HOMOGENEOUS VECTOR BUNDLES OVER ABELIAN VARIETIES VIA REPRESENTATION THEORY

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ABSTRACT. Let A be an abelian variety over a field. The homogeneous (or translation-invariant) vector bundles over A form an abelian category HVec_A; the Fourier-Mukai transform yields an equivalence of HVec_A with the category of coherent sheaves with finite support on the dual abelian variety. In this paper, we develop an alternative approach to homogeneous vector bundles, based on the equivalence of HVec_A with the category of finite-dimensional representations of a commutative affine group scheme (the "affine fundamental group" of A). This displays remarkable analogies between homogeneous vector bundles over abelian varieties and representations of split reductive algebraic groups.

CONTENTS

1. INTRODUCTION

Consider an abelian variety A over a field k. A vector bundle over A is said to be homogeneous, if it is invariant under pullback by translations in A; for instance, the homogeneous line bundles are parameterized by $\mathrm{Pic}^0(A) = \widehat{A}(k)$, where $\widehat{A}$ denotes the dual abelian variety. The homogeneous vector bundles have been extensively studied when k is algebraically closed. As

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shown by Mukai, they form an abelian category HVec_A which is equivalent to the category of coherent sheaves with finite support on $\widehat{A}$ via the Fourier-Mukai transform. Moreover, any homogeneous vector bundle has a canonical decomposition

$$(1.1) \qquad\qquad E = \bigoplus_{L \in \widehat{A}(k)} L \otimes U_L,$$

where each U_L is a unipotent vector bundle, i.e., an iterated extension of trivial line bundles; the unipotent bundles form an abelian subcategory UVec_A, equivalent to that of coherent sheaves on $\widehat{A}$ with support at the origin (see [Mu78, §4] for these results). Also, the homogeneous vector bundles can be characterized as the iterated extensions of algebraically trivial line bundles (a result of Miyanishi, see [Mi73, §2]), or as the numerically flat vector bundles (this is due to Langer, see [La12, §6]).

In this paper, we develop an alternative approach to homogeneous vector bundles via representation theory, over an arbitrary ground field k. The starting point is again a result of Miyanishi in [Mi73]: every such bundle is an associated vector bundle $G \times^H V \to G/H = A$, for some extension of algebraic groups $1 \to H \to G \to A \to 1$ with H affine and some finite-dimensional H-module V. To handle all homogeneous vector bundles simultaneously, we take the inverse limit of all such extensions; this yields the universal extension of A by an affine k-group scheme,

$$(1.2) \qquad\qquad 1 \longrightarrow H_A \longrightarrow G_A \longrightarrow A \longrightarrow 1,$$

where G_A is a quasi-compact k-group scheme (see [EGA, IV.8.2.3]). We show that the associated vector bundle construction $V \mapsto G_A \times^{H_A} V$ yields an equivalence of categories

$$(1.3) \qquad\qquad \mathcal{L}_A : H_A\text{-mod} \xrightarrow{\simeq} \mathrm{HVec}_A,$$

where H_A-mod denotes the category of finite-dimensional H_A-modules (Theorem 2.9). Moreover, $\mathcal{L}_A$ induces isomorphisms of extension groups

$$(1.4) \qquad\qquad \mathrm{Ext}^i_{H_A}(V, W) \xrightarrow{\simeq} \mathrm{Ext}^i_A(\mathcal{L}_A(V), \mathcal{L}_A(W))$$

for all $i \geq 0$ and all finite-dimensional H_A-modules V, W (Theorem 2.12). In particular, homogeneity is preserved under extensions of vector bundles.

The k-group schemes G_A and H_A are not of finite type, but their structure is rather well-understood (see [Br18, §3.3] and §2.2). In particular, G_A is commutative and geometrically integral; its formation commutes with base change under arbitrary field extensions. Moreover, H_A is an extension of a unipotent group scheme U_A by the group scheme of multiplicative type M_A with Cartier dual $\widehat{A}(\bar{k})$. Also, U_A is a vector group of dimension $g = \dim(A)$ if k has characteristic 0; in positive characteristics, U_A is profinite (see

§2.2 again). By Theorem 4.8, the equivalence of categories (1.3) induces an equivalence

$$U_A\text{-mod} \xrightarrow{\simeq} \mathrm{UVec}_A.$$

Homogeneous vector bundles are also preserved under tensor product and duality; clearly, these operations on HVec_A correspond via (1.3) to the tensor product and duality of H_A-modules (the description of these operations in terms of Fourier-Mukai transform is more involved, see [Mu78, Thm. 4.12]). Also, homogeneity is preserved by pushforward and pullback under isogenies; we show that these correspond again to natural operations on modules, in view of a remarkable invariance property of the extension (1.2). More specifically, let $\varphi : A \to B$ be an isogeny of abelian varieties, with kernel N. Then φ induces an isomorphism $G_A \xrightarrow{\simeq} G_B$ and an exact sequence

$$(1.5) \qquad\qquad 0 \longrightarrow H_A \longrightarrow H_B \longrightarrow N \longrightarrow 0$$

(Proposition 4.1). Moreover, the pushforward (resp. pullback) under φ yields an exact functor $\varphi_* : \mathrm{HVec}_A \to \mathrm{HVec}_B$ (resp. $\varphi^* : \mathrm{HVec}_B \to \mathrm{HVec}_A$), which may be identified with the induction, $\mathrm{ind}_{H_A}^{H_B}$ (resp. the restriction, $\mathrm{res}_{H_A}^{H_B}$) via the equivalences of categories $\mathcal{L}_A$, $\mathcal{L}_B$ (Theorem 4.2).

In particular, the homogeneous vector bundle $\varphi_*(\mathcal{O}_A)$ on B corresponds to the H_B-module $\mathcal{O}(N)$, where H_B acts via the regular representation of its quotient group N; thus, the indecomposable summands of $\varphi_*(\mathcal{O}_A)$ correspond to the blocks of the finite group scheme N. When applied to the nth relative Frobenius morphism in characteristic p, this yields a refinement of a recent result of Sannai and Tanaka ([ST16, Thm. 1.2], see Remark 4.11 for details).

More generally, the category H_A-mod decomposes into blocks associated with irreducible representations of H_A, since every extension of two non-isomorphic such representations is trivial (see §4.1). This translates into a decomposition of the category HVec_A which generalizes (1.1); the line bundles $L \in \widehat{A}(k)$ are replaced with irreducible homogeneous vector bundles, parameterized by the orbits of the absolute Galois group in $\widehat{A}(\bar{k})$ (see §4.1 again). We obtain a description of the block $\mathrm{HVec}_{A,x}$ associated with a point $x \in \widehat{A}(\bar{k})$, which takes a simpler form when the residue field $K := k(x)$ is separable over k: then $\mathrm{HVec}_{A,x} \simeq \mathrm{UVec}_{A_K} = \mathrm{HVec}_{A_K,0}$ (Theorem 4.8).

In addition to these results, the paper contains developments in two independent directions. Firstly, we consider the universal affine covers of abelian varieties in the setting of quasi-compact group schemes (§2.1). These covers are interesting objects in their own right; over an algebraically closed field, they were introduced and studied by Serre as projective covers in the abelian category of commutative proalgebraic groups (see [Se60, §9.2], and [Br18, §3.3] for a generalization to an arbitrary ground field). They also occur in very recent work of Ayoub on motives (see [Ay18, §5.9]); moreover, the corresponding

affine fundamental group, i.e., the group H_A in the exact sequence (1.2), coincides with the S-fundamental group of A as defined in [La11] (see [La12, §6]). We show that the group G_A is the projective cover of A in the abelian category of commutative quasi-compact group schemes; as a consequence, that category has enough projectives (Proposition 2.3). This builds on results of Perrin about (not necessarily commutative) quasi-compact group schemes, see [Pe75, Pe76], and follows a suggestion of Grothendieck in [EGA, IV.8.13.6]. [1]

The second development, in Section 3 (which can be read independently of the rest of the paper), investigates the representation theory of a commutative affine group scheme H over a field k. This is easy and well-known when k is algebraically closed (see e.g. [Ja03, I.3.11]); the case of a perfect field k follows readily by using Galois descent and the splitting $H = U \times M$, where U is unipotent and M of multiplicative type. But the affine fundamental group H_A over an imperfect field admits no such splitting (Lemma 2.6); this motivates our study of a topic which seems to have been unexplored.

This representation-theoretic approach displays remarkable analogies between homogeneous vector bundles over an abelian variety A and over a full flag variety $X = G/B$, where G is a split reductive algebraic group and B a Borel subgroup: G_A, H_A play similar roles as G, B; the group $\widehat{A}(\bar{k})$ is replaced with the weight lattice, and the Galois group with the Weyl group... But these analogies are incomplete, as the combinatorics associated with root data of reductive groups have no clear counterpart on the side of abelian varieties; also, the block decompositions behave very differently.

Notation and conventions. Throughout this paper, we fix a ground field k of characteristic $p \geq 0$. We choose an algebraic closure $\bar{k}$ of k, and denote by k_s the separable closure of k in $\bar{k}$. The Galois group of k_s/k is denoted by Γ.

We consider schemes over k unless otherwise mentioned; morphisms and products of schemes are understood to be over k as well. Given a scheme S and a field extension K/k, we denote by S_K the scheme obtained from S by the corresponding base change. We freely identify line bundles (resp. vector bundles) over a scheme with invertible sheaves (resp. locally free sheaves).

A *variety* is a separated, geometrically integral scheme of finite type. An *algebraic group* is a group scheme of finite type.

We denote by A a nonzero abelian variety, and set $g := \dim(A)$; as usual, the group law of A is denoted additively, with neutral element $0 \in A(k)$. We use the notation n_A for the multiplication by an integer n in A, and $A[n]$ for its (scheme-theoretic) kernel. The dual abelian variety is denoted by $\widehat{A}$.

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[1] "Les seuls pro-groupes algébriques rencontrés en pratique jusqu'à présent étant en fait essentiellement affines, il y aura sans aucun doute avantage à substituer à l'étude des groupes pro-algébriques généraux (introduits et étudiés par Serre [Se60]) celle des schémas en groupes quasi-compacts sur k, dont la définition est conceptuellement plus simple."

2. Universal affine covers and homogeneous bundles

2.1. **Commutative quasi-compact group schemes.** Let $\mathcal{C}$ be the category with objects the commutative algebraic groups, and morphisms the homomorphisms of group schemes; then $\mathcal{C}$ is an abelian category (see [SGA3, VIA.5.4.2]). Denote by $\mathcal{L}$ the full subcategory of $\mathcal{C}$ with objects the affine (or equivalently, linear) algebraic groups; then $\mathcal{L}$ is an abelian subcategory of $\mathcal{C}$, stable under taking subobjects, quotients and extensions. We say that $\mathcal{L}$ is a *Serre subcategory* of $\mathcal{C}$. The full subcategory $\mathcal{F}$ of $\mathcal{C}$ with objects the finite group schemes is a Serre subcategory of $\mathcal{L}$.

Next, consider the category $\widetilde{\mathcal{C}}$ with objects the commutative quasi-compact group schemes; the morphisms in $\widetilde{\mathcal{C}}$ are still the homomorphisms of group schemes. By [Pe75, V.3.6], $\widetilde{\mathcal{C}}$ is an abelian category. The monomorphisms in $\widetilde{\mathcal{C}}$ are exactly the closed immersions; the epimorphisms are exactly the faithfully flat morphisms (see [Pe75, V.3.2, V.3.4]). Also, note that the quotient morphism $G \to G/H$ is a fpqc torsor for any $G \in \widetilde{\mathcal{C}}$ and any subgroup scheme $H \subset G$ (see [SGA3, IV.5.1.7.1]). Using fpqc descent, it follows that $\mathcal{C}$ is a Serre subcategory of $\widetilde{\mathcal{C}}$.

Every commutative affine group scheme is an object of $\widetilde{\mathcal{C}}$; these objects form a full subcategory $\widetilde{\mathcal{L}}$, which is again a Serre subcategory in view of [SGA3, VIB.9.2, VIB.11.17]. Likewise, the profinite group schemes form a Serre subcategory $\widetilde{\mathcal{F}}$ of $\widetilde{\mathcal{L}}$.

By the main theorem of [Pe75], every quasi-compact group scheme G (possibly non-commutative) is the filtered inverse limit of algebraic quotient groups G_i with transition functions affine for large i; equivalently, G is *essentially affine* in the sense of [EGA, IV.8.13.4]. We now record a slightly stronger version of this result in our commutative setting:

Lemma 2.1. *Let $G \in \widetilde{\mathcal{C}}$.*

(i) *For any epimorphism $f : G \to H$ in $\widetilde{\mathcal{C}}$, where H is affine (resp. profinite), there exists an affine (resp. profinite) subgroup scheme $H' \subset G$ such that the composition $H' \to G \to H$ is an epimorphism.*

(ii) *There is an exact sequence in $\widetilde{\mathcal{C}}$*

$$0 \longrightarrow H \longrightarrow G \longrightarrow A \longrightarrow 0,$$

where H is affine and A is an abelian variety.

(iii) *G is the filtered inverse limit of its quotients G/H, where H runs over the affine subgroup schemes of G such that G/H is algebraic.*

Proof. (i) We may choose an affine subgroup scheme $L \subset G$ such that G/L is algebraic. Then f induces an epimorphism $G/L \to H/f(L)$, and $H/f(L)$ is affine. By the lifting property for the pair $(\mathcal{C}, \mathcal{L})$ (see [Br18, Lem. 3.1]),

there exists an affine subgroup scheme $K \subset G/L$ such that the composition $K \to G/L \to H/f(L)$ is an epimorphism. Consider the preimage H' of K in G. Then H' is affine (as an extension of K by L) and the composition $H' \to G \to H$ is an epimorphism. This shows the assertion for affine quotients. The assertion for profinite quotients follows similarly from the lifting property for the pair $(\mathcal{C}, \mathcal{F})$, obtained in [Br15, Thm. 1.1] (see also [Lu17, Thm. 3.2]).

(ii) Consider the neutral component G^0 of G; this is a connected subgroup scheme and the quotient G/G^0 is pro-étale (see [Pe75, V.4.1]). Moreover, by [Pe75, V.4.3.1], there is an exact sequence in $\widetilde{\mathcal{C}}$

$$0 \longrightarrow H' \longrightarrow G^0 \longrightarrow A \longrightarrow 0,$$

where H' is affine and A is an abelian variety. In view of (i), there is a profinite subgroup scheme $H'' \subset G$ such that the composition $H'' \to G \to G/G^0$ is an epimorphism. Then $H := H' + H''$ is an affine subgroup scheme of G, and G/H is a quotient of A, hence an abelian variety.

