Homogeneous projective bundles over abelian varieties

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Abstract

We consider those projective bundles (or Brauer-Severi varieties) over an abelian variety that are homogeneous, i.e., invariant under translation. We describe the structure of these bundles in terms of projective representations of commutative group schemes; the irreducible bundles correspond to Heisenberg groups and their standard representations. Our results extend those of Mukai on semi-homogeneous vector bundles, and yield a geometric view of the Brauer group of abelian varieties.

1 Introduction

The objects of the present article are those projective bundles (or Brauer-Severi varieties) over an abelian variety X that are homogeneous, i.e., isomorphic to their pull-backs under all translations. Among these bundles, the projectivizations of vector bundles are well understood by work of Mukai (see [22]). Indeed, the vector bundles with homogeneous projectivization are exactly the semi-homogeneous vector bundles of [loc. cit.]. Those that are simple (i.e., their global endomorphisms are just scalars) admit several nice characterizations; for example, they are all obtained as direct images of line bundles under isogenies. Moreover, every indecomposable semi-homogeneous vector bundle is the tensor product of a unipotent bundle and of a simple semi-homogeneous bundle.

In this article, we obtain somewhat similar statements for the structure of homogeneous projective bundles. We build on the results of our earlier paper [9] about homogeneous principal bundles under an arbitrary algebraic group; here we consider of course the projective linear group PGL_n. In loose terms, the approach of [loc. cit.] reduces the classification of homogeneous bundles to that of commutative subgroup schemes of PGL_n. The latter, carried out in Section 2, is based on the classical construction of Heisenberg groups and their irreducible representations.

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In Section 3, we introduce a notion of irreducibility for homogeneous projective bundles, which is equivalent to the group scheme of bundle automorphisms being finite. (The projectivization of a semi-homogeneous vector bundle E is irreducible if and only if E is simple). We characterize those projective bundles that are homogeneous and irreducible, by the vanishing of all the cohomology groups of their adjoint vector bundle (Proposition 3.7). Also, we show that the homogeneous irreducible bundles are classified by the pairs (H, e), where H is a finite subgroup of the dual abelian variety, and $e : H \times H \to \mathbb{G}_m$ a non-degenerate alternating bilinear pairing (Proposition 3.1). Finally, we obtain a characterization of those homogeneous projective bundles that are projectivizations of vector bundles, first in the irreducible case (Proposition 3.10; it states in loose terms that the pairing e originates from a line bundle on X) and then in the general case (Theorem 3.11).

The irreducible homogeneous projective bundles over an elliptic curve are exactly the projectivizations of indecomposable vector bundles with coprime rank and degree, as follows from classical work of Atiyah (see [1]). But any abelian variety X of dimension at least 2 admits many homogeneous projective bundles that are not projectivizations of vector bundles. In fact, any class in the Brauer group $\mathrm{Br}(X)$ is represented by a homogeneous bundle (as shown by Elencwajg and Narasimhan in the setting of complex tori, see [14, Theorem 1]). Also, our approach yields a geometric view of a description of $\mathrm{Br}(X)$ due to Berkovich (see [5]); this is developed in Remark 3.13.

Spaces of effective divisors on an abelian variety afford natural examples of homogeneous projective bundles. These are presented in Section 4, which can be read independently of the rest of the paper. The final Section 5, which is rather an appendix, develops the analogy between abelian varieties and flag varieties to obtain an analogue of the classical theorem of Tannaka: any linear algebraic group over a field of characteristic 0 can be reconstructed from its finite-dimensional representations (see e.g. [30, Theorem 2.5.3]). Here we show how to reconstruct a semi-simple algebraic group of adjoint type from its irreducible finite-dimensional projective representations, by viewing them as spaces of effective divisors on the associated flag variety (Theorem 5.1). This should be compared with a theorem of Larsen (see [19, Theorem 5.4]): every semi-simple Lie algebra is determined by the multiset of dimensions of its irreducible representations, up to (non-unique) isomorphism.

Throughout this article, the base field k is algebraically closed, of arbitrary characteristic $p \geq 0$; almost all of our results hold for $\mathbb{P}^{n-1}$-bundles under the assumption that n is not a multiple of p. Indeed, the structure of commutative subgroup schemes of PGL_n is much more complicated when p divides n (see [20]); it would be interesting to extend our results to these 'bad' characteristics. In another direction, the 'self-dual' homogeneous projective bundles can be analyzed along similar lines when $p \neq 2$, see [10].

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Notation and conventions. We use the book [12] by Demazure and Gabriel as a general reference for group schemes. Our reference for abelian varieties is Mumford's book [24]; we generally follow its notation. In particular, X stands for a fixed abelian variety, of dimension g; the group law of X is denoted additively, and the multiplication by an integer n is denoted by n_X, with kernel ${}_nX$. For any point $x \in X$, we denote by $T_x : X \to X$ the translation $y \mapsto x + y$. The dual abelian variety is denoted by $\widehat{X}$.

2 Structure

Recall that a *projective bundle* over X is a variety P equipped with a proper flat morphism

(1) $$f : P \longrightarrow X$$

with fibers at all closed points isomorphic to projective space $\mathbb{P}^{n-1}$ for some integer $n \geq 1$; then f is a $\mathbb{P}^{n-1}$-bundle for the étale topology by [18, Section I.8].

Also, recall from [loc. cit.] that the $\mathbb{P}^{n-1}$-bundles are in a one-to-one correspondence with the torsors (or principal bundles)

(2) $$\pi : Y \longrightarrow X$$

under the projective linear group

$$\mathrm{PGL}_n = \mathrm{Aut}(\mathbb{P}^{n-1}).$$

Specifically, P is the associated bundle $Y \times^{\mathrm{PGL}_n} \mathbb{P}^{n-1}$, and Y is the bundle of isomorphisms $X \times \mathbb{P}^{n-1} \to P$ over X. Thus, any representation $\rho : \mathrm{PGL}_n \to \mathrm{GL}(V)$ defines the associated vector bundle $Y \times^{\mathrm{PGL}_n} V$ over X. In particular, the conjugation representation of PGL_n in the space M_n of $n \times n$ matrices yields a 'matrix bundle' on X; its sheaf of local sections is an Azumaya algebra of rank n^2 over X,

$$\mathcal{A} := (\pi_*(\mathcal{O}_Y) \otimes \mathrm{M}_n)^{\mathrm{PGL}_n},$$

viewed as a sheaf of non-commutative $\mathcal{O}_X$-algebras over $\pi_*(\mathcal{O}_Y)^{\mathrm{PGL}_n} = \mathcal{O}_X$. In particular, $\mathcal{A}$ yields a central simple algebra of degree n over the function field $k(X)$. By [18, Corollaire I.5.11], the assignement $P \mapsto \mathcal{A}$ yields a one-to-one correspondence between

$\mathbb{P}^{n-1}$-bundles and Azumaya algebras of rank n^2. The quotient of $\mathcal{A}$ by $\mathcal{O}_X$ is the sheaf of local sections of the *adjoint bundle* $\mathrm{ad}(P)$, the vector bundle associated with the adjoint representation of PGL_n in its Lie algebra $\mathfrak{pgl}_n$ (the quotient of the Lie algebra M_n by the scalar matrices).

The variety P is easily seen to be projective; hence its automorphism group functor is represented by a group scheme $\mathrm{Aut}(P)$, locally of finite type. Via the correspondence with PGL_n-torsors, $\mathrm{Aut}(P)$ is identified with the group scheme $\mathrm{Aut}^{\mathrm{PGL}_n}(Y)$ of equivariant automorphisms. This defines a morphism of group schemes

$$f_* : \mathrm{Aut}(P) \longrightarrow \mathrm{Aut}(X)$$

with kernel the subgroup scheme $\mathrm{Aut}_X(P) \cong \mathrm{Aut}_X^{\mathrm{PGL}_n}(Y)$ of bundle automorphisms. Moreover, $\mathrm{Aut}_X(P)$ is affine of finite type, and its Lie algebra is $H^0(X, \mathrm{ad}(P))$ (these results follow e.g. from [8, Section 4]).

We say that a $\mathbb{P}^{n-1}$-bundle (1) is *homogeneous*, if the image of f_* contains the subgroup $X \subset \mathrm{Aut}(X)$ of translations; equivalently, the bundle (1) is isomorphic to its pull-backs under all translations. This amounts to the vector bundle $\mathrm{ad}(P)$ being homogeneous (see [9, Corollary 2.15]; if (1) is the projectivization of a vector bundle, this also follows from [22, Theorem 5.8]).

The structure of homogeneous projective bundles is described by the following:

THEOREM 2.1. (i) *A $\mathbb{P}^{n-1}$-bundle $f : P \to X$ is homogeneous if and only if there exist an exact sequence of group schemes*

$$1 \longrightarrow H \longrightarrow G \xrightarrow{\ \gamma\ } X \longrightarrow 1, \tag{3}$$

where G is anti-affine (i.e., $\mathcal{O}(G) = k$), and a faithful homomorphism

$$\rho : H \longrightarrow \mathrm{PGL}_n$$

such that P is the associated bundle $G \times^H \mathbb{P}^{n-1} \to G/H = X$, where H acts on $\mathbb{P}^{n-1}$ via ρ.

Then the exact sequence (3) is unique; the group scheme G is smooth, connected, and commutative (in particular, H is commutative), and the projective representation ρ is unique up to conjugacy in PGL_n. Moreover, the corresponding PGL_n-torsor is the associated bundle $G \times^H \mathrm{PGL}_n \to X$, and the corresponding Azumaya algebra satisfies

$$\mathcal{A} \cong (\gamma_*(\mathcal{O}_G) \otimes \mathrm{M}_n)^H \tag{4}$$

as a sheaf of algebras over $\gamma_*(\mathcal{O}_G)^H \cong \mathcal{O}_X$.

(ii) *For P as in* (i), *we have an isomorphism*

$$\mathrm{Aut}_X(P) \cong \mathrm{PGL}_n^H \tag{5}$$

(the centralizer of H in PGL_n*). As a consequence,*

$$H^0(X, \mathrm{ad}(P)) = \mathfrak{pgl}_n^H. \tag{6}$$

(iii) *The homogeneous projective sub-bundles of P are exactly the bundles $G \times^H S \to X$, where $S \subset \mathbb{P}^{n-1}$ is an H-stable linear subspace.*

Proof. (i) follows readily from Theorem 3.1 in [9], and (ii) from Proposition 3.6 there.

