Local properties of algebraic group actions

Michel Brion

Abstract

These are notes for a mini-course at the 3rd Swiss-French workshop on algebraic geometry. They present some fundamental results on actions of linear algebraic groups on algebraic varieties: linearization of line bundles and local properties of such actions. They are based on work of Mumford [MFK94], Sumihiro [Su74, Su75] and Knop-Kraft-Luna-Vust [KKV89, KKLV89].

1 Basic notions and examples

1.1 Notation and preliminary results

Throughout these notes, we fix a base field k, assumed to be algebraically closed.

An (algebraic) *variety* is a reduced separated scheme of finite type over k; in particular, varieties need not be irreducible. By a point of a variety X, we mean a closed (or equivalently, k-rational) point. We will identify X with its set of points, equipped with the Zariski topology and with the structure sheaf $\mathcal{O}_X$. The algebra of global sections of $\mathcal{O}_X$ will be called the algebra of global regular functions on X, and denoted by $\mathcal{O}(X)$. We will denote by $\mathcal{O}(X)^*$ the group of units (or invertible elements) of $\mathcal{O}(X)$, and by $k(X)$ the field of rational functions on X. We will use the standard notation $\mathrm{Pic}(X)$ for the Picard group of X, that is, the set of isomorphism classes of line bundles on X, equipped with a commutative group law via the tensor product of such bundles.

An *algebraic group* G is a variety equipped with a group structure such that the multiplication map $m : G \times G \to G$, $(g, h) \mapsto gh$ and the inverse map $i : G \to G$, $g \mapsto g^{-1}$ are morphisms.

An (algebraic) *action* of an algebraic group G on a variety X is a morphism of varieties $\alpha : G \times X \to X$ such that $\alpha(e_G, x) = x$ for all $x \in X$, where e_G denotes the neutral element of G, and $\alpha(g, \alpha(h, x)) = \alpha(gh, x)$ for all $g, h \in G$ and $x \in X$. We then say that X is a *G-variety*; for simplicity, we denote $\alpha(g, x)$ by $g \cdot x$.

A G-action on X is called *faithful* if its scheme-theoretic kernel is trivial, i.e., for any k-algebra R, every nontrivial $g \in G(R)$ acts nontrivially on $X(R)$; in characteristic 0, this is equivalent to the condition that any nontrivial $g \in G$ acts nontrivially on X. We then say that G is an *algebraic group of automorphisms* of X.

A finite-dimensional vector space V is called a (rational) *G-module* if V is equipped with a linear action of G, i.e., the map $V \to V$, $v \mapsto g \cdot v$ is linear for any $g \in G$. Equivalently, G acts on V (viewed as an affine space) via a homomorphism of algebraic groups

$\rho : G \to \mathrm{GL}(V)$. The G-module V is called *faithful* if ρ is an immersion; equivalently, its scheme-theoretic kernel is trivial.

The notion of G-module extends to an arbitrary vector space V, not necessarily of finite dimension, by requiring that V be equipped with a linear action of the abstract group G such that every $v \in V$ is contained in some finite-dimensional G-stable subspace on which G acts algebraically.

Lemma 1.1. *Let X be a G-variety, and consider the linear action of G on $\mathcal{O}(X)$ via $(g \cdot f)(x) := f(g^{-1} \cdot x)$ for all $g \in G$, $f \in \mathcal{O}(X)$ and $x \in X$. Then $\mathcal{O}(X)$ is a G-module.*

Proof. Let $f \in \mathcal{O}(X)$. Then $f \circ \alpha \in \mathcal{O}(G \times X) \cong \mathcal{O}(G) \otimes_k \mathcal{O}(X)$. Thus, we have $f(g \cdot x) = \sum_{i \in I} \varphi_i(g)\psi_i(x)$ for some $\varphi_i \in \mathcal{O}(G)$ and $\psi_i \in \mathcal{O}(X)$, where i runs over a finite set I. Equivalently, $g \cdot f = \sum_{i \in I} \varphi_i(g^{-1})\psi_i$. Thus, the translates $g \cdot f$, where $g \in G$, span a finite-dimensional subspace $V = V(f) \subset \mathcal{O}(X)$, which is obviously G-stable. To show that G acts algebraically on V, choose a linear form on that space and extend it to a linear form ℓ on $\mathcal{O}(X)$. Then for any $h \in G$, the function $g \mapsto \ell(gh \cdot f) = \sum_{i \in I} \varphi_i(h^{-1}g^{-1})\ell(\psi_i)$ is regular on G. Thus, the group homomorphism $G \to \mathrm{GL}(V)$ is a morphism of varieties. $\square$

Proposition 1.2. *Let X be an affine G-variety. Then there exists a closed immersion $\iota : X \to V$, where V is a finite-dimensional G-module and ι is G-equivariant.*

Proof. We may choose a finite-dimensional subspace $V \subset \mathcal{O}(X)$ which generates that algebra. By Lemma 1.1, V is contained in some finite-dimensional G-submodule $W \subset \mathcal{O}(X)$. Thus, the algebra $\mathcal{O}(X)$ is G-equivariantly isomorphic to the quotient of the symmetric algebra $\mathrm{Sym}(W)$ by a G-stable ideal I. This means that X is equivariantly isomorphic to the closed G-stable subvariety of the dual G-module $W^\vee$ defined by I. $\square$

An algebraic group G is called *linear* if it is isomorphic to a closed subgroup of the general linear group GL_n for some n; equivalently, G admits a faithful finite-dimensional module. Proposition 1.2 implies readily the following:

Corollary 1.3. *Any algebraic group of automorphisms of an affine variety is linear.*

1.2 G-quasiprojective varieties and G-linearized line bundles

In this subsection, we fix an algebraic group G.

Given a finite-dimensional vector space V, we denote by $\mathbb{P}(V)$ the projective space of lines in V; the line spanned by a nonzero $v \in V$ will be denoted by $[v] \in \mathbb{P}(V)$. When V is a G-module, $\mathbb{P}(V)$ is a G-variety. Moreover, the G-action on $\mathbb{P}(V)$ lifts to an action on the tautological line bundle $\mathcal{O}_{\mathbb{P}(V)}(-1)$. Indeed, we may realize that bundle as the subvariety $L \subset \mathbb{P}(V) \times V$ consisting of those pairs (x, v) such that v lies on the line x; then the projection $\pi : \mathcal{O}_{\mathbb{P}(V)}(-1) \to \mathbb{P}(V)$ is the restriction of the first projection $p_1 : \mathbb{P}(V) \times V \to \mathbb{P}(V)$. Clearly, the diagonal G-action on $\mathbb{P}(V) \times V$ stabilizes the subvariety L, and π is equivariant. Moreover, G acts linearly on the fibers of L, i.e., every $g \in G$ induces a linear map $L_x \to L_{g \cdot x}$ for any $x \in \mathbb{P}(V)$; equivalently, the G-action on L commutes with the $\mathbb{G}_m$-action by multiplication on fibers.

