NORMALIZED ENTROPY VERSUS VOLUME FOR PSEUDO-ANOSOVS

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Abstract. Thanks to a recent theorem by Jean-Marc Schlenker, we establish an explicit linear inequality between the normalized entropies of pseudo-Anosov automorphisms and the hyperbolic volumes of their mapping tori. As its corollaries, we give an improved lower bound for values of entropies of pseudo-Anosovs on a surface with fixed topology, and an alternative proof of a result of Farb, Leininger and Margalit, and independently of Agol, on finiteness of cusped manifolds generating surface automorphisms with small normalized entropies.

1. Introduction

Let $\Sigma = \Sigma_{g,n}$ be an orientable surface of genus g with n punctures. We will suppose that $3g - 3 + n \geq 1$ so that Σ admits a Riemannian metric of constant curvature -1, a hyperbolic structure of finite area, which, by Gauss-Bonnet, satisfies $\mathrm{Area}\,\Sigma = 2\pi|\chi(\Sigma)| = 2\pi(2g - 2 + n)$ with respect to the hyperbolic metric.

The isotopy classes of orientation preserving automorphisms of Σ, called mapping classes, were classified into three families by Nielsen and Thurston [20], namely periodic, reducible and pseudo-Anosov. Choose a representative h of a mapping class φ, and consider its mapping torus,

$$\Sigma \times [0,1]/(x,1) \sim (h(x),0).$$

Since the topology of the mapping torus depends only on the mapping class φ, we denote its topological type by N_φ

A celebrated theorem by Thurston [21] asserts that N_φ admits a hyperbolic structure iff φ is pseudo-Anosov. By Mostow-Prasad rigidity a hyperbolic structure of finite volume in dimension 3 is unique and geometric invariants are in fact topological invariants. In [11], Kin, Takasawa and the first named author compared the hyperbolic volume of N_φ, denoted by $\mathrm{vol}\,N_\varphi$, with the entropy of φ, denoted by $\mathrm{ent}\,\varphi$, where it is defined by the infimum of topological entropies of automorphisms in φ. In particular, they proved

Theorem 1 (Kin, Kojima and Takasawa [11])**.** *There is a constant $C(g,n) > 0$ depending only on the topology of Σ such that*

$$\mathrm{ent}\,\varphi \geq C(g,n)\,\mathrm{vol}N_\varphi.$$

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They asserts only an existence of a constant $C(g,n)$ since their argument is based on a theorem of Brock [5] involving several constants which are difficult to make precise. On the other hand, $C(g,n)$ is necessarily decreasing to 0 when $g+n \to \infty$, since the set of volumes of hyperbolic 3-manifolds is bounded away from 0 by Jørgensen-Thurston theory [19], while Penner showed in [17] that the entropy of a pseudo-Anosov could be arbitrarily small if a surface becomes complicated.

In this paper, we make $C(g,n)$ more explicit by

Theorem 2. *The inequality,*

$$\operatorname{ent}\varphi \geq \frac{1}{3\pi|\chi(\Sigma)|}\operatorname{vol}N_\varphi, \tag{1.1}$$

or equivalently,

$$2\pi|\chi(\Sigma)|\operatorname{ent}\varphi \geq \frac{2}{3}\operatorname{vol}N_\varphi \tag{1.2}$$

holds for any pseudo-Anosov φ.

The left hand side of (1.2) or its variant is called a *normalized entropy* sometime. It is the main claim of this paper that the normalized entropy over the volume is bounded from below by a constant depending on the normalization in fact, but not on the topology of Σ.

Unfortunately, the constant we present here is quite unlikely sharp. For example, when $g=1$ and $n=1$, since $|\chi(\Sigma_{1,1})|=1$, the inequality (1.1) gives us

$$\frac{\operatorname{ent}\varphi}{\operatorname{vol}N_\varphi} \geq \frac{1}{3\pi} = 0.10610\ldots$$

However, it is conjectured in this particular case (see Conjecture 6.10 in [11] with some supporting evidence) that

$$\frac{\operatorname{ent}\varphi}{\operatorname{vol}N_\varphi} \geq \frac{\log\frac{3+\sqrt{5}}{2}}{2v_3} = 0.47412\ldots,$$

where $v_3 = 1.01494\ldots$ is the volume of the hyperbolic regular ideal simplex. The conjectured constant above is known to be attained by the figure eight knot complement which admits a unique $\Sigma_{1,1}$-fibration. The difference of these values makes us to predict some limitation of our method.

