BORROMEAN SURGERY AND THE CASSON INVARIANT

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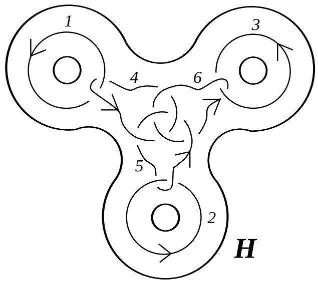
Abstract. This note is based on a talk given at the conference *Intelligence of low dimensionnal Topology*, held in Osaka in November 2005. All results are taken from [M] (except for §2), where detailed proof can be found.

1. Motivations

Let M be a closed oriented 3-manifold. A *Borromean surgery* move on M is defined as the surgery along a link L obtained by embedding in M a genus 3 handlebody H containing a copy of the 6-component framed oriented link depicted below (we make use of the blackboard framing convention).



We call L a *Borromean surgery link*. This notion was first introduced by S. Matveev [Mt] in slightly different terms. Matveev showed that two closed oriented 3-manifolds are *Borromean equivalent*, i.e. are related by a sequence of such surgery moves, if and only if they have the same homology and linking form. As a consequence, every oriented integral homology 3-sphere is obtained from S^3 by surgery along claspers. It is thus a natural problem to give easily computable formulas for the variation of the Casson invariant λ under such a surgery move.

Recall that two integral homology spheres are always related by a sequence of (± 1)-framed surgeries along knots. A. Casson expressed the variation of the Casson invariant λ under such a surgery move as a coefficient of the Alexander-Conway polynomial Δ:

$$\lambda(M_{K_{\pm 1}}) - \lambda(M) = \pm\frac{1}{2}\Delta''_K(1),$$

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where $M_{K_{\pm 1}}$ denotes the result of $(\pm 1)-$framed surgery along a knot K in an integral homology sphere M. The aim of this note is to provide such an easily-computable formula for the variation of λ under Borromean surgery.

Notations 1.1. A Borromean surgery link $L = \cup_{1\leq i\leq 6} L_i$ in a closed oriented 3-manifold M inherits its orientation and ordering from the 6-component link depicted above. We denote by f_i the framing of L_i, and by l_{ij} the linking number $lk(L_i, L_j)$, $1 \leq i \neq j \leq 6$.

2. First method: using Lescop's formula

In this section, we compute the variation of the Casson invariant under a Borromean surgery move using a global surgery formula of C. Lescop [L2]. Three distinct formula are actually presented in Lescop's monograph ; here we consider the first one [L2, 1.4.8], which involves the multivariable Alexander polynomial Δ.

2.1. **Statement of Lescop's formula.** Let $K = K_1 \cup ... \cup K_n$ be an oriented link with <u>integral</u> framing in an <u>integral</u> homology sphere M. We denote by k_{ii} the framing of K_i, and $k_{ij} := lk(K_i, K_j)$ for $i \neq j$. Let $N = \{1, 2, ..., n\}$. For every non-empty subset $I \subset N$, let $K_I := \cup_{i\in I} K_i$.

The variation $\lambda(M_K) - \lambda(M)$ is given by the formula

$$(-1)^{b_-(K)} \sum_{\substack{I\subset N\\ I\neq\emptyset}} \det\left(E(K_{N\setminus I})\right) \left(\zeta(K_I) + \frac{(-1)^{|I|}}{24} M_8(K_I)\right) + \frac{b_+(K) - b_-(K)}{8}, \tag{2.1}$$

where, for every non-empty $I \subset N$,

- $E(K_{N\setminus I})$ is the linking matrix of $K \setminus K_I = K_{N\setminus I}$,
- $b_+(K)$ (resp. $b_-(K)$) is the number of positive (resp. negative) eigenvalues of the linking matrix of K,
- $|I|$ is the cardinality of I,
-
$$\zeta(K_I) = \begin{cases} \frac{1}{2}\Delta''_{K_I}(1) - \frac{1}{24} & \text{if } I = \{i\}, \\ \frac{\partial^{|I|}}{\partial t_1 ... \partial t_{|I|}} \Delta_{K_I}(1, ..., 1) & \text{if } |I| > 2, \end{cases}$$
-
$$M_8(K_I) = \begin{cases} k_{ii} + 1 & \text{if } I = \{i\}, \\ \sum_{(j,\sigma)\in I\times\sigma_I} k_{j\sigma(1)} k_{j\sigma(1)\sigma(2)} ... k_{\sigma(|I|-1)\sigma(|I|)} k_{\sigma(|I|)j} & \text{if } |I| > 2, \end{cases}$$

  where σ_i denotes the set of all bijections from $\{1, ..., |I|\}$ to I.