Definition 1.4. We say that a G-variety X is *G-quasiprojective* if there exists a (locally closed) immersion $\iota : X \to \mathbb{P}(V)$, where V is a finite-dimensional G-module and ι is G-equivariant.

Definition 1.5. Let X be a G-variety, and $\pi : L \to X$ a line bundle. A *G-linearization* of L is an action of G on the variety L, such that π is equivariant and the action on fibers is linear.

Lemma 1.6. *Let X be a G-variety, and L, M G-linearized line bundles on X. Then the dual line bundle L^{-1} and the tensor product $L \otimes M$ are equipped with G-linearizations.*

Proof. We use the correspondence between line bundles and $\mathbb{G}_m$-torsors (or principal bundles) over X. Denote by $L^\times \subset L$ the complement of the zero section; this is an open subset of L, stable by the (commuting) actions of $\mathbb{G}_m$ and G. Moreover, the projection $\pi : L \to X$ restricts to a $\mathbb{G}_m$-torsor $L^\times \to X$, and L is the fiber bundle over X associated with this torsor and with the $\mathbb{G}_m$-module of dimension 1 and weight 1. As a consequence, the restriction to $L^\times$ yields an isomorphism $\mathrm{Aut}^{\mathbb{G}_m}(L) \cong \mathrm{Aut}^{\mathbb{G}_m}(L^\times)$. Likewise, L^{-1} is the fiber bundle associated with the $\mathbb{G}_m$-module of dimension 1 and weight -1. Thus, the G-action on $L^\times \cong (L^{-1})^\times$ extends to an action on L^{-1} which commutes with the $\mathbb{G}_m$-action. This yields the G-linearization of L^{-1}. To obtain that of $L \otimes M$, we argue similarly: with an obvious notation, $L \otimes M$ is the line bundle associated with the $\mathbb{G}_m$-torsor $(L^\times \times_X M^\times)/\mathbb{G}_m$, where the fiber product $L^\times \times_X M^\times$ is a $\mathbb{G}_m \times \mathbb{G}_m$-torsor over X, and $\mathbb{G}_m$ is embedded in $\mathbb{G}_m \times \mathbb{G}_m$ as the anti-diagonal, via $t \mapsto (t, t^{-1})$. Thus, $(L \otimes M)^\times$ is equipped with a G-action, which commutes with the $\mathbb{G}_m$-action and extends to an action on $L \otimes M$. □

Lemma 1.7. *Let X be a G-variety, and L a G-linearized line bundle on X. Then the space of global sections, $\Gamma(X, L)$, has a natural structure of G-module.*

Proof. By Lemma 1.6, the dual line bundle L^{-1} is equipped with a G-linearization; in particular, L^{-1} is a G-variety. Moreover, the space of global regular functions on L^{-1} which are homogeneous of degree 1 for the $\mathbb{G}_m$-action is identified with $\Gamma(X, L)$. So the statement follows from Lemma 1.1. □

Proposition 1.8. *Let X be a G-variety. Then X is G-quasiprojective if and only if it admits an ample G-linearized line bundle.*

Proof. Assume that X is G-quasiprojective and choose an immersion $\iota : X \to \mathbb{P}(V)$ as in Definition 1.4. Then $\iota^*\mathcal{O}_{\mathbb{P}(V)}(1)$ is an ample G-linearized line bundle on X.

Conversely, assume that X has such a bundle L. Replacing L with a positive power, we may assume that L is generated by its global sections. Then we may choose a finite-dimensional subspace $V \subset \Gamma(X, L)$ such that for any $x \in X$, there exists $s \in V$ which does not vanish at x. Hence we obtain a morphism $f_V : X \to \mathbb{P}(V^\vee)$ by assigning to each $x \in X$, the hyperplane of V consisting of sections vanishing at x. Replacing again L with a positive power, we may assume that L is very ample, i.e., f_V is a (locally closed) immersion. By Lemma 1.7, V is contained in some finite-dimensional G-submodule $W \subset \Gamma(X, L)$. Then the corresponding rational map $f_W : X \dashrightarrow \mathbb{P}(W^\vee)$ is defined everywhere, G-equivariant, and factors through f_V. Hence f_W is the desired locally closed immersion. □

Remark 1.9. Let G be an algebraic group of automorphisms of a variety X. If X is G-quasiprojective, then G is linear. Indeed, G is isomorphic to a closed subgroup of some projective linear group PGL_n and the latter group is linear (it may be identified with the automorphism group of the algebra of $n \times n$ matrices).

Example 1.10. Let V be a vector space of finite dimension n, and consider the projective space $X := \mathbb{P}(V) \cong \mathbb{P}^{n-1}$ equipped with the action of its full automorphism group $G := \mathrm{PGL}(V) \cong \mathrm{PGL}_n$. Then one can show that the line bundle $L := \mathcal{O}_{\mathbb{P}(V)}(1)$ is not G-linearizable (see Example 2.12). But L is linearizable relative to the action of $\mathrm{GL}(V)$, and hence to $\mathrm{SL}(V)$. The natural homomorphism $\mathrm{SL}(V) \to \mathrm{PGL}(V)$ is the quotient by the center Z of $\mathrm{SL}(V)$; moreover, Z is isomorphic to the group scheme μ_n of nth roots of unity, acting on L by scalar multiplication. Thus, Z acts trivially on $L^{\otimes n}$, and hence this line bundle is G-linearizable. The corresponding immersion is just the nth Segre embedding $\mathbb{P}(V) \to \mathbb{P}(\mathrm{Sym}^n V)$.

Example 1.11. Let X be the nodal curve obtained from the projective line $\mathbb{P}^1$ by identifying the points 0 and ∞ to the node x. The action of $G := \mathbb{G}_m$ on $\mathbb{P}^1$ by multiplication fixes 0 and ∞, and hence yields an action on X for which x is the unique fixed point. The curve X is projective (it may be realized as a cubic curve in $\mathbb{P}^2$) but not G-projective: otherwise, X is isomorphic to the orbit closure $\overline{G \cdot x} \subset \mathbb{P}(V)$ for some finite-dimensional G-module V. Let $V = \oplus_{n \in \mathbb{Z}} V_n$ be the decomposition of V into weight spaces, where $t \cdot v = t^n v$ for all $t \in \mathbb{G}_m$ and $v \in V_n$. Write accordingly $x = [v_{n_0} + \cdots + v_{n_1}]$, where $n_0 < n_1$ and $v_{n_0} \neq 0 \neq v_{n_1}$. Then $[v_{n_0}]$ and $[v_{n_1}]$ are distinct G-fixed points in $\overline{G \cdot x}$, a contradiction. We will show in Example 2.13 that X admits no G-linearizable line bundle of nonzero degree.

