{array}{ccc}
A_1(P) & \xrightarrow{\psi_1} & \mathcal{T}_1(\Sigma_{g,1}) \\
\downarrow & & \downarrow \beta \\
A_1(e) & \xrightarrow{} & A_1(\mathcal{B}^{(3)}_g, \mathcal{T})
\end{array}
\]

Proof. According to the definition of $\beta$ we gave in Proposition 3.17, this is a direct consequence of Eq. (3.9).

We still denote by

\[
(H, 0) \xrightarrow{\otimes \mathbb{Z}_2} (H_{(2)}, 0) \text{ and } \left( B^{(1)}_g, \mathcal{T} \right) \xrightarrow{\kappa} (H_{(2)}, 0),
\]

the maps which appear in the pullback diagram for $P$ (see §2.2). Then, as a consequence of the two preceding lemmas, $A_1(\kappa)\beta\psi_1 = A_1(\kappa\kappa) = A_1(\kappa)\left( -\otimes \mathbb{Z}_2 \right)p = A_1(\kappa)\left( -\otimes \mathbb{Z}_2 \right)\eta_1\psi_1$. Since $\psi_1$ is an epimorphism, we get: $A_1(\kappa)\beta = A_1(\kappa)\left( -\otimes \mathbb{Z}_2 \right)\eta_1$.

Construct the following pull-back:

\[
\begin{array}{ccc}
A_1(H, 0) \times A_1(H_{(2)}, 0) & \xrightarrow{A_1(\mathcal{B}^{(3)}_g, \mathcal{T})} & A_1(\mathcal{B}^{(3)}_g, \mathcal{T}) \\
\downarrow & & \downarrow \beta \\
A_1(H, 0) & \xrightarrow{} & A_1(H_{(2)}, 0)
\end{array}
\]

which, through the above mentioned identifications, is essentially the pull-back diagram for $\Lambda^3H \times \Lambda^3H_{(2)} B^{(3)}_g$ appearing in §1.3. By the universal property of the pull-backs, there is then a homomorphism

\[
\mathcal{T}_1(\Sigma_{g,1}) \xrightarrow{(\eta_1, \beta)} A_1(H, 0) \times A_1(H_{(2)}, 0) A_1(\mathcal{B}^{(3)}_g, \mathcal{T}) \simeq \Lambda^3H \times \Lambda^3H_{(2)} B^{(3)}_g.
\]

Moreover, we also have by functoriality another natural map

\[
A_1\left( (H, 0) \times (H_{(2)}, 0) \left( B^{(3)}_g, \mathcal{T} \right) \right) \xrightarrow{\rho} A_1(H, 0) \times A_1(H_{(2)}, 0) A_1(\mathcal{B}^{(3)}_g, \mathcal{T}).
\]

Lemmas 4.22 and 4.23 can then be summarized in the commutativity of the following diagram:

\[
\begin{array}{ccc}
A_1(P) & \xrightarrow{\psi_1} & \mathcal{T}_1(\Sigma_{g,1}) \\
\downarrow \rho & \downarrow \rho \times \rho & \downarrow \rho \times \rho \\
\Lambda^3H \times \Lambda^3H_{(2)} B^{(3)}_g
\end{array}
\]
The following lemma will be the final step in proving Theorem 1.3.

**Lemma 4.24.** The map \( \rho : A_1(P) \to \Lambda^3 H \times \Lambda^3 H_{(2)} B_g^{(3)} \) is an isomorphism.

Assume Lemma 4.24. Then, from the previous commutative diagram, it follows that \( \psi_1 \) is injective, and so is an isomorphism: as a consequence, the same holds for \((\eta_1, \beta)\). The commutativity of

\[
\begin{array}{ccc}
C & \to & \Lambda^3 H \times \Lambda^3 H_{(2)} B_g^{(3)} \\
\downarrow & & \downarrow \\
T_g \to & \nu_g \to & \nu_g
\end{array}
\]

follows from Remarks 3.15 and 3.18. In particular, when \( g \geq 3 \), \( C \) is an isomorphism because \((\eta_1, \beta) : T_{g,1}/T_{g,1}' \to \Lambda^3 H \times \Lambda^3 H_{(2)} B_g^{(3)} \) is so by [10].