On the other hand, we can improve Penner's lower bound,

$$\operatorname{ent}\varphi \geq \frac{\log 2}{4(3g-3+n)},$$

for the entropy of pseudo-Anosovs in [17] provided that a surface has at least one puncture.

Corollary 3. *Let φ be a pseudo-Anosov on $\Sigma_{g,n}$ with $n \geq 1$. Then*

$$\operatorname{ent}\varphi \geq \frac{2v_3}{3\pi|\chi(\Sigma)|} = \frac{2v_3}{3\pi(2g-2+n)}.$$

Proof. It is known by Cao and Meryerhoff in [7] that the smallest volume of an orientable noncompact hyperbolic 3-manifold is attained by the figure eight knot complement and it is $2v_3$. Thus replacing $\operatorname{vol}N_\varphi$ in (1.1) by $2v_3$, we obtain the estimate. $\square$

Remark 4. *If a manifold admits a fibration over the circle, its first Betti number is necessarily positive. It is conjectured that the smallest volume of a hyperbolic* 3*-manifold with positive first Betti number is also* $2v_3$*. If it were true, then we could drop the assumption on* n *in* Corollary 3.

As another immediate corollary to Theorem 2, we give an alternative proof of a theorem by Farb, Leininger and Margarlit, and independently by Agol.

Corollary 5 (Farb, Leininger and Margalit [9], Agol [1])**.** *For any* $C > 0$*, there are finitely many cusped hyperbolic* 3*-manifolds* M_k *such that any pseudo-Anosov* φ *on* Σ *with* $|\chi(\Sigma)| \operatorname{ent} \varphi < C$ *can be realized as the monodromy of a fibration on a manifold obtained from one of the* M_k *by an appropriate Dehn filling.*

Proof. If a version of a normalized entropy, $|\chi(\Sigma)| \operatorname{end} \varphi$, is bounded from above by a constant C, then it certainly bounds the volume of N_φ by Theorem 2. The conclusion follows from the fact, due to Jørgensen and Thurston [19], that the thick part of such manifolds admit only finitely many topological types. □

The proof of Theorem 2 heavily depends on Krasnov and Schlenker's work on the renormalization of volume in [13, 16]. We review their results in the next section and prove Theorem 2 in the last section.

2. PRELIMINARIES

2.1. **Differentials.** Let R be a Riemann surface and $T^{1,0}R$ and $T^{0,1}R$ denote respectively the holomorphic and the anti-holomorophic parts of the complex cotangent bundle, a canonical bundle, over R respectively.

A Beltrami differential μ on R is a section of the line bundle $T^{0,1}R \otimes (T^{1,0})^*R$ and it represents an infinitesimal deformation of complex structure on R. As such μ can be viewed as a tangent vector to the Teichmüller space $\mathcal{T}$ of R, but the corresponding deformation might be trivial. Let $L_\infty(R)$ be the space of uniformly bounded Beltrami differentials, namely $L_\infty(R) = \{\mu\,;\, ||\mu||_\infty < \infty\}$.

Dual to the Beltrami differentials is the space of quadratic differentials: a quadratic differential on R is a section of the line bundle $(T^{1,0}R)^{\otimes 2}$. Let $Q(R)$ be the space of holomorphic quadratic differentials with bounded L^1-norm,

$$||q||_1 = \int_R |q| < \infty.$$

By Riemann-Roch, the dimension of $Q(R)$ is just $3g-3+n$. There is a natural pairing between Beltrami differentials and quadratic differentials defined by