For details, and for a more general statement, we refer the reader to [L2, pp. 10–13].

2.2. **Lescop's Borromean surgery formula.** From now on, let us consider in M a Borromean surgery link L as defined in §1.2, and let us use the notations introduced there. We have $b_\pm(L) = 3$.

For any non-empty $I \subset \{1, ..., 6\}$, set

$$T_I := \zeta(L_I) + \frac{(-1)^{|I|}}{24} M_8(L_I).$$

By computing all the determinants in (2.1), we obtain that $\lambda(M_L) - \lambda(M)$ equals

$$f_1.T_4 + f_2.T_5 + f_3.T_6 + T_{14} + T_{25} + T_{36}$$

$$+ \begin{vmatrix} f_1 & l_{12} \\ l_{12} & f_2 \end{vmatrix} .T_{45} - \begin{vmatrix} f_1 & l_{13} \\ l_{13} & f_3 \end{vmatrix} .T_{46} - \begin{vmatrix} f_2 & l_{23} \\ l_{23} & f_3 \end{vmatrix} .T_{56} - \begin{vmatrix} f_1 & l_{12} & l_{13} \\ l_{12} & f_2 & l_{23} \\ l_{13} & l_{23} & f_3 \end{vmatrix} .T_{456}$$

$$-f_2.T_{145} - f_3.T_{146} - f_1.T_{245} - f_3.T_{256} - f_1.T_{346} - f_2.T_{356}$$

$$+ \begin{vmatrix} f_1 & l_{12} \\ l_{12} & f_2 \end{vmatrix} .T_{3456} + \begin{vmatrix} f_1 & l_{13} \\ l_{13} & f_3 \end{vmatrix} .T_{2456} + \begin{vmatrix} f_2 & l_{23} \\ l_{23} & f_3 \end{vmatrix} .T_{1456}$$

$$-T_{1245} - T_{1346} - T_{2356} + f_1.T_{23456} + f_2.T_{13456} + f_3.T_{12456} + T_{123456}.$$

Before computing the various T_I's, we make two observations which greatly simplify the above formula.

Lemma 2.1. $f_1.T_4 + T_{14} = f_2.T_5 + T_{25} = f_3.T_6 + T_{36} = 0.$

Proof. Let us prove that $f_1.T_4 + T_{14} = 0$. Consider in M the 2-component link $L_1 \cup L_4$: as L_4 is a 0-framed copy of a meridian of L_1, it follows from the Kirby calculus that $M_{L_1 \cup L_4} \simeq M$. Therefore the variation of the Casson invariant $\lambda(M_{L_1 \cup L_4}) - \lambda(M)$ must be 0. By (2.1), this variation equals $f_1.T_4 + T_{14}$. The two remaining equalities are proved in exactly the same way. □

Lemma 2.2. $$\begin{vmatrix} f_1 & l_{12} \\ l_{12} & f_2 \end{vmatrix} .T_{45} - f_2.T_{145} - f_1.T_{245} - T_{1245} = 0,$$

$$\begin{vmatrix} f_1 & l_{13} \\ l_{13} & f_3 \end{vmatrix} .T_{46} - f_3.T_{146} - f_1.T_{346} - T_{1346} = 0,$$

$$\begin{vmatrix} f_2 & l_{23} \\ l_{23} & f_3 \end{vmatrix} .T_{56} - f_3.T_{256} - f_2.T_{356} - T_{2356} = 0.$$