Example 1.12. Let X be the cuspidal curve obtained from $\mathbb{P}^1$ by sending $\mathrm{Spec}(\mathcal{O}_{\mathbb{P}^1,\infty}/\mathfrak{m}^2)$ (a fat point of order 2) to the cusp x. The action of $G := \mathbb{G}_a$ on $\mathbb{P}^1$ by translation fixes ∞, and hence yields an action on X for which x is the unique fixed point. Also, X is projective, since it may again be realized as a cubic curve in $\mathbb{P}^2$.

If $\mathrm{char}(k) = 0$, then one can show that X is not G-projective, see Example 2.14. But if $\mathrm{char}(k) = p > 0$, then X is G-projective: indeed, when $p \geq 3$, the morphism

$$f : \mathbb{P}^1 \longrightarrow \mathbb{P}^{p-1}, \quad [x : y] \longmapsto [x^p : x^{p-2}y^2 : x^{p-3}y^3 : \cdots : y^p]$$

factors through an immersion $F : X \to \mathbb{P}^{p-1}$. Moreover, f is equivariant for the G-action on $\mathbb{P}^1$ given by $t \cdot (x, y) = (x + ty, y)$, and for the induced action on $\mathbb{P}^{p-1} \subset \mathbb{P}(k[x, y]_p)$, where $k[x, y]_p$ denotes the space of homogeneous polynomials of degree p in x, y; the hyperplane of that space spanned by $x^p, x^{p-2}y^2, x^{p-3}y^3, \cdots, y^p$ is stable under this action. When $p = 2$, the above morphism f has degree 2; we replace it with the birational morphism

$$g : \mathbb{P}^1 \longrightarrow \mathbb{P}^3, \quad [x : y] \longmapsto [x^4 : x^2y^2 : xy^3 : y^4],$$

and argue similarly.

1.3 Invariant line bundles and lifting groups

Let X be a variety, and $\pi : L \to X$ a line bundle. We denote by $\mathrm{Aut}^{\mathbb{G}_m}(L)$ the group of automorphisms of L which commute with the action of $\mathbb{G}_m$ by multiplication on fibers.

Every such automorphism γ preserves the zero section of L, and hence defines an automorphism g of X.

Lemma 1.13. (i) *With the above notation, the diagram*

$$\begin{array}{ccc} L & \xrightarrow{\gamma} & L \\ {\scriptstyle\pi}\big\downarrow & & {\scriptstyle\pi}\big\downarrow \\ X & \xrightarrow{g} & X \end{array}$$

commutes; in particular, g sits in the subgroup $\mathrm{Aut}(X)_L \subset \mathrm{Aut}(X)$ *that stabilizes L.*
(ii) *The assignement*

$$\pi_* : \mathrm{Aut}^{\mathbb{G}_m}(L) \longrightarrow \mathrm{Aut}(X)_L, \quad \gamma \longmapsto g$$

yields an exact sequence of groups

$$1 \longrightarrow \mathcal{O}(X)^* \longrightarrow \mathrm{Aut}^{\mathbb{G}_m}(L) \longrightarrow \mathrm{Aut}(X)_L \longrightarrow 1. \tag{1}$$

(iii) *For any $\gamma \in$* $\mathrm{Aut}^{\mathbb{G}_m}(L)$ *and $f \in \mathcal{O}(X)^*$, we have $\gamma f \gamma^{-1} = \pi_*(\gamma) \cdot f$, where the conjugation in the left-hand side takes place in* $\mathrm{Aut}^{\mathbb{G}_m}(L)$*, and the action in the right-hand side arises from the natural action of* $\mathrm{Aut}(X)$ *on $\mathcal{O}(X)$.*

Proof. (i) Since the fibers of π are exactly the $\mathbb{G}_m$-orbit closures, γ sends the fiber L_x to $L_{g(x)}$ for any $x \in X$. This yields the assertion.

(ii) Clearly, π_* is a group homomorphism. To show that it is surjective, let $g \in \mathrm{Aut}(X)_L$. Then there exists an isomorphism $\varphi : L \to g^*(L)$. Moreover, $g^*(L)$ sits in a cartesian square

$$\begin{array}{ccc} g^*(L) & \xrightarrow{\psi} & L \\ \big\downarrow & & {\scriptstyle\pi}\big\downarrow \\ X & \xrightarrow{g} & X \end{array}$$

for some $\mathbb{G}_m$-equivariant morphism of varieties ψ. Thus, $\gamma := \psi \circ \varphi$ is a $\mathbb{G}_m$-equivariant automorphism of L that lifts g.

Also, the kernel of π_* consists of the automorphisms of L viewed as a line bundle, i.e., of the multiplications by regular invertible functions on X.

(iii) Since γ is linear on fibers, we have $\gamma(f(z)) = f(\pi(z))\,\gamma(z)$ for any $z \in L$. Thus, $\gamma f \gamma^{-1}(z) = f(\pi(\gamma^{-1}(z)) = f(\pi_*(\gamma)^{-1})\,\pi(z)$. This yields the statement. □

Next, let G be an algebraic group acting on X. We say that L is (pointwise) *G-invariant* if $g^*(L) \cong L$ for all $g \in G$. Equivalently, the image of the homomorphism $G \to \mathrm{Aut}(X)$ is contained in $\mathrm{Aut}(X)_L$. We may then take the pull-back of the exact sequence (1) by the resulting homomorphism $G \to \mathrm{Aut}(X)_L$; this yields an exact sequence of abstract groups

$$1 \longrightarrow \mathcal{O}(X)^* \longrightarrow \mathcal{G}(L) \longrightarrow G \longrightarrow 1, \tag{2}$$

for some subgroup $\mathcal{G}(L) \subset \mathrm{Aut}^{\mathbb{G}_m}(X)$, which may be called the *lifting group* associated with L. When G is an abelian variety acting on itself by translation, $\mathcal{G}(L)$ is the theta group considered by Mumford (see [Mu70, Chap. III, §23]).

Note that L is G-linearizable if and only if it satisfies the following two conditions : L is G-invariant, and the extension (2) admits a splitting which defines an algebraic action of G on L.