(iii) Let H be as in (ii). Then the subgroup schemes $H' \subset H$ such that H/H' is algebraic form a filtered inverse system, and their intersection is trivial. Moreover, for any such subgroup scheme H', the quotient G/H' is algebraic (as an extension of A by H/H'). This yields the assertion in view of [Pe75, II.3.1.1]. □

In particular, every commutative quasi-compact group scheme G is a filtered inverse limit of algebraic quotient groups with affine transition morphisms.

We now consider the *pro category* $\mathrm{Pro}(\mathcal{C})$: its objects (the *pro-algebraic groups*) are the filtered inverse systems of objects of $\mathcal{C}$, and the morphisms are defined by

$$\mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(\varprojlim G_i, \varprojlim H_j) := \varprojlim_{\leftarrow,j} \varinjlim_{\rightarrow,i} \mathrm{Hom}_{\mathcal{C}}(G_i, H_j).$$

Recall that $\mathrm{Pro}(\mathcal{C})$ is an abelian category having enough projectives; moreover, the natural functor

$$F : \mathcal{C} \longrightarrow \mathrm{Pro}(\mathcal{C})$$

yields an equivalence of $\mathcal{C}$ with the Serre subcategory $\mathcal{C}'$ of $\mathrm{Pro}(\mathcal{C})$ consisting of artinian objects (see e.g. [Oo66, I.4] for these facts). Also, recall that for any object X of $\mathrm{Pro}(\mathcal{C})$, the artinian quotients X_i of X form a filtered inverse system, and the resulting map $X \to \varprojlim X_i$ is an isomorphism (see e.g. [DG70, V.2.2]). As a consequence, the restriction of F to $\mathcal{L}$ extends to an equivalence of categories $\widetilde{\mathcal{L}} \xrightarrow{\simeq} \mathrm{Pro}(\mathcal{L})$; also, note that $\mathrm{Pro}(\mathcal{L})$ is a Serre subcategory of $\mathrm{Pro}(\mathcal{C})$.

Denote by $\widetilde{\mathcal{C}}'$ the full subcategory of $\mathrm{Pro}(\mathcal{C})$ consisting of those objects X such that there exists an exact sequence in $\mathrm{Pro}(\mathcal{C})$

$$(2.1) \qquad\qquad\qquad 0 \longrightarrow Y \longrightarrow X \longrightarrow A \longrightarrow 0,$$

where $Y \in \mathrm{Pro}(\mathcal{L})$ and A is (the image under F of) an abelian variety.

Lemma 2.2. (i) *The objects of $\widetilde{\mathcal{C}}'$ are exactly the pro-algebraic groups that are isomorphic to essentially affine objects.*

   (ii) *$\widetilde{\mathcal{C}}'$ is a Serre subcatgory of $\mathrm{Pro}(\mathcal{C})$.*

  (iii) *Sending each $G \in \widetilde{\mathcal{C}}$ to the filtered inverse system of its algebraic quotients extends to an exact functor $S : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{C}}'$, which is an equivalence of categories.*

Proof. (i) Let $X \in \mathrm{Pro}(\mathcal{C})$ be an essentially affine object. Then there is an exact sequence in $\mathrm{Pro}(\mathcal{C})$

$$0 \longrightarrow Y \longrightarrow X \longrightarrow Z \longrightarrow 0,$$

where Y is affine and Z is algebraic. Thus, Z is an extension of an abelian variety by an affine group scheme. Hence so is X; it follows that X is an object of $\widetilde{\mathcal{C}}'$.

Conversely, every object of $\mathrm{Pro}(\mathcal{C})$ that lies in an extension (2.1) is isomorphic to the essentially affine object consisting of the inverse system of quotients X/Y', where Y' is a subobject of Y.

(ii) Let $X \in \widetilde{\mathcal{C}}'$ and consider a subobject X' of X in $\mathrm{Pro}(\mathcal{C})$. Then we have a commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & Y' & \longrightarrow & X' & \longrightarrow & A' & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & A & \longrightarrow & 0,
\end{array}
$$

where the vertical arrows are monomorphisms. It follows that $Y' \in \mathrm{Pro}(\mathcal{L})$, and A' is an extension of an abelian variety by a finite group scheme (Lemma 2.1). As a consequence, $X' \in \widetilde{\mathcal{C}}'$. Moreover, X/X' is an extension of A/A' by Y/Y', and hence is an object of $\widetilde{\mathcal{C}}'$ as well. So $\widetilde{\mathcal{C}}'$ is stable by subobjects and quotients. To show the stability by extensions, it suffices to check that $\widetilde{\mathcal{C}}'$ contains all objects X of $\mathrm{Pro}(\mathcal{C})$ which lie in an exact sequence

$$0 \longrightarrow A \longrightarrow X \longrightarrow Y \longrightarrow 0,$$

where A is an abelian variety and $Y \in \mathrm{Pro}(\mathcal{L})$. But this follows from the fact that the pair $(\mathrm{Pro}(\mathcal{C}), \mathrm{Pro}(\mathcal{L}))$ satisfies the lifting property (see [Br18, Cor. 2.12]).

(iii) Note that $S(G)$ is essentially affine for any $G \in \widetilde{\mathcal{C}}$. Next, we define $S(f)$ for any morphism $f : G \to H$ in $\widetilde{\mathcal{C}}$. If $H \in \mathcal{C}$, then f factors through a morphism $f' : G' \to H$ for any sufficiently large algebraic quotient G' of G; we then define $S(f)$ as the image of f' in

$$\varinjlim \mathrm{Hom}_{\mathcal{C}}(G', H) = \mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(S(G), S(H)).$$

This extends to an arbitrary $H \in \widetilde{\mathcal{C}}$ by using the equality

$$\mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(G, \varprojlim H') = \varprojlim \mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(G, H').$$

To show that S satisfies our assertions, recall from [EGA, IV.8.2.3] that for any essentially affine object $X = (X_i)$ of $\mathrm{Pro}(\mathcal{C})$, the inverse limit of the X_i exists in $\widetilde{\mathcal{C}}$; denote this quasi-compact group scheme by $L(X)$. Moreover, L extends to a equivalence from the full subcategory of $\mathrm{Pro}(\mathcal{C})$ consisting of the essentially affine objects, to the category $\widetilde{\mathcal{C}}$ (see [EGA, IV.8.13.5, IV.8.13.6]). By construction, $L \circ S$ is isomorphic to the identity functor of $\widetilde{\mathcal{C}}$. This yields the desired statement, except for the exactness of S. By [EGA, IV.8.13.6] again, L commutes with products; equivalently, L is additive. It follows that S is additive as well. To show that it is exact, it suffices to check that S preserves kernels and cokernels. In turn, it suffices to show that S preserves monomorphisms and epimorphisms. But this follows readily from the stability of $\widetilde{\mathcal{C}}'$ under subobjects and quotients. $\qquad\square$

Proposition 2.3.   (i) *For any object G of $\widetilde{\mathcal{C}}'$, the projective cover of G in* $\mathrm{Pro}(\mathcal{C})$ *is an object of $\widetilde{\mathcal{C}}'$ as well.*
   (ii) *Every indecomposable projective of $\mathrm{Pro}(\mathcal{C})$ is an object of $\widetilde{\mathcal{C}}'$.*
  (iii) *$\widetilde{\mathcal{C}}$ has enough projectives.*

Proof. (i) View G as an extension of an abelian variety A by an affine group scheme H. This readily yields an isomorphism of projective covers $P(G) \simeq P(H) \oplus P(A)$ with an obvious notation. Moreover, $P(H)$ is affine, and $P(A)$ is an extension of A by an affine group scheme (see [Br18, Prop. 3.3]). Thus, $P(G) \in \widetilde{\mathcal{C}}'$.

(ii) This follows from the fact that every indecomposable projective object of $\mathrm{Pro}(\mathcal{C})$ is the projective cover of an algebraic group (see e.g. [DG70, V.2.4.3]).

(iii) By (i), the abelian category $\widetilde{\mathcal{C}}'$ has enough projectives. Hence so does $\widetilde{\mathcal{C}}$ in view of Lemma 2.2. $\qquad\square$

2.2. **Universal affine covers of abelian varieties.** Let A be an abelian variety. By Proposition 2.3, A has a projective cover G_A in $\widetilde{\mathcal{C}}$. The resulting exact sequence,

$$(2.2) \qquad\qquad 0 \longrightarrow H_A \longrightarrow G_A \xrightarrow{f_A} A \longrightarrow 0,$$

is the universal extension of A by an affine group scheme, as follows from [Br18, §3.3] combined with Lemma 2.2. More specifically, for any commutative affine group scheme H, there is an isomorphism

$$(2.3) \qquad\qquad \mathrm{Hom}_{\widetilde{\mathcal{L}}}(H_A, H) \xrightarrow{\simeq} \mathrm{Ext}^1_{\widetilde{\mathcal{C}}}(A, H)$$

given by pushout of the extension (2.2). Also, by Lemma 2.2 again and [Br18, §§2.3, 3.4], the projective objects of $\widetilde{\mathcal{C}}$ are exactly the products $P \times G_B$, where P is a projective object of $\widetilde{\mathcal{L}}$ and B is an abelian variety.

Note that $f_A : G_A \to A$ is the filtered inverse limit of all *anti-affine extensions* of A, i.e., of all algebraic groups G equipped with an epimorphism $f : G \to A$ such that $\mathrm{Ker}(f)$ is affine and $\mathcal{O}(G) = k$ (see [Br18, Lem. 2.14]). In particular, $\mathcal{O}(G_A) = k$ as well. Using [Pe75, 4.2.2], it follows that G_A is geometrically integral.

Next, we show that (2.2) is the universal extension of A by a (possibly non-commutative) affine group scheme:

Theorem 2.4. *Let G be a quasi-compact group scheme, and $f : G \to A$ a faithfully flat morphism of group schemes with affine kernel H. Then there exist unique morphisms of group schemes $\varphi : G_A \to G$, $\psi : H_A \to H$ such that we have a commutative diagram with exact rows*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H_A & \longrightarrow & G_A & \xrightarrow{\ f_A\ } & A & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
1 & \longrightarrow & H & \longrightarrow & G & \xrightarrow{\ f\ } & A & \longrightarrow & 1.
\end{array}
$$

Moreover, φ factors through the center of the neutral component G^0.

Proof. As G is quasi-compact, we may freely use the results of [Pe75, V.3] on the representability of the fpqc quotients of G and its subgroup schemes. Also, recall the affinization theorem (see [Pe75, 4.2.2]): G has a largest normal subgroup scheme N such that G/N is affine; moreover, N is geometrically integral and contained in the center of G^0. In particular, N is an object of $\widetilde{\mathcal{C}}$. Let B denote the scheme-theoretic image of N under $f : G \to A$. Then B is an abelian subvariety of A; moreover, A/B is isomorphic to a quotient of G/N, and hence is affine. So $B = A$, and hence f restricts to an epimorphism $g : N \to A$ in $\widetilde{\mathcal{C}}$ with affine kernel. This readily yields the existence of φ, ψ.

For the uniqueness, just note that φ factors through N, since every affine quotient of G_A is trivial. $\qquad\square$

Proposition 2.5. *The formation of G_A commutes with base change under arbitrary field extensions.*

Proof. Let k'/k be a field extension. For any anti-affine extension $f : G \to A$, the base change $f_{k'} : G_{k'} \to A_{k'}$ is an anti-affine extension again, as $\mathcal{O}(G_{k'}) = \mathcal{O}(G) \otimes_k k'$ (see e.g. [SGA3, VIB.11.1]). Also, since $G_A = \lim_{\leftarrow} G$ (where G runs over the above anti-affine extensions of A) and base change commutes

with filtered inverse limits, we obtain $(G_A)_{k'} = \lim_{\leftarrow} G_{k'}$. This yields a commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & (H_A)_{k'} & \longrightarrow & (G_A)_{k'} & \longrightarrow & A_{k'} & \longrightarrow & 0 \\
& & \downarrow{\psi} & & \downarrow{\varphi} & & \downarrow{\mathrm{id}} & & \\
0 & \longrightarrow & H_{A_{k'}} & \longrightarrow & G_{A_{k'}} & \longrightarrow & A_{k'} & \longrightarrow & 0.
\end{array}
$$

As a consequence, $\mathrm{Coker}(\varphi) \simeq \mathrm{Coker}(\psi)$ is affine. Since $(G_A)_{k'}$ is anti-affine, it follows that φ is an epimorphism. As $G_{A_{k'}}$ is projective in $\widetilde{\mathcal{C}}_{k'}$, this yields an isomorphism $(G_A)_{k'} \simeq G_{A_{k'}} \times \mathrm{Ker}(\varphi)$. In particular, $\mathrm{Ker}(\varphi)$ is a quotient of $(G_A)_{k'}$. But $\mathrm{Ker}(\varphi) \simeq \mathrm{Ker}(\psi)$ is affine, and hence trivial. $\qquad\square$

The commutative affine group scheme H_A lies in a unique exact sequence

$$(2.4) \qquad\qquad 0 \longrightarrow M_A \longrightarrow H_A \longrightarrow U_A \longrightarrow 0,$$

where M_A is of multiplicative type and U_A is unipotent; when k is perfect, this exact sequence has a unique splitting (see [DG70, IV.3.1.1]). We now describe the structure of M_A:

Lemma 2.6.   (i) *For any field extension k'/k, there is a natural isomorphism $\mathrm{Hom}_{\widetilde{\mathcal{L}}}(H_{A_{k'}}, \mathbb{G}_{m,k'}) \simeq \widehat{A}(k')$.*

(ii) *The Cartier dual of M_A is the Γ-module $\widehat{A}(\bar{k})$.*

(iii) *The group scheme H_A is not algebraic. When k is imperfect, the exact sequence (2.4) does not split.*

Proof. (i) By Proposition 2.5, we may assume that $k' = k$. Then (2.3) yields a natural isomorphism $\mathrm{Hom}_{\widetilde{\mathcal{L}}}(H_A, \mathbb{G}_m) \simeq \mathrm{Ext}^1_{\widetilde{\mathcal{C}}}(A, \mathbb{G}_m)$, which implies the asssertion in view of the Weil-Barsotti formula (see e.g. [Oo66, III.17, III.18]).

(ii) This follows readily from (i) by taking $k' = \bar{k}$.

(iii) Assume that H_A is an algebraic group. Then $\mathrm{Hom}_{\widetilde{\mathcal{L}}}(H_{A_{\bar{k}}}, \mathbb{G}_{m,\bar{k}})$ is a finitely generated abelian group. But $\widehat{A}(\bar{k})$ is not finitely generated, since it has nonzero ℓ-torsion for any prime $\ell \neq p$; a contradiction.