(iii) Any projective sub-bundle $f' : P' \to X$ defines a reduction of structure group of the PGL_n-torsor (2) to a $\mathrm{PGL}_{n,n'}$-torsor $\pi' : Y' \to X$, where $\mathrm{PGL}_{n,n'} \subset \mathrm{PGL}_n$ denotes the stabilizer of a linear subspace $S \cong \mathbb{P}^{n'-1}$ of $\mathbb{P}^{n-1}$. If f is homogeneous, then by [9, Theorem 3.1] again, we have a $\mathrm{PGL}_{n,n'}$-equivariant isomorphism

$$Y' \cong G' \times^{H'} \mathrm{PGL}_{n,n'}$$

for an exact sequence $0 \to H' \to G' \to X \to 0$ with G' anti-affine, and a faithful homomorphism $\rho' : H' \to \mathrm{PGL}_{n,n'}$. Thus,

$$Y \cong Y' \times^{\mathrm{PGL}_{n,n'}} \mathrm{PGL}_n \cong G' \times^{H'} \mathrm{PGL}_n$$

equivariantly for the action of PGL_n. Since $\mathcal{O}(G') = k$, it follows that

$$\mathcal{O}(Y) \cong \mathcal{O}(G' \times \mathrm{PGL}_n)^{H'} \cong \mathcal{O}(\mathrm{PGL}_n)^{H'} \cong \mathcal{O}(\mathrm{PGL}_n/H')$$

as algebras equipped with an action of PGL_n. But also $\mathcal{O}(Y) \cong \mathcal{O}(G \times^H \mathrm{PGL}_n) \cong \mathcal{O}(\mathrm{PGL}_n/H)$. Moreover, the varieties PGL_n/H and PGL_n/H' are both quasi-affine, since H and H' are commutative (see [9, Remark 2.10 and Lemma 2.11]). Thus, $\mathrm{PGL}_n/H \cong \mathrm{PGL}_n/H'$ as PGL_n-varieties, i.e., H' is conjugate to H in PGL_n. We may therefore assume that $H' = H \subset G'$. Then $G' = G$ (the largest anti-affine subgroup of $\mathrm{Aut}^{\mathrm{PGL}_n}(Y)$) and hence H stabilizes S.

Conversely, any H-stable linear subspace obviously yields a homogeneous projective sub-bundle. □

Remark 2.2. There is a natural operation of *product* on projective bundles: to any $\mathbb{P}^{n_i-1}$-bundles $f_i : P_i \to X$ ($i = 1, 2$), with associated PGL_{n_i}- bundles $\pi_i : Y_i \to X$, one associates the $\mathbb{P}^{n_1 n_2 - 1}$-bundle

$$f : P_1 P_2 \longrightarrow X$$

that corresponds to the $\mathrm{PGL}_{n_1 n_2}$-torsor obtained from the $\mathrm{PGL}_{n_1} \times \mathrm{PGL}_{n_2}$-torsor

$$(\pi_1, \pi_2) : Y_1 \times_X Y_2 \longrightarrow X$$

by the extension of structure groups

$$\mathrm{PGL}_{n_1} \times \mathrm{PGL}_{n_2} = \mathrm{PGL}(k^{n_1}) \times \mathrm{PGL}(k^{n_2}) \longrightarrow \mathrm{PGL}(k^{n_1} \otimes k^{n_2}) = \mathrm{PGL}_{n_1 n_2}.$$

So $P_1 P_2$ contains the fibered product $P_1 \times_X P_2$.

Likewise, any projective bundle $f : P \to X$ has a *dual* bundle

$$f^* : P^* \longrightarrow X,$$

where P^* is the same variety as P, but the action of PGL_n is twisted by the automorphism arising from inverse transpose.

Via the correspondence between projective bundles and Azumaya algebras, the product of bundles corresponds to the tensor product of algebras, and the duality to the opposite (see [18, Section I.8]).

Clearly, taking products and duals preserves homogeneity. Moreover, if P_i corresponds to an extension $1 \to H_i \to G_i \to X \to 1$ and a projective representation $\rho_i : H_i \to \mathbb{P}^{n_i}$, then the $\mathrm{PGL}_{n_1 n_2}$-torsor that corresponds to $P_1 P_2$ is the associated bundle

$$(G_1 \times_X G_2) \times^{H_1 \times H_2} \mathrm{PGL}_{n_1 n_2} \longrightarrow (G_1 \times_X G_2)/(H_1 \times H_2) = X,$$

where the homomorphism $H_1 \times H_2 \to \mathrm{PGL}_{n_1 n_2}$ is given by the tensor product $\rho_1 \otimes \rho_2$. Thus, $P_1 P_2$ is the homogeneous bundle classified by the extension $1 \to H \to G \to X \to 1$, where $G \subset G_1 \times_X G_2$ denotes the largest anti-affine subgroup and $H = (H_1 \times H_2) \cap G$, and the projective representation $(\rho_1 \otimes \rho_2)|_H$. As a consequence, the mth power P^m *corresponds to the same extension as P, and to the mth tensor power of its projective representation. Likewise, the dual of a homogeneous bundle is the homogeneous bundle associated with the same extension and with the dual projective representation.*

The anti-affine algebraic groups are classified in [7] and independently [27], and the anti-affine extensions (3) in [9, Section 3.3]. We now describe the other ingredients of Theorem 2.1, i.e., the commutative subgroup schemes $H \subset \mathrm{PGL}_n$. Any such subgroup scheme has a canonical decomposition

$$H = H_u \times H_s,$$

where H_u is unipotent and H_s is diagonalizable. Thus, H_s fits in an exact sequence

$$1 \longrightarrow H_s^0 \longrightarrow H_s \longrightarrow F \longrightarrow 1,$$

where H_s^0 is a connected diagonalizable group scheme (the neutral component of H_s), and the group of components F is finite, diagonalizable and of order prime to p (in particular, smooth); this exact sequence is unique and splits non-canonically. In turn, H_s^0 is an extension of a finite diagonalizable group scheme of order a power of p, by a torus (the reduced neutral component); this extension is also unique and splits non-canonically.

Denote by $\widetilde{H} \subset \mathrm{GL}_n$ the preimage of $H \subset \mathrm{PGL}_n$. This yields a central extension

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \widetilde{H} \longrightarrow H \longrightarrow 1, \tag{7}$$

where the multiplicative group $\mathbb{G}_m$ is viewed as the group of invertible scalar matrices. We say that $\widetilde{H}$ is the *theta group* of H, and define similarly $\widetilde{H_u}, \widetilde{H_s}$ and $\widetilde{H_s^0}$ (the latter is the neutral component of $\widetilde{H_s}$).

Given two S-valued points $\tilde{x}, \tilde{y}$ of $\widetilde{H}$, where S denotes an arbitrary scheme, the commutator $\tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^{-1}$ is a S-valued point of $\mathbb{G}_m$ and depends only on the images of $\tilde{x}, \tilde{y}$ in H. This defines a morphism

$$e : H \times H \longrightarrow \mathbb{G}_m \tag{8}$$

which is readily seen to be bilinear (i.e., we have $e(xy, z) = e(x, z)\, e(y, z)$ and $e(x, yz) = e(x, z)\, e(y, z)$ for all S-valued points x, y, z of H) and alternating (i.e., $e(x, x) = 1$ for all x). We say that e is the *commutator pairing* of the extension (7).

Note that the dual bundle P^* (as defined in Remark 2.2) has pairing e^{-1}; moreover, the power P^m, where m is a positive integer, has pairing e^m.

The center $Z(\widetilde{H})$ fits in an exact sequence of group schemes

$$1 \longrightarrow \mathbb{G}_m \longrightarrow Z(\widetilde{H}) \longrightarrow H^{\perp} \longrightarrow 1, \tag{9}$$

where the S-valued points of $H^{\perp}$ are those points of H such that $e(x, y) = 1$ for all S'-valued points y of H and all schemes S' over S. In particular, $\widetilde{H}$ *is commutative if and only if* $e = 1$.

We now show that the obstruction for being the projectivization of a homogeneous vector bundle is just the commutator pairing. The obstruction for being the projectivization of an arbitrary vector bundle will be determined in Theorem 3.11.

Proposition 2.3. *With the above notation, the following conditions are equivalent:*

(i) *P is the projectivization of a homogeneous vector bundle.*

(ii) *The extension (7) splits.*

(iii) $e = 1$.

Proof. (i)⇒(ii) By [9, Theorem 3.11], any homogeneous vector bundle of rank n over X is of the form

$$E = G \times^H k^n \longrightarrow G/H = X$$

for some anti-affine extension $1 \to H \to G \to X \to 1$ and some faithful representation $\sigma : H \to \mathrm{GL}_n$. Since H is commutative, k^n contains eigenvectors of H; thus, twisting σ by a character of H (which does not change the projectivization $\mathbb{P}(E)$), we may assume that k^n contains non-zero fixed points of H. Then σ defines a faithful projective representation $\rho : H \to \mathrm{PGL}_n$. Hence G and ρ are the data associated with the homogeneous projective bundle $\mathbb{P}(E) \to X$, and σ splits the extension (7).

(ii)⇒(i) Any splitting of that extension yields a homomorphism $\sigma : H \to \mathrm{GL}_n$ that lifts ρ. Then the associated bundle $G \times^H k^n \to X$ is a homogeneous vector bundle with projectivization P.

(ii)⇒(iii) is obvious. Conversely, if $e = 1$, then $\widetilde{H}$ is commutative. It follows that $\widetilde{H} \cong U \times \widetilde{H_s}$, where the unipotent part U is isomorphic to H_u via the homomorphism $\widetilde{H} \to H$, and $\widetilde{H_s}$ fits in an exact sequence of diagonalizable group schemes $1 \to \mathbb{G}_m \to \widetilde{H_s} \to H_s \to 1$. But any such sequence splits, since so does the dual exact sequence of character groups. □

Next, we assume that n is not divisible by the characteristic p, and obtain a very useful structure result for H:

Proposition 2.4. *Keep the above notation, and assume that* $(n, p) = 1$.

(i) *The extension* $1 \to \mathbb{G}_m \to \widetilde{H_u} \to H_u \to 1$ *has a unique splitting, and the corresponding lift of* H_u *(that we still denote by* H_u*) is central in* $\widetilde{H}$. *Also, the extension* $1 \to \mathbb{G}_m \to \widetilde{H_s^0} \to H_s^0 \to 1$ *splits non-canonically, and* $\widetilde{H_s^0}$ *is central in* $\widetilde{H}$.

(ii) *We have canonical decompositions of group schemes*

$$\widetilde{H} = H_u \times \widetilde{H_s}, \quad Z(\widetilde{H}) = H_u \times Z(\widetilde{H_s}).$$

Moreover, $Z(\widetilde{H_s})$ *is diagonalizable and fits in an exact sequence*

$$1 \longrightarrow \widetilde{H_s^0} \longrightarrow Z(\widetilde{H_s}) \longrightarrow F^{\perp} \longrightarrow 1$$

which splits non-canonically.

(iii) *The commutator pairing* e *factors through a bilinear alternating morphism*

$$e_F : F \times F \longrightarrow \mathbb{G}_m. \tag{10}$$

Proof. Since any commutator in GL_n has determinant 1, we see that e takes values in the subgroup scheme $\mu_n = \mathbb{G}_m \cap \mathrm{SL}_n$ of nth roots of unity. In other terms, e factors through the pairing

$$se : H \times H \longrightarrow \mu_n$$

defined by the central extension

$$1 \longrightarrow \mu_n \longrightarrow S\widetilde{H} \longrightarrow H \longrightarrow 1,$$

where $S\widetilde{H} := H \cap \mathrm{SL}_n$. Note that μ_n is smooth, since $(n,p) = 1$. Moreover, se restricts trivially to $nH \times H$, where nH denotes the image of the multiplication by n in the commutative group scheme H.

We claim that $H_u \subset nH$. This is clear if $p = 0$, since H_u is then isomorphic to the additive group of a vector space. If $p \geq 1$, then the commutative unipotent group scheme H_u is killed by some power of p. Using again the assumption that $(n,p) = 1$, it follows that $H_u = nH_u \subset nH$.

By that claim, se restricts trivially to $H_u \times H$, and hence $\widetilde{H_u} \subset Z(\widetilde{H})$; in particular, $\widetilde{H_u}$ is commutative. Thus, $\widetilde{H_u} \cong H_u \times \mathbb{G}_m$; this proves the assertion about H_u.