Also, note that the extension (2) is generally not central: by Lemma 1.13 (iii), the action of $\mathcal{G}(L)$ on $\mathcal{O}(X)^*$ by conjugation factors through the action of G via its natural action on $\mathcal{O}(X)$. Also, $\mathcal{O}(X)^*$ is not necessarily (the group of k-rational points of) an algebraic group: for example, if $X = \mathbb{G}_m^n$, then $\mathcal{O}(X) = k[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]$ and hence

$$\mathcal{O}(X)^* = \{ct_1^{a_1} \cdots t_n^{a_n} \mid c \in k^*, (a_1, \dots, a_n) \in \mathbb{Z}^n\} \cong k^* \times \mathbb{Z}^n.$$

The structure of the unit group $\mathcal{O}(X)^*$, where X is an irreducible variety, will be analyzed in Subsection 2.1.

Lemma 1.14. *With the above notation and assumptions, denote by* $\mathrm{Pic}(X)^G$ *the subgroup of* $\mathrm{Pic}(X)$ *consisting of classes of G-invariant line bundles, and by* $\mathrm{Ext}(G, \mathcal{O}(X)^*)$ *the group of isomorphism classes of extensions (of abstract groups). Then assigning with L the extension (2) yields a homomorphism* $\epsilon : \mathrm{Pic}(X)^G \to \mathrm{Ext}(G, \mathcal{O}(X)^*)$.

Proof. Let L, M be G-invariant line bundles on X. With the notation of the proof of Lemma 1.6, there is a natural homomorphism

$$\mathrm{Aut}^{\mathbb{G}_m}(L^\times) \times_{\mathrm{Aut}(X)} \mathrm{Aut}^{\mathbb{G}_m}(M^\times) \longrightarrow \mathrm{Aut}^{\mathbb{G}_m \times \mathbb{G}_m}(L^\times \times_X M^\times)$$

of groups over $\mathrm{Aut}(X)$. Also, recall that $\mathrm{Aut}^{\mathbb{G}_m}(L) \cong \mathrm{Aut}^{\mathbb{G}_m}(L^\times)$, and $(L \otimes M)^\times$ is the quotient of $L^\times \times_X M^\times$ by the anti-diagonal in $\mathbb{G}_m \times \mathbb{G}_m$. Thus, we obtain a homomorphism

$$\mathrm{Aut}^{\mathbb{G}_m}(L) \times_{\mathrm{Aut}(X)} \mathrm{Aut}^{\mathbb{G}_m}(M) \longrightarrow \mathrm{Aut}^{\mathbb{G}_m}(L \otimes M)^\times) \cong \mathrm{Aut}^{\mathbb{G}_m}(L \otimes M)$$

of groups over $\mathrm{Aut}(X)$. By the definition of the lifting groups, this yields a homomorphism $\mathcal{G}(L) \times_G \mathcal{G}(M) \to \mathcal{G}(L \otimes M)$ of groups over G, which restricts to the product map $\mathcal{O}(X)^* \times \mathcal{O}(X)^* \to \mathcal{O}(X)^*$. Since the kernel of the latter map is the anti-diagonal, we obtain a homomorphism from the extension $1 \to \mathcal{O}(X)^* \to \mathcal{G}(L) \oplus \mathcal{G}(M) \to G \to 1$ (where $\oplus$ denotes the sum of extensions) to the extension $1 \to \mathcal{O}(X)^* \to \mathcal{G}(L \otimes M) \to G \to 1$. As every homomorphism of extensions is an isomorphism, this yields the desired statement. $\square$

We now consider actions of finite groups; we may then use standard results about extensions and cohomology.

Proposition 1.15. *Let G be a finite group acting on a variety X, and L a line bundle on X. If L is G-invariant, then some positive power $L^{\otimes n}$ admits a G-linearization; we may take for n the order of G.*

Proof. Recall that $\mathrm{Ext}(G, M) \cong H^2(G, M)$ for any $\mathbb{Z}G$-module M (see e.g. [Br94, Thm. 3.12]); also, the cohomology groups $H^i(G, M)$, where $i \geq 1$, are killed by the order n of G (see [Br94, Cor. 10.2]). Thus, $\epsilon(L^{\otimes n}) = 0$ with the notation of Lemma 1.14. Since the kernel of ϵ consists of the classes of G-linearizable line bundles, this yields the statement. $\square$

Remark 1.16. The assumption that L is G-invariant cannot be omitted in the above result. Consider indeed $X := \mathbb{P}^1 \times \mathbb{P}^1$ equipped with the involution $(x, y) \mapsto (y, x)$. Let $L := \mathcal{O}_{\mathbb{P}^1}(m) \boxtimes \mathcal{O}_{\mathbb{P}^1}(n)$, where $m \neq n$. Then no power of L is invariant, and hence linearizable.

Corollary 1.17. *Let X be a quasiprojective variety equipped with the action of a finite group G. Then X is G-quasiprojective. Moreover, X is covered by G-stable affine open subsets.*

Proof. By assumption, X admits an ample line bundle L. Then the line bundle $M := \bigotimes_{g\in G} g^*(L)$ is also ample, and clearly G-invariant. In view of Proposition 1.15, there exists a positive integer n such that the ample line bundle $M^{\otimes n}$ is G-linearizable. So X is G-quasiprojective in view of Proposition 1.8.

To show the second assertion, we may assume that X is a G-stable locally closed subvariety of $\mathbb{P}(V)$, where V is a finite-dimensional G-module. Then $\overline{X} \setminus X$ is a closed G-stable subvariety of $\mathbb{P}(V)$, and hence corresponds to a closed G-stable subvariety $Y \subset V$, also stable by the $\mathbb{G}_m$-action via scalar multiplication. It suffices to show that for any $x = [v] \in X$, there exists $f \in \mathcal{O}(V)^G$ homogeneous such that $f(v) \neq 0$ and f vanishes identically on Y: then the open subset of $\overline{X}$ on which $f \neq 0$ is affine, G-stable, contains x, and is contained in X. Since the orbit Gv is a finite set and does not meet the cone Y, we may find $F \in \mathcal{O}(V)$ homogeneous such that $F|_Y = 0$ and $F(g \cdot v) \neq 0$ for all $g \in G$. Then $f := \prod_{g\in G} g \cdot F$ satisfies the desired properties. □

Remark 1.18. More generally, the following conditions are equivalent for a variety X equipped with an action of a finite group G:

(i) X is covered by G-stable open affine subsets.

(ii) Every G-orbit is contained in some open affine subset of X.

(iii) There exists a quotient morphism $\pi : X \to X/G$, where X/G is a variety.

(This follows e.g. from [Mu70, Chap. II, §7]). If X is quasiprojective, then (ii) is satisfied since every finite set of points is contained in some open affine subset.