Next, assume that the extension (2.4) splits. Then the resulting isomorphism $H_A \simeq M_A \times U_A$ and Proposition 2.5 yield an isomorphism

$$\mathrm{Hom}_{\widetilde{\mathcal{L}}}(H_{A_{k'}}, \mathbb{G}_{m,k'}) \simeq \mathrm{Hom}_{\widetilde{\mathcal{L}}}((M_A)_{k'}, \mathbb{G}_{m,k'})$$

for any field extension k'/k. As M_A is of multiplicative type, it follows that the natural map $\mathrm{Hom}_{\widetilde{\mathcal{L}}}(H_{A_{k_s}}, \mathbb{G}_{m,k_s}) \to \mathrm{Hom}_{\widetilde{\mathcal{L}}}(H_{A_{\bar{k}}}, \mathbb{G}_{m,\bar{k}})$ is an isomorphism. In view of (i), this yields the equality $\widehat{A}(k_s) = \widehat{A}(\bar{k})$. Choose a non-empty open affine subscheme U of $\widehat{A}$; then $U(k_s) = U(\bar{k})$ as well. By Noether normalization, there exists a finite surjective morphism $F : U \to \mathbb{A}_k^g$. If k is imperfect, choose $t \in \bar{k} \setminus k_s$; then the fiber of F at $(t, 0, \ldots, 0) \in \mathbb{A}_k^g(\bar{k})$ contains no k_s-rational point, a contradiction. $\qquad\square$

We now turn to the structure of U_A. If $p = 0$, then U_A is the vector group associated with the dual vector space of $\mathrm{H}^1(A, \mathcal{O}_A)$ (see [Br18, Lem. 3.8]); thus, $U_A \simeq (\mathbb{G}_a)^g$. If $p > 0$, then U_A is profinite; more specifically, U_A is the largest unipotent quotient of the profinite fundamental group of A, $\lim_{\leftarrow} A[n]$ (as follows from [Br18, Thm. 3.10]). Thus, U_A is the largest unipotent quotient of its pro-p part, $A[p^\infty] := \lim_{\leftarrow} A[p^n]$.

Next, recall from [Mu08, §15] that

$$(2.5) \qquad A[p^n]_{\bar{k}} \simeq (\mathbb{Z}/p^n\mathbb{Z})^r_k \times (\mu_{p^n})^r_k \times G_n,$$

where $r = r_A$ is an integer independent of n (called the p-rank of A), and G_n is a unipotent infinitesimal $\bar{k}$-group scheme of order $p^{2n(g-r)}$. The abelian variety A is said to be *ordinary* if it has maximal p-rank, i.e., $r = g$ (see [ST16, §2.1] for further characterizations of ordinary abelian varieties). It follows that

$$(U_A)_{\bar{k}} \simeq \lim_{\leftarrow} (\mathbb{Z}/p^n\mathbb{Z})^r_k \times G_n n \simeq (\mathbb{Z}_p)^r_k \times \lim_{\leftarrow} G_n.$$

As a consequence, A is ordinary if and only if U_A is pro-étale; then

$$(2.6) \qquad U_A(k_s) \simeq (\mathbb{Z}_p)^g.$$

Remark 2.7. When k is algebraically closed, the groups $\mathrm{Ext}^1_{\mathcal{C}}(A, H)$, where H is a commutative affine algebraic group, have been determined by Wu in [Wu86]. His results may be recovered from the above description of H_A in view of the isomorphism (2.3).

Also, this description can be interpreted in terms of formal groups associated with $\widehat{A}$, via Cartier duality which yields an anti-equivalence of $\widetilde{\mathcal{L}}$ with the category of commutative formal k-groups (see [SGA3, VIIB.2], that we will freely use as a general reference for formal groups). Under that equivalence, the exact sequence (2.4) corresponds to an exact sequence of formal groups,

$$0 \longrightarrow \mathrm{D}(U_A) \longrightarrow \mathrm{D}(H_A) \longrightarrow \mathrm{D}(M_A) \longrightarrow 0,$$

where $\mathrm{D}(U_A)$ is infinitesimal and $\mathrm{D}(M_A)$ is étale; this exact sequence splits if and only if k is perfect. By Lemma 2.6 (ii), $\mathrm{D}(M_A)$ corresponds to the Γ-group $\widehat{A}(\bar{k})$ under the equivalence of étale formal groups with Γ-groups.

Also, if $p = 0$, then $\mathrm{D}(U_A)$ is the infinitesimal formal group associated with the commutative Lie algebra $\mathrm{H}^1(A, \mathcal{O}_A)$. Since the latter is the Lie algebra of $\widehat{A}$, we may identify $\mathrm{D}(U_A)$ with the infinitesimal formal neighborhood of the origin in $\widehat{A}$.

This still holds if $p > 0$: then the above infinitesimal formal group is isomorphic to $\lim_{\rightarrow} \mathrm{Ker}(\mathrm{F}^n_{\widehat{A}})$, where $\mathrm{F}^n_{\widehat{A}} : \widehat{A} \to \widehat{A}^{(p^n)}$ denotes the nth relative Frobenius morphism. Moreover, we have canonical isomorphisms

$$\mathrm{D}(\lim_{\rightarrow} \mathrm{Ker}(\mathrm{F}^n_{\widehat{A}})) \simeq \lim_{\leftarrow} \mathrm{D}(\mathrm{Ker}(\mathrm{F}^n_{\widehat{A}})) \simeq \lim_{\leftarrow} \mathrm{Ker}(\mathrm{V}^n_A),$$

where $V_A^n : A^{(p^n)} \to A$ denotes the nth Verschiebung, and the second isomorphism follows from duality between Frobenius and Verschiebung. Also, we have a canonical exact sequence

$$0 \longrightarrow \mathrm{Ker}(F_A^n) \longrightarrow A[p^n] \longrightarrow \mathrm{Ker}(V_A^n) \longrightarrow 0$$

for any $n \geq 1$; as a consequence, $\lim_{\leftarrow} \mathrm{Ker}(V_A^n)$ may be identified with the largest unipotent quotient of $A[p^\infty]$, i.e., with U_A.

2.3. **Applications to homogeneous vector bundles.** Let $\pi : E \to A$ be a vector bundle. For any scheme S, consider the set $G(S)$ consisting of the pairs (φ, a) satisfying the following conditions:

(i) $\varphi : E_S \to E_S$ is an isomorphism of S-schemes, and $a \in A(S)$.

(ii) The diagram

$$
\begin{array}{ccc}
E_S & \xrightarrow{\;\varphi\;} & E_S \\
\pi_S \downarrow & & \downarrow \pi_S \\
A_S & \xrightarrow{\;\tau_a\;} & A_S
\end{array}
$$

commutes, where τ_a denotes the translation by a in A_S.

(iii) The isomorphism of A_S-schemes $E_S \to \tau_a^*(E_S)$ induced by φ, is an isomorphism of vector bundles.

Alternatively, we may view $G(S)$ either as the set of pairs (a, ψ), where $a \in A(S)$ and $\psi : E_S \to \tau_a^*(E_S)$ is an isomorphism of vector bundles (see [Mi73, p. 72]), or as the group of automorphisms of the S-scheme E_S which lift translations in A_S and commute with the action of $\mathbb{G}_{m,S}$ by multiplication on fibers.

Clearly, $G(S)$ is a group for componentwise multiplication of pairs (φ, a); moreover, the assignment $S \mapsto G(S)$ extends to a group functor that we still denote by G, or G_E to emphasize its dependence in E. The second projection yields a morphism of group functors $f : G \to A$; the kernel of f is the group functor $H = H_E$ of automorphisms of the vector bundle E.

Lemma 2.8. (i) H *is a smooth connected affine algebraic group.*

(ii) G *is an algebraic group.*

(iii) E *is equipped with a G-linearization.*

Proof. (i) Note that H is the group of invertibles of the monoid functor of endomorphisms of the vector bundle E, and this monoid functor is represented by an affine space. So the assertion follows from [DG70, II.2.3.6].

(ii) The group functor of $\mathbb{G}_m$-equivariant automorphisms of E is represented by a group scheme $\mathrm{Aut}_E^{\mathbb{G}_m}$, locally of finite type; moreover, we have an exact sequence of group schemes

$$1 \longrightarrow H \longrightarrow \mathrm{Aut}_E^{\mathbb{G}_m} \longrightarrow \mathrm{Aut}_A$$

(see [BSU13, Prop. 6.3.2]). Viewing A as a subgroup scheme of Aut_A via its action by translation, we may identify G with the pullback of A in $\mathrm{Aut}_E^{\mathbb{G}_m}$; this yields the assertion.

(iii) This follows e.g. from [SGA3, I.6.5.3]. $\qquad\square$

By Lemma 2.8, there is an exact sequence of algebraic groups

$$(2.7) \qquad\qquad 1 \longrightarrow H_E \longrightarrow G_E \xrightarrow{\ f\ } A.$$

We say that E is *homogeneous* if f is faithfully flat; equivalently, (2.7) is right exact. Since f is a morphism of algebraic groups and A is smooth, this amounts to the following condition: for any $a \in A(\bar{k})$, the translation by a in $A_{\bar{k}}$ lifts to an automorphism of the $\bar{k}$-variety $E_{\bar{k}}$, linear on fibers. Equivalently, $E_{\bar{k}} \simeq \tau_a^*(E_{\bar{k}})$ for any $a \in A(\bar{k})$.

As a consequence, if E is homogeneous, then so is the dual vector bundle $E^\vee$; if in addition F is a homogeneous vector bundle over A, then $E \oplus F$ and $E \otimes F$ are homogeneous as well. Thus, the homogeneous vector bundles form a full subcategory of the category Vec_A of vector bundles over A, stable under finite direct sums, duals and tensor products. We denote this additive tensor subcategory by HVec_A.

Let E be a homogeneous vector bundle over A. Then the algebraic group G_E is smooth and connected, since so are H_E and A. This group acts transitively on A via f, and the stabilizer of the origin 0 equals H_E. The G_E-linearization of E restricts to an action of H_E on the fiber E_0 via a representation

$$\rho : H_E \longrightarrow \mathrm{GL}(E_0).$$

Moreover, the morphism $G_E \times E_0 \to E$ given by the action, factors through an isomorphism

$$G_E \times^{H_E} E_0 \xrightarrow{\ \simeq\ } E,$$

where the left-hand side denotes the quotient of $G_E \times E_0$ by the action of H_E via $h \cdot (g, x) := (gh^{-1}, \rho(h)x)$; this is the vector bundle over A associated with the H_E-torsor $G_E \to A$ and the H_E-representation in E_0.

Also, by Theorem 2.4, we have a commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & H_A & \longrightarrow & G_A & \xrightarrow{\ f_A\ } & A & \longrightarrow & 1 \\
 & & \downarrow{\psi} & & \downarrow{\varphi} & & \downarrow{\mathrm{id}} & & \\
1 & \longrightarrow & H_E & \longrightarrow & G_E & \xrightarrow{\ f\ } & A & \longrightarrow & 1
\end{array}
$$

for unique morphisms $\varphi = \varphi_E$, $\psi = \psi_E$; moreover, φ_E factors through the center of G_E. This yields a representation $\rho \circ \psi : H_A \to \mathrm{GL}(E_0)$ and an isomorphim $G_A \times^{H_A} E_0 \simeq E$. In particular, E is G_A-linearized.

Conversely, given a finite-dimensional representation $\rho : H_A \to \mathrm{GL}(V)$, the quotient $H := H_A/\mathrm{Ker}(\rho)$ is algebraic and lies in an exact sequence

$$0 \longrightarrow H \longrightarrow G \longrightarrow A \longrightarrow 0,$$

where $G := G_A/\mathrm{Ker}(\rho)$ is algebraic as well. Thus, we may form the associated vector bundle $\mathcal{L}_{G/H}(V)$ (see [Ja03, I.5.8, I.5.15]): this is a G-linearized vector bundle on A, and hence is homogeneous. We denote this vector bundle by

$$\mathcal{L}_A(V) = G_A \times^{H_A} V \longrightarrow A.$$

Its fiber at 0 is V, on which H_A acts via the representation ρ.

Denote by H_A-mod the category of finite-dimensional representations of H_A; this is an abelian tensor category. We may now state:

Theorem 2.9. *The above assignments $E \mapsto E_0$, $V \mapsto G_A \times^{H_A} V$ extend to exact functors*

$$\mathcal{M}_A : \mathrm{HVec}_A \longrightarrow H_A\text{-mod}, \quad \mathcal{L}_A : H_A\text{-mod} \longrightarrow \mathrm{HVec}_A$$

which are quasi-inverse equivalences of additive tensor categories.

Proof. Let E, F be homogeneous vector bundles over A. Since they both are G_A-linearized, the finite-dimensional vector space $\mathrm{Hom}_{\mathrm{Vec}_A}(E, F)$ (consisting of the morphisms of vector bundles $\gamma : E \to F$) is equipped with a linear representation of G_A (see [SGA3, I.6.6.2]). As $\mathcal{O}(G_A) = k$, this representation is trivial, i.e., every γ as above is G_A-equivariant. As a consequence, the restriction $\gamma_0 : E_0 \to F_0$ is H_A-equivariant, where H_A acts on E_0 and F_0 via the above representations. The assignment $\gamma \mapsto \gamma_0$ defines the functor $\mathcal{M}_A$ on morphisms.

Given E, F as above, consider the image G of the product morphism

$$(\varphi_E, \varphi_F) : G_A \longrightarrow G_E \times G_F.$$

Then G is an algebraic quotient of G_A, and lies in an exact sequence of algebraic groups

$$0 \longrightarrow H \longrightarrow G \longrightarrow A \longrightarrow 0,$$

for some algebraic quotient H of H_A. Moreover, $E \simeq G \times^H E_0$, $F \simeq G \times^H F_0$, and G acts trivially on $\mathrm{Hom}_{\mathrm{Vec}_A}(E, F)$. This yields an isomorphism

$$\mathrm{Hom}_{\mathrm{Vec}_A}(E, F) \simeq \mathrm{Hom}_{\mathrm{Vec}_A}^G(G \times^H E_0, G \times^H F_0).$$

The right-hand side is contained in the set $\mathrm{Hom}^G(G \times^H E_0, G \times^H F_0)$ of G-equivariant morphisms of schemes, which is identified with $\mathrm{Hom}^H(E_0, G \times^H F_0)$ via restriction; this identifies the subset $\mathrm{Hom}_{\mathrm{Vec}_A}^G(G \times^H E_0, G \times^H F_0)$ with the subset of H-equivariant linear maps $E_0 \to F_0$. As a consequence, the functor $\mathcal{M}_A$ is fully faithful. It is essentially surjective, since $\mathcal{M}_A(\mathcal{L}_A(V)) = V$ for any finite-dimensional H_A-module V.