Proof. These three equalities are proved by exactly the same argument as for Lemma 2.1, by considering in M the 4-component links $L_1 \cup L_2 \cup L_4 \cup L_5$, $L_1 \cup L_3 \cup L_4 \cup L_6$ and $L_2 \cup L_3 \cup L_5 \cup L_6$ respectively. □

So we obtain

Theorem 2.3. *Let L be a Borromean surgery link in $M \in \mathbb{Z}HS$. Then $\lambda(M_L) - \lambda(M)$ equals*

$$-\begin{vmatrix} f_1 & l_{12} & l_{13} \\ l_{12} & f_2 & l_{23} \\ l_{13} & l_{23} & f_3 \end{vmatrix}.\zeta(B) - \sum_{\circlearrowleft_{1,2,3}} \begin{vmatrix} f_1 & l_{12} \\ l_{12} & f_2 \end{vmatrix}.\zeta(B \cup L_3) - \sum_{\circlearrowleft_{1,2,3}} f_1.\zeta(B \cup L_2 \cup L_3) - \zeta(L),$$

where the sum is over all cyclic permutations of the indices $(1, 2, 3)$ *and where* B *denotes the Borromean link* $L_4 \cup L_5 \cup L_6$.

One can easily check that $\zeta(B) = 1$, and *a priori* there remain 7 multivariable Alexander polynomials to be computed. We will see later that this formula can be further simplified – see §4.4.

3. Second method: using claspers

3.1. Clasper theory for 3-manifolds.

Definition 3.1. Let M be a compact connected oriented 3-manifold. A *clasper* G in M is an embedding

$$G : F \longrightarrow M$$

of a surface F which is a thickening of a (non-necessarily connected) uni-trivalent graph having a copy of S^1 attached to each of its univalent vertices (here we do not allow connected components without trivalent vertex).
The (thickened) circles are called the *leaves* of G, the trivalent vertices are called the *nodes* of G and we still call the thickened edges of the graph the *edges* of G.
The *degree* of a clasper G is the minimal number of nodes of its connected components.

Given a clasper G in M, there is a precise procedure to construct, in a regular neighbourhood of the clasper, an associated framed link L_G: *surgery along the clasper* G simply means surgery along L_G. Though the procedure for the construction of L_G will not be explained here, it is well illustrated by the two examples of Figure 3.1.
We respectively call these two particular types of claspers $\mathbf{Y}$*-graphs* and $\mathbf{H}$*-graphs*. Ob-

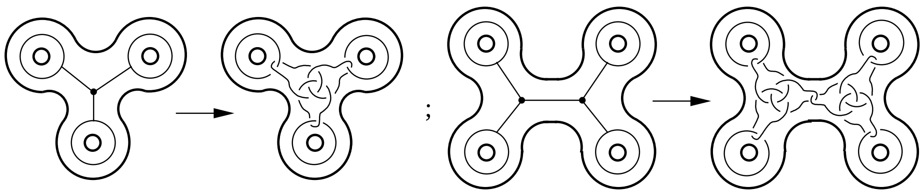


Figure 3.1. A degree 1 and a degree 2 clasper and the associated framed links in their regular neighbourhoods.

viously, surgery along a $\mathbf{Y}$-graph is a Borromean surgery.

3.2. **Claspers and finite type invariants.** Let $\mathcal{S}$ denote the free $\mathbf{Z}$-module generated by orientation-preserving diffeomorphism classes of integral homology spheres. In $\mathcal{S}$, put

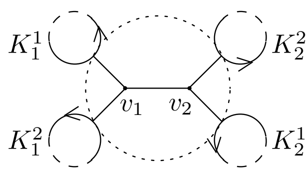
$$[M;F]=\sum_{F'\subset F}(-1)^{|F'|}S^3_{F'},$$

where the sum ranges over all possible subset of F (starting from the empty set). Denote by $\mathcal{S}_k$ the submodule generated by the elements $[M;F]$ for all claspers $F=F_1\cup...\cup F_p$ in an integral homology sphere M such that $\sum_i degF_i=k$.
We call *finite type invariant of degree* k any map $f:\mathcal{S}\longrightarrow A$, where A is an Abelian group, which vanishes on $\mathcal{S}_{k+1}$.