Finally, we obtain an existence result for linearizations in the setting of complete varieties:

Proposition 1.19. *Let G be a linear algebraic group, X a complete G-variety, and L a G-invariant line bundle on X. Then there exists a positive integer n such that $L^{\otimes n}$ is G-linearizable.*

Proof. Note that $\mathcal{O}(X)^* = k^*$, since every regular function on X is constant. Also, by [Ra64, Cor. 2], the group $\mathrm{Aut}^{\mathbb{G}_m}(L)$ has a structure of algebraic group (possibly with infinitely many components) of automorphisms of L. Since G has finitely many connected components, it follows that $\mathcal{G}(L)$ is an algebraic group acting algebraically on L; moreover, (2) yields a central extension of algebraic groups

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \mathcal{G}(L) \longrightarrow G \longrightarrow 1. \tag{3}$$

It follows that the variety $\mathcal{G}(L)$ is affine (as the total space of a $\mathbb{G}_m$-torsor over an affine variety), and hence the algebraic group $\mathcal{G}(L)$ is linear. As another consequence, we obtain a central extension of neutral components

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \mathcal{G}(L)^0 \longrightarrow G^0 \longrightarrow 1, \tag{4}$$

and hence of connected reductive groups

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \mathcal{G}(L)^0/R_u(\mathcal{G}(L)) \longrightarrow G^0/R_u(G) \longrightarrow 1,$$

where $R_u(H)$ denotes the unipotent radical of an algebraic group H. Also, we have an isomorphism of groups of components

$$\mathcal{G}(L)/\mathcal{G}(L)^0 \xrightarrow{\cong} G/G^0.$$

In view of the structure of connected reductive groups (see e.g. [Bo91]), there exists a homomorphism of algebraic groups $\chi : \mathcal{G}(L)^0/R_u(G) \to \mathbb{G}_m$ which restricts nontrivially to the central $\mathbb{G}_m$; thus, there exists a nonzero integer n such that $\chi(t) = t^n$ for all $t \in \mathbb{G}_m$. We may view χ as a homomorphism $\mathcal{G}(L)^0 \to \mathbb{G}_m$. Replacing χ with the product of its conjugates under the group of components, we may assume that χ is invariant under that group. The extension (4) is trivialized by the push-out via the homomorphism $\mathbb{G}_m \to \mathbb{G}_m$, $t \mapsto t^n$, i.e., the class of this extension is n-torsion. Replacing L with L^n and using Lemma 1.14, we may thus assume that (4) is split compatibly with the action of G/G^0. Fix such a splitting, $\mathcal{G}(L)^0 \cong \mathbb{G}_m \times G^0$. Then G^0 is identified with a closed normal subgroup of $\mathcal{G}(L)$, and hence we get a central extension

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \mathcal{G}(L)/G^0 \longrightarrow G/G^0 \longrightarrow 1.$$

Replacing L with a positive power again, we may also assume that this extension splits. This yields a closed normal subgroup $H \subset \mathcal{G}(L)$ containing G^0, which is sent isomorphically to G/G^0. Thus, H is sent isomorphically to G, and hence the extension (3) is split. $\square$

This result is a version of [MFK94, Prop. 1.5]; we will show in Theorem 3.5 that it also holds for all line bundles on normal G-varieties (not necessarily complete), when G is connected.

2 Linearization of line bundles

2.1 Unit groups of irreducible varieties

Lemma 2.1. *Let X be an irreducible variety. Then the quotient group $\mathcal{O}(X)^*/k^*$ is free of finite rank, where k^* is embedded in $\mathcal{O}(X)^*$ as the subgroup of nonzero constant functions.*

Proof. Since $\mathcal{O}(X)^* \subset \mathcal{O}(U)^*$ for any nonempty open subset $U \subset X$, we may assume that X is affine. Next, since the normalization map $\tilde{X} \to X$ identifies $\mathcal{O}(X)^*/k^*$ with a subgroup of $\mathcal{O}(\tilde{X})^*/k^*$, we may assume in addition that X is normal. Choose a locally closed immersion $X \subset \mathbb{P}(V)$ for some finite-dimensional vector space V, and denote by $\bar{X}$ the normalization of the closure of X in $\mathbb{P}(V)$. Then $\bar{X}$ is a normal complete irreducible variety containing X as an open subset. Denote by $D_1, \ldots, D_r$ the irreducible components of $X \backslash U$ of codimension 1 in X (actually, X is projective and $X \backslash U$ has pure codimension 1; we will not need these facts). Then we may view $\mathcal{O}(X)$ as the algebra of rational functions on $\bar{X}$ having poles along $D_1, \ldots, D_r$ only; this identifies $\mathcal{O}(X)^*$ with the multiplicative group of rational functions on X having zeroes and poles along $D_1, \ldots, D_r$ only. Let $v_1, \ldots, v_r$ be the valuations of the function field $k(\bar{X}) = k(X)$ associated with $D_1, \ldots, D_r$, so that $v_i(f)$ is the order of the zero or pole of f along D_i for any $f \in k(\bar{X})$. Then the map

$$\mathcal{O}(X)^* \longrightarrow \mathbb{Z}^r, \quad f \longmapsto (v_1(f), \ldots, v_r(f))$$

is a group homomorphism with kernel k^*, since every rational function on $\bar{X}$ having no zero or pole is constant. Thus, $\mathcal{O}(X)^*/k^*$ is isomorphic to a subgroup of $\mathbb{Z}^r$. □

Lemma 2.2. *Let X, Y be irreducible varieties. Then the product map $\mathcal{O}(X)^* \times \mathcal{O}(Y)^* \to \mathcal{O}(X \times Y)^*$ is surjective.*

Proof. We adapt the argument of [FI73, Lem. 2.1]. It suffices to show that every $f \in \mathcal{O}(X \times Y)^*$ can be written as $(x, y) \mapsto g(x)h(y)$ for some $g \in k(X)$ and $h \in k(Y)$. Indeed, the function $f_y : x \mapsto f(x, y)$ is regular and invertible on X for any $y \in Y$. Taking y such that h is defined at y, we see that $g \in \mathcal{O}(X)^*$; likewise, $h \in \mathcal{O}(Y)^*$.