$$\langle q, \mu \rangle = \int_R \mu q.$$

If we let

$$K = \{\mu\,;\, \langle \mu, q\rangle = 0 \text{ for all } q \in Q(R)\},$$

then $L_\infty(R)/K$ can be identified with the tangent space $T_R\mathcal{T}$ of the Teichmüller space $\mathcal{T}$ at R, and moreover the natural pairing induces an isomorphism of $Q(R)^*$ with $L_\infty(R)/K \cong T_R\mathcal{T}$. Thus, $Q(R)$ can be regarded as a cotangent space of $\mathcal{T}$ at R, $T_R^*\mathcal{T}$, and $\langle\ ,\ \rangle$ is a duality pairing,

$$\langle\ ,\ \rangle : Q(R) \times L_\infty(R)/K \to \mathbb{C}. \tag{2.1}$$

2.2. **Projective Structures.** A projective structure on a surface Σ is a special type of complex structure being locally modeled on the geometry of the complex projective line $(\widehat{\mathbb{C}}, \mathrm{PSL}(2,\mathbb{C}))$, where $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere. The projective structure on Σ is given by an atlas such that each transition map is the restriction of some element in $\mathrm{PSL}(2,\mathbb{C})$. Clearly these transition maps are holomorphic and so there is a unique complex structure naturally associated to a given projective structure. Let X be a surface homeomorphic to Σ together with a projective structure, and let R be its underlying Riemann surface. When $\chi(\Sigma) < 0$ there is a bijection between projective structures and holomorphic quadratic differentials. To see this, recall that by the Uniformization Theorem the universal cover of X can be identified with the Pounceré disk $\mathbb{D} = \{z \in \mathbb{C}\,;\, |z| < 1\}$. The developing map $f : \widetilde{X} \to \widehat{\mathbb{C}}$ can be regarded as a meromorphic function on $\mathbb{D}$. Now the Schwarzian derivative $S(f)$ of f defines a holomorphic quadratic differential q on R. Conversely, given a holomorphic quadratic differential q on R, the Schwarzian differential equation

$$S(f) = q$$

has a solution which gives rise to a developing map of some complex projective structure on R. Thus, there is a one-to-one correspondence between the set of all complex projective structures on R and $Q(R)$.

2.3. **Quasi-Fuchsian Manifolds.** A quasi-Fuchsian group is defined to be a discrete subgroup Γ of $\mathrm{PSL}(2,\mathbb{C})$ such that its limit set L_Γ of Γ on the boundary $\partial\mathbb{H}^3 = S^2_\infty$ is either a circle or a quasi-circle (an embedded copy of the circle with Hausdorff dimension strictly greater than 1). This definition implies many consequences. For example, the domain of discontinuity $\Omega_\Gamma = S^2_\infty - L_\Gamma$ consists of exactly two simply-connended domains, denoted Ω^+_Γ and Ω^-_Γ, the quotients of $\Omega^\pm_\Gamma$ by Γ are the same 2-orbifold O but with opposite orientations, and $\mathbb{H}^3/\Gamma$ is a geometrically finite hyperbolic 3-orbifold homeomorphic to $O \times \mathbb{R}$. If Γ is torsion free, then O becomes a surface Σ and, in particular, Γ is isomorphic to the fundamental group of Σ. In this case we say that the quotient of $\mathbb{H}^3$ by the quasi-Fuchsian group Γ is a *quasi-Fuchsian manifold.*

The action of Γ on Ω_Γ is holomorphic so $X = \Omega^+_\Gamma/\Gamma$ and $Y = \Omega^-_\Gamma/\Gamma$ are marked Riemann surfaces. Thus, there is a well defined map taking a quasi-Fuchsian manifold $\mathbb{H}^3/\Gamma$ to a pair of marked Riemann surfaces X, Y in the Teichmüller space $\mathcal{T}$ of Σ. In [2] Bers showed that this map has an inverse. He obtains as a corollary a parametrization of the set of quasi-Fuchsian manifolds by the $\mathcal{T} \times \overline{\mathcal{T}}$. In other words, for any pair $(X, Y) \in \mathcal{T} \times \overline{\mathcal{T}}$, there is a unique quasi-Fuchsian manifold $QF(X, Y)$. As noted above, the limit set of a quasi-Fuchsian group is either a circle or a quasi-circle, and the quotient of its convex hull in $\mathbb{H}^3$ by Γ, denoted $C(X, Y)$, is homemorphic to Σ or $\Sigma \times [0, 1]$ accordingly, called the *convex core.* It is known to be the smallest convex subset homotopy equivalent to $QF(X, Y)$.