We already saw that the extension $1 \to \mathbb{G}_m \to \widetilde{H_s^0} \to H_s^0 \to 1$ splits. Also, $H_s^0 \cong T \times E$, where T is a torus (the reduced neutral component), and E is a local diagonalizable group scheme; then E is killed by some power of p. As above, it follows that $H_s^0 \subset nH$, and that $\widetilde{H_s^0}$ is central in $\widetilde{H}$. This completes the proof of (i).

The decompositions in (ii) are direct consequences of (i). The assertion on $Z(\widetilde{H_s})$ follows from the exact sequence $1 \to \widetilde{H_s^0} \to \widetilde{H_s} \to F \to 1$, since $\widetilde{H_s^0} \subset Z(\widetilde{H_s})$. Finally, (iii) also follows readily from (i). □

Remark 2.5. With the notation and assumptions of the above proposition, the group scheme $\mathrm{Aut}_X(P)$ is smooth, as follows from the isomorphism (5) together with [15, Theorem 1.1]. Moreover, $\mathrm{Aut}(P)$ is smooth as well: indeed, we have an exact sequence of group schemes

$$1 \longrightarrow \mathrm{Aut}_X(P) \longrightarrow \mathrm{Aut}(P) \xrightarrow{f_*} \mathrm{Aut}_P(X) \longrightarrow 1,$$

where $\mathrm{Aut}_P(X)$ is a subgroup scheme of $\mathrm{Aut}(X)$ containing the group X of translations. Since $\mathrm{Aut}(X) = X \ltimes \mathrm{Aut}_{\mathrm{gp}}(X)$, where the group scheme of automorphisms of algebraic groups $\mathrm{Aut}_{\mathrm{gp}}(X)$ is étale (possibly infinite), it follows that $\mathrm{Aut}_P(X)$ is smooth, and hence so is $\mathrm{Aut}(X)$.

Still assuming that $(n,p) = 1$, we say that the theta group $\widetilde{H} \subset \mathrm{GL}_n$ is *non-degenerate*, if $Z(\widetilde{H}) = \mathbb{G}_m$. By Proposition 2.4, this is equivalent to the assertions that H is a finite commutative group of order prime to p, and the homomorphism

$$\epsilon : H \longrightarrow \mathcal{X}(H), \quad x \longmapsto (y \mapsto e(x,y)) \tag{11}$$

is faithful, where $\mathcal{X}(H) := \mathrm{Hom}_{\mathrm{gp}}(H, \mathbb{G}_m)$ denotes the character group of H. It follows that ϵ is an isomorphism.

We now recall from [23, Section 1] the structure of non-degenerate theta groups. Choose a subgroup $K \subset H$ that is totally isotropic for the commutator pairing e, and maximal with this property. Then

$$\widetilde{H} \cong \mathbb{G}_m \times K \times \mathcal{X}(K),$$

where the group law on the right-hand side is given by

$$(t, x, \chi) \cdot (t', x', \chi') = (tt'\chi'(x), x + x', \chi + \chi'), \tag{12}$$

the group laws on K and $\mathcal{X}(K)$ being denoted additively. Such a group is called the *Heisenberg group* associated with the finite group K; we denote it by $\mathcal{H}(K)$ and identify the group K (resp. $\chi(K)$) with its lift $1 \times K \times 0$ (resp. $1 \times 0 \times \chi(K)$).

Also, recall that $\mathcal{H}(K)$ has a unique irreducible representation on which $\mathbb{G}_m$ acts via $t \mapsto t\,\mathrm{id}$: the *standard representation* (also called the *Schrödinger representation*) in the space $\mathcal{O}(K)$ of functions on K with values in k, on which $\widetilde{H}$ acts via

$$((t, x, \chi) \cdot f)(y) = t\,\chi(y)\,f(x + y).$$

The corresponding commutator pairing e is given by

$$e((x, \chi), (x', \chi')) = \chi'(x)\,\chi(x')^{-1}.$$

In particular, the standard representation $V(K)$ contains a unique line of K-fixed points and has dimension $n = \#(K)$; moreover, the group H is killed by n and has order n^2. Any finite-dimensional representation V of $\mathcal{H}(K)$ on which $\mathbb{G}_m$ acts by scalar multiplication is a direct sum of m copies of $V(K)$, where $m := \dim(V)^K$. Such a representation is called *of weight* 1.

For later use, we record the following result, which is well-known in the setting of theta structures on ample line bundles over complex abelian varieties (see [6, Lemma 6.6.6 and Exercise 6.10.14]):

Lemma 2.6. *Assume that $(n, p) = 1$ and let $\widetilde{H} \subset \mathrm{GL}_n$ be a non-degenerate theta group.*

(i) *The algebra M_n has a basis $(u_h)_{h \in H}$ such that every u_h is an eigenvector of H (acting by conjugation) with weight $\epsilon(h)$, and*

$$u_{x,\chi}\,u_{x',\chi'} = \chi'(x)\,u_{x+x',\chi+\chi'}$$

for all $h = (x, \chi)$ and $h' = (x', \chi')$ in H. In particular, the representation of H in M_n by conjugation is isomorphic to the regular representation.

(ii) *H is its own centralizer in PGL_n.*

(iii) *The image in $\mathrm{Aut}(H)$ of the normalizer of H in PGL_n is the 'symplectic group' $\mathrm{Aut}(H, e)$.*

PROOF. (i) We may view H as a subset of M_n via $(x,\chi) \mapsto u_{x,\chi} := (1,x,\chi) \in \widetilde{H} \subset \mathrm{GL}_n$. Then the assertions follow readily from the formula (12) for the group law of $\widetilde{H}$.

(ii) We just saw that the fixed points of H acting on $\mathbb{P}(M_n)$ by conjugation are exactly the points of $H \subset \mathrm{PGL}_n$.

(iii) If $g \in \mathrm{PGL}_n$ normalizes H, then one readily checks that the conjugation $\mathrm{Int}(g)|_H$ preserves the pairing e. Conversely, let $g \in \mathrm{Aut}(H,e)$; then composing the inclusion $\rho : H \to \mathrm{PGL}_n$ with g, we obtain a projective representation ρ_g with the same commutator pairing. Thus, ρ_g lifts to a representation $\tilde{\rho}_g : \widetilde{H} \to \mathrm{GL}_n$ which is isomorphic to the standard representation. It follows that g extends to the conjugation by some $\tilde{g} \in \mathrm{GL}_n$ that normalizes H. □

Returning to an arbitrary theta group $\widetilde{H} \subset \mathrm{GL}_n$, where $(n,p)=1$, we now describe the representation of $\widetilde{H}$ in $k^n =: V$. Consider the decomposition

$$V = \bigoplus_{\lambda} V_\lambda \tag{13}$$

into weight spaces of the diagonalizable group $Z(\widetilde{H_s})$, where λ runs over the characters of weight 1 of that group (those that restrict to the identity character of $\mathbb{G}_m$). By Proposition 2.4, each V_λ is stable under $\widetilde{H}$.

PROPOSITION 2.7. *With the above notation, each quotient $\widetilde{H_s}/\ker(\lambda)$ is isomorphic to the Heisenberg group $\mathcal{H}(K/F^\perp)$, where K denotes a maximal totally isotropic subgroup scheme of F relative to e_F.*

Moreover, we have an isomorphism of representations of $\widetilde{H} \cong H_u \times \widetilde{H_s}$:

$$V_\lambda \cong U_\lambda \otimes V(K/F^\perp),$$

where U_λ is a representation of H_u, and $V(K/F^\perp)$ is the standard representation of $\widetilde{H_s}/\ker(\lambda)$.

PROOF. Note that λ yields a splitting of (9), and an isomorphism $Z(\widetilde{H_s})/\ker(\lambda) \cong \mathbb{G}_m$. Also, $\widetilde{H_s}/Z(\widetilde{H_s}) \cong \widetilde{H}/Z(\widetilde{H}) \cong F/F^\perp$ by Proposition 2.4. Thus, the exact sequence

$$1 \longrightarrow Z(\widetilde{H_s})/\ker(\lambda) \longrightarrow \widetilde{H_s}/\ker(\lambda) \longrightarrow \widetilde{H_s}/Z(\widetilde{H_s}) \longrightarrow 1$$

may be identified with the central extension

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \widetilde{H_s}/\ker(\lambda) \longrightarrow F/F^\perp \longrightarrow 1$$

and the corresponding commutator pairing is induced by e_F. This shows that $\widetilde{H_s}/\ker(\lambda)$ is a non-degenerate theta group. Now the first assertion follows from the structure of these groups.

Also, V_λ is a representation of $\widetilde{H_s}/\ker(\lambda)$ on which the center $\mathbb{G}_m$ acts with weight 1, and hence a direct sum of copies of the standard representation. This implies the second assertion, in view of Proposition 2.4 again. □

Corollary 2.8. *With the above notation, the representation of $\widetilde{H}$ in k^n is an iterated extension of irreducible representations of the same dimension,*

$$d := [K : F^\perp] = \sqrt{[F : F^\perp]} = \sqrt{[H : H^\perp]}. \tag{14}$$

In particular, n is a multiple of d, with equality if and only if $\widetilde{H}$ is a Heisenberg group acting via its standard representation.

We say that d is the *homogeneous index* of the bundle (1); this is the minimal rank of a homogeneous sub-bundle of P, in view of Theorem 2.1. (One can show that the homogeneous index of P is a multiple of the index of the associated central simple algebra over $k(X)$). Note that $F/F^\perp$ is killed by d, and hence $e_F^d = 1$. In view of Proposition 2.3, it follows that the dth power P^d *is the projectivization of a homogeneous vector bundle.*

As another application of the above results, we now describe the *indecomposable* homogeneous projective bundles, i.e., those $\mathbb{P}^{n-1}$-bundles P that admit no proper disjoint $\mathbb{P}^{n_i-1}$-sub-bundles ($i = 1, 2$) such that $n_1 + n_2 = n$.

Proposition 2.9. *With the notation and assumptions of Proposition 2.4, the following conditions are equivalent for a homogeneous $\mathbb{P}^{n-1}$-bundle $f : P \to X$:*

(i) *P is indecomposable.*

(ii) *The associated representation $\tilde{\rho} : \widetilde{H} \to \mathrm{GL}_n$ is indecomposable.*

(iii) *$\widetilde{H_s}$ is a Heisenberg group and $k^n \cong U \otimes V$ as representations of $H \cong H_u \times \widetilde{H_s}$, where U is an indecomposable representation of H_u, and V is the standard representation of $\widetilde{H_s}$.*

(iv) *The neutral component $\mathrm{Aut}^0_X(P)$ is unipotent.*

Proof. (i)⇒(ii) is obvious.

(ii)⇒(i) If P is decomposable, then the associated (homogeneous) PGL_n-torsor $\pi : Y \to X$ admits a reduction of structure group to some $\mathrm{P}(\mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2})$-bundle $\pi' : Y' \to X$, i.e., there exists a PGL_n-equivariant morphism $\varphi : Y \to \mathrm{PGL}_n/\mathrm{P}(\mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2})$. Since the latter homogeneous space is affine, φ is invariant under the largest anti-affine subgroup G of $\mathrm{Aut}(P) = \mathrm{Aut}^{\mathrm{PGL}_n}(Y)$ (see [9, Proposition 2.8]). It follows that P admits a decomposition into homogeneous sub-bundles; this yields a decomposition of $\tilde{\rho}$, in view of Theorem 2.1 (iii).

(ii)⇔(iii) follows readily from Proposition 2.7.