Therefore, we may replace X and Y with any non-empty open subsets, and hence assume that they are both affine and normal. Choose a normal completion $\bar{X}$ of X as in the proof of Lemma 2.1, and denote by $D_1, \ldots, D_r$ the irreducible components of codimension 1 of $\bar{X} \setminus X$. Let $f \in \mathcal{O}(X \times Y)^*$ and view f as a rational function on $\bar{X} \times Y$. Then the divisor $\mathrm{div}(f)$ is supported in $(\bar{X} \setminus X) \times Y$, and hence we have $\mathrm{div}(f) = \sum_{i=1}^r n_j D_j \times Y$ for some integers $n_1, \ldots, n_r$. Then $\mathrm{div}(f_y) = \sum_{i=1}^r n_j D_j$ for all y in a nonempty open subset $V \subset Y$. Choose $y_0 \in V$; then $\mathrm{div}(f_y f_{y_0}^{-1}) = 0$ for all $y \in V$. Since $\bar{X}$ is complete and normal, it follows that $f_y f_{y_0}^{-1}$ is constant (as a function of x). Thus, $f(x, y) = f(x, y_0)h(y)$ for some function h on Y; then $h \in k(Y)$. □

Remark 2.3. For any scheme X, denote by $\mathrm{U}(X)$ the quotient group $\mathcal{O}(X)^*/k^*$. By definition, we have an exact sequence

$$0 \longrightarrow k^* \longrightarrow \mathcal{O}(X)^* \longrightarrow \mathrm{U}(X) \longrightarrow 0.$$

Moreover, any k-rational point $x \in X$ defines a splitting of that sequence, since the subgroup of $\mathcal{O}(X)^*$ consisting of functions with value 1 at x is sent isomorphically to $\mathrm{U}(X)$.

Lemma 2.1 asserts that the abelian group $\mathrm{U}(X)$ is free of finite rank for any irreducible variety X. Moreover, $\mathrm{U}(X \times Y) \cong \mathrm{U}(X) \times \mathrm{U}(Y)$ for any irreducible varieties X, Y, as follows from Lemma 2.2. But this isomorphism does not hold for arbitrary schemes X, Y, nor is the group $\mathrm{U}(X)$ finitely generated.

For example, if X is any scheme and $Y = \mathrm{Spec}(k[\varepsilon])$, where $\varepsilon^2 = 0$, then $\mathcal{O}(X \times Y) \cong \mathcal{O}(X) \oplus \varepsilon\mathcal{O}(X)$, so that

$$\mathcal{O}(X \times Y)^* = \mathcal{O}(X)^*(1 + \varepsilon\mathcal{O}(X)) \cong \mathcal{O}(X)^* \times \mathcal{O}(X).$$

In particular, $\mathcal{O}(Y)^* \cong k^* \times k$, and $\mathrm{U}(Y) \cong k$ while $\mathrm{U}(X \times Y) \cong \mathrm{U}(X) \times \mathcal{O}(X)$.

This example also shows that the unit group functor,

$$S \longmapsto \mathcal{O}(X \times S)^*/p_2^*\mathcal{O}(S)^* = \mathrm{U}(X \times S)/p_2^*\mathrm{U}(S),$$

is represented by no scheme locally of finite type, when $\mathcal{O}(X)$ is infinite-dimensional as a vector space.

We now apply Lemmas 2.1 and 2.2 to the (multiplicative) *characters* of an algebraic group G, that is, to the homomorphisms of algebraic groups $\chi : G \to \mathbb{G}_m$. These characters form a multiplicative subgroup of $\mathcal{O}(G)^*$ that we denote by $\widehat{G}$.

Lemma 2.4. *Let G be a connected algebraic group, and $f \in \mathcal{O}(G)^*$ such that $f(e_G) = 1$. Then $f \in \widehat{G}$.*

Proof. The function $(g, h) \mapsto f(gh)$ lies in $\mathcal{O}(G \times G)^*$. By Lemma 2.2 (which may be applied, since G is an irreducible variety), it follows that there exist $\varphi, \psi \in \mathcal{O}(G)^*$ such that $f(gh) = \varphi(g)\psi(h)$ for all $g, h \in G$; replacing φ with a scalar multiple, we may assume that $\varphi(e_G) = 1$. Taking $g = e_G$, we obtain that $\psi = f$; then taking $h = e_G$ yields that $\varphi = f$. Thus, $f(gh) = f(g)f(h)$. □

It follows that $\widehat{G} \cong \mathrm{U}(G)$, and hence is free of finite rank in view of Lemma 2.1; moreover, the product map $k^* \times \widehat{G} \to \mathcal{O}(G)^*$ is an isomorphism. We will need the following generalization of these facts:

Lemma 2.5. *Let G be a connected algebraic group, and X an irreducible variety. Then the product map $\widehat{G} \times \mathcal{O}(X)^* \to \mathcal{O}(G \times X)^*$ is an isomorphism.*

If X is equipped with a G-action, then for any $f \in \mathcal{O}(X)^*$, there exists $\chi = \chi(f) \in \widehat{G}$ such that $f(g \cdot x) = \chi(g)f(x)$ for all $g \in G$ and $x \in X$. Moreover, the assignement $f \mapsto \chi(f)$ defines an exact sequence

$$0 \longrightarrow \mathcal{O}(X)^{*G} \longrightarrow \mathcal{O}(X)^* \xrightarrow{\chi} \widehat{G}, \tag{5}$$

where $\mathcal{O}(X)^{*G}$ denotes the subgroup of G-invariants in $\mathcal{O}(X)^*$.

Proof. The first assertion is a consequence of Lemmas 2.2 and 2.4. The second assertion follows similarly by considering the map $(g, x) \mapsto f(g \cdot x)$. □

2.2 Linearization of the trivial bundle

Throughout this subsection, G denotes a connected algebraic group.

Lemma 2.6. *Let X be an irreducible G-variety. Then every G-linearization of the trivial line bundle $p_1 : X \times \mathbb{A}^1 \to X$ is of the form $g \cdot (x, z) = (g \cdot x, \chi(g)z)$ for a unique $\chi \in \widehat{G}$.*

Proof. Let $\beta : X \times \mathbb{A}^1 \to X \times \mathbb{A}^1$ be a G-linearization. Since β lifts the G-action on X and commutes with the $\mathbb{G}_m$-action by multiplication on $\mathbb{A}^1$, we must have

$$\beta(g, x, z) = (g \cdot x, f(g, x)z)$$

for some $f \in \mathcal{O}(G \times X)^*$; moreover, $f(e_G, x) = 1$ for all $x \in X$, since $\beta(e_G, x, z) = z$ for all $z \in \mathbb{A}^1$. Using Lemma 2.5, it follows that $f(g, x) = \chi(g)$ for some character χ of G. □

Given $\chi \in \widehat{G}$, we denote by $\mathcal{O}_X(\chi)$ the trivial line bundle on X equipped with the G-linearization as in Lemma 2.6. We have isomorphisms of G-linearized line bundles $\mathcal{O}_X(-\chi) \cong \mathcal{O}_X(\chi)^*$ and $\mathcal{O}_X(\chi) \otimes \mathcal{O}_X(\delta) \cong \mathcal{O}_X(\chi + \delta)$ for all $\chi, \delta \in \widehat{G}$.