Proposition 3.2. [O1, GGP] *The Casson invariant is a finite type invariant of degree* 2.

As an immediate consequence, $\lambda(M_G)-\lambda(M)=0$ for all clasper G of degree at least 3 in an integral homology sphere M.

3.3. **Variation of the Casson invariant under Y_2-surgery.** Let G be a connected degree 2 clasper in M, whose vertices are denoted by v_1 and v_2. By [H, Prop. 2.7, move 2], we can freely assume that G is a $\mathbf{H}$-graph. Up to isotopy, we can always suppose that there exists a 3-ball in M which intersects the edges and nodes of G as shown below.



G is an oriented surfaces, so at each vertex the two adjacent leaves are naturally ordered: we denote by K_i^1 and K_i^2 the two leaves adjacent to v_i $(i=1,2)$. The leaves are oriented as depicted.

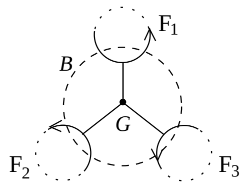
Theorem 3.3. *Let M be an integral homology sphere, and G be a connected degree* 2 *clasper in M. Then*

$$\lambda(M_G)-\lambda(M)=-2\big(lk(K_1^1,K_2^2).lk(K_1^2,K_2^1)-lk(K_1^1,K_2^1).lk(K_1^2,K_2^2)\big).$$

This formula is essentially well-known. It comes as a consequence of [GR, Prop. 3.4], which expresses the degree $2n$ of the Aarhus integral of $S^3_G-S^3$, where G is a connected degree $2n$ clasper. It can also be seen as a direct corollary of the sum formula of C. Lescop [L2]. Alternatively, one can prove it using elementary properties of the Casson invariant and standard calculus of claspers [M].

4. Variation of the Casson invariant under Y_1-surgery

4.1. **Statement of the main result.** Let G be a connected degree 1 clasper in an integral homology sphere M. Up to isotopy one can always assume that there is a 3-ball B in M which intersects G as depicted below.



Fix an orientation of the vertex of G. We denote the three leaves of G by F_1, F_2 and F_3 according to this orientation. Denote by f_i the framing of F_i, and by l_{ij} the linking number $lk(F_i, F_j)$, $1 \leq i \neq j \leq 3$ (the leaves are oriented as in the figure above).

Theorem 4.1. *The difference $\lambda(M_G) - \lambda(M)$ is given by the formula*

$$-f_1.f_2.f_3 - 2.l_{12}.l_{13}.l_{23} - 2.\mu_{123}(G) - 2.l_{12}.l_{23} + \sum_{\circlearrowleft 1,2,3} l_{23}.(l_{23}+1).f_1,$$

where the sum is over all cyclic permutations of the indices $(1, 2, 3)$ *and where* $\mu_{123}(G)$ *denotes* Milnor's triple linking number of G.

Indeed, as explained in the next subsection, there is a 3-component string link σ_G naturally associated to G.[1] $\mu_{123}(G)$ is defined as Milnor's triple linking number of this string link – see below.

4.2. **Milnor invariants for Y-graphs.** For a definition of Milnor's $\overline{\mu}$-invariant for links, see [Mi1, Mi2]. It is well-known that Milnor's $\overline{\mu}$-invariants of lenght n are only well-defined modulo some indeterminacy coming from Milnor's invariants of length $< n$. In particular, for $n = 3$, *Milnor's triple linking number* $\overline{\mu}_{ijk}(L)$ is defined modulo the greatest commun divisor of the linking numbers of components i, j, k of the link L. This indeterminacy comes from the indeterminacy of representing the link as the closure of a string link [HL].

Observe that the leaves of a **Y**-graph G in a homology sphere M form a 3-component link with a natural 'basing' (a base point on each component) given by the attaching region of the edges. There is thus a natural way of representing this link as the closure of a string link. Consider a 3-ball B as in §4.1. By isotopying the leaves of G, we can regard $G \cap (S^3 \setminus B)$ as a framed 3-component string link σ_G in the homology ball $M \setminus B$ – see Fig.4.1. We call σ_G the *string link representation of* G.