(iii)⇒(iv) Since $(n, p) = 1$, we have $\mathrm{M}_n = k\mathrm{id} \oplus \mathfrak{pgl}_n$ as representations of PGL_n acting by conjugation. In view of (6), this yields

$$\mathrm{Lie}\,\mathrm{Aut}_X(P) = \mathrm{M}_n^H / k\,\mathrm{id} = \mathrm{End}^H(U \otimes V)/k\,\mathrm{id}.$$

Moreover, $\mathrm{End}^H(U \otimes V) \cong \mathrm{End}^{H_u}(U)$ by Schur's lemma, and hence

$$\mathrm{Lie}\,\mathrm{Aut}_X(P) \cong \mathrm{End}^{H_u}(U)/k\,\mathrm{id}.$$

This isomorphism of Lie algebras arises from the natural homomorphism

$$\mathrm{GL}(U)^{H_u}/\mathbb{G}_m\,\mathrm{id} \longrightarrow \mathrm{Aut}_X(P).$$

Since $\mathrm{Aut}_X(P)$ is smooth (Remark 2.5), we see that its neutral component is a quotient of $\mathrm{GL}(U)^{H_u}/\mathbb{G}_m\,\mathrm{id}$. But the latter group is unipotent, since U is indecomposable.

(iv)⇒(iii) Observe that the weight space decomposition (13) is trivial: otherwise, $\mathrm{Aut}_X(P)$ contains a copy of $\mathbb{G}_m$ that fixes some weight space pointwise and acts by scalar multiplication on all the other weight spaces. Thus, $k^n \cong U \otimes V$, where U is unipotent and V is irreducible. Moreover, U is indecomposable; otherwise, $\mathrm{Aut}_X(P)$ contains a copy of $\mathbb{G}_m$ by the above argument. □

Remarks 2.10. (i) For an arbitrary homogeneous projective bundle P, each representation U_λ (with the notation of Proposition 2.7) is a direct sum of indecomposable representations with multiplicities; moreover, these indecomposable summands and their multiplicities are uniquely determined up to reordering, in view of the Krull-Schmidt theorem. Thus, the representation of $\widetilde{H}$ in k^n decomposes into a direct sum (with multiplicities) of tensor products $U \otimes V$, where U is an indecomposable representation of H_u, and V an irreducible representation of $\widetilde{H_s}$.

Let $L \subset \mathrm{PGL}_n$ denote the stabilizer of such a decomposition. Then L is a Levi subgroup, uniquely determined up to conjugation; moreover, the PGL_n-torsor $\pi : Y \to X$, associated with P, admits a reduction of structure group to an L-torsor $\pi_L : Y_L \to X$. Arguing as in the proof of (iii)⇒(iv) above, one may check that the natural homorphism $Z(L) \to \mathrm{Aut}_X^L(Y_L)$ (where $Z(L)$ denotes the center of L, and $\mathrm{Aut}_X^L(Y_L)$ the group of bundle automorphisms of Y_L) yields an isomorphism of the reduced neutral component $Z(L)^0_{\mathrm{red}}$ to a maximal torus of $\mathrm{Aut}_X^L(Y_L)$. Thus, the torsor $\pi_L : Y_L \to X$ is *L-indecomposable* in the sense of [2, Definition 2.1]. Moreover, this torsor is the unique reduction of $\pi : P \to X$ to an L-indecomposable torsor for a Levi subgroup, by Theorem 3.4 of [loc. cit.] (the latter result is obtained there in characteristic zero, and generalized to arbitrary characteristics in [4]; see also [3]).

(ii) The above results do not extend readily to the case where p divides n, e.g., there exists a non-degenerate theta group $\widetilde{H} \subset \mathrm{GL}_p$ with H unipotent and local. Consider indeed the group scheme α_p (the kernel of the pth power map of $\mathbb{G}_a$) and the duality pairing

$$u : \alpha_p \times \alpha_p \longrightarrow \mathbb{G}_m, \quad (x, y) \longmapsto \sum_{i=0}^{p-1} \frac{x^i}{i!}.$$

This yields a bilinear alternating pairing e on $H := \alpha_p \times \alpha_p$, via

$$e((x, y), (x', y')) := u(x, y')\, u(x', y)^{-1}.$$

Then we may take for $\widetilde{H}$ the associated Heisenberg group scheme (with $K = \alpha_p \times 0$ and $\mathcal{X}(K) = 0 \times \alpha_p$), equipped with its standard representation in $\mathcal{O}(\alpha_p) \cong k^p$.

Note that the above group scheme H is contained in some abelian variety, e.g., in a product of two supersingular elliptic curves. More generally, any finite commutative group scheme is contained in some abelian variety (see [26, Section 15.4]).

3 Irreducible bundles

Throughout this section, we consider $\mathbb{P}^{n-1}$-bundles $f : P \to X$, and call them *bundles* for simplicity; we still assume that $(n, p) = 1$.

We say that a homogeneous bundle P is *irreducible*, if so is the projective representation $\rho : H \to \mathrm{PGL}_n$ associated with P via Theorem 2.1. By Proposition 2.7, this means that $\widetilde{H}$ is a Heisenberg group acting on k^n via its standard representation.

We now parametrize the irreducible homogeneous bundles, and describe their associated Azumaya algebra as well as the adjoint bundle and automorphism group:

PROPOSITION 3.1. (i) *The irreducible homogeneous $\mathbb{P}^{n-1}$-bundles are classified by the pairs (H, e), where $H \subset {}_nX$ is a subgroup of order n^2, and $e : H \times H \to \mathbb{G}_m$ is a non-degenerate alternating pairing. In particular, such bundles exist for any given n, and they form only finitely many isomorphism classes.*
(ii) *For the bundle P corresponding to (H, e), the associated Azumaya algebra $\mathcal{A}$ admits a grading by the group H, namely,*

$$\mathcal{A} \cong \bigoplus_{\mathcal{L} \in H} \mathcal{L},$$

where each element of $H \subset \widehat{X}$ is viewed as an invertible sheaf on X. In particular, we have a decomposition

$$\mathrm{ad}(P) \cong \bigoplus_{\mathcal{L} \in H, \mathcal{L} \neq 0} \mathcal{L}.$$

(iii) *For P as in* (ii)*, we have that $\mathrm{Aut}_X(P) \cong H$; the neutral component $\mathrm{Aut}^0(P)$ is the extension of X by H dual to the inclusion $\mathcal{X}(H) \cong H \subset \widehat{X}$, and $\mathrm{Aut}(P)/\mathrm{Aut}^0(P)$ is isomorphic to the subgroup of $\mathrm{Aut}_{\mathrm{gp}}(X) \cong \mathrm{Aut}_{\mathrm{gp}}(\widehat{X})$ that preserves H and e.*

Proof. (i) By the results of Section 2, the irreducible homogeneous bundles are classified by the pairs consisting of an isogeny $1 \to H \to G \to X \to 1$ and a non-degenerate alternating pairing e on H; then e provides an isomorphism (11) of H with its character group. The assertion now follows from duality of isogenies.

(ii) follows from the isomorphism of $\mathcal{O}_X$-algebras (4) together with the isomorphism of $\mathcal{O}_X$-H-algebras $\gamma_*(\mathcal{O}_G) \cong \bigoplus_{\mathcal{L} \in \mathcal{X}(H)} \mathcal{L}$ and with the decomposition $\mathrm{M}_n \cong \bigoplus_{h \in H} k\, u_h$ obtained in Lemma 2.6 (iii).

(iii) Combining the isomorphism (5) and Lemma 2.6 (i), we see that the natural map $H \to \mathrm{Aut}_X(P)$ is an isomorphism. In view of the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & X & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{Aut}_X(P) & \longrightarrow & \mathrm{Aut}(P) & \xrightarrow{f_*} & \mathrm{Aut}(X) & & \end{array}$$

and of the isomorphism $\mathrm{Aut}(X) \cong X \ltimes \mathrm{Aut}_{\mathrm{gp}}(X)$, where $\mathrm{Aut}_{\mathrm{gp}}(X)$ is étale, it follows that the natural map $G \to \mathrm{Aut}^0(P)$ is an isomorphism as well. The structure of $\mathrm{Aut}(P)/\mathrm{Aut}^0(P)$ follows from Theorem 2.1 together with Lemma 2.6 (ii). □

Remark 3.2. Recall from [23, Section 1] that every finite commutative group H of order prime to p, equipped with a non-degenerate alternating pairing e, admits a decomposition

$$H = H_{n_1} \times \cdots \times H_{n_r}, \quad e = (e_{d_1}, \ldots, e_{d_r})$$

such that

$$H_{n_i} = \mathbb{Z}/n_i\mathbb{Z} \times \mathcal{X}(\mathbb{Z}/n_i\mathbb{Z}) \cong (\mathbb{Z}/n_i\mathbb{Z})^2, \quad e_{d_i}((x,\chi),(x',\chi')) = \chi'(x)^{d_i}\,\chi(x')^{-d_i},$$

where the n_i, d_j are integers satisfying $n_{i+1}|n_i$, $0 \le d_i < n_i$, and $(d_i, n_i) = 1$ for all i. Moreover, $n_1, \ldots, n_r$ are uniquely determined by H. Since H is a subgroup of ${}_n\widehat{X} \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$, we see that $r \le g$; conversely, any product of r cyclic groups of order prime to p can be embedded into ${}_n\widehat{X}$, provided that $r \le g$.

It follows that every homogeneous irreducible bundle admits a decomposition into a product

$$P = P_1 \cdots P_r,$$

where each P_i corresponds to (H_{n_i}, e_{d_i}). Moreover, the P_i are exactly the irreducible homogeneous bundles associated with a product of two copies of a cyclic group; we call these bundles *cyclic*.

Equivalently, the associated Azumaya algebra satisfies

$$\mathcal{A} = \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_r,$$

where again, $\mathcal{A}_i$ corresponds to (H_{n_i}, e_{d_i}). Moreover, the $\mathcal{O}_X$-algebra $\mathcal{A}_i$ is generated by two invertible sheaves $\mathcal{L}$, $\mathcal{M}$ (associated with the natural generators of $(\mathbb{Z}/n_i\mathbb{Z})^2$), with relations $x^{n_i} = \xi$, $y^{n_i} = \eta$, $xy = \zeta^{d_i} yx$ for any local generators $x \in \mathcal{L}$, $y \in \mathcal{M}$, where ξ (resp. η) denotes a local trivialization of $\mathcal{L}^{\otimes n}$ (resp. $\mathcal{M}^{\otimes n}$), and ζ is a fixed primitive d_ith root of unity (this follows by combining the isomorphism of algebras (4) with the description of the H_{n_i}-algebra M_{n_i} obtained in Lemma 2.6 (iii)). In particular, $\mathcal{A}_i$ yields a cyclic division algebra over $k(X)$.

Example 3.3. Let X be an elliptic curve. Then X is canonically isomorphic to $\widehat{X}$, and the finite subgroups of X admitting a non-degenerate alternating pairing are exactly the n-torsion subgroups ${}_nX$. In view of the above remark, it follows that the irreducible homogeneous bundles over X are exactly the cyclic bundles. By a result of Atiyah (see [1, Theorem 10]), they are exactly the projectivizations of the indecomposable vector bundles of coprime rank and degree, i.e. of the simple vector bundles.