Next, note that every morphism of H_A-modules $u : V \to W$ yields a morphism of associated vector bundles $\mathcal{L}_A(u) : \mathcal{L}_A(V) \to \mathcal{L}_A(W)$ such that $\mathcal{L}_A(u)_0 = u$, as may be checked by factoring both representations $H_A \to \mathrm{GL}(V)$, $H_A \to \mathrm{GL}(W)$ through a common algebraic quotient of H_A. This defines the functor $\mathcal{L}_A$ on morphisms, and shows that it is quasi-inverse to $\mathcal{M}_A$. Clearly, $\mathcal{L}_A$ and $\mathcal{M}_A$ are exact and preserve finite direct sums, duals and tensor products. $\qquad\square$

Corollary 2.10. *The subcategory* HVec_A *of* Vec_A *is abelian and stable under direct summands.*

Next, we obtain a key vanishing result for the coherent cohomology of G_A:

Proposition 2.11. $\mathrm{H}^i(G_A, \mathcal{O}_{G_A}) = 0$ *for any* $i \geq 1$.

Proof. To simplify the notation, we set $G := G_A$, $H := H_A$ and $f := f_A$. Since the morphism $f : G \to A$ is affine, we have $R^i f_*(\mathcal{O}_G) = 0$ for all $i \geq 1$ in view of [EGA, III.1.3.2]. This yields isomorphisms

$$\mathrm{H}^i(G, \mathcal{O}_G) \cong \mathrm{H}^i(A, f_*(\mathcal{O}_G)) \quad (i \geq 0).$$

Next, recall that $G \simeq \lim_{\leftarrow} G'$, where the limit is taken over the filtered inverse system of anti-affine extensions $f' : G' \to A$. Thus, we have an isomorphism of quasi-coherent sheaves of $\mathcal{O}_A$-algebras

$$f_*(\mathcal{O}_G) \simeq \lim_{\rightarrow} f'_*(\mathcal{O}_{G'}).$$

Since cohomology commutes with direct limits, this yields in turn isomorphisms

$$\mathrm{H}^i(G, \mathcal{O}_G) \simeq \lim_{\rightarrow} \mathrm{H}^i(A, f'_*(\mathcal{O}_{G'})) \simeq \lim_{\rightarrow} \mathrm{H}^i(G', \mathcal{O}_{G'}) \quad (i \geq 0).$$

As each G' is anti-affine, the coherent cohomology ring $\mathrm{H}^*(G', \mathcal{O}_{G'})$ is the exterior algebra over $\mathrm{H}^1(G', \mathcal{O}_{G'})$ (see [Br13, Thm. 1.1]). As a consequence, $\mathrm{H}^*(G, \mathcal{O}_G)$ is the exterior algebra over $\mathrm{H}^1(G, \mathcal{O}_G)$. Thus, it suffices to show the vanishing of $\mathrm{H}^1(G, \mathcal{O}_G)$. For this, we may assume that k is algebraically closed, in view of Proposition 2.5.

By adapting the argument of [To13, Lem. 9.2] (see also [Se97, Thm. VII.5]), one may check that the canonical map

$$\mathrm{Ext}^1_{\mathcal{C}}(G', \mathbb{G}_a) \longrightarrow \mathrm{H}^1(G', \mathcal{O}_{G'})^{G'}$$

is an isomorphism, where the right-hand side denotes the subspace of G'-invariants in $\mathrm{H}^1(G', \mathcal{O}_{G'})$. As $\mathrm{H}^1(G', \mathcal{O}_{G'})$ is a (rational) G'-module (see e.g. [Br13, Lem. 2.1]) and G' is anti-affine, this yields compatible isomorphisms $\mathrm{Ext}^1_{\mathcal{C}}(G', \mathbb{G}_a) \simeq \mathrm{H}^1(G', \mathcal{O}_{G'})$, and hence an isomorphism

$$\mathrm{Ext}^1_{\mathrm{Pro}(\mathcal{C})}(G, \mathbb{G}_a) \simeq \mathrm{H}^1(G, \mathcal{O}_G),$$

by using [DG70, V.2.3.9]. The desired vanishing follows from this, since G is projective in $\mathrm{Pro}(\mathcal{C})$.

Alternatively, the vanishing of $\mathrm{H}^1(G, \mathcal{O}_G)$ can be obtained as follows. Let again $f' : G' \to A'$ be an anti-affine extension and consider the Albanese morphism $\alpha : G' \to A'$; then f' is the composition of α with an isogeny $A' \to A$. If $p = 0$ and α lifts to an epimorphism $G' \to E(A')$ (the universal vector extension of A'), then $H^1(G', \mathcal{O}_{G'}) = 0$ by [Br13, Prop. 4.1, Prop. 4.3]; this yields the desired vanishing. On the other hand, if $p > 0$, then the pullback $\alpha^* : \mathrm{H}^*(A', \mathcal{O}_{A'}) \to \mathrm{H}^*(G', \mathcal{O}_{G'})$ is an isomorphism by [Br13, Cor. 4.2]. The commutative diagram of multiplication maps

$$
\begin{array}{ccccc}
G & \longrightarrow & G' & \xrightarrow{\ \alpha\ } & A' \\
\downarrow{\scriptstyle p_G} & & \downarrow{\scriptstyle p_{G'}} & & \downarrow{\scriptstyle p_{A'}} \\
G & \longrightarrow & G' & \xrightarrow{\ \alpha\ } & A'
\end{array}
$$

yields a commutative diagram of pullbacks

$$
\begin{array}{ccccc}
\mathrm{H}^1(A', \mathcal{O}_{A'}) & \xrightarrow{\ \simeq\ } & \mathrm{H}^1(G', \mathcal{O}_{G'}) & \longrightarrow & \mathrm{H}^1(G, \mathcal{O}_G) \\
\downarrow{\scriptstyle p_{A'}^*} & & \downarrow{\scriptstyle p_{G'}^*} & & \downarrow{\scriptstyle p_G^*} \\
\mathrm{H}^1(A', \mathcal{O}_{A'}) & \xrightarrow{\ \simeq\ } & \mathrm{H}^1(G', \mathcal{O}_{G'}) & \longrightarrow & \mathrm{H}^1(G, \mathcal{O}_G).
\end{array}
$$

Moreover, p_G is an isomorphism by [Br18, Lem. 3.4]; hence so is p_G^*. Thus, it suffices to show that $p_{A'}^* = 0$. But this follows from the isomorphism $\mathrm{Ext}_{\mathcal{C}}^1(A', \mathbb{G}_a) \cong \mathrm{H}^1(A', \mathcal{O}_{A'})$ (see [Se97, Thm. VII.5]) together with the equality $p_{A'}^* = (p_{\mathbb{G}_a})_* = 0$ in $\mathrm{Ext}_{\mathcal{C}}^1(A', \mathbb{G}_a)$. $\qquad\square$

Theorem 2.12. *Let V, W be finite-dimensional H_A-modules. Then the map*

$$
\mathrm{Ext}_{H_A}^i(V, W) \longrightarrow \mathrm{Ext}_A^i(\mathcal{L}_A(V), \mathcal{L}_A(W))
$$

is an isomorphism for any $i \geq 0$, where the right-hand side denotes the higher extension group of coherent sheaves on A.

Proof. Denoting by $V^\vee$ the dual H_A-module of V, the statement can be reformulated as follows: the map

$$
\mathrm{H}^i(H_A, V^\vee \otimes_k W) \longrightarrow \mathrm{H}^i(A, \mathcal{L}_A(V^\vee \otimes_k W))
$$

is an isomorphism for any $i \geq 0$ (use [Ja03, I.4.2, I.4.4] and the compatibility of $\mathcal{L}_A$ with duals and tensor products). This holds for $i = 0$ in view of Theorem 2.9. Thus, it suffices to extend $\mathcal{L}_A$ to an exact functor from H_A-modules to quasi-coherent sheaves on A, which takes injective modules to acyclic sheaves; then the statement will follow from a degenerate case of Grothendieck's spectral sequence.

We first check that $\mathcal{L}_A$ extends uniquely to an exact functor, commuting with filtered direct limits, from the category H_A-Mod of all H_A-modules (not necessarily finite-dimensional), to the category QCoh_A of quasi-coherent sheaves on A. Indeed, H_A-Mod is a Grothendieck category and its noetherian objects are exactly those of H_A-mod; as a consequence, H_A-Mod is equivalent to the ind category $\mathrm{Ind}(H_A\text{-mod})$. Likewise, QCoh_A is equivalent to $\mathrm{Ind}(\mathrm{Coh}_A)$. So the desired assertion follows from [KS05, 6.1.9, 8.6.8].

We still denote the extended functor by

$$\mathcal{L}_A : H_A\text{-Mod} \longrightarrow \mathrm{QCoh}_A.$$

We now check that $\mathcal{L}_A$ takes injectives to acyclics. By [Ja03, I.3.10], every injective H_A-module is a direct summand of a direct sum of copies of $\mathcal{O}(H_A)$; hence it suffices to show that $\mathrm{H}^i(A, \mathcal{L}_A(\mathcal{O}(H_A))) = 0$ for all $i \geq 1$. In view of (the proof of) Proposition 2.11, it suffices in turn to check that

$$(2.8) \qquad \mathcal{L}_A(\mathcal{O}(H_A)) \simeq f_*(\mathcal{O}_{G_A}),$$

where H_A acts on $\mathcal{O}(H_A)$ via the regular representation.

Recall that $\mathcal{O}(H_A) = \lim_{\leftarrow} \mathcal{O}(H)$ and $f_*(\mathcal{O}_{G_A}) = \lim_{\rightarrow} f'_*(\mathcal{O}_G)$, where both limits run over all anti-affine extensions $0 \longrightarrow H \longrightarrow G \xrightarrow{f} A \longrightarrow 0$. Also, recall that $\mathcal{L}_A(V)$ is defined as the associated sheaf $\mathcal{L}_{G/H}(V)$ for any finite-dimensional H-module V, and hence for any H-module. By [Ja03, I.5.18], there are compatible isomorphisms

$$\mathcal{L}_A(\mathcal{O}(H)) \xrightarrow{\simeq} f'_*(\mathcal{O}_G).$$

This yields the desired isomorphism (2.8). $\qquad\qquad\square$

Corollary 2.13. *The subcategory* HVec_A *of* Vec_A *is stable under extensions.*

3. Representations of commutative affine group schemes

Throughout this section, we consider linear representations of a fixed commutative affine k-group scheme H. We use the book [Ja03] as a general reference for representation theory.

3.1. **Irreducible representations.** The aim of this subsection is to construct the irreducible representations of H (Proposition 3.1), and classify them up to equivalence (Proposition 3.3).

Define a *character* of H as a morphism of $\bar{k}$-group schemes

$$\chi : H_{\bar{k}} \longrightarrow \mathbb{G}_{m,\bar{k}}.$$

The characters form an abelian group that we denote by $\mathrm{X}(H)$. Every such character χ may be viewed as an element of $\mathcal{O}(H_{\bar{k}}) = \mathcal{O}(H) \otimes_k \bar{k}$. Thus, χ is defined over a smallest finite subextension K/k of $\bar{k}/k$: the *field of definition*, $K = k(\chi)$ (generated by the coordinates of χ in a $\bar{k}$-basis of $\mathcal{O}(H_{\bar{k}})$ consisting

of elements of $\mathcal{O}(H)$). So χ defines a morphism of K-group schemes $H_K \to \mathbb{G}_{m,K}$, or equivalently a morphism of k-group schemes

$$(3.1) \qquad\qquad \rho(\chi) : H \longrightarrow \mathrm{R}_{K/k}(\mathbb{G}_{m,K}),$$

where $\mathrm{R}_{K/k}$ denotes the Weil restriction of scalars: for any scheme S, the group of S-points $\mathrm{R}_{K/k}(\mathbb{G}_{m,K})(S)$ is the unit group of the K-algebra $\mathcal{O}(S) \otimes_k K$ (see [CGP15, A.5] for generalities on Weil restriction). Thus, we may view $\rho(\chi)$ as a linear representation of H in the k-vector space K, in which $\mathrm{R}_{K/k}(\mathbb{G}_{m,K})$ acts by multiplication.

Proposition 3.1.   (i) *The representation $\rho(\chi)$ is irreducible, with commutant algebra $k(\chi)$.*

   (ii) *Every irreducible representation of H is isomorphic to $\rho(\chi)$ for some $\chi \in \mathrm{X}(H)$.*

Proof. We argue as in the proof of [Wa79, 9.4]. Let $\rho : H \to \mathrm{GL}(V)$ be a finite-dimensional representation. Consider the dual representation $\rho^\vee$ of H in $V^\vee$, and the corresponding comodule map

$$\Delta_{\rho^\vee} : V^\vee \longrightarrow V^\vee \otimes_k \mathcal{O}(H).$$

Since H is commutative, $\Delta_{\rho^\vee}$ is equivariant for the H-representation on $V^\vee$, and that on $V^\vee \otimes_k \mathcal{O}(H)$ via its action on $V^\vee$. Also, the matrix coefficients of $\rho^\vee$ span a subspace $C = C(\rho^\vee) \subset \mathcal{O}(H)$ which is stable under the comultiplication $\Delta : \mathcal{O}(H) \to \mathcal{O}(H) \otimes_k \mathcal{O}(H)$; hence C is a finite-dimensional sub-coalgebra. Note that C is co-commutative and has a co-unit, since these properties hold for $\mathcal{O}(H)$. Thus, the dual vector space $C^\vee$ is a finite-dimensional commutative algebra with unit, and V is a $C^\vee$-module. Moreover, $C^\vee$ acts on V by H-invariant endomorphisms, since the transpose map $V^\vee \to V^\vee \otimes_k C$ is equivariant for the H-representations as above. Also, by [DG70, II.2.2.4], a subspace $W \subset V^\vee$ is H-stable if and only if $\Delta_{\rho^\vee}(W) \subset W \otimes_k C$.

We now assume that ρ is irreducible; then so is $\rho^\vee$, and hence $V^\vee$ is a simple C-comodule. Equivalently, V is a simple $C^\vee$-module. This yields an isomorphism of $C^\vee$-modules $V \simeq K$ for some quotient field K of $C^\vee$; moreover, H acts linearly on K and commutes with the K-action by multiplication. Thus, ρ is identified with a morphism of k-group schemes $H \to \mathrm{R}_{K/k}(\mathbb{G}_{m,K})$, which corresponds to a morphism of K-group schemes $H_K \to \mathbb{G}_{m,K}$, or equivalently to a character $\chi \in \mathrm{X}(H)$, defined over K. If χ is defined over a smaller subfield L containing k, then the corresponding morphism of L-group schemes $H_L \to \mathbb{G}_{m,L}$ yields a factorization of ρ through a morphism $H \to \mathrm{R}_{L/k}(\mathbb{G}_{m,L})$. In particular, H stabilizes the subspace L of K. As ρ is irreducible, it follows that $L = K$, i.e., $K = k(\chi)$ and $\rho = \rho(\chi)$. This shows (ii).