Since the action of a quasi-Fuchsian group Γ on Ω_Γ is linear fractional, the Riemann surfaces X and Y are equipped not only with complex structures but also with complex projective structures. Thus we have associated holomorphic quadratic differentials q_X and q_Y. Let q denote the unique holomorphic quadratic differential on $X \sqcup Y$ such that its restriction to X is q_X and to Y is q_Y.

The notations q_X, q_Y may be a bit misleading since they both could vary even if one of complex structures of X or Y stays constant. However, as long as discussing quasi-Fuchsian deformations, we regard q_X as a complex projective structure of the Riemann surface on the left and q_Y on the right. This convention would resolve confusion of notations.

2.4. **Renormalization of Volume.** Renormalization of the volume of convex cocompact hyperbolic 3-manifolds were studied extensively by Krasnov and Schlenker in [13]. Though it is assumed throughout [13] that the quasi-Fuchsian manifold in question is convex cocompact, almost without exception the important arguments are based on local computations of first and second fundamental forms (see next paragraph) so carry over to the convex cofinite volume case. In the following paragraphs we recall Krasnov and Schlenker's results focusing on quasi-Fuchsian case. Note that the surface at infinity Ω_Γ/Γ has 2 connected components.

Let M be a quasi-Fuchsian manifold $\mathbb{H}^3/\Gamma$ homeomorphic to $\Sigma \times \mathbb{R}$. Following [13] we say that a codimension-zero smooth convex submanifold $N \subset M$ of finite volume is *strongly convex* if the normal hyperbolic Gauss map from $\partial N = \partial_+ N \sqcup \partial_- N$ to the boundary at infinity Ω_Γ/Γ is a homeomorphism. For example, a closed ε-neighborhood of the convex core of a quasi-Fuchsian manifold is strongly convex. Let $S_0 = \partial N$ then there is a family of surfaces $\{S_r\}_{r\geq 0}$ equidistant to S_0 foliating the ends of M. If g_r denotes the induced metric on S_r then there is a metric at infinity associated to the family $\{S_r\}_{r\geq 0}$ given by

$$g = \lim_{r\to\infty} e^{-2r} g_r.$$

The resulting metric g in fact belongs to the *conformal class at infinity* that is the conformal structure (determined by the complex structure) on Ω_Γ/Γ, and moreover, the area of each component is equal to $2\pi|\chi(\Sigma)|$. Conversely, if g is a Riemannian metric in the conformal class at infinity where each component has Area $= 2\pi|\chi(\Sigma)|$, then Theorem 5.8 in [13] shows that there is a unique foliation of the ends of M by equidistant surfaces such that the associated metric at infinity is equal to g. The construction of a foliation is due to Epstein [8].

A natural quantity to study in the context of strongly convex submanifolds $N \subset M$ is the *W-volume* defined by the right hand side of (2.2) below. In fact, a simple computation which can be found in [16], shows that the W-volume depends only on the metric at infinity g justifying the notation:

$$W(M, g) := \operatorname{vol} N - \frac{1}{4}\int_{\partial N} H \, da, \tag{2.2}$$

where H is the mean curvature of ∂N. The renormalized volume of M is now defined by taking the supremum of W-volume over all metrics g in the conformal class at infinity such that the area of each surface at infinity Ω_Γ/Γ with respect to g is $2\pi|\chi(\Sigma)|$,

$$\operatorname{Rvol}(M) := \sup_g W(M, g).$$

Section 7 in [13] presents an argument, based on the variational formula stated in Corollary 6.2 in [13], that the supremum is in fact uniquely attained by the metric of constant curvature -1.

If we take N to be a very small neighborhood of the convex core $C(X,Y)$ for a quasi-Fuchsian manifold $QF(X,Y)$, then it is strongly convex and the W-volume of N should be quite close to

$$\operatorname{vol} C(X,Y) - \frac{1}{4}\,\ell(\nu),$$

where ν is the lamination with bending measure of the boundary of $C(X,Y)$, and $\ell(\nu)$ is the length of ν with respect to the induced metric. One half of Theorem 1.1 in [16] asserts that this will give in fact an estimate. Since, by Bridgeman [4], the term $\ell(\nu)$ is bounded above by a constant depending only on the topology of Σ, we can now state, in a slightly modified form, one half of Schlenker's theorem as follows.