Example 3.4. Returning to an arbitrary abelian variety X, we recall from [23, Section 1] a geometric construction of Heisenberg groups. Let L be a line bundle on X, and $K(L)$ the kernel of the polarization homomorphism

$$\varphi_L : X \longrightarrow \widehat{X}, \quad x \longmapsto T_x^*(L) \otimes L^{-1}. \tag{15}$$

Denoting by $\mathcal{G}(L)$ the group scheme of automorphisms of the variety L that commute with the action of $\mathbb{G}_m$ by multiplication on fibers, we have a central extension

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \mathcal{G}(L) \longrightarrow K(L) \longrightarrow 1.$$

The associated commutator pairing on $K(L)$ is denoted by e^L.

Also, recall that an effective line bundle L is ample if and only if φ_L is an isogeny; equivalently, $K(L)$ is finite. Then the theta group $\mathcal{G}(L)$ is non-degenerate, and acts on the space of global sections $H^0(X, L)$ via its standard representation. Thus, $K(L)$ acts on the associated projective space

$$|L| := \mathbb{P}(H^0(X, L)),$$

and the natural map

$$f : X \times^{K(L)} |L| \longrightarrow X/K(L) \cong \widehat{X}$$

is an irreducible homogeneous bundle.

As will be shown in detail in Section 4, this bundle is the projectivization of a natural vector bundle E over $\widehat{X}$. Moreover, if X is an elliptic curve (so that $X \cong \widehat{X}$) and L has degree n, then E has rank n and degree -1.

We now obtain several criteria for a homogeneous bundle to be irreducible:

Proposition 3.5. *The following conditions are equivalent for a homogeneous bundle P:*

(i) P *is irreducible.*

(ii) P *admits no proper homogeneous sub-bundle.*

(iii) $\mathrm{ad}(P)$ *splits into a direct sum of non-zero algebraically trivial line bundles.*

(iv) $H^0(X, \mathrm{ad}(P)) = 0$.

(v) $\mathrm{Aut}_X(P)$ *is finite.*

If P is the projectivization of a (semi-homogeneous) vector bundle E, then P is irreducible if and only if E is simple.

Proof. (i)$\Leftrightarrow$(ii) follows from Theorem 2.1 (iii), and (i)$\Rightarrow$(iii) from Proposition 3.1 (ii).

(iii)$\Rightarrow$(iv) holds since $H^0(X, L) = 0$ for any non-zero $L \in \widehat{X}$.

(iv)$\Rightarrow$(v) follows from the fact that $\mathrm{Lie}\,\mathrm{Aut}_X(P) = H^0(X, \mathrm{ad}(P))$.

(v)$\Rightarrow$(i) By Proposition 2.9, P is indecomposable and the quotient $\mathrm{GL}(U)^{H_u}/\mathbb{G}_m\mathrm{id}$ is finite, where U is the indecomposable representation of H_u given by that proposition. But $\mathrm{GL}(U)^{H_u}/\mathbb{G}_m\mathrm{id}$ is of positive dimension for any unipotent subgroup scheme $H_u \subset \mathrm{GL}(U)$, unless $\dim(U) = 1$; in the latter case, P is clearly irreducible.

The final assertion follows from the isomorphism

$$H^0(X, \mathrm{ad}(\mathbb{P}(E))) \cong H^0(X, \mathrm{End}(E))/k\,\mathrm{id}.$$

□

Remark 3.6. The indecomposable homogeneous bundles are exactly the products $\mathbb{P}(U)\, I$, where U is an indecomposable unipotent vector bundle, and I an irreducible homogeneous bundle (as follows from Remark 2.2 and Proposition 2.9).

In particular, the indecomposable homogeneous bundles over an elliptic curve X are exactly the projectivizations $\mathbb{P}(U \otimes E)$, where U is as above, and E is a simple vector bundle (as in Example 3.3).

In fact, any indecomposable vector bundle over X is isomorphic to $U \otimes E \otimes L$ for U, E as above and L a line bundle. Moreover, U is uniquely determined by its rank, and E is uniquely determined by its (coprime) rank and degree, up to tensoring with a line bundle of degree 0 (see [1]).

Next, we obtain a cohomological criterion for a bundle to be homogeneous and irreducible, thereby extending a result of Mukai about simple semi-homogeneous vector bundles (see [22, Theorem 5.8]):

PROPOSITION 3.7. *A bundle P is homogeneous and irreducible if and only if we have $H^0(X,\mathrm{ad}(P)) = H^1(X,\mathrm{ad}(P)) = 0$; then $H^i(X,\mathrm{ad}(P)) = 0$ for all $i \geq 0$.*

PROOF. Recall that $H^i(X,L) = 0$ for all $i \geq 0$ and all non-zero $L \in \widehat{X}$. By Proposition 3.1 (ii), the same holds with L replaced with $\mathrm{ad}(P)$, if P is homogeneous and irreducible.

For the converse, observe that $\mathrm{ad}(P) = \pi_*(T_{Y/X})^{\mathrm{PGL}_n}$, where $\pi : Y \to X$ denotes the PGL_n-torsor associated to P, and $T_{Y/X}$ the relative tangent bundle. Thus, $\mathrm{ad}(P)$ fits in an exact sequence

$$0 \longrightarrow \mathrm{ad}(P) \longrightarrow \pi_*(T_Y)^{\mathrm{PGL}_n} \longrightarrow T_X \longrightarrow 0$$

obtained from the standard exact sequence $0 \to T_{Y/X} \to T_Y \to \pi^*T_X \to 0$ by taking the invariant direct image under π. If $H^1(X,\mathrm{ad}(P)) = 0$, then the natural map

$$H^0(Y,T_Y)^{\mathrm{PGL}_n} = H^0(X,\pi_*(T_Y)^{\mathrm{PGL}_n}) \longrightarrow H^0(X,T_X)$$

is surjective. But $H^0(Y,T_Y)^{\mathrm{PGL}_n} \cong \mathrm{Lie}(\mathrm{Aut}^{\mathrm{PGL}_n}(Y))$ and $H^0(X,T_X) \cong \mathrm{Lie}(\mathrm{Aut}(X))$; moreover, $\mathrm{Aut}^{\mathrm{PGL}_n}(Y) = \mathrm{Aut}(P)$ is smooth by Remark 2.5, and $\mathrm{Aut}(X)$ is smooth as well. Hence the homomorphism $\mathrm{Aut}^{\mathrm{PGL}_n}(Y) \to \mathrm{Aut}(X)$ is surjective on neutral components, i.e., Y is homogeneous. Thus, P is homogeneous, too. If in addition $H^0(X,\mathrm{ad}(P)) = 0$, then P is irreducible by Proposition 3.5. □

REMARK 3.8. The above argument shows that a bundle P is homogeneous if it satisfies $H^1(X,\mathrm{ad}(P)) = 0$. This may also be seen as follows: observe that $\mathrm{ad}(P) = f_*(T_{P/X})$ (as follows e.g. by considering an étale trivialization of P). Moreover, $R^if_*(T_{P/X}) = 0$ for all $i \geq 1$, since $H^i(\mathbb{P}^{n-1},T_{\mathbb{P}^{n-1}}) = 0$ for all such i. As a consequence, $H^1(P,T_{P/X}) = 0$. Then f is rigid as a morphism with target X, in view of [28, Corollary 3.4.9]. It follows readily that P is homogeneous.

The converse statement does not hold, e.g, if X is an elliptic curve in characteristic zero, U_n the indecomposable unipotent vector bundle of rank $n \geq 2$, and $P = \mathbb{P}(U_n)$, then

$$\mathrm{ad}(P) \cong (U_n \otimes U_n^*)/k\,\mathrm{id} \cong U_{2n-1} \oplus U_{2n-3} \oplus \cdots \oplus U_3$$

and hence $H^0(X,\mathrm{ad}(P))$ has dimension $n-1$. By the Riemann-Roch theorem, the same holds for $H^1(X,\mathrm{ad}(P))$.

Given a positive integer m, not divisible by p, we say that a bundle P is *trivialized by* m_X (the multiplication by m in X) if the pull-back bundle $m_X^*(P) \to X$ is trivial.

PROPOSITION 3.9. (i) *A bundle P is trivialized by m_X if and only if $P \cong X \times^{{}_mX} \mathbb{P}^{n-1}$ as bundles over $X \cong X/{}_mX$, for some action of ${}_mX$ on $\mathbb{P}^{n-1}$. In particular, every such bundle is homogeneous.*

(ii) *Any irreducible homogeneous $\mathbb{P}^{n-1}$-bundle is trivialized by n_X.*

PROOF. (i) If P is trivialized by m_X, then we have a cartesian square

$$\begin{array}{ccc} X \times \mathbb{P}^{n-1} & \xrightarrow{p_1} & X \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle m_X} \\ P & \xrightarrow{f} & X, \end{array}$$

where p_1 denotes the first projection. Thus, the action of ${}_mX$ by translations on X lifts to an action on $X \times \mathbb{P}^{n-1}$ such that q is invariant. This action is of the form

$$x \cdot (y, z) = (x + y, \varphi(x, y) \cdot z)$$

for some morphism $\varphi : {}_mX \times X \to \mathrm{Aut}(\mathbb{P}^{n-1}) = \mathrm{PGL}_n$. But every morphism from the abelian variety X to the affine variety PGL_n is constant. Thus, φ is independent of y, i.e., φ yields an action of ${}_mX$ on $\mathbb{P}^{n-1}$. Moreover, the ${}_mX$-invariant morphism q factors through a morphism of $\mathbb{P}^{n-1}$-bundles $X \times^{{}_mX} \mathbb{P}^{n-1} \to P$, which is the desired isomorphism.

The converse implication is obvious.

(ii) Write $P = G \times^H \mathbb{P}^{n-1}$ as in Theorem 2.1; then H is killed by n, in view of the structure of non-degenerate theta groups. In other words, the homomorphism $\gamma : G \to X$ is an isogeny with kernel killed by n. Thus, there exists a unique isogeny $\tau : X \to G$ such that $\gamma\tau = n_X$. Then ${}_nX = \tau^{-1}(H)$ and hence $X = X \times^{{}_nX} \mathbb{P}^{n-1}$, where ${}_nX$ acts on $\mathbb{P}^{n-1}$ via the surjective homomorphism $\tau_{|{}_nX} : {}_nX \to H$. □

By the above proposition, a bundle P trivialized by m_X defines an alternating bilinear map

$$e_{P,m} : {}_mX \times {}_mX \longrightarrow \mu_m.$$

Moreover, the irreducible homogeneous bundles are classified by those maps such that $[{}_mX : {}_mX^{\perp}] = m^2$ (as follows from Proposition 3.5). Also, one easily checks that *the assignement $P \mapsto e_{P,m}$ is multiplicative*, i.e., $e_{P_1P_2,m} = e_{P_1,m}\, e_{P_2,m}$ and $e_{P^*,m} = e_{P,m}^{-1}$.

PROPOSITION 3.10. *With the above notation and assumptions, P is the projectivization of a vector bundle if and only if there exists a line bundle L on X such that $e_{P,m} = e^{L^{\otimes m}}|_{{}_mX}$ (this makes sense as $K(L^{\otimes m})$ contains ${}_mX$).*

PROOF. Assume that $P = \mathbb{P}(E)$ for some vector bundle E of rank n on X. Since $m_X^*\mathbb{P}(E)$ is trivial, we have

$$m_X^*(E) \cong M^{\oplus n}$$

for some line bundle M on X. Replacing E with $E \otimes N$, where N is a symmetric line bundle on X, leaves $\mathbb{P}(E)$ unchanged and replaces $m_X^*(E)$ with $m_X^*(E) \otimes N^{\otimes m^2}$, and hence M with $M \otimes N^{\otimes m^2}$. Taking for N a large power of an ample symmetric line bundle, we may assume that M is very ample.