To prove (i), we may assume that $\rho(\chi)$ is a monomorphism, and hence view H as a k-subgroup scheme of $\mathrm{R}_{K/k}(\mathbb{G}_{m,K})$, where $K = k(\chi)$. For the

representation of $R_{K/k}(\mathbb{G}_{m,K})$ in K, the co-algebra of matrix coefficients is easily checked to be $K^\vee$ (the k-linear dual of K). As a consequence, the co-algebra C of matrix coefficients of $\rho(\chi)$ is a quotient of $K^\vee$; equivalently, $C^\vee$ is a k-subalgebra of K, and hence a subfield, say L. Moreover, we have a commutative diagram of morphisms of k-algebras

$$
\begin{array}{ccc}
\mathrm{Sym}(K^\vee) & \longrightarrow & \mathcal{O}(R_{K/k}(\mathbb{G}_{m,K})) \\
\downarrow & & \downarrow \\
\mathrm{Sym}(L^\vee) & \longrightarrow & \mathcal{O}(H),
\end{array}
$$

which translates into a commutative diagram of morphisms of k-schemes

$$
\begin{array}{ccc}
H & \longrightarrow & R_{L/k}(\mathbb{A}^1_L) \\
\downarrow & & \downarrow \\
R_{K/k}(\mathbb{G}_{m,K}) & \longrightarrow & R_{K/k}(\mathbb{A}^1_K),
\end{array}
$$

where the vertical arrows and the bottom horizontal arrow are immersions. Hence so is the top horizontal arrow. As a consequence, H acts linearly on K via a morphism to $R_{L/k}(\mathbb{G}_{m,L})$, and χ is defined over L. Thus, $L = K = C^\vee$. Using again the characterization of the H-stable subspaces of $V^\vee$ in terms of the comodule map, it follows that V is irreducible. In particular, its commutant algebra is a division algebra D of finite dimension over k, say d. We have $d \geq [K : k]$ as $D \supset K$, and $d \leq [K : k]$ as K is a D-module. So equality holds, and $D = K$. $\qquad\square$

Next, consider the absolute Galois group

$$
\Gamma := \mathrm{Gal}(k_s/k) = \mathrm{Aut}(\bar{k}/k),
$$

and its continuous action on the character group $\mathrm{X}(H)$; denote this action by $(\gamma, \chi) \mapsto \gamma \cdot \chi$. Note that $k(\gamma \cdot \chi) = \gamma k(\chi)$ for any $\gamma \in \Gamma$ and $\chi \in \mathrm{X}(H)$.

Also, recall that H lies in a unique exact sequence of commutative affine group schemes

$$
(3.2) \qquad 0 \longrightarrow M \longrightarrow H \longrightarrow U \longrightarrow 0,
$$

where M is of multiplicative type and U is unipotent; the formation of M, U commutes with base change under field extensions. For any such extension k'/k with k' perfect, the induced exact sequence

$$
0 \longrightarrow M_{k'} \longrightarrow H_{k'} \longrightarrow U_{k'} \longrightarrow 0
$$

has a unique splitting; in particular, (3.2) splits over $\bar{k}$. Since every character of U is trivial, this yields a Γ-equivariant isomorphism

$$
\mathrm{X}(H) \longrightarrow \mathrm{X}(M), \quad \chi \longmapsto \chi|_M,
$$

where the Γ-group $X(M)$ may be identified with the Cartier dual $D(M)$. Finally, recall from [DG70, II.3.3.7, II.3.4.3] that every representation of M is completely reducible.

Lemma 3.2. *Let $\chi \in X(H)$ and denote by Γ_χ its stabilizer in Γ. Set $K := k(\chi)$, $\eta := \chi|_M$ and $L := k(\eta)$.*

(i) $L = K \cap k_s$.
(ii) L is the fixed subfield of Γ_χ in k_s.
(iii) $\mathrm{res}_M^H \rho(\chi) \simeq [K : L]\, \rho(\eta)$.

Proof. (i) If $p = 0$ then the assertion follows readily from the isomorphism $H \simeq M \times U$. Thus, we may assume $p > 0$.

Clearly, η is defined over K; that is, $L \subset K$. Also, as M is of multiplicative type, its characters are all defined over k_s; in particular, L is separable over k. To complete the proof, it suffices to show that $K^{p^n} \subset L$ for $n \gg 0$. For this, we may replace k with L, and hence assume that $\eta \in \mathrm{Hom}_{\widetilde{\mathcal{L}}}(M, \mathbb{G}_m)$. Since $\mathrm{Hom}_{\widetilde{\mathcal{L}}}(U, \mathbb{G}_m) = 0$, we have an exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\widetilde{\mathcal{L}}}(H, \mathbb{G}_m) \longrightarrow \mathrm{Hom}_{\widetilde{\mathcal{L}}}(M, \mathbb{G}_m) \longrightarrow \mathrm{Ext}^1_{\widetilde{\mathcal{L}}}(U, \mathbb{G}_m).$$

Also, U is the filtered inverse limit of its algebraic quotients U'; in view of [DG70, V.2.3.9], this yields

$$\mathrm{Ext}^1_{\widetilde{\mathcal{L}}}(U, \mathbb{G}_m) \simeq \varinjlim \mathrm{Ext}^1_{\widetilde{\mathcal{L}}}(U', \mathbb{G}_m).$$

Since each U' is a unipotent algebraic group, $\mathrm{Ext}^1_{\widetilde{\mathcal{L}}}(U', \mathbb{G}_m)$ is killed by a power of p. Thus, so is every element of $\mathrm{Ext}^1_{\widetilde{\mathcal{L}}}(U, \mathbb{G}_m)$; hence $p^n \eta$ extends to a morphism $H \to \mathbb{G}_m$ for $n \gg 0$. This implies the assertion.

(ii) We view η as an element of $\mathcal{O}(M_{k_s}) = \mathcal{O}(M) \otimes_k k_s$. Write accordingly $\eta = \sum_{i=1}^n f_i \otimes z_i$, where $f_1, \ldots, f_n \in \mathcal{O}(M)$ are linearly independent over k and $z_1, \ldots, z_n \in k_s$. Then $k(\eta) = k(z_1, \ldots, z_n)$. Also, the stabilizer in Γ of χ equals that of η, and hence consists of the γ such that $\gamma z_i = z_i$ for $i = 1, \ldots, n$. Thus, $\gamma \in \Gamma_\chi$ if and only if γ fixes L pointwise. So the assertion follows from Galois theory.

(iii) This is a direct consequence of the definition of $\rho(\chi)$. $\square$

Proposition 3.3. *Let $\chi, \chi' \in X(H)$. Then we have $\rho(\chi') \simeq \rho(\chi)$ if and only if $\chi' \in \Gamma \cdot \chi$.*

Proof. Assume that $\chi' = \gamma \cdot \chi$ for some $\gamma \in \Gamma$. We may choose a finite subextension K'/k which is stable under Γ and contains $k(\chi)$. Then Γ acts by automorphisms on the k-group scheme $R_{K'/k}(\mathbb{G}_{m,K'})$, and γ restricts to an isomorphism

$$R_{k(\chi)/k}(\mathbb{G}_{m,k(\chi)}) \xrightarrow{\simeq} R_{k(\chi')/k}(\mathbb{G}_{m,k(\chi')})$$

which intertwines $\rho(\chi)$ and $\rho(\chi')$.

Conversely, assume that $\rho(\chi')$ is isomorphic to $\rho(\chi)$. Then the commutant algebras are isomorphic as well, hence $k(\chi') \simeq k(\chi)$ as extensions of k (Proposition 3.1). In view of Lemma 3.2, it follows that $\rho(\chi'|_M) \simeq \rho(\chi|_M)$. Consider the diagonalizable k_s-group scheme M_{k_s}; then its representation $\rho(\chi|_M)_{k_s}$ decomposes into the direct sum of one-dimensional representations with weights the $\gamma \cdot \chi|_M$, where $\gamma \in \Gamma$ (since the algebra $k(\chi|_M) \otimes_k k_s$ decomposes into the direct sum of twists of k_s by elements of Γ). Using the analogous decomposition for $\rho(\chi'|_M)_{k_s}$, it follows that $\Gamma \cdot \chi'|_M = \Gamma \cdot \chi|_M$. Hence $\chi' \in \Gamma \cdot \chi$. $\qquad\square$

3.2. **Blocks.** In this subsection, we study the decomposition of the category H-mod into blocks; the main result (Proposition 3.9) describes the structure of each block in terms of twisted representations of the largest unipotent quotient U. We start with a key observation:

Proposition 3.4. *Let* $\chi, \chi' \in \mathrm{X}(H)$. *If* $\chi' \notin \Gamma \cdot \chi$ *then* $\mathrm{Ext}^1_H(k(\chi), k(\chi')) = 0$.

Proof. By Lemma 3.2, we have $\mathrm{res}^H_M k(\chi) \simeq n\, k(\chi|_M)$, $\mathrm{res}^H_M k(\chi') \simeq n'\, k(\chi'|_M)$ for positive integers n, n'; moreover, the simple M-modules $k(\chi|_M)$, $k(\chi'|_M)$ are non-isomorphic. Thus, every extension of H-modules

$$0 \longrightarrow k(\chi') \longrightarrow E \longrightarrow k(\chi) \longrightarrow 0$$

has a unique splitting as an extension of M-modules; in particular, the space $\mathrm{Hom}_M(k(\chi), E)$ is nonzero. As this space is a module under the unipotent group scheme $U = H/M$, it contains a nonzero U-invariant; hence $\mathrm{Hom}_H(k(\chi), E) \neq 0$. As a consequence, the above extension splits. $\qquad\square$

Combining Proposition 3.4 with [Ja03, II.7.1], we obtain a decomposition of the category of finite-dimensional H-modules into *blocks*,

$$(3.3) \qquad\qquad H\text{-mod} = \bigoplus_{\chi \in \mathrm{X}(H)/\Gamma} H\text{-mod}_\chi,$$

where H-mod$_\chi$ denotes the full subcategory of H-mod consisting of the modules having all their composition factors isomorphic to $k(\chi)$. Each H-mod$_\chi$ is an abelian category; it has a unique simple object, $k(\chi)$, which is its own commutant algebra (Proposition 3.1).

Taking for χ the trivial character, we obtain the *principal block* H-mod$_0$, consisting of the modules with trivial composition factors. By Lemma 3.2, these are exactly the modules fixed pointwise by M, or equivalently the modules under $H/M = U$. This yields an equivalence

$$(3.4) \qquad\qquad\qquad H\text{-mod}_0 \simeq U\text{-mod},$$

where the right-hand side is the category of *unipotent representations*.

Remark 3.5. Lemma 3.4 and the block decomposition (3.3) easily imply that $\operatorname{Ext}_H^i(k(\chi), k(\chi')) = 0$ for all $i \geq 1$ and $\chi, \chi' \in \mathrm{X}(H)$ such that $\chi' \notin \Gamma \cdot \chi$. Also, we have canonical isomorphisms

$$\operatorname{Ext}_H^i(k(\chi), k(\chi)) \simeq \mathrm{H}^i(H, k(\chi)^\vee \otimes k(\chi)) \simeq \mathrm{H}^i(U, \operatorname{End}_M k(\chi)),$$

where the first equality holds by [Ja03, I.4.4], and the second one follows from the Hochschild-Serre spectral sequence (see [Ja03, I.6.6]) in view of the complete reducibility of representations of M.

If $k(\chi)$ is separable over k, then $k(\chi)$ is irreducible as an M-module, and hence $\operatorname{End}_M k(\chi) = k(\chi)$ by Proposition 3.1. This yields canonical isomorphisms

$$\operatorname{Ext}_H^i(k(\chi), k(\chi)) \cong \mathrm{H}^i(U, k) \otimes_k k(\chi) \quad (i \geq 0),$$

which also follow from Theorem 3.11 below. As $\mathrm{H}^1(U, k) \neq 0$ for any nontrivial unipotent group U (see e.g. [DG70, II.3.7, IV.2.5]), it follows that the category $H\text{-mod}_\chi$ is semisimple if and only if $U = 0$.

For an arbitrary character χ, it would be interesting to explicitly describe $\operatorname{Ext}_H^1(k(\chi), k(\chi))$, and to deduce an effective criterion for its vanishing (which is equivalent to the category $H\text{-mod}_\chi$ being semisimple). We will obtain a characterization of the semisimplicity of $H\text{-mod}_\chi$ by an alternative approach, in Lemma 3.7.

The block decomposition (3.3) extends to a decomposition of the category of H-modules,

$$H\text{-Mod} = \bigoplus_{\chi \in \mathrm{X}(H)/\Gamma} H\text{-Mod}_\chi,$$

where $H\text{-Mod}_\chi$ consists of the direct limits of objects of $H\text{-mod}_\chi$. In particular, we obtain a decomposition of the regular representation,

$$\mathcal{O}(H) = \bigoplus_{\chi \in \mathrm{X}(H)/\Gamma} \mathcal{O}(H)_\chi.$$

Lemma 3.6.   (i) *For any H-module V, we have a natural isomorphism of H-modules*

$$\bigoplus_{\chi \in \mathrm{X}(H)/\Gamma} \operatorname{Hom}_M(k(\chi|_M), V) \xrightarrow{\simeq} V, \quad f \longmapsto f(1),$$

  where H acts on $\operatorname{Hom}_M(k(\chi|_M), V)$ via its action on V.
   (ii) *Each $\operatorname{Hom}_M(k(\chi|_M), V)$ is an object of $H\text{-Mod}_\chi$.*
   (iii) *Each $\mathcal{O}(H)_\chi$ is the injective hull of $k(\chi)$ in $H\text{-Mod}_\chi$.*

Proof. (i) As the M-module $\operatorname{res}_M^H V$ is semisimple and the simple M-modules are exactly the $k(\chi|_M)$, where χ is uniquely determined up to the Γ-action,

we have an isomorphism

$$\bigoplus_{\chi \in \mathrm{X}(H)/\Gamma} \mathrm{Hom}_M(k(\chi|_M), V) \otimes_{\mathrm{End}_M(k(\chi|_M))} k(\chi|_M) \xrightarrow{\simeq} V,$$

which takes every $f \otimes v$ to $f(v)$. Moreover, $\mathrm{End}_M(k(\chi|_M))$ is identified with $k(\chi|_M)$ acting by multiplication, in view of Proposition 3.1. This yields the assertion.