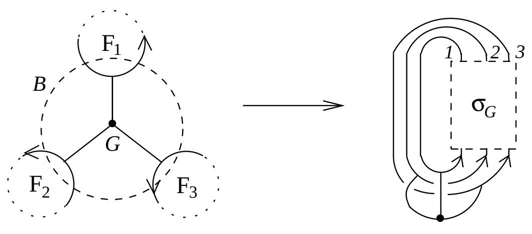


FIGURE 4.1. The string link representation of a **Y**-graph.

[1]Recall that a string link is a pure tangle, without closed components.

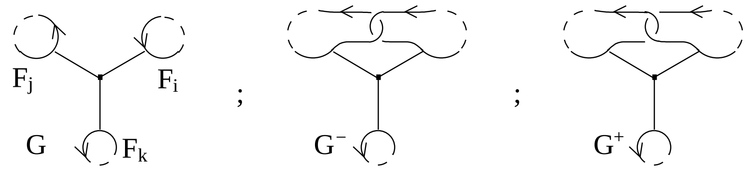
By *Milnor's triple linking number of G* $\mu_{ijk}(G)$, we mean the Milnor invariant μ_{ijk} of σ_G (recall that the definition of Milnor's invariant for string links extends to string links in any homology ball).

It is worth recalling that (at least in S^3) Milnor's triple linking is easily computed using Polyak's Skein relations, which only involve the linking number [P].

4.3. **Sketch of the proof of the main result.** For simplicity, let us just consider here the case where G is a $\mathbf{Y}$-graph in $M = S^3$. The proof goes in three steps, where we simplify the clasper until we obtain an elementary situation where the Casson invariant is easily computed. At each step, we perform simple transformations on the clasper, so that we can compute how the various invariants involved in the statement vary. The reader is refered to [M] for a detailed proof.

Step 1. *Killing the linking numbers.* The following lemma allows us to modify the linking numbers of the leaves.

Lemma 4.2. [M, Lem.4.4] *Let* G^+ *and* G^- *be obtained from* G *as depicted below.*



We have

$$\lambda(S^3_G) = \lambda(S^3_{G^-}) - 2.l_{ik}.l_{jk} + 2.f_k.l_{ij},\ \lambda(S^3_G) = \lambda(S^3_{G^+}) + 2.l_{ik}.l_{jk} - 2.f_k.(l_{ij}+1).$$

By suitably applying $|l_{23}| + |l_{13}| + |l_{12}|$ times this lemma to G, we obtain a $\mathbf{Y}$-graph G_0 whose leaves have zero linking numbers and such that

$$\lambda(S^3_G) = \lambda(S^3_{G_0}) - 2.l_{12}.l_{13}.l_{23} + \sum_{\circlearrowleft_{1,2,3}} f_1.l_{23}.(l_{23}+1). \tag{4.1}$$

On the other hand, the framing on the leaves is unchanged, and one can check, using the Skein formulas in [P], that

$$\mu_{123}(G) - \mu_{123}(G_0) = -l_{12}.l_{23}. \tag{4.2}$$

Step 2. *Unknotting and pairwise unlinking the leaves.* Consider any pair of leaves of G_0, regarded as a 2-component link in S^3. As it has zero linking number, this link is Δ-equivalent to the 2-component unlink by [MN]. Each Δ-move is achieved by surgery along a $\mathbf{Y}$-graph whose leaves are small meridians of these two leaves [H, §7.1]. See the left part of Fig. 4.2 for an illustration. There is thus a sequence of n Δ-moves (of the type described above)

$$G_0 = G^1 \mapsto G^2 \mapsto \ldots \mapsto G^n = G^u_0,$$

such that the leaves of G^u_0 form pairwise a 2-component unlink.
Now, by using some calculus of clasper and Prop. 3.2, we have, for each $1 \le k \le n-i$,

$$\lambda(S^3_{G^{k+1}}) = \lambda(S^3_{G^k}) + \lambda(S^3_{H^k}),$$

where H^k is a degree 2 clasper as on the right part of Fig. 4.2. By Thm.3.3 we clearly have $\lambda(S^3_{H^k}) = 0$. Thus

$$\lambda(S^3_{G_0}) = \lambda(S^3_{G_0^u}).$$

Also, these moves do not change the framings, linking numbers and triple linking number.