The pull-back $m_X^*(E)$ is equipped with an ${}_mX$-linearization. Equivalently, the action of ${}_mX$ by translations on X lifts to an action on $M^{\oplus n}$ which is linear on fibers. In particular, $T_x^*(M^{\oplus n}) \cong M^{\oplus n}$ for any $x \in {}_mX$. This isomorphism is given by an $n \times n$ matrix of maps $T_x^* M \to M$; thus, $H^0(X, T_x^*(M^{-1}) \otimes M) \neq 0$. Since $T_x^*(M^{-1}) \otimes M \in \widehat{X}$, it follows that this line bundle is trivial. In other words, ${}_mX \subset K(M)$; this is equivalent to the existence of a line bundle L in X such that $M = L^{\otimes m}$. Moreover, we have a representation of ${}_mX$ in $H^0(X, M^{\oplus n}) \cong H^0(X, M) \otimes k^n$ that lifts the homomorphism

$$\phi : {}_mX \longrightarrow \mathrm{PGL}(H^0(X, M)) \times \mathrm{PGL}_n \tag{16}$$

given by the ${}_mX$-action on $\mathbb{P}(H^0(X, M))$ as a subgroup of $K(M)$, and the ${}_mX$-action on $\mathbb{P}^{n-1}$ that defines P. It follows that $e^M \, e_{P,m} = 1$ on ${}_mX$; equivalently, $e_{P,m}$ is the restriction to ${}_mX$ of $e^{M^{\otimes(-1)}} = e^{L^{\otimes(-m)}} = e^{L^{\otimes m(m-1)}}$ (since $e^{L^{\otimes m^2}} = 1$).

To show the converse, we reduce by inverting the above arguments to the case that $e^M \, e_{P,m} = 1$ on ${}_mX$, for some line bundle M on X such that ${}_mX \subset K(M)$; we may also assume that M is very ample. Then ${}_mX$ acts on $H^0(X, M^{\oplus n})$ by lifting the homomorphism (16). Moreover, the evaluation morphism

$$\mathcal{O}_X \otimes H^0(X, M^{\oplus n}) = \mathcal{O}_X \otimes H^0(X, M) \otimes k^n \longrightarrow M \otimes k^n = M^{\oplus n}$$

is surjective, and its kernel is stable under the induced action of ${}_mX$ (since the analogous morphism $\mathcal{O}_X \otimes H^0(X, M) \to M$ is equivariant with respect to the theta group of ${}_mX \subset K(M)$). Thus, ${}_mX$ acts on $M^{\oplus n}$ by lifting its action on X via translation. Now $M^{\oplus n}$ descends to a vector bundle on $X/{}_mX \cong X$, with projectivization P. □

We now extend the statement of Proposition 3.10 to all homogeneous bundles P. We use the notation of Section 2; in particular, the associated pairing e_F introduced in Proposition 2.4. Then e_F factors through a non-degenerate pairing on $F/F^{\perp} \cong H/H^{\perp}$, and this group is killed by the homogeneous index $d = d(H)$ defined by (14). Thus, the isogeny $G/H^{\perp} \to G/H = X$ has its kernel killed by d; as in the proof of Proposition 3.9 (ii), this yields a canonical surjective homomorphism ${}_dX \to H/H^{\perp}$ and, in turn, a bilinear alternating pairing e_P on ${}_dX$.

Theorem 3.11. *With the preceding notation, P is the projectivization of a vector bundle if and only if $e_P = e^{L^d}|_{{}_dX}$ for some line bundle L on X.*

Proof. Choose a linear subspace $S \subset \mathbb{P}^{n-1}$ which is H-stable, and minimal for this property. Then S yields a homogeneous irreducible $\mathbb{P}^{d-1}$-sub-bundle, and the associated pairing on ${}_dX$ is just e_P. Now the statement is a consequence of Proposition 3.10 together with the following observation.

Lemma 3.12. *Let $f : P \to Z$ be a projective bundle over a non-singular variety, and $f' : P' \to Z$ a projective sub-bundle. Then P is the projectivization of a vector bundle if and only if so is P'.*

Proof. Clearly, if $P = \mathbb{P}(E)$ for some vector bundle E over Z, then $P' = \mathbb{P}(E')$ for some sub-bundle $E' \subset E$. To show the converse, consider the PGL_n-torsor $\pi : Y \to Z$ associated with P. As in the proof of Theorem 2.1 (iii), the sub-bundle P' yields a reduction of structure group to a $\mathrm{PGL}_{n,n'}$-torsor $\pi' : Y' \to Z$, where $\mathrm{PGL}_{n,n'} \subset \mathrm{PGL}_n$ denotes the stabilizer of $\mathbb{P}^{n'-1} \subset \mathbb{P}^n$. Moreover, we have an exact sequence of algebraic groups

$$1 \longrightarrow G_{n,n'} \longrightarrow \mathrm{PGL}_{n,n'} \xrightarrow{\ r\ } \mathrm{PGL}_{n'} \longrightarrow 1,$$

where r denotes the restriction to $\mathbb{P}^{n'-1}$, and $G_{n,n'} = \mathrm{M}_{n',n-n'} \ltimes \mathrm{GL}_{n-n'}$, where $G_{n,n'}$ acts naturally on the space of matrices $M_{n',n-n'}$. Also, π' factors as

$$Y' \xrightarrow{\ \varphi\ } Y'/G_{n,n'} \xrightarrow{\ \psi\ } Z$$

where φ is a $G_{n,n'}$-torsor, and ψ is the $\mathrm{PGL}_{n'}$-torsor associated with P'. By assumption, $P' = \mathbb{P}(E')$ for some vector bundle E'; this is equivalent to ψ being locally trivial, in view of [29, Proposition 18]. But φ is locally trivial as well, since the algebraic group $G_{n,n'}$ is special by the results of [loc. cit., 4.3, 4.4]. Thus, π' is locally trivial, and hence so is π. We conclude that $P = \mathbb{P}(E)$ for some vector bundle E.

Alternatively, one may use the fact that P is the projectivization of a vector bundle if and only if f has a rational section ([29, Proposition 18] again), and conclude by applying [16, Proposition 5.3.1]. □

□

Remark 3.13. We now relate Proposition 3.10 to a description of the Brauer group $\mathrm{Br}(X)$, due to Berkovich. Recall from [18, Section I.8.4] that $\mathrm{Br}(X)$ may be viewed as the set of equivalence classes of projective bundles over X, where two such bundles P_1, P_2 are equivalent if there exist vector bundles E_1, E_2 such that $\mathbb{P}(E_1)P_1 \cong \mathbb{P}(E_2)P_2$; the group law arises from the operations of product and duality. By [5, Section 3], we have an exact sequence for any positive integer n:

$$0 \longrightarrow \mathrm{Pic}(X)/n\mathrm{Pic}(X) \xrightarrow{\ \varphi\ } \mathrm{Hom}(\Lambda^2{}_nX, \mu_n) \xrightarrow{\ \psi\ } {}_n\mathrm{Br}(X) \longrightarrow 0,$$

where $\mathrm{Hom}(\Lambda^2{}_nX, \mu_n)$ consists of the bilinear alternating pairings ${}_nX \times {}_nX \to \mu_n$, and ${}_n\mathrm{Br}(X) \subset \mathrm{Br}(X)$ denotes the n-torsion subgroup; the map φ sends the class of $L \in \mathrm{Pic}(X)$ to the pairing $e^{L^{\otimes n}}|_{{}_nX}$, and ψ sends e to the class of the Azumaya algebra

$$\mathcal{A} := \bigoplus_{\alpha \in {}_n\widehat{X},\, \sigma \in {}_nX} \mathcal{L}_\alpha \, e_\sigma,$$

where $\mathcal{L}_\alpha$ denotes the invertible sheaf associated with α, and the multiplication is defined by

$$f_\alpha\, e_\sigma \cdot f_\beta\, e_\tau = \bar{e}_n(\beta,\sigma)\, a_{\sigma,\tau}\, f_\alpha\, f_\beta\, e_{\sigma+\tau}.$$

Here f_α (resp. f_β) is a local section of $\mathcal{L}_\alpha$ (resp. $\mathcal{L}_\beta$); $\bar{e}_n$ is the canonical pairing between ${}_n\widehat{X}$ and ${}_nX$, and $\{a_{\sigma,\tau}\} \in Z^2({}_nX, \mathbb{G}_m)$ is a 2-cocycle such that $e(\sigma,\tau) = a_{\sigma,\tau}\, a_{\tau,\sigma}^{-1}$. (The class of $\mathcal{A}$ in the Brauer group does not depend on the choice of the representative $\{a_{\sigma,\tau}\}$ of e viewed as an element of $H^2({}_nX, \mathbb{G}_m)$.) Thus,

$$\mathcal{L} := \bigoplus_{\alpha \in {}_n\widehat{X}} \mathcal{L}_\alpha\, e_0$$

is a maximal étale subalgebra of $\mathcal{A}$ in the sense of [18, Définition 5.6]; note that $\mathcal{L} \cong (n_X)_*\mathcal{O}_X$ as $\mathcal{O}_X$-algebras. Moreover, the left $\mathcal{L}$-module $\mathcal{A}$ is free with basis $(a_\sigma)_{\sigma \in {}_nX}$. By [loc. cit., Corollaire 5.5], it follows that $n_X^*(\mathcal{A}) \cong \mathrm{M}_m(\mathcal{O}_X)$, where $m := \#({}_nX) = n^{2g}$. In other words, the projective bundle associated with $\mathcal{A}$ is trivialized by n_X. In view of Proposition 3.9, it follows that the associated projective bundle is homogeneous.

In fact, *any class in* ${}_n\mathrm{Br}(X)$ *is represented by an irreducible homogeneous bundle.* Indeed, given any homogeneous bundle P, we may choose an irreducible sub-bundle Q; then the product QQ^* is a sub-bundle of PQ^*, and is the projectivization of a vector bundle. By Lemma 3.12, it follows that the class of PQ^* in $\mathrm{Br}(X)$ is trivial; equivalently, P and Q have the same class there.

Also, recall that the natural map $\mathrm{Br}(X) \to \mathrm{Br}(k(X))$ is injective (see [18, Section II.1]). As a very special case of a theorem of Merkurjev and Suslin (see [16, Theorem 2.5.7]), each class in ${}_n\mathrm{Br}(k(X))$ can be represented by a tensor product of cyclic algebras. So the decomposition of classes in ${}_n\mathrm{Br}(X)$ obtained in Remark 3.2 may be viewed as a global analogue of that result, for abelian varieties.

Finally, note that Proposition 3.10 is equivalent to the assertion that *the image of* φ *consists of those pairings associated with projectivizations of semi-homogeneous vector bundles.* In loose terms, the Brauer group is generated by homogeneous bundles, and the relations arise from semi-homogeneous vector bundles.

4 Examples

Let λ be an effective class in the Néron-Severi group $NS(X)$, viewed as the group of divisors on X modulo algebraic equivalence. The effective divisors on X with class λ are parametrized by a projective scheme $\mathrm{Div}^\lambda(X)$. Indeed, the Hilbert polynomial of any such divisor D, relative to a fixed ample line bundle on X, depends only on λ; thus, $\mathrm{Div}^\lambda(X)$ is a union of connected components of the Hilbert scheme $\mathrm{Hilb}(X)$.