Lemma 2.7. *Let $\chi, \delta \in \widehat{G}$. Then the G-linearized line bundles $\mathcal{O}_X(\chi), \mathcal{O}_X(\delta)$ are isomorphic if and only if $\delta - \chi = \chi(f)$ for some $f \in \mathcal{O}(X)^*$.*

Proof. Let $F : \mathcal{O}_X(\chi) \to \mathcal{O}_X(\delta)$ be an isomorphism of G-linearized line bundles. Then F is in particular an automorphism of the trivial bundle, i.e., the multiplication by some $f \in \mathcal{O}(X)^*$. Recall from Lemma 2.5 that we then have $f(g \cdot x) = \chi(g)f(x)$ for all $g \in G$ and $x \in X$. Thus, $\chi - \delta = \chi(f)$. The converse follows by reversing this argument. $\square$

2.3 A criterion for linearizability

Lemma 2.8. *Let G be an algebraic group, X a G-variety, and $\pi : L \to X$ a line bundle. Denote by $\alpha : G \times X \to X$ the action, and by $p_2 : G \times X \to X$ the projection. Then there is a bijective correspondence between the G-linearizations of L and those isomorphisms $\Phi : \alpha^*(L) \to p_2^*(L)$ of line bundles on $G \times X$ such that the restrictions $\Phi_g : g^*(L) \to L$ satisfy the cocycle condition $\Phi_{gh} = \Phi_h \circ h^*(\Phi_g)$ for all $g, h \in G$.*

Proof. Let $\beta : G \times L \to L$ be a G-linearization. Then the diagram

$$\begin{array}{ccc} G \times L & \xrightarrow{\beta} & L \\ {\scriptstyle \mathrm{id}_G \times \pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ G \times X & \xrightarrow{\alpha} & X \end{array}$$

is commutative, and hence induces a morphism $\gamma : G \times L \to \alpha^*(L)$ of varieties over $G \times X$ (recall that the pull-back line bundle $\alpha^*(L)$ is the fiber product $(G \times X) \times_X L$). Since β is linear on fibers, so is γ; hence we may view γ as a morphism of line bundles $p_2^*(L) \to \alpha^*(L)$. In fact, γ is an isomorphism, since so are the induced morphisms $\gamma_g : L \to g^*(L)$ for all $g \in G$. Moreover, the associativity property of the action β on $G \times L$ translates into the condition that $\gamma_{gh} = h^*(\gamma_g) \circ \gamma_h$ for all $g, h \in G$. Thus, $\Phi := \gamma^{-1}$ is an isomorphism satisfying the desired cocycle condition. Conversely, any such isomorphism yields a linearization by reversing the above arguments. $\square$

Lemma 2.9. *Let G be a connected algebraic group, X an irreducible G-variety, and $\pi : L \to X$ a line bundle on X. Then L admits a G-linearization if and only if the line bundles $\alpha^*(L)$ and $p_2^*(L)$ on $G \times X$ are isomorphic.*

Proof. If L admits a linearization, then $\alpha^*(L) \cong p_2^*(L)$ by Lemma 2.8. For the converse, let $\Phi : \alpha^*(L) \to p_2^*(L)$ be an isomorphism. Since $\alpha(e_G, x) = p_2(e_G, x) = x$ for all $g \in G$ and $x \in X$, the pull-back of Φ to $\{e_G\} \times X$ is identified with an automorphism of the line bundle L, i.e., with the multiplication by some $f \in \mathcal{O}(X)^*$. Replacing Φ with $\Phi \circ p_2^*(f^{-1})$, we may assume that $f = 1$. Then, as in the proof of Lemma 2.8, Φ corresponds to a morphism $\beta : G \times L \to L$ such that the diagram

$$\begin{array}{ccc} G \times L & \xrightarrow{\beta} & L \\ {\scriptstyle \mathrm{id}_G \times \pi}\big\downarrow & & {\scriptstyle \pi}\big\downarrow \\ G \times X & \xrightarrow{\alpha} & X \end{array}$$

commutes; moreover, $\beta(e_G, z) = z$ for all $z \in L$. It remains to show that β satisfies the associativity condition of a group action. But the obstruction to associativity is an automorphism of the line bundle $\mathrm{id}_{G\times G} \times \pi : G\times G\times L \to G\times G\times X$, i.e., the multiplication by some $\varphi \in \mathcal{O}(G \times G \times X)^*$. Moreover, since $\beta(g, \beta(e_G, z) = \beta(g, z) = \beta(e_G, \beta(g, z))$ for all $g \in G$ and $z \in L$, we have $\varphi(g, e_G, x) = 1 = \varphi(e_G, g, x)$ for all $g \in G$ and $x \in X$.

We now show that $\varphi = 1$. By Lemma 2.5, there exists $\chi \in \widehat{G \times G}$ and $\psi \in \mathcal{O}(X)^*$ such that $\varphi(g, h, x) = \chi(g, h)\psi(x)$ for all $g, h \in G$ and $x \in X$. Then $\chi(g, e_G)\psi(x) = \chi(e_G, g)\psi(x) = 1$ for all such g and x. Since $\widehat{G \times G} = \widehat{G} \times \widehat{G}$, it follows that $\chi = \psi = 1$. Thus, $\varphi(g, h, x) = 1$ as desired. $\square$

2.4 The equivariant Picard group

Let G be an algebraic group, and X a G-variety. By Lemma 1.6, the isomorphism classes of G-linearized line bundles on X form an abelian group relative to the tensor product. We call that group the *equivariant Picard group*, and denote it by $\mathrm{Pic}_G(X)$. It comes with a natural homomorphism

$$\varphi : \mathrm{Pic}_G(X) \longrightarrow \mathrm{Pic}(X)$$

which forgets the linearization. Also, recall the homomorphisms

$$\gamma : \widehat{G} \longrightarrow \mathrm{Pic}_G(X), \quad \chi \longmapsto \mathcal{O}_X(\chi)$$

(defined in Subsection 2.2), and

$$\chi : \mathcal{O}(X)^* \longrightarrow \widehat{G}, \quad f \longmapsto \chi(f)$$

(defined in Subsection 2.1). We may now state one of the main results of these notes:

Theorem 2.10. *Let G be a connected algebraic group, and X an irreducible G-variety. Then there is an exact sequence*

$$0 \to \mathcal{O}(X)^{*G} \to \mathcal{O}(X)^* \xrightarrow{\chi} \widehat{G} \xrightarrow{\gamma} \mathrm{Pic}_G(X) \xrightarrow{\varphi} \mathrm{Pic}(X) \xrightarrow{\alpha^* - p_2^*} \mathrm{Pic}(G \times X). \tag{6}$$