(ii) By construction, $\mathrm{res}_M^H \mathrm{Hom}_M(k(\chi|_M), V)$ is a direct sum of copies of $k(\chi|_M)$. This implies the statement in view of Lemma 3.2 (iii).

(iii) Since $\mathcal{O}(H)$ is an injective object of H-Mod, we see that $\mathcal{O}(H)_\chi$ is an injective object of H-Mod$_\chi$. So it suffices to show that $\mathcal{O}(H)_\chi$ contains a unique copy of $k(\chi)$. But

$$\mathrm{Hom}_H(k(\chi), \mathcal{O}(H)_\chi) = \mathrm{Hom}_H(k(\chi), \mathcal{O}(H)) \simeq k(\chi)^\vee,$$

where the isomorphism holds by Frobenius reciprocity (see [Ja03, I.3.7]). Thus, $\mathrm{Hom}_H(k(\chi), \mathcal{O}(H)_\chi)$ is a k-vector space of dimension $[k(\chi) : k]$, and hence a vector space of dimension 1 under $\mathrm{End}_H k(\chi) = k(\chi)$. This yields the desired assertion. $\square$

Next, choose $\chi \in \mathrm{X}(H)$ and use the notation K, η, L of Lemma 3.2. Recall from that lemma that L is the separable closure of k in K; in particular, the degree $[K : L]$ is a power of the characteristic exponent of k. We view K as a simple H-module via $\rho(\chi)$; likewise, L is a simple M-module via $\rho(\eta)$.

Lemma 3.7. (i) $\mathcal{O}(H)_\chi \simeq \mathrm{ind}_M^H(L)$ as H-modules.
(ii) The category H-mod$_\chi$ is semisimple if and only if the group scheme U is finite of order $[K : L]$.

Proof. (i) We argue as in the proof of Lemma 3.6 (iii). By Frobenius reciprocity (see [Ja03, I.3.4]), we have

$$\mathrm{Hom}_H(V, \mathrm{ind}_M^H L)) \simeq \mathrm{Hom}_M(\mathrm{res}_M^H V, L)$$

for any H-module V. Since every M-module is semisimple, it follows that $\mathrm{ind}_M^H L$ is injective in H-Mod. Also, if V is simple and not isomorphic to K, then $\mathrm{Hom}_M(\mathrm{res}_M^H L, V) = 0$ by Lemma 3.2 (iii). Thus, $\mathrm{ind}_M^H L$ is an object of H-Mod$_\chi$. Finally,

$$\mathrm{Hom}_H(K, \mathrm{ind}_M^H L) \simeq \mathrm{Hom}_M(K, L) = \mathrm{Hom}_L(K, L),$$

where the equality follows from the fact that $\mathrm{End}_M(L) = L$ (Proposition 3.1). Thus, $\mathrm{Hom}_H(K, \mathrm{ind}_M^H(L))$ is a L-vector space of dimension $[K : L]$, and hence a k-vector space of dimension $[K : k]$. Since the k-vector space $\mathrm{End}_H(K)$ has dimension $[K : k]$ as well (Proposition 3.1 again), this yields the assertion in view of [Ja03, I.3.18].

(ii) The semisimplicity of $H\text{-mod}_\chi$ is equivalent to that of $H\text{-Mod}_\chi$, and the latter holds if and only if the injective cogenerator $\mathcal{O}(H)_\chi$ is semi-simple; by Lemma 3.6 (iii), this is equivalent to the equality $\dim \mathcal{O}(H)_\chi = [K : k]$. Also,

$$\dim \mathcal{O}(H)_\chi = \dim(\mathcal{O}(H) \otimes_k L)^M = \dim_{\bar{k}}(\mathcal{O}(H_{\bar{k}}) \otimes_{\bar{k}} L_{\bar{k}})^{M_{\bar{k}}} = \dim_{\bar{k}} \mathcal{O}(U_{\bar{k}}) \otimes_{\bar{k}} L_{\bar{k}},$$

where the latter equality follows from the isomorphism $H_{\bar{k}} \simeq U_{\bar{k}} \times M_{\bar{k}}$. Thus, $\dim \mathcal{O}(H)_\chi = [L : k] \dim \mathcal{O}(U)$; this implies the assertion. $\qquad\square$

Example 3.8. Assume that k is imperfect and separably closed. We may then choose $t \in k \setminus k^p$. Let $V \subset \mathbb{G}_a \times \mathbb{G}_a$ be the zero subscheme of $y^p - x - tx^p$. Then V is a nontrivial k-form of $\mathbb{G}_a$; in view of [To13, Lem. 9.4], it follows that there exists a nontrivial extension

$$\xi : \quad 0 \longrightarrow \mathbb{G}_m \longrightarrow E \longrightarrow V \longrightarrow 0.$$

On the other hand, the projection $x : V \to \mathbb{G}_a$ lies in an exact sequence

$$0 \longrightarrow \alpha_p \longrightarrow V \longrightarrow \mathbb{G}_a \longrightarrow 0,$$

where α_p denotes the kernel of the Frobenius endomorphism of $\mathbb{G}_a$. The pullback of ξ by $\alpha_p \to V$ yields an extension

$$0 \longrightarrow \mathbb{G}_m \longrightarrow H \longrightarrow \alpha_p \longrightarrow 0,$$

which is nontrivial as well; indeed, the pullback map

$$\operatorname{Ext}^1_{\mathcal{C}}(V, \mathbb{G}_m) \longrightarrow \operatorname{Ext}^1_{\mathcal{C}}(\alpha_p, \mathbb{G}_m)$$

is injective in view of the vanishing of $\operatorname{Ext}^1_{\mathcal{C}}(\mathbb{G}_a, \mathbb{G}_m)$ (which follows e.g. from [DG70, III.6.2.5, III. 6.5.1]). Thus, the canonical character of $\mathbb{G}_m$ extends uniquely to a character χ of H, which is not defined over k. But χ is defined over $K := k(t^{1/p})$, since $V_K \simeq (\mathbb{G}_a)_K$ and hence ξ splits over K. As $[K : k] = p$, it follows that $k(\chi) = K$. Since $k(\chi|_{\mathbb{G}_m}) = k$ and α_p has order p, Lemma 3.7 (ii) yields that the category $H\text{-mod}_\chi$ is semisimple.

The exact sequence (3.2) yields an exact sequence of L-group schemes

$$0 \longrightarrow M_L \longrightarrow H_L \longrightarrow U_L \longrightarrow 0.$$

Viewing η as a morphism $M_L \to \mathbb{G}_{m,L}$, this yields in turn a pushout diagram of such group schemes

$$(3.5) \qquad \begin{array}{ccccccccc}
0 & \longrightarrow & M_L & \longrightarrow & H_L & \longrightarrow & U_L & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \eta} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & \mathbb{G}_{m,L} & \longrightarrow & H_\eta & \longrightarrow & U_L & \longrightarrow & 0
\end{array}$$

Also, every object V of $H\text{-Mod}_\chi$ may be identified with $\operatorname{Hom}_M(L, V)$ in view of Lemma 3.6. Thus, V has the structure of a L-vector space equipped with a compatible action of H, i.e., of an H_L-module. Since M_L acts on V via η,

the representation of H_L in V factors uniquely through a representation of the pushout H_η. Denote by $F(V)$ the corresponding H_η-module, and by H_η-mod$_1$ the block of H_η-mod associated with the identity character of $\mathbb{G}_{m,L}$. We may view this block as that of twisted U_L-modules, where the twist is defined by the extension in the bottom line of (3.5).

Proposition 3.9. *The assignment $V \mapsto F(V)$ extends to an exact functor, which yields an equivalence of categories*

$$F : H\text{-mod}_\chi \longrightarrow H_\eta\text{-mod}_1.$$

Proof. Consider a morphism of H-modules $f : V \to W$. This defines a map

$$F(f) : \text{Hom}_M(L, V) \longrightarrow \text{Hom}_M(L, W), \quad u \longmapsto f \circ u.$$

We claim that $F(f)$ is a morphism of H_η-modules. It suffices to check that $F(f)$ is L-linear, where L acts on $\text{Hom}_M(L, V)$, $\text{Hom}_M(L, W)$ via multiplication on itself. But this follows by identifying the L-modules V, W with direct sums of copies of L, and using the fact that $\text{End}_M(L) = L$ (Proposition 3.1).

The claim defines the functor F on morphisms; one may readily check that F is exact, fully faithful and essentially surjective. $\qquad\square$

Next, we introduce a class of characters for which the associated block has an especially simple structure. We say that $\chi \in \text{X}(H)$ is *separable*, if its field of definition is a separable extension of k. Also, recall that an H-module V is said to be *absolutely semisimple*, if the $H_{\bar{k}}$-module $V_{\bar{k}} := V \otimes_k \bar{k}$ is semisimple.

Lemma 3.10. *With the notation of Lemma 3.2, the following conditions are equivalent for a character χ:*

(i) χ *is separable.*
(ii) $L = K$.
(iii) *The representation $\text{res}^H_M \rho(\chi)$ is irreducible.*
(iv) *The extension $0 \to \mathbb{G}_{m,L} \to H_\eta \to U_L \to 0$ splits.*
(v) *The H-module K is absolutely semisimple.*

Proof. The equivalences (i)⇔(ii)⇔(iii) are obvious. Also, in view of the commutative diagram (3.5), the splittings of the extension in (iv) may be identified with the morphisms $H_L \to \mathbb{G}_{m,L}$ extending the identity character of $\mathbb{G}_{m,L}$ i.e., with the morphisms $H \to \text{R}_{L/k}(\mathbb{G}_{m,L})$ extending $\rho(\eta) : M \to \text{R}_{L/k}(\mathbb{G}_{m,L})$; this implies the equivalence (ii)⇔(iv).

We now show the equivalence (i)⇔(v). If χ is separable, then the $\bar{k}$-algebra $K \otimes_k \bar{k}$ is isomorphic to a product of copies of $\bar{k}$. This yields a decomposition of the corresponding $H_{\bar{k}}$-module into $H_{\bar{k}}$-stable lines; in particular, this module is semisimple.

Conversely, if the H-module K is absolutely semisimple, then so is its commutant algebra. By Proposition 3.1, it follows that the field extension K/k is separable. $\qquad\square$

Theorem 3.11. *Let χ be a separable character of H, with field of definition K. Then the abelian category H-mod_χ is equivalent to U_K-mod.*

Proof. We may replace k with $K = L$ and H with H_η in view of Proposition 3.9. Then Lemma 3.10 yields an isomorphism $H \simeq \mathbb{G}_m \times U$. Denote by k_1 the 1-dimensional H-module with weight 1; then the assignment $V \mapsto V \otimes k_1$ extends to the desired equivalence

$$U\text{-mod} = H\text{-mod}_0 \xrightarrow{\simeq} H\text{-mod}_1.$$

$\square$

4. Isogenies

4.1. **Functorial properties of universal affine covers.** Consider a morphism of abelian varieties $\varphi : A \to B$.

Proposition 4.1.   (i) *There are unique morphisms $G(\varphi) : G_A \to G_B$, $H(\varphi) : H_A \to H_B$ such that the diagram of extensions*

$$(4.1) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & H_A & \longrightarrow & G_A & \xrightarrow{f_A} & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle H(\varphi)} & & \downarrow{\scriptstyle G(\varphi)} & & \downarrow{\scriptstyle \varphi} & & \\ 0 & \longrightarrow & H_B & \longrightarrow & G_B & \xrightarrow{f_B} & B & \longrightarrow & 0 \end{array}$$

commutes. If $\varphi = n_A$ for some integer n, then $G(\varphi) = n_{G_A}$, $H(\varphi) = n_{H_A}$. Also, the formations of $G(\varphi)$, $H(\varphi)$ commute with base change under field extensions.

   (ii) *For any morphism of abelian varieties $\psi : B \to C$, we have $G(\psi \circ \varphi) = G(\psi) \circ G(\varphi)$, $H(\psi \circ \varphi) = H(\psi) \circ H(\varphi)$.*

   (iii) *If φ is an isogeny, then $G(\varphi)$ is an isomorphism. Moreover, there is an exact sequence of commutative affine group schemes*

$$(4.2) \qquad\qquad 0 \longrightarrow H_A \xrightarrow{H(\varphi)} H_B \longrightarrow N \longrightarrow 0,$$

where $N := \mathrm{Ker}(\varphi)$.

Proof. (i) The existence of the morphisms $G(\varphi)$, $H(\varphi)$ follows from the fact that G_A is projective in the category $\widetilde{\mathcal{C}}$ of commutative quasi-compact group schemes. For the uniqueness, just note that every morphism $G_A \to H_B$ is zero, since $\mathcal{O}(G_A) = k$ and H_B is affine. This uniqueness property implies the assertion on multiplication maps. Finally, the assertion on field extensions follows from Proposition 2.5.

   (ii) This follows again from the uniqueness in (i).

   (iii) There exists an isogeny $\psi : B \to A$ such that $\psi \circ \varphi$ is the multiplication map n_A for some positive integer n, and $\varphi \circ \psi = n_B$. Then $G(\psi) \circ G(\varphi) = n_{G_A}$ by (ii); moreover, n_{G_A} is an isomorphism in view of [Br18, Lem. 3.4]. Likewise,

$G(\varphi) \circ G(\psi) = n_{G_B}$ is an isomorphism. This yields the assertion on $G(\varphi)$, and in turn that on $H(\varphi)$ by applying the snake lemma to the commutative diagram (4.1). $\qquad\square$

Next, assume that $\varphi : A \to B$ is an isogeny and let $N := \mathrm{Ker}(\varphi)$. Then the exact sequence (4.2) defines induction and restriction functors (see [Ja03, I.3] for details),

$$\mathrm{ind} = \mathrm{ind}_{H_A}^{H_B} : H_A\text{-mod} \to H_B\text{-mod}, \quad \mathrm{res} = \mathrm{res}_{H_A}^{H_B} : H_B\text{-mod} \to H_A\text{-mod}.$$

Theorem 4.2. (i) *For any homogeneous vector bundle F on B, the pullback $\varphi^*(F)$ is a homogeneous vector bundle on A. Moreover, the assignment $F \mapsto \varphi^*(F)$ yields an exact functor $\varphi^* : \mathrm{HVec}_B \to \mathrm{HVec}_A$ which fits in a commutative square*

$$
\begin{array}{ccc}
H_B\text{-mod} & \xrightarrow{\ \mathrm{res}\ } & H_A\text{-mod} \\
\downarrow{\scriptstyle \mathcal{L}_B} & & \downarrow{\scriptstyle \mathcal{L}_A} \\
\mathrm{HVec}_B & \xrightarrow{\ \varphi^*\ } & \mathrm{HVec}_A.
\end{array}
$$

(ii) *For any homogeneous vector bundle E on A, the push-forward $\varphi_*(E)$ is a homogeneous vector bundle on B. Moreover, the assignment $E \mapsto \varphi_*(E)$ yields an exact functor $\varphi_* : \mathrm{HVec}_A \to \mathrm{HVec}_B$ which fits in a commutative square*

$$
\begin{array}{ccc}
H_A\text{-mod} & \xrightarrow{\ \mathrm{ind}\ } & H_B\text{-mod} \\
\downarrow{\scriptstyle \mathcal{L}_A} & & \downarrow{\scriptstyle \mathcal{L}_B} \\
\mathrm{HVec}_A & \xrightarrow{\ \varphi_*\ } & \mathrm{HVec}_B.
\end{array}
$$

Proof. (i) Let $a \in A(\bar{k})$. Since φ is a group homomorphism, we have $\tau_a^* \varphi^*(F) \simeq \varphi^* \tau_{\varphi(a)}^*(F)$. As F is homogeneous, it follows that $\tau_a^* \varphi^*(F) \simeq \varphi^*(F)$. So $\varphi^*(F)$ is homogeneous as well.