Also, recall that the line bundles on X with class λ are parametrized by the Picard variety $\mathrm{Pic}^{\lambda}(X)$. Choosing L in that variety, we have

$$\mathrm{Pic}^{\lambda}(X) = L \otimes \mathrm{Pic}^{0}(X) = L \otimes \widehat{X}.$$

On $X \times \mathrm{Pic}^{\lambda}(X)$ we have a universal bundle: the Poincaré bundle $\mathcal{P}$, uniquely determined up to the pull-back of a line bundle under the second projection

$$\pi : X \times \mathrm{Pic}^{\lambda}(X) \longrightarrow \mathrm{Pic}^{\lambda}(X).$$

The universal family on $\mathrm{Div}^{\lambda}(X)$ yields a morphism

$$f : \mathrm{Div}^{\lambda}(X) \longrightarrow \mathrm{Pic}^{\lambda}(X), \quad D \longmapsto \mathcal{O}_X(D). \tag{17}$$

Note that X acts on $\mathrm{Div}^{\lambda}(X)$ and on $\mathrm{Pic}^{\lambda}(X)$ via its action on itself by translations; moreover, f is equivariant. Also, the isotropy subgroup scheme in X of any point of $\mathrm{Pic}^{\lambda}(X)$ is the group scheme $K(L)$ that occured in Example 3.4.

If λ is ample, then $\mathrm{Pic}^{\lambda}(X)$ is the X-orbit $X \cdot L \cong X/K(L)$. Thus, f is a homogeneous fiber bundle over $X/K(L)$; the latter abelian variety is isomorphic to $\widehat{X}$ via the polarization homomorphism (15).

Proposition 4.1. *Let $\lambda \in \mathrm{NS}(X)$ be an ample class, and $L \in \mathrm{Pic}^{\lambda}(X)$.*

(i) *We have an isomorphism*

$$\mathrm{Div}^{\lambda}(X) \cong X \times^{K(L)} |L|$$

of homogeneous bundles over $X/K(L)$. In particular, $\mathrm{Div}^{\lambda}(X)$ is a homogeneous projective bundle over $\widehat{X}$.

(ii) *The sheaf $\mathcal{E} := \pi_*(\mathcal{P})$ is locally free, and the morphism (17) is the projectivization of the corresponding vector bundle.*

(iii) *The group scheme $\mathrm{Aut}(\mathrm{Div}^{\lambda}(X))$ is the semi-direct product of X (acting by translations) with the subgroup of $\mathrm{Aut}_{\mathrm{gp}}(X)$ that preserves $K(L)$ and e^L.*

Proof. (i) Clearly, the set-theoretic fiber of f at L is the projective space $|L|$, and its dimension $h^0(L) - 1 = \chi(L) - 1$ is independent of $L \in \mathrm{Pic}^{\lambda}(X)$. As a consequence, the scheme $\mathrm{Div}^{\lambda}(X)$ is irreducible of dimension $\dim(X) + h^0(L) - 1$.

To complete the proof, it suffices to show that the differential of f at any $D \in |L|$ is surjective with kernel of dimension $h^0(L) - 1$. Identifying $\mathrm{Div}^{\lambda}(X)$ with a union of components of $\mathrm{Hilb}(X)$, and $\mathrm{Pic}^{\lambda}(X)$ with $\widehat{X}$, the differential

$$T_D f : T_D \mathrm{Div}^{\lambda}(X) \longrightarrow T_L \mathrm{Pic}^{\lambda}(X)$$

is identified with the boundary map $\partial : H^0(D, L_{|D}) \to H^1(X, \mathcal{O}_X)$ of the long exact sequence of cohomology associated with the short exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow L \longrightarrow L_{|D} \longrightarrow 0$$

(see [28, Proposition 3.3.6]). Since $H^1(X, L) = 0$, this long exact sequence begins with

$$0 \longrightarrow k \longrightarrow H^0(X, L) \longrightarrow H^0(D, L_{|D}) \xrightarrow{\partial} H^1(X, \mathcal{O}_X) \longrightarrow 0$$

which yields the desired assertion.

(ii) The vanishing of $H^1(X, L)$ also implies that $\mathcal{E}$ is locally free and satisfies $\mathcal{E}(L) \cong H^0(X, L)$. Thus, it suffices to check that the associated projective bundle $\mathbb{P}(\mathcal{E})$ is homogeneous. But for any $x \in X$, there exists an invertible sheaf L_x on $\mathrm{Pic}^\lambda(X)$ such that

$$(T_x, T_x)^*(\mathcal{P}) \cong \mathcal{P} \otimes \pi^* L_x,$$

in view of the universal property of the Poincaré bundle $\mathcal{P}$. Since $\pi_*(T_x, T_x)^*(\mathcal{P}) \cong T_x^*(\pi_*(\mathcal{P})) = T_x^*(\mathcal{E})$, this yields an isomorphism

$$T_x^*(\mathcal{E}) \cong \mathcal{E} \otimes L_x.$$

In other words, $\mathcal{E}$ is semi-homogeneous.

(iii) is checked by arguing as in the proof of Proposition 3.1 (iii). □

The case of an arbitrary effective class λ reduces to the ample case in view of the following:

PROPOSITION 4.2. *Let $\lambda \in \mathrm{NS}(X)$ be an effective class, $L \in \mathrm{Pic}^\lambda(X)$, and $q : X \to \bar{X}$ the quotient map by the reduced neutral component $K(L)^0_{\mathrm{red}} \subset K(L)$. Then $\lambda = q^*(\bar{\lambda})$ for a unique ample class $\bar{\lambda} \in \mathrm{NS}(\bar{X})$, and $f : \mathrm{Div}^\lambda(X) \to \mathrm{Pic}^\lambda(X)$ may be identified with $\bar{f} : \mathrm{Div}^{\bar{\lambda}}(\bar{X}) \to \mathrm{Pic}^{\bar{\lambda}}(\bar{X})$.*

PROOF. We claim that any $D \in \mathrm{Div}^\lambda(X)$ equals $q^*(\bar{D})$ for some ample effective divisor $\bar{D}$ on $\bar{X}$.

To see this, recall that nD is base-point-free for any $n \geq 2$; this yields morphisms

$$\gamma_n : X \longrightarrow \mathbb{P}(H^0(X, L^{\otimes n})^*) \quad (n \geq 2)$$

which are equivariant for the action of $K(L)$. The abelian variety $K(L)^0_{\mathrm{red}}$ acts trivially on each projective space $\mathbb{P}(H^0(X, L^{\otimes n})^*)$; thus, each γ_n is invariant under $K(L)^0_{\mathrm{red}}$. In the Stein factorization of γ_n as

$$X \xrightarrow{\varphi_n} Y_n \xrightarrow{\psi_n} \mathbb{P}(H^0(X, L^{\otimes n})^*),$$

where $(\varphi_n)_*\mathcal{O}_X = \mathcal{O}_{Y_n}$ and ψ_n is finite, the morphism φ_n is the natural map

$$\varphi : X \longrightarrow \operatorname{Proj} \bigoplus_{m=0}^{\infty} H^0(X, L^{\otimes m}) =: Y.$$

In particular, φ_n is independent of n, and invariant under $K(L)^0_{\mathrm{red}}$. Moreover, since nD is the pull-back of a hyperplane under γ_n for any $n \geq 2$, we see that $D = 3D - 2D = \varphi^*(E)$ for some Cartier divisor E on Y. Then E is effective and $H^0(X, L^{\otimes n}) \cong H^0(Y, M^{\otimes n})$ for all n, where $M := \mathcal{O}_Y(E)$; it follows that E is ample. Consider the factorization

$$\bar{\varphi} : \bar{X} := X/K(L)^0_{\mathrm{red}} \longrightarrow Y,$$

the effective divisor $\bar{D} := \bar{\varphi}^*(E)$, and the associated invertible sheaf $\bar{L} = \bar{\varphi}^*(M)$. Then $L = q^*(\bar{L})$. Thus, the group scheme $K(\bar{L}) = K(L)/K(L)^0_{\mathrm{red}}$ is finite and $\bar{L}$ has non-zero global sections; hence $\bar{L}$ is ample. Thus, $\bar{\varphi}$ is finite. But $\bar{\varphi}_*\mathcal{O}_{\bar{X}} = \mathcal{O}_Y$; it follows that $\bar{\varphi}$ is an isomorphism, and identifies φ with q. This proves the claim.

As a consequence, $\lambda = q^*(\bar{\lambda})$ for a unique ample class $\bar{\lambda}$. We now show that the morphism

$$q^* : \operatorname{Div}^{\bar{\lambda}}(\bar{X}) \longrightarrow \operatorname{Div}^{\lambda}(X)$$

is an isomorphism. By the first step, q^* is bijective. In view of Proposition 4.1, it follows that the scheme $\operatorname{Div}^{\lambda}(X)$ is irreducible of dimension $\dim(\bar{X}) + h^0(\bar{X}, \bar{L}) - 1$. On the other hand, the Zariski tangent space of $\operatorname{Div}^{\lambda}(X)$ at D equals

$$H^0(D, L_{|D}) \cong H^0(\bar{D}, \bar{L}_{|\bar{D}}) = T_{\bar{D}}\operatorname{Div}^{\bar{\lambda}}(\bar{X}).$$

Thus, q^* is étale, and hence is an isomorphism. □

In the above construction, one may replace the abelian variety X with any smooth projective variety; for example, a curve C. Then an effective class in $\mathrm{NS}(C) \cong \mathbb{Z}$ is just a non-negative integer d. Moreover, $\operatorname{Div}^d(C)$ is the symmetric product $C^{(d)}$, a smooth projective variety of dimension d equipped with a morphism

$$f = f_d : C^{(d)} \longrightarrow \operatorname{Pic}^d(C). \tag{18}$$

Choosing a point of C, we may identify $\operatorname{Pic}^d(C)$ with the Jacobian variety $J = J(C)$.

If $d > 2g - 2$, where g denotes of course the genus of C, then f is the projectivization of a vector bundle $E = E_d$ on $\operatorname{Pic}^d(C)$, the direct image of the Poincaré bundle on $C \times \operatorname{Pic}^d(C)$ under the second projection. Moreover, E has rank $n := d - g + 1$.

Proposition 4.3. *With the above notation, the projective bundle (18) is homogeneous if and only if $g \leq 1$.*

Proof. Assume that (18) is homogeneous. Then E is semi-homogeneous; in view of [22, Lemma 6.11], we then have an isomorphism of vector bundles on J

$$n_J^*(E) \cong \det(E)^{\otimes n} \otimes F$$

for some homogeneous vector bundle F. Moreover, the Chern classes of F are algebraically trivial by [22, Theorem 4.17]. Thus, the total Chern class of E satisfies

$$n_J^*(c(E)) = (1 + nc_1(E))^n$$

in the cycle ring of J modulo algebraic equivalence. Since $n_J^*(c_1(E)) = n^2 c_1(E)$ in that ring, this yields

$$c(E) = (1 + \frac{c_1(E)}{n})^n. \tag{19}$$

We now recall a formula for $c(E)$ due to Mattuck (see [21, Theorem 3]). Denoting by W_i the image of p_i for $0 \leq i \leq g$, we have

$$c(E) = \sum_{i=0}^{g} (-1)^i [W_{g-i}^-],$$

where W_j^- denotes the image of W_j under the involution $(-1)_J$ and the equality holds again modulo algebraic equivalence. In particular,

$$c_1(E) = -[W_{g-1}^-] = -\theta,$$

where θ denotes the Chern class of the theta divisor, and

$$c_g(E) = (-1)^g e,$$

where e denotes the class of a point. In view of (19), this yields

$$e = \binom{n}{g} \frac{\theta^g}{n^g}.$$

Since $\theta^g = g!\, e$, we obtain $n^g = n(n-1)\cdots(n-g+1)$ and hence $g \leq 1$.