Proof. In view of Lemma 2.5, it suffices to show that the sequence (6) is exact at $\widehat{G}$, $\mathrm{Pic}_G(X)$ and $\mathrm{Pic}(X)$. The exactness at $\widehat{G}$ follows from Lemma 2.7. Since the kernel of φ consists of the isomorphism classes of G-linearized line bundles which are trivial as line bundles, the exactness at $\mathrm{Pic}_G(X)$ follows from Lemmas 2.6 and 2.7. Finally, the exactness at $\mathrm{Pic}(X)$ is equivalent to Lemma 2.9. $\square$

Proposition 2.11. *With the notation and assumptions of Theorem 2.10, we may replace the map $\alpha^* - p_2^* : \mathrm{Pic}(X) \to \mathrm{Pic}(G \times X)$ with the map*

$$\psi : \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(G \times X)/p_2^*\mathrm{Pic}(X), \quad L \longmapsto \alpha^*(L) \pmod{p_2^*\mathrm{Pic}(X)}$$

in the exact sequence (6).

Proof. Consider the morphism $e_G \times \mathrm{id}_X : X \to G \times X, \quad x \mapsto (e_G, x)$. As already noted in the proof of Lemma 2.9, we have

$$\alpha \circ (e_G \times \mathrm{id}_X) = p_2 \circ (e_G \times \mathrm{id}_X) = \mathrm{id}_X.$$

Thus, $(e_G \times \mathrm{id}_X)^* \circ (\alpha^* - p_2^*) = 0$ on $\mathrm{Pic}(G \times X)$. Also, since $e_G \times \mathrm{id}_X$ is a section of p_2, we see that $p_2^* : \mathrm{Pic}(X) \to \mathrm{Pic}(G \times X)$ is a section of $(e_G \times \mathrm{id}_X)^*$; as a consequence, the kernel of $(e_G \times \mathrm{id}_X)^* : \mathrm{Pic}(G \times X) \to \mathrm{Pic}(X)$ is sent isomorphically to $\mathrm{Pic}(G \times X)/p_2^*\mathrm{Pic}(X)$ by the quotient map $\mathrm{Pic}(G \times X) \to \mathrm{Pic}(G \times X)/p_2^*\mathrm{Pic}(X)$. Combining these observations yields the statement. $\square$

Example 2.12. Continuing with Example 1.10, we show that $L = \mathcal{O}_{\mathbb{P}(V)}(1)$ admits no linearization relative to $G = \mathrm{PGL}(V)$. Indeed, L admits an $\mathrm{SL}(V)$-linearization, which is unique as the character group is trivial. If L is G-linearized, then G acts on $\Gamma(\mathbb{P}(V), L) = V^\vee$ by lifting the natural action of $\mathrm{SL}(V)$. This yields a section of the quotient homomorphism $\mathrm{SL}(V) \to \mathrm{PGL}(V)$, a contradiction.

Example 2.13. Continuing with Example 1.11, we show that no line bundle of nonzero degree on X is G-linearizable. Consider indeed the normalization $\eta : \mathbb{P}^1 \to X$, so that $\eta^{-1}(x) = \{0, \infty\}$ (as schemes), and η restricts to an isomorphism $\mathbb{P}^1 \setminus \{0, \infty\} \to X \setminus \{x\}$. For any line bundle L on X, the pull-back $\eta^*(L)$ is equipped with isomorphisms of fibers

$$\eta^{-1}(L)_0 \cong \eta^{-1}(L)_\infty \cong L_x.$$

If L is G-linearized, then G acts on these fibers and the above isomorphisms are G-equivariant. Thus, the G-actions on the lines $\eta^{-1}(L)_0$ and $\eta^{-1}(L)_\infty$ have the same weight. On the other hand, the G-linearized line bundle $\eta^{-1}(L)$ on $\mathbb{P}^1$ is isomorphic to some $\mathcal{O}_{\mathbb{P}^1}(n)$ equipped with its natural linearization twisted by some character (or weight) m, as follows from Theorem 2.10 (then n is of course the degree of L). Thus, $\eta^{-1}(L)_0$ has weight $n + m$, and $\eta^{-1}(L)_\infty$ has weight m. So we conclude that $n = 0$ if L is linearizable.

Example 2.14. Continuing with Example 1.12, we show again that no line bundle of nonzero degree is G-linearizable, if $\mathrm{char}(k) = 0$; it follows that X is not G-projective. We adapt the argument of Example 2.13: the normalization $\eta : \mathbb{P}^1 \to X$ satisfies $\eta^{-1}(x) = \mathrm{Spec}(\mathcal{O}_{\mathbb{P}^1,\infty}/\mathfrak{m}^2) =: Z$ (as schemes), and η restricts to an isomorphism $\mathbb{P}^1 \setminus \{\infty\} \to X \setminus \{x\}$. The pull-back of any line bundle L on X restricts to the trivial bundle on Z, and this also holds for a G-linearized line bundle. On the other hand, $\eta^{-1}(L) \cong \mathcal{O}_{\mathbb{P}^1}(n)$ equipped with its natural linearization, since $G = \mathbb{G}_a$ has trivial character group. If $n \neq 0$, then we may assume that $n \geq 1$ by replacing L with L^{-1}. Note that Z is the zero scheme of the section y^2 of $\mathcal{O}_{\mathbb{P}^1}(2)$; thus, we have an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(n-2) \xrightarrow{y^2} \mathcal{O}_{\mathbb{P}^1}(n) \longrightarrow \mathcal{O}_Z(n) \longrightarrow 0$$

and hence an exact sequence

$$0 \longrightarrow \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n-2)) \xrightarrow{y^2} \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n)) \longrightarrow \Gamma(Z, \mathcal{O}_Z(n)) \longrightarrow 0,$$

since $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n-2)) = 0$. This yields an isomorphism of G-modules

$$\Gamma(Z, \mathcal{O}_Z(n)) \cong k[x,y]_n / y^2 k[x,y]_{n-2},$$

where G acts on $k[x,y]_n$ via $t \cdot (x,y) = (x+ty, y)$ as in Example 1.12. Thus, the G-invariant subspace of $\Gamma(Z, \mathcal{O}_Z(n))$ is the line spanned by the image of $x^{n-1}y$. Likewise, the fiber $\mathcal{O}_{\mathbb{P}^1}(n)_\infty$ is the line spanned by the image of x^n. It follows that $\mathcal{O}_Z(n)$ has no G-invariant trivialization, and hence that L is not G-linearizable.

In characteristic $p > 0$, the action of G on $\Gamma(Z