**Proof of Lemma 4.24.** We proceed as in Lemma 4.21. It suffices to construct an epimorphism

\[
\Lambda^3 H \times \Lambda^3 H_{(2)} B_g^{(3)} \to A_1(P)
\]

such that \( \rho \circ \epsilon \) is the identity.

Pick a basis \((e_i)_{i=1}^{2g}\) of \( H \): we have seen in the proof of Lemma 4.21 that this choice determines an isomorphism between \( B_g^{(3)} \) and \( \Lambda^3 H_{(2)} \oplus \Lambda^3 H_{(2)} \oplus H_{(2)} \oplus \mathbb{Z}_2 \). Thus, it also defines an isomorphism between \( \Lambda^3 H \times \Lambda^3 H_{(2)} B_g^{(3)} \) and \( \Lambda^3 H \oplus \Lambda^3 H_{(2)} \oplus H_{(2)} \oplus \mathbb{Z}_2 \). We now define \( \epsilon \) by putting

(i) \( \epsilon(e_i \wedge e_j \wedge e_k) = Y [(e_i, \bar{e}_1, (e_j, \bar{e}_j), (e_k, \bar{e}_k)] \), with \( 1 \leq i < j < k \leq 2g \),
(ii) \( \epsilon(e_i \wedge e_j) = Y [(e_i, \bar{e}_i), (e_j, \bar{e}_j), (0, \bar{0})] \), with \( 1 \leq i < j \leq 2g \),
(iii) \( \epsilon(e_i) = Y [(e_i, \bar{e}_i), (0, \bar{0}), (0, \bar{0})] \), with \( 1 \leq i \leq 2g \),
(iv) \( \epsilon(1) = Y [(0, \bar{0}), (0, \bar{0}), (0, \bar{0})] \).

Here, elements of \( P \) are denoted as in Remark 2.8. This assignment well defines \( \epsilon \) because (i) determines \( \epsilon \) on a basis of the free group \( \Lambda^3 H \), while (ii),(iii) and (iv) assign elements of \( A_1(P) \) of order at most 2 to each element basis of the \( \mathbb{Z}_2 \)-vector space \( \Lambda^2 H_{(2)} \oplus H_{(2)} \oplus \mathbb{Z}_2 \). Obviously, \( \epsilon \) followed by \( \rho \) gives the identity. Take now any generator \( Y[z_1, z_2, z_3] \) of \( A_1(P) \). For \( i = 1, 2, 3 \), \( z_i \in P \) can be written as a linear combination of some \((e_j, \bar{e}_j)\) and \((0, \bar{0})\). The multilinearity, AS and slide relation allow us to conclude that \( Y[z_1, z_2, z_3] \) is realized by \( \epsilon \). Thus, \( \epsilon \) is surjective.

\[\square\]

### 4.3. The closed case: proof of Theorem 1.4

An isomorphism

\[
A_1(P) \xrightarrow{\rho} \Lambda^3 H \times \Lambda^3 H_{(2)} B_g^{(3)}
\]
is defined formally in the same way as in the boundary case (see Lemma 4.24). Recall that $S$ stands for the subgroup of the pullback $\Lambda^3 H \times_{\Lambda^2 H(z)} B_g^{(3)}$ corresponding to $\omega \wedge H \subset \Lambda^3 H$ and $\alpha \cdot B_g^{(1)} \subset B_g^{(3)}$. Then, $\rho^{-1}(S)$ is the subgroup of $A_1(P)$ comprising the elements

$$\sum_{i=1}^{g} Y [(x_i, \overline{x_i}), (y_i, \overline{y_i}), z], \quad \text{where } z \text{ is any element of } P.$$ 

**Lemma 4.25.** In the closed case, the surgery map $\psi_1$ defined in §2.3 vanishes on the subspace $\rho^{-1}(S)$.

As mentioned in the introduction, these symplectic relations $\rho^{-1}(S)$ appears in [6] for higher degrees.