Theorem 6 (Theorem 1.1 in [16]). *There exists a constant $D = D(\Sigma) > 0$ depending only on the topology of Σ such that the inequality,*

$$\operatorname{vol} C(X,Y) \leq \operatorname{Rvol} QF(X,Y) + D, \tag{2.3}$$

holds for any X, $Y \in \mathcal{T}$.

2.5. **Variational Formula.** In [16] the metric at infinity of a quasi-Fuchsian manifold M which attains the renormalized volume is denoted by I^*. This is consistent with the standard notation for the first fundamental form. There is also an analogous notion II^* of the second fundamental form on the surface at infinity. More precisely, there is a unique bundle morphism B^* of the tangent space of boundary, corresponding to the shape operator, which is self-adjoint for I^* and such that

$$\mathrm{II}^* = \mathrm{I}^*(B^*\cdot,\cdot)$$

Krasnov and Schlenker found a remarkable relation between II_0^*, the trace free part of II^*, and the holomorophic quadratic differential q corresponding to the projective structure of the boundary. Their identity below follows directly from explicit formulae for the holomorphic quadratic differential q in question. Another more geometric proof can also be found in the appendix of [13].

Lemma 7 (Lemma 8.3 in [13]).

$$\mathrm{II}_0^* = -\operatorname{Re}(q).$$

Thus, the trace free part of the second fundamental form at infinity can be regarded as a cotangent vector of the Teichmüller space. The variational formula of the W-volume stated in Cororally 6.2 in [13] implies

Lemma 8 (Proposition 3.10 [16]). *Under a first-order deformation of the hyperbolic structure on N,*

$$d\operatorname{Rvol} = -\frac{1}{4}\langle \mathrm{II}_0^*, \dot{\mathrm{I}}^*\rangle$$

holds. Here $\langle\ ,\ \rangle$ is the duality pairing (2.1).

2.6. **Rvol versus Teichmüller Distance.** We start with a quasi-Fuchsian manifold $\mathbb{H}^3/\Gamma = QF(X,Y)$. Fix a conformal structure X the left boundary component, and regard a conformal structure of Y as a variable. To each Y, we assign an associated complex projective structure on X and therefore a holomorphic quadratic differential q_X. This defines a map,

$$B_X : \mathcal{T} \to Q(X),$$

called a Bers embedding.

Using the hyperbolic metric in the conformal class of X, we can measure at each point of Σ the norm of q_X. Let $Q^\infty(X)$ be $Q(X)$ endowed with the L^∞ norm, namely,

$$||q||_\infty = \sup_{x\in X} \frac{|q(x)|}{\rho^2(x)},$$

where $\rho|dz|$ defines the hyperbolic metric.

The following theorem, due to Nehari, can be found in a standard text book of the Teichmüller theory such as Theorem 1, p.134 in [10].

Theorem 9 (Nehari [15])**.** *The image of B_X in $Q^\infty(X)$ is contained in the ball of radius* 6*.*

Then, consider now the L^1-norm on $Q(X)$ and denote by $Q^1(X)$ the vector space $Q(X)$ endowed with the L^1 norm.

Corollary 10. *The image of B_X in $Q^1(X)$ is contained in the ball of radius $12\pi|\chi(\Sigma)|$.*

Proof. The inequality

$$||q||_1 = \int_X |q| = \int_X \frac{|q|}{\rho^2}\rho^2 \le ||q||_\infty \int_X \rho^2 = 2\pi|\chi(\Sigma)|\, ||q||_\infty$$

immediately implies the conclusion. □

The proof of the following comparison result is the same as that of Theorem 1.2 in [16] with a different norm.

Proposition 11. *The inequality,*

$$\operatorname{Rvol} QF(X,Y) \le 3\pi|\chi(\Sigma)|d_{\mathrm{T}}(X,Y),$$

holds for any quasi-Fuchsian manifold $QF(X,Y)$, where d_{T} is the Teichmüller distance on $\mathcal{T}$.