Clearly, the assignment $F \mapsto \varphi^*(F)$ extends to an exact functor $\mathrm{Vec}_B \to \mathrm{Vec}_A$, and hence to an exact functor $\varphi^* : \mathrm{HVec}_B \to \mathrm{HVec}_A$. The commutativity of the displayed square follows readily from the definitions.

(ii) Since φ is finite and flat, it yields an exact functor $\varphi_* : \mathrm{Vec}_A \to \mathrm{Vec}_B$. Let $b \in B(\bar{k})$, and choose $a \in A(\bar{k})$ such that $b = \varphi(a)$; then the diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\ \tau_a\ } & A \\
\downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi} \\
B & \xrightarrow{\ \tau_b\ } & B
\end{array}
$$

commutes, where the horizontal arrows are isomorphisms. It follows that $\tau_b^* \varphi_*(E) \simeq \varphi_* \tau_a^*(E)$. Thus, $\varphi_*(E)$ is homogeneous.

By the projection formula, φ_* is right adjoint to φ^*. Also, recall that ind is right adjoint to res (see [Ja03, I.3.4]). So the desired isomorphism of functors $\mathcal{L}_B \circ \mathrm{ind} \simeq \varphi_* \circ \mathcal{L}_A$ follows from the isomorphism $\mathcal{L}_A \circ \mathrm{res} \simeq \varphi^* \circ \mathcal{L}_B$ in view of the uniqueness of adjoints. $\qquad\square$

Next, recall the dual exact sequence

$$(4.3) \qquad\qquad 0 \longrightarrow \mathrm{D}(N) \longrightarrow \widehat{B} \xrightarrow{\widehat{\varphi}} \widehat{A} \longrightarrow 0,$$

where $\widehat{\varphi}$ denotes the dual isogeny, and $\mathrm{D}(N)$ the Cartier dual of the finite group scheme N. Also, recall the natural isomorphism $\mathrm{Hom}_{\widetilde{\mathcal{L}}}(H_{A_{k'}}, \mathbb{G}_{m,k'}) \simeq \widehat{A}(k')$ for any field extension k'/k (Lemma 2.6). Thus, we may identify the character group $\mathrm{X}(H_A)$ with $\widehat{A}(\bar{k})$, and the field of definition of any $x \in \mathrm{X}(H_A)$ with the residue field $k(x)$. In particular, the separable characters (as defined in §3.2) correspond to the points of $\widehat{A}(k_s)$. Also, the morphism $H(\varphi) : H_A \to H_B$ defines a pullback map $H(\varphi)^* : \mathrm{X}(H_B) \to \mathrm{X}(H_A)$.

Lemma 4.3. *For any $y \in \widehat{B}(\bar{k})$, we have $H(\varphi)^*(y) = \widehat{\varphi}(y)$ in $\widehat{A}(\bar{k})$.*

Proof. Consider first the case where $y \in \widehat{B}(k)$. We then have a commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H_A & \longrightarrow & G_A & \xrightarrow{f_A} & A & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle H(\varphi)} & & \downarrow{\scriptstyle G(\varphi)} & & \downarrow{\scriptstyle \varphi} & & \\
0 & \longrightarrow & H_B & \longrightarrow & G_B & \xrightarrow{f_B} & B & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle y} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & \mathbb{G}_m & \longrightarrow & G & \longrightarrow & B & \longrightarrow & 0,
\end{array}
$$

where the bottom line is obtained by pushout. This yields a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Hom}_{\widetilde{\mathcal{L}}}(H_B, \mathbb{G}_m) & \xrightarrow{\partial(B)} & \mathrm{Ext}^1_{\mathcal{C}}(B, \mathbb{G}_m) \\
\downarrow{\scriptstyle H(\varphi)^*} & & \downarrow{\scriptstyle \varphi^*} \\
\mathrm{Hom}_{\widetilde{\mathcal{L}}}(H_A, \mathbb{G}_m) & \xrightarrow{\partial(A)} & \mathrm{Ext}^1_{\mathcal{C}}(A, \mathbb{G}_m),
\end{array}
$$

where the horizontal arrows are isomorphisms given by pushout. Also, the right vertical arrow may be identified with $\widehat{\varphi} : \widehat{B}(k) \to \widehat{A}(k)$. So we may

rewrite the above square as

$$\begin{CD} \operatorname{Hom}_{\widetilde{\mathcal{L}}}(H_B, \mathbb{G}_m) @>\simeq>> \widehat{B}(k) \\ @VH(\varphi)^*VV @VV\widehat{\varphi}V \\ \operatorname{Hom}_{\widetilde{\mathcal{L}}}(H_A, \mathbb{G}_m) @>\simeq>> \widehat{A}(k), \end{CD}$$

which yields the assertion in that case.

In the general case, we argue similarly by replacing k with the residue field $k(y)$, and using compatibility with field extensions (Proposition 4.1). $\qquad\square$

Still considering $x \in \mathrm{X}(H_A) = \widehat{A}(\bar{k})$ with residue field $k(x)$, the representation of H_A in $k(x)$ (constructed in §3.1) yields an associated homogeneous vector bundle $E(x) := \mathcal{L}_A(k(x))$ over A, of rank $[k(x) : k]$. By Theorem 2.9 and Proposition 3.1, $E(x)$ is irreducible; moreover, every irreducible homogeneous vector bundle on A is obtained in this way. Also, given $x' \in \widehat{A}(\bar{k})$, we have $E(x') \simeq E(x)$ if and only if $x' \in \Gamma \cdot x$ (Proposition 3.3). Otherwise, $\operatorname{Ext}_A^i(E(x), E(x')) = 0$ for any $i \geq 0$, by Theorem 2.12 and Proposition 3.4.

If $x \in \widehat{A}(k)$, then $E(x)$ is just the corresponding (algebraically trivial) line bundle on A. For an arbitrary residue field $k(x) =: K$, we obtain a line bundle L on A_K, and hence a vector bundle $\mathrm{R}_{K/k}(L)$ on $\mathrm{R}_{K/k}(A_K)$, of rank $[K : k]$; one may check that $E(x)$ is the pullback of that vector bundle under the canonical immersion $j_A : A \to \mathrm{R}_{K/k}(A_K)$ (see [CGP15, A.5.7]).

Likewise, we have irreducible homogeneous vector bundles $F(y)$ on B, indexed by the Γ-orbits in $\widehat{B}(\bar{k})$. We now determine their pullback to A:

Proposition 4.4. *Let $y \in \widehat{B}(\bar{k})$ and set $x := \widehat{\varphi}(y)$. Then we have $k(x) \subset k(y)$ and $\varphi^* F(y) \simeq [k(y) : k(x)]\, E(x)$.*

Proof. Clearly, $k(x) \subset k(y)$. Also, by Theorem 4.2, $\varphi^* F(y)$ is the homogeneous vector bundle associated with the representation of H_A in $k(y)$ via the composition

$$H_A \xrightarrow{H(\varphi)} H_B \xrightarrow{\rho(y)} \mathrm{R}_{k(y)/k}(\mathbb{G}_{m,k(y)}).$$

The associated character $(H_A)_{k(y)} \to \mathbb{G}_{m,k(y)}$ equals x in view of Lemma 4.3. Thus, H_A acts on L via the corresponding morphism $H_A \to \mathrm{R}_{k(x)/k}(\mathbb{G}_{m,k(x)})$. It follows that $\operatorname{res}_{H_A}^{H_B} \rho(y) \simeq [k(y) : k(x)]\, \rho(x)$; this translates into the desired isomorphism. $\qquad\square$

The block decomposition (3.3) of H_A-mod yields a decomposition

$$\mathrm{HVec}_A = \bigoplus_{x \in \widehat{A}(\bar{k})/\Gamma} \mathrm{HVec}_{A,x}.$$

As a direct consequence of Proposition 4.4, we obtain:

Corollary 4.5. $\varphi^* : \mathrm{HVec}_B \to \mathrm{HVec}_A$ *takes* $\mathrm{HVec}_{B,y}$ *to* $\mathrm{HVec}_{A,\widehat{\varphi}(y)}$ *for any* $y \in \widehat{B}(\bar{k})$. *Moreover,* φ^* *preserves semisimplicity.*

Next, we describe the pushforward of irreducible homogeneous bundles:

Proposition 4.6.   (i) *Let* $x \in \widehat{A}(\bar{k})$ *and consider the block decomposition*

$$\varphi_* E(x) = \bigoplus_{y \in \widehat{B}(\bar{k})/\Gamma} F_{x,y}.$$

  Then $F_{x,y} \neq 0$ *if and only if* $\widehat{\varphi}(y) \in \Gamma \cdot x$. *Under this assumption,* $F_{x,y}$ *contains a unique copy of* $F(y)$; *in particular,* $F_{x,y}$ *is indecomposable.*
  (ii) φ_* *preserves semisimplicity if and only if* $\widehat{\varphi}$ *is separable. Under this assumption,* $F_{x,y}$ *is irreducible for all* x, y.

Proof. (i) Let $y \in \widehat{B}(\bar{k})$. Then

$$\mathrm{Hom}_{\mathrm{HVec}_B}(F(y), \varphi_* E(x)) \simeq \mathrm{Hom}_{\mathrm{HVec}_A}(\varphi^* F(y), E(x))$$

$$\simeq [k(y) : k(\widehat{\varphi}(y)] \, \mathrm{Hom}_{\mathrm{HVec}_A}(E(\widehat{\varphi}(y)), E(x)),$$

where the first isomorphism holds by adjunction, and the second one follows from Proposition 4.4. Thus,

$$\dim_k \mathrm{Hom}_{\mathrm{HVec}_B}(F(y), \varphi_* E(x)) = [k(y) : k(\gamma \cdot x)] \, [k(\gamma \cdot x) : k] = [k(y) : k]$$

if $\widehat{\varphi}(y) = \gamma \cdot x$ for some $\gamma \in \Gamma$, and this dimension is zero otherwise. Since the k-vector space $\mathrm{Hom}_{\mathrm{HVec}_B}(F(y), \varphi_* E(x))$ is a module under $\mathrm{End}_{\mathrm{HVec}_B} k(y) \simeq k(y)$, we see that $F_{x,y}$ contains a unique copy of $F(y)$ if $\widehat{\varphi}(y) \in \Gamma \cdot x$, and is zero otherwise.

(ii) Assume that φ_* preserves semisimplicity; in particular, the homogeneous bundle $\varphi_*(\mathcal{O}_A)$ is semisimple. By Theorems 2.9 and 4.2, it follows that the H_B-module $\mathcal{O}(H_B/H_A) = \mathcal{O}(N)$ is semisimple as well. Equivalently, N is of multiplicative type (see [DG70, II.2.2.2, IV.3.3.6]), i.e., the Cartier dual $D(N)$ is étale. In view of the exact sequence (4.3), this means that $\widehat{\varphi}$ is separable.

Conversely, assume that $\widehat{\varphi}$ is separable; then N is of multiplicative type, as seen by reverting the above arguments. Note that φ_* preserves semisimplicity if and only if so does $\mathrm{ind}_{H_A}^{H_B}$ (Theorem 4.2). We also need a general observation: let H be an affine group scheme and V an H-module; then V is semisimple if and only if the H_{k_s}-module V_{k_s} is semisimple. Indeed, V is a module under the finite-dimensional algebra $C^\vee$ constructed in the proof of Proposition 3.1; moreover, V is semisimple as an H-module if and only if it is semisimple as a $C^\vee$-module (as follows from [DG70, II.2.2.4]). Since the formation of $C^\vee$ commutes with field extensions, this implies the observation by using the invariance of semisimplicity under separable extensions (see [Bo58, VIII.13.4]).

In view of this observation, we may assume that k is separably closed. The fiber of $\widehat{\varphi}$ at any $x \in \widehat{A}(\bar{k})$ has n distinct $\bar{k}$-points $y_1, \ldots, y_n$, where

$n = \deg(\widehat{\varphi}) = \deg(\varphi)$. Moreover, $k(y_i) = k(x)$ for $i = 1, \ldots, n$: indeed, $k(y_i)$ is a separable extension of $k(x)$ as $\widehat{\varphi}$ is étale, and $k(y_i)/k(x)$ is purely inseparable as $k = k_s$. By (i), $\varphi_* E(x)$ contains a sub-bundle isomorphic to $\oplus_{i=1}^n F(y_i)$. Since

$$\mathrm{rk}\, \varphi_* E(x) = n\,[k(x) : k] = \sum_{i=1}^n [k(y_i) : k],$$

it follows that $\varphi_* E(x) \simeq \oplus_{i=1}^n F(y_i)$. This completes the proof of preservation of semisimplicity under φ_*. The irreducibility of $F_{x,y}$ follows in view of (i). $\square$

Remark 4.7. To determine the pushforward of irreducible homogeneous vector bundles under an arbitrary isogeny φ, consider the exact sequence

$$0 \longrightarrow M \longrightarrow N \longrightarrow U \longrightarrow 0,$$

where M is of multiplicative type and U is unipotent. This yields a factorization $\varphi = \varphi_u \circ \varphi_m$, where $\varphi_m : A \to A/M$ has kernel M and $\varphi_u : A/M \to B$ has kernel U, and a dual factorization $\widehat{\varphi} = \widehat{\varphi}_m \circ \widehat{\varphi}_u$, where $\widehat{\varphi}_m$ is separable and $\widehat{\varphi}_u$ is purely inseparable. As the pushforward $(\varphi_m)_* E(x)$ is described by Proposition 4.6, we may replace φ with φ_u, and hence assume that $N = U$ is unipotent. Then $\widehat{\varphi}$ is bijective on $\bar{k}$-points; by Proposition 4.6 again, it follows that $\varphi_* E(x)$ is indecomposable for any $x \in \widehat{A}(\bar{k})$.