**Proof of Lemma 4.25.** Let $z \in P$, we aim to show that

$$\sum_{i=1}^{g} \psi_1 (Y [(x_i, \overline{x_i}), (y_i, \overline{y_i}), z]) = 0 \in C_1(\Sigma_g).$$ 

(4.11)

Consider in $1_{\Sigma_g}$ a basic clover $G$ with one trivial leaf $f$, and the other leaf $f'$ satisfying $t_{f'} = z \in P$. Then, $f$ being trivial, $(1_{\Sigma_g})_G$ is diffeomorphic to $1_{\Sigma_g}$. Furthermore, $f$ can be seen as a push-off of $\partial D$ where $D$ is a 2-disk in $\Sigma_g^+$; in particular, $f$ bounds the push-off of $\Sigma_g^+ \setminus D$ which is an embedded genus $g$ surface. By applying Habiro’s moves 7 and 5, $f$ can be split in $g$ pieces so that $G$ is equivalent to the union of $g$ basic clovers denoted by $G_1, \ldots, G_g$. See Fig. 8. Each clover $G_i$ has

a leaf which bounds a genus 1 surface; by applying Habiro’s move 10, it is seen to be equivalent to a $Y$-graph $G'_i$. According to Remark 2.8, the leaves of $G'_i$ represent $(x_i, \overline{x_i}), (y_i, \overline{y_i})$ and $z$ in $P$, so that $(1_{\Sigma_g})_{G'_i} = \psi_1 (Y [(x_i, \overline{x_i}), (y_i, \overline{y_i}), z]) \in C_1(\Sigma_g)$. Equation (4.11) then follows. \qed
By the same arguments, appropriate versions of Lemmas 4.22 and 4.23 hold in the closed case: $A_1(p) = \eta_1 \circ \psi_1$ and $A_1(e) = \beta \circ \psi_1$. This leads us to a commutative diagram

$$
\begin{array}{ccc}
A_1(P) & \xrightarrow{\psi_1} & \mathcal{C}_1(\Sigma_g) \\
\rho^{-1}(S) & \xrightarrow{\rho} & \mathcal{C}_1(\Sigma_g) \\
\xrightarrow{\sim} & (\eta_1, \beta) & \xrightarrow{\sim} \\
\Lambda^3 H \times \Lambda^3 H(2) & \xrightarrow{\lambda(3)} & \Lambda^3 H \times \left( \frac{\Lambda^3 H(2)}{\omega \wedge H} \right) \times \left( \frac{B_0(3)}{\Sigma_{(2)} H(2)} \right) \\
S & \xrightarrow{\alpha \cdot B_0(3)} & \alpha \cdot B_0(3)
\end{array}
$$

from which it follows that $\psi_1$, and then $(\eta_1, \beta)$, are isomorphisms. The commutativity of the right triangle in Theorem 1.4 is still given by Remarks 3.15 and 3.18.

### 4.4. Finite type invariants of degree 1: proof of Corollary 1.5

The equivalence (a)$\Rightarrow$(b) immediately results from the existence of the universal degree one primitive invariant $\upsilon$ introduced in Remark 2.12. The equivalence (c)$\Rightarrow$(a) is a direct consequence of Theorems 1.4 and 1.3.

### 4.5. From the boundary case to the closed case

In this last paragraph, we fix an isomorphism

$$H_1(\Sigma_{g,1}; \mathbb{Z}) \xrightarrow{\phi} H_1(\Sigma_g; \mathbb{Z}) .$$

It allows us to identify the sets $H$, $\Omega_g \simeq Spin(\Sigma)$, $B_g$ and $P$ corresponding to $\Sigma_{g,1}$ with those of $\Sigma_g$.

Moreover, let $j : \Sigma_{g,1} \hookrightarrow \Sigma_g$ be an embedding such that $j_\ast = \phi$ at the level of $H_1(-; \mathbb{Z})$. Recall from §4.1 the filling-up map, which can be restricted to

$$\mathcal{C}_1(\Sigma_{g,1}) \xrightarrow{j} \mathcal{C}_1(\Sigma_g) .$$

Note that it is compatible with the “extending by the identity” map $T_{g,1} \longrightarrow T_g$ defined by $j$, and that it induces a group homomorphism $\overline{\mathcal{C}}_1(\Sigma_{g,1}) \longrightarrow \overline{\mathcal{C}}_1(\Sigma_g)$. The latter can be verified to be independent on the choice of the embedding $j$ such that $j_\ast = \phi$, and so can be denoted by

$$\overline{\mathcal{C}}_1(\Sigma_{g,1}) \xrightarrow{\phi} \overline{\mathcal{C}}_1(\Sigma_g) .$$

The commutativity of the following diagram is easily proved from the various definitions:
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