Conversely, if $g = 0$ then $C^{(d)} = \mathbb{P}^d$ and there is nothing to prove; if $g = 1$ then the assertion follows from Proposition 4.1. □

Remark 4.4. By [13], the vector bundle E is stable with respect to the principal polarization of J. In particular, E is simple, i.e., $\mathrm{Aut}_J(P)$ is finite. This yields examples of simple vector bundles on abelian varieties which are not semi-homogeneous (see [25] for the first construction of bundles satisfying these properties).

5 Reconstructing adjoint semi-simple groups from their irreducible projective representations

Let G be a simply-connected semi-simple algebraic group and $B \subset G$ a Borel subgroup; let $X = G/B$ be the full flag variety of G. Recall that $\mathrm{Pic}(X)$ is isomorphic to the weight lattice Λ (the character group of B) by assigning to each weight λ, the homogeneous line bundle L_λ on G/B associated with $-\lambda$. Moreover, L_λ has non-zero global sections if and only if λ is dominant; then $H^0(X, L_\lambda)$ is a finite-dimensional G-module, that we denote by $H^0(\lambda)$ for simplicity. The projective space $\mathbb{P}(H^0(\lambda))$ parametrizes the effective divisors on X with class λ in the Picard group of X (which is also the group of divisors up to algebraic equivalence).

In characteristic zero, the assignement $\lambda \mapsto H^0(\lambda)$ yields a bijective correspondence from the monoid Λ^+ of dominant weights to the set of isomorphism classes of irreducible representations of G. The latter may also be viewed as the irreducible projective representations of the associated adjoint group $G_{\mathrm{ad}} := G/Z(G)$.

Next, denote by $\mathrm{Div}^+(X)$ the set of all effective divisors on X. Then $\mathrm{Div}^+(X)$ is a monoid under addition, and is the disjoint union of the $\mathrm{Div}^\lambda(X) := \mathbb{P}(H^0(\lambda))$, where $\lambda \in \Lambda^+$. Moreover, the monoid structure is algebraic in the sense that the addition is given by morphisms

$$\varphi_{\lambda,\mu} : \mathbb{P}(H^0(\lambda)) \times \mathbb{P}(H^0(\mu)) \longrightarrow \mathbb{P}(H^0(\lambda+\mu))$$

for any $\lambda, \mu \in \Lambda^+$. Indeed, $\varphi_{\lambda,\mu}$ arises from the multiplication of sections, a morphism of G-modules

$$m_{\lambda,\mu} : H^0(\lambda) \otimes H^0(\mu) \longrightarrow H^0(\lambda+\mu), \tag{20}$$

and the product of any two non-zero sections is non-zero. The action of the adjoint group G_{ad} on X yields an action on $\mathrm{Div}^+(X)$ that preserves the structure of algebraic monoid as well as each component $\mathrm{Div}^\lambda(X)$; this action is clearly faithful. Conversely, we have:

THEOREM 5.1. *With the above notation, any automorphism of the algebraic monoid* $\mathrm{Div}^+(X)$ *that preserves each* $\mathrm{Div}^\lambda(X)$ *is given by an element of* G_{ad}.

PROOF. An automorphism as in the statement is a family $u = (u_\lambda)_{\lambda\in\Lambda^+}$ such that $u_\lambda \in \mathrm{PGL}(H^0(\lambda))$ and the diagrams

$$\begin{array}{ccc}
\mathbb{P}(H^0(\lambda)) \times \mathbb{P}(H^0(\mu)) & \xrightarrow{\varphi_{\lambda,\mu}} & \mathbb{P}(H^0(\lambda+\mu)) \\
{\scriptstyle (u_\lambda,u_\mu)}\downarrow & & {\scriptstyle u_{\lambda+\mu}}\downarrow \\
\mathbb{P}(H^0(\lambda)) \times \mathbb{P}(H^0(\mu)) & \xrightarrow{\varphi_{\lambda,\mu}} & \mathbb{P}(H^0(\lambda+\mu))
\end{array}$$

commute for all $\lambda, \mu \in \Lambda^+$. Since $\varphi_{\lambda,\mu}^* \mathcal{O}(1) = \mathcal{O}(1,1)$, we obtain a map

$$H^0(\lambda+\mu)^* = H^0(\mathbb{P}H^0(\lambda+\mu), \mathcal{O}(1)) \to H^0(\mathbb{P}H^0(\lambda) \times \mathbb{P}H^0(\mu), \mathcal{O}(1,1)) = H^0(\lambda)^* \otimes H^0(\mu)^*$$

which is the transpose of the multiplication map (20). Thus, choosing lifts $g_\lambda \in \mathrm{GL}(H^0(\lambda))$ of u_λ for all $\lambda \in \Lambda^+$, we see that each diagram

$$\begin{array}{ccc} H^0(\lambda) \otimes H^0(\mu) & \xrightarrow{m_{\lambda,\mu}} & H^0(\lambda+\mu) \\ {\scriptstyle g_\lambda \otimes g_\mu} \downarrow & & {\scriptstyle g_{\lambda+\mu}} \downarrow \\ H^0(\lambda) \otimes H^0(\mu) & \xrightarrow{m_{\lambda,\mu}} & H^0(\lambda+\mu) \end{array} \tag{21}$$

commutes up to a scalar multiple. Since $m_{\lambda,\mu}$ is surjective, it follows that $g_{\lambda+\mu}$ is determined up to a scalar by g_λ and g_μ. As a consequence, g_λ is determined up to a scalar by $g_{\omega_1}, \ldots, g_{\omega_r}$, where $\omega_1, \ldots, \omega_r$ denote the fundamental weights.

Each map $g_\lambda \otimes g_\mu$ preserves the kernel $K_{\lambda,\mu}$ of the multiplication map (20). Thus, $g_{\omega_1}, \ldots, g_{\omega_r}$ define an automorphism of the symmetric algebra

$$S := \mathrm{Sym}(H^0(\omega_1) \oplus \cdots \oplus H^0(\omega_r))$$

that preserves the ideal I generated by the K_{ω_i,ω_j}, where $1 \le i, j \le r$. But the algebra

$$R := S/I$$

is the 'total coordinate ring' of X, i.e.,

$$R = \bigoplus_{\lambda \in \Lambda^+} H^0(\lambda)$$

with multiplication given by the $m_{\lambda,\mu}$. So $g_{\omega_1}, \ldots, g_{\omega_r}$ yield an automorphism γ of the Λ^+-graded algebra R. In other words, we have maps $\gamma_\lambda \in \mathrm{GL}(H^0(\lambda))$ ($\lambda \in \Lambda^+$) such that all the diagrams (21) commute (with g's replaced by γ's) and that $\gamma_{\omega_i} = g_{\omega_i}$ for all i. It follows that we may choose $g_\lambda = \gamma_\lambda$ for all λ. By the lemma below, replacing each g_{ω_i} with a scalar multiple, we may further assume that $g \in G$. Then the image of g in G_{ad} acts on $\mathrm{Div}^+(X)$ as u. □

Lemma 5.2. *The automorphism group of the Λ^+-graded algebra R equals $(G \times T)/Z(G)$, where T denotes a maximal torus of B, acting on R via its action on each $H^0(\lambda)$ through the character λ, and the center $Z(G)$ is embedded in $G \times T$ via $z \mapsto (z, z)$.*

Proof. The G-variety

$$Y := \mathrm{Spec}(R)$$

may be identified with a closed G-stable subvariety of the G-module

$$V := V(\omega_1) \oplus \cdots \oplus V(\omega_r),$$

where we set $V(\omega_i) := H^0(\omega_i)^*$. In fact, Y is the closure of the G-orbit of

$$v := v_1 + \cdots + v_r,$$

where $v_i \in V(\omega_i)$ denotes an eigenvector of B. Moreover, the isotropy group G_v is the unipotent part $U := B_u$. Also, the boundary

$$\partial Y := Y \setminus G \cdot v$$

is the union of the zero sets of $H^0(\omega_1), \ldots, H^0(\omega_r)$ in Y, and the restriction map $R = \mathcal{O}(Y) \to \mathcal{O}(G \cdot v)$ is an isomorphism (these facts follow e.g. from the results in [17, Section 12]). Thus, ∂Y is stable under any automorphism γ of the Λ^+-graded algebra R, and hence γ restricts to a T-equivariant automorphism of G/U, where T acts by multiplication on the right. The so defined map

$$\mathrm{Aut}_{\mathrm{gr}}(R) = \mathrm{Aut}^T(Y) \longrightarrow \mathrm{Aut}^T(G \cdot v) = \mathrm{Aut}^T(G/U)$$

is an isomorphism. On the other hand, the principal T-bundle

$$f : G/U \longrightarrow G/B = X$$

yields an exact sequence

$$1 \longrightarrow \mathrm{Aut}^T_X(G/U) \longrightarrow \mathrm{Aut}^T(G/U) \xrightarrow{f_*} \mathrm{Aut}(X).$$

Moreover, the group $\mathrm{Aut}^T_X(G/U)$ of bundle automorphisms may be identified with

$$\mathrm{Hom}^T(Y, T) = \mathrm{Hom}(X, T) = T$$

(see e.g. [8, Section 4] for these facts). Also, the image of f_* consists of those automorphisms of X that act trivially on Λ^+, or equivalently on Λ; in view of [11, Théorème 1], these automorphisms are induced by elements of G_{ad}. In other words, we have an exact sequence

$$1 \longrightarrow T \longrightarrow \mathrm{Aut}_{\mathrm{gr}}(R) \longrightarrow G_{\mathrm{ad}} \longrightarrow 1.$$

It follows that the natural homomorphism $G \times T \to \mathrm{Aut}_{\mathrm{gr}}(R)$ is surjective with kernel $Z(G)$. □

REMARKS 5.3. (i) More generally, *any automorphism u of the algebraic monoid* $\mathrm{Div}^+(X)$ *comes from an automorphism of* G_{ad} (or equivalently, of X). Indeed, u acts on Λ^+ by an automorphism v of that monoid, which preserves the dimensions of (projective) representations. One can show that any such automorphism arises from an automorphism of the Dynkin diagram of G, and hence from an automorphism φ of G_{ad}. Thus, $u\varphi^{-1}$ preserves each $\mathrm{Div}^+(X)$, and thus comes from an element of G_{ad}.

(ii) One may also consider a partial flag variety $Y = G/P$, where $P \supset B$ is an arbitrary parabolic subgroup. Then $\mathrm{Div}^+(Y)$ is still an algebraic monoid, the disjoint union of the $\mathrm{Div}^\lambda(Y) = \mathbb{P}(H^0(\lambda))$, where λ runs over the dominant weights that extend to characters of P. These form a submonoid of Λ^+, freely generated by the corresponding fundamental weights. Now one shows as above that *any automorphism of* $\mathrm{Div}^+(Y)$ *that preserves each* $\mathrm{Div}^\lambda(Y)$ *is given by an element of* $\mathrm{Aut}^0(Y)$ (the neutral component of the automorphism group of Y). Recall that $\mathrm{Aut}^0(Y)$ is a semi-simple group of adjoint type, and the natural map $G_{\mathrm{ad}} \to \mathrm{Aut}^0(Y)$ is generally surjective, the exceptional cases being described in [11].

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