4.2. **Unipotent vector bundles.** Recall that a vector bundle E over A is *unipotent* if it admits a filtration $0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$, where each E_i is a sub-bundle and $E_i/E_{i-1} \simeq \mathcal{O}_A$ for $i = 1, \ldots, n$. The unipotent vector bundles form a full subcategory UVec_A of Vec_A.

Theorem 4.8.   (i) $\mathrm{UVec}_A = \mathrm{HVec}_{A,0} \simeq U_A\text{-mod}$; *in particular,* UVec_A *has a unique (up to isomorphism) simple object,* $\mathcal{O}_A$. *Moreover, we have isomorphisms of graded algebras*

$$\mathrm{Ext}^*_{\mathrm{UVec}_A}(\mathcal{O}_A, \mathcal{O}_A) \simeq \mathrm{H}^*(A, \mathcal{O}_A) \simeq \Lambda^* \mathrm{H}^1(A, \mathcal{O}_A).$$

  (ii) UVec_A *is an abelian tensor subcategory of* Vec_A, *stable under extensions and direct summands.*

  (iii) *We have an equivalence of abelian categories* $\mathrm{HVec}_{A,x} \simeq \mathrm{UVec}_{A_{k(x)}}$ *for any* $x \in \widehat{A}(k_s)$.

Proof. (i) By Corollary 2.13, every unipotent vector bundle is homogeneous. This yields the equality $\mathrm{UVec}_A = \mathrm{HVec}_{A,0}$; the latter category is equivalent to U_A-mod by Theorem 2.9 and (3.4). This shows the first assertion. The second assertion is obtained by combining Theorem 2.12 and [Se97, Thm. VII.10].

  (ii) This follows from (i) by using Corollary 2.10.

  (iii) This is a direct consequence of Theorem 3.11. $\square$

Remark 4.9. The unipotent vector bundles over an elliptic curve A have been determined by Atiyah (when k is algebraically closed), in the process of his description of all vector bundles over A; see [At57]. In particular, there is a unique indecomposable unipotent bundle of rank r for any integer $r \geq 0$. The decomposition of the tensor product of any two such bundles has been determined in [At57] when $p = 0$; the case where $p > 0$ has been treated much more recently by Schroer (see [Sc10]).

Returning to an abelian variety A over an arbitrary field k, recall that $U_A \simeq (\mathbb{G}_a)^g$ if $p = 0$. As a consequence, the category UVec_A is equivalent to the category with objects the tuples $(r, X_1, \ldots, X_g)$, where r is a non-negative integer and $X_1, \ldots, X_g$ are commuting nilpotent $r \times r$ matrices with coefficients in k; the morphisms from $(r, X_1, \ldots, X_g)$ to $(s, Y_1, \ldots, Y_g)$ are the $s \times r$ matrices Z with coefficients in k such that $ZX_i = Y_iZ$ for $i = 1, \ldots, g$. An explicit description of the isomorphism classes of such tuples is well-known for $g = 1$, via the Jordan canonical form (which gives back the above results of Atiyah). But the higher-dimensional case is quite open; see [HH19] for a study of the moduli space of "regular" tuples.

This description of UVec_A in terms of linear algebra extends to an ordinary abelian variety A over a separably closed field k of characteristic $p > 0$, since we then have $U_A \simeq (\mathbb{Z}_p)_k^g$ by (2.6).

Proposition 4.10. *Let $\varphi : A \to B$ be an isogeny with kernel N.*

   (i) *$\varphi^* : \mathrm{HVec}_B \to \mathrm{HVec}_A$ takes UVec_B to UVec_A. If φ is separable, then this yields an equivalence of categories $\mathrm{UVec}_B \simeq \mathrm{UVec}_A$.*

   (ii) *$\varphi_* : \mathrm{HVec}_A \to \mathrm{HVec}_B$ takes UVec_A to $\bigoplus_{y \in \mathrm{X}(N)/\Gamma} \mathrm{HVec}_{B,y}$. In the resulting decomposition of $\varphi_*(\mathcal{O}_A)$, each summand is indecomposable.*

Proof. (i) The first assertion holds since φ^* is exact and takes $\mathcal{O}_B$ to $\mathcal{O}_A$. The second assertion follows from Theorem 4.2 in view of the exact sequence (4.2), which yields an isomorphism $U_A \xrightarrow{\simeq} U_B$.

(ii) This follows similarly from the exactness of φ_* and Proposition 4.6. □

Remark 4.11. Assume that $p > 0$; then the nth relative Frobenius morphism F_A^n is a purely inseparable isogeny of degree p^{ng}. By Theorem 4.2 and Proposition 4.10, the decomposition of $(\mathrm{F}_A^n)_* \mathcal{O}_A$ into indecomposable summands corresponds to the block decomposition of $\mathcal{O}(N)$, where $N := \mathrm{Ker}(\mathrm{F}_A^n)$ is an infinitesimal group scheme of order p^{ng}. Denoting by r the p-rank of A, the largest subgroup scheme of multiplicative type of N is a k-form of $(\mu_{p^n})^r$ in view of (2.5). When (say) k is separably closed, it follows that each block of $\mathcal{O}(N)$ has dimension $p^{n(g-r)}$; equivalently, $(\mathrm{F}_A^n)_* \mathcal{O}_A$ is the direct sum of p^{nr} indecomposable summands of dimension $p^{n(g-r)}$. This gives back a recent result of Sannai and Tanaka (see [ST16, Thm. 1.2]).

The analogous decomposition of $(\mathrm{F}_A^n)_*(L)$, obtained in [ST16, Thm. 5.3] for any $L \in \widehat{A}(k)$, can also be derived from Proposition 4.6 and Remark 4.7.

Still assuming $p > 0$, and denoting by $\mathrm{V}_A^n : A^{(p^n)} \to A$ the nth Verschiebung, we obtain a characterization of unipotent vector bundles which refines a result of Miyanishi (see [Mi73, Rem. 2.4]):

Proposition 4.12. *Let E be a vector bundle over A. Then E is unipotent if and only if $(\mathrm{V}_A^n)^* E$ is trivial for $n \gg 0$.*

Proof. Assume that E is unipotent. Then there exists a finite-dimensional representation $\rho : H_A \to \mathrm{GL}(V)$ such that $E = \mathcal{L}_A(V)$ and $\mathrm{Ker}(\rho) \supset M_A$. Thus, $H_A/\mathrm{Ker}(\rho)$ is a commutative unipotent algebraic group, and hence there exists n_0 such that $\mathrm{V}_{H_A/\mathrm{Ker}(\rho)}^n = 0$ for $n \geq n_0$ (see [DG70, IV.3.4.11]). As a consequence, $\rho \circ \mathrm{V}_{H_A}^n$ is trivial for $n \geq n_0$. In view of the commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H_A^{(p^n)} & \longrightarrow & G_A^{(p^n)} & \xrightarrow{f_A^{(p^n)}} & A^{(p^n)} & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \mathrm{V}_{H_A}^n} & & \downarrow{\scriptstyle \mathrm{V}_{G_A}^n} & & \downarrow{\scriptstyle \mathrm{V}_A^n} & & \\
0 & \longrightarrow & H_A & \longrightarrow & G_A & \xrightarrow{f_A} & A & \longrightarrow & 0,
\end{array}
$$

it follows that $(\mathrm{V}_A^n)^* E$ is trivial for $n \geq n_0$.

Conversely, assume that $(\mathrm{V}_A^n)^* E$ is trivial for $n \geq n_0$. By Theorem 4.2 (i), it follows that $E \simeq G_A \times^{H_A} V$, where V is a finite-dimensional H_A-module which restricts trivially to $H_A^{(p^n)}$ via $H(\mathrm{V}_A^n)$. Thus,

$$
E \simeq G_A/H_A^{(p^n)} \times^{H_A/H_A^{(p^n)}} V \simeq A^{(p^n)} \times^{\mathrm{Ker}(\mathrm{V}_A^n)} V.
$$

Since $\mathrm{Ker}(\mathrm{V}_A^n)$ is unipotent, it follows that E is unipotent as well. $\square$

Remark 4.13. The *essentially finite* vector bundles E over A, i.e., those such that $f^*(E)$ is trivial for some torsor $f : X \to A$ under a finite group scheme, admit a similar characterization. Indeed, as shown by Nori in [No83], for any such bundle E, there exists $n > 1$ such that $n_a^*(E)$ is trivial. As a consequence, the essentially finite vector bundles are exactly the iterated extensions of irreducible homogeneous vector bundles associated with torsion points of $\widehat{A}(\bar{k})$.

Also, U_A is canonically isomorphic to the "nilpotent fundamental group scheme" $U(A, 0)$ introduced and studied by Nori in [No82, Chap. IV] in the more general setting of pointed schemes of finite type. This follows from [loc. cit., Prop. 1]; note that the "nilpotent group schemes" considered there are exactly the unipotent group schemes in the sense of [DG70, IV.2.2].

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References

[At57] M. F. Atiyah, *Vector bundles on an elliptic curve*, Proc. London Math. Soc. **7** (1957),
        414–452.

[Ay18] J. Ayoub, *Topologie feuilletée et théorie de Galois différentielle*, preprint, 2018.

[Bo58] N. Bourbaki, *Éléments de mathématique. Algèbre*, Hermann, Paris, 1958.

[Br13] M. Brion, *The coherent cohomology ring of an algebraic group*, Algebr. Represent. Theory **16** (2013), 1449–1467.

[Br15] M. Brion, *On extensions of algebraic groups with finite quotient*, Pacific J. Math.
        279 (2015), 135–153.

[BSU13] M. Brion, P. Samuel, V. Uma, *Lectures on the structure of algebraic groups and
        geometric applications*, Hindustan Book Agency, New Dehli, 2013.

[Br18] M. Brion, *On the fundamental groups of commutative algebraic groups*,
        arXiv:1805.09525, to appear at Annales Henri Lebesgue.

[CGP15] B. Conrad, O. Gabber, G. Prasad, *Pseudo-reductive groups. Second edition*, New
        Math. Monogr. **26**, Cambridge Univ. Press, Cambridge, 2015.

[DG70] M. Demazure, P. Gabriel, *Groupes algébriques*, Masson, Paris, 1970.

[EGA]   A. Grothendieck, *Éléments de géométrie algébrique (rédigés avec la collaboration de
        J. Dieudonné)*, Pub. Math. I.H.É.S. **11** (1961), **28** (1966).

[HH19] W. Haboush, D. Hyeon, *Conjugacy classes of commuting nilpotents*, Trans. Amer.
        Math. Soc. **372** (2019), 4293–4311.

[Ja03] J. C. Jantzen, *Representations of algebraic groups. Second edition*, Math. Surveys
        Monogr. **107**, Amer. Math. Soc., Providence, 2003.

[KS05] M. Kashiwara, P. Schapira, *Categories and sheaves*, Grundlehren Math. Wiss. **332**,
        Springer-Verlag, Berlin-New York, 2005.

[La11] A. Langer, *On the S-fundamental group scheme*, Ann. Inst. Fourier (Grenoble) **26**
        (2011), no. 5, 2077–2119.

[La12] A. Langer, *On the S-fundamental group scheme. II*, J. Inst. Math. Jussieu **11** (2012),
        no. 4, 835–854.

[Lu17] G. Lucchini Arteche, *Extensions of algebraic groups with finite quotient and non-
        abelian 2-cohomology*, J. Algebra **492** (2017), 102–129.

[Mi73] M. Miyanishi, *Some remarks on algebraic homogeneous vector bundles*, Number Theory, Algebraic Geometry and Commutative Algebra, in honor of Y. Akizuki, Kinokuniya, Tokyo, Japan, (1973), 71–93.

[Mu78] S. Mukai, *Semi-homogeneous vector bundles on an abelian variety*, J. Math. Kyoto
        Univ. **18** (1978), no. 2, 239–272.

[Mu08] D. Mumford, *Abelian varieties.* With appendices by C. P. Ramanujam and Yuri
        Manin. Corrected reprint of the 2nd ed., Hindustan Book Agency, New Dehli, 2008.

[No82] M. V. Nori, *The fundamental group-scheme*, Proc. Indian Acad. Sci. (Math. Sci.)
        91 (1982), 73–122.

[No83] M. V. Nori, *The fundamental group-scheme of an Abelian variety*, Math. Ann. **263**
        (1983), 263–266.

[Oo66] F. Oort, *Commutative group schemes*, Lecture Notes in Math. **15**, Springer-Verlag, Berlin-New York, 1966.

[Pe75] D. Perrin, *Schémas en groupes quasi-compacts sur un corps et Groupes henséliens*, Publications mathématiques d'Orsay **165** (1975).

[Pe76] D. Perrin, *Approximation des schémas en groupes quasi-compacts sur un corps*, Bull. Soc. math. France **104** (1976), 323–335.

[ST16] A. Sannai, H. Tanaka, *A characterization of ordinary abelian varieties by the Frobenius push-forward of the structure sheaf*, Math. Ann. **336** (2016), no. 3-4, 1067–1087.

[Sc10] S. Schroer, *On the ring of unipotent vector bundles on elliptic curves in positive characteristics*, J. London Math. Soc. **82** (2010), no. 1, 110–124.

[SGA3] M. Demazure, A. Grothendieck, *Séminaire de Géométrie Algébrique du Bois Marie, 1962–64, Schémas en groupes (SGA 3)*, Lect. Notes Math. **151, 152, 153**, Springer-Verlag, Berlin-New York, 1970.

[Se60] J.-P. Serre, *Groupes proalgébriques*, Publ. Math. IHÉS **7** (1960).

[Se97] J.-P. Serre, *Algebraic groups and class fields*, Graduate Texts Math. **117**, Springer-Verlag, Berlin-New York, 1997.

[SP19] The Stacks Project Authors, *Stacks Project*, `http://stacks.math.columbia.edu`, 2019.

[To13] B. Totaro, *Pseudo-abelian varieties*, Ann. Scient. Éc. Norm. Sup. (4) **46** (2013), no. 5, 693–721.

[Wa79] W. Waterhouse, *Introduction to affine group schemes*, Graduate Texts Math. **66**, Springer-Verlag, Berlin-New York, 1979.

[Wu86] X. L. Wu, *On the extensions of abelian varieties by affine group schemes*, Group theory (1984), Lect. Notes Math. **1185**, Springer-Verlag, Berlin-New York, 1986.