Proof. Let $Y : [0,d] \to \mathcal{T}$ be the unit speed Teichmüller geodesic joining X and Y, so that, in particular, $Y(0) = X$, $Y(d) = Y$ and $d = d_{\mathrm{T}}(X,Y)$. Then consider a one-parameter family of quasi-Fuchsian manifolds $\{QF(X,Y(t))\}_{0\le t\le d}$. The variational formula in Theorem 8 under the first-order deformation at time t says

$$d\operatorname{Rvol} = \frac{1}{4}(\langle \operatorname{Re}(q_X(t)), \dot X\rangle + \langle \operatorname{Re}(q_{Y(t)}(t)), \dot Y(t)\rangle),$$

where $\dot X$, $\dot Y(t)$ are tangent vectors of the deformation of complex structures on each boundary. Since $\dot X = 0$, integrating the variation of Rvol along the path

$Y(t)$ $(t \in [0,d])$, we obtain

$$\mathrm{Rvol}\, QF(X,Y) = \frac{1}{4}\int_{t=0}^{d} \langle \mathrm{Re}(q_{Y(t)}(t)), \dot{Y}(t)\rangle dt.$$

On the other hand,

$$\begin{aligned}\langle \mathrm{Re}(q_{Y(t)}(t)), \dot{Y}(t)\rangle &= \int_{Y(t)} \mathrm{Re}(q_{Y(t)}(t))\, \dot{Y}(t) \\ &\leq \|q_{Y(t)}(t)\|_1 \, \|\dot{Y}(t)\|_\infty\end{aligned}$$

where $\|\cdot\|_\infty$ is the supremum norm on $L_\infty(Y(t))$ which is the dual to the L^1-norm on $Q(Y(t))$ and hence an infinitesimal form of the Teichmüller metric. Then, Corollary 10 says,

$$\|q_{Y(t)}(t)\|_1 \leq 12\pi|\chi(\Sigma)|$$

holds for all $t \in [0,d]$, and also we have $\|\dot{Y}(t)\|_\infty = 1$ by definition. Thus, the rest is obvious. □

3. Proof

We duplicate the argument in [5]. Let φ be a pseudo-Anosov automorphism on Σ, and X any marked Riemann surface in $\mathcal{T}$. Then, φ acts naturally on $\mathcal{T}$ and consider $QF((\varphi^{-n})^*X, (\varphi^n)^*X)$. This manifold is quite close to the infinite cyclic covering space of N_φ if n is sufficiently large. Fixing the topology of Σ, we obtain from Theorem 6 and Proposition 11 that

$$\begin{aligned}\mathrm{vol}\, C((\varphi^{-n})^*X, (\varphi^n)^*X) &\leq \mathrm{Rvol}\, QF((\varphi^{-n})^*X, (\varphi^n)^*X) + D \\ &\leq 3\pi|\chi(\Sigma)|\, d_{\mathrm{T}}((\varphi^{-n})^*X, (\varphi^n)^*X) + D.\end{aligned}$$

Divide three sides by $2n$ and take n to ∞, then the left hand side converges to the volume of N_φ by inflexibility, originally due to McMullen [14] for bounded geometry case, and due to Brock and Bromberg [6] in general eventually.

Thus we obtain

$$\mathrm{vol}\, N_\varphi \leq 3\pi|\chi(\Sigma)|\, \|\varphi\|_{\mathrm{T}}, \tag{3.1}$$

where $\|\varphi\|_{\mathrm{T}}$ is the Teichmüller translation distance of φ defined by

$$\|\varphi\|_{\mathrm{T}} = \inf_{R\in\mathcal{T}} d_{\mathrm{T}}(R, \varphi(R)).$$

Now, recall

Proposition 12 (Bers [3], cf, [12])**.** *For any mapping class φ of Σ,*

$$\|\varphi\|_{\mathrm{T}} = \mathrm{ent}\, \varphi \tag{3.2}$$

holds.

Proof of Theorem 2. The inequality in Theorem 2 follows immediately from (3.1) and (3.2). □

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