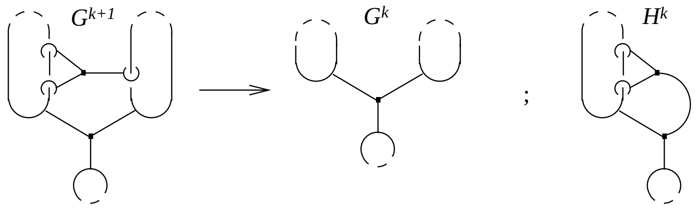
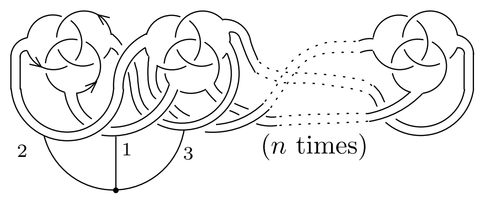


FIGURE 4.2. A Δ-move which doesn't affect the Casson invariant.

Step 3. *Killing the triple linking number and the framings.* For simplicity, let us suppose that $n := \mu_{123}(G_0^u) > 0$. Let $G_{(n)}$ be the **Y**-graph depicted below

$G_{(n)}$ 

where the three leaves are f_i-framed ($1 \leq i \leq 3$). Clearly $\mu_{123}(S^3_{G_{(n)}}) = n$. By [TY] there is a sequence of *clasp-pass moves*

$$G_0^u = G_1 \mapsto G_2 \mapsto ... \mapsto G_p = G_{(n)},$$

where each clasp-pass move is realized by surgery along a degree 2 clasper whose leaves are all copies of a small meridians of a leaf of G_k. By strictly the same arguments as in Step 2, we have

$$\lambda(S^3_{G_0^u}) = \lambda(S^3_{G_0}) = \lambda(S^3_{G_{(n)}}).$$

Now we have to compute $\lambda(G_m^\varepsilon)$. By the calculus of clasper and Prop. 3.2, we have

$$\lambda(S^3_{G_{(n)}}) = n.\lambda(S^3_{G_{(1)}}) + \lambda(S^3_{G_f}),$$

where the leaves of G_f are three f_i-framed trivial knots ($1 \leq i \leq 3$).
It is not hard to check that $\lambda(S^3_{G_{(1)}}) = -2$ and $\lambda(S^3_{G_f}) = -f_1.f_2.f_3$ (see [M, Lem. 3.1 and 4.3]). We thus obtain

$$\lambda(S^3_{G_{(n)}}) = -f_1.f_2.f_3 - 2.\mu_{123}(G_0^u). \tag{4.3}$$

Theorem 4.1 (for $M = S^3$) follows from (4.1), (4.2) and (4.3).

4.4. **More on Lescop's Borromean surgery formula.** Let L denote the 6-component link associated to a $\mathbf{Y}$-graph G in M. We make use of Notations 1.1.
By the same arguments as in Step 2 of §4.3, we can suppose for $1 \le i, j \le 3$ that each L_i is an unknot and each pair $L_i \cup L_j$ is a band-sum of $|l_{ij}|$ copies of the Hopf link (with the appropriate sign). Using these models, we are able to compute the two sums in the formula of Theorem 2.3. We obtain

$$\zeta(B \cup L_i) = 0 \quad \text{and} \quad \zeta(B \cup L_i \cup L_j) = -l_{ij}, \qquad \text{for all } 1 \le i \neq j \le 3.$$

It follows that $\lambda(M_G) - \lambda(M)$ is given by

$$-f_1.f_2.f_3 - 2.l_{12}.l_{13}.l_{23} - \zeta(L) + \sum_{\circlearrowleft_{1,2,3}} l_{23}.(l_{23}+1).f_1.$$

In particular we obtain

$$\frac{\partial^6}{\partial t_1 ... \partial t_6} \Delta_L(1, ..., 1) = 2.\mu_{123}(G) + 2.l_{12}.l_{23}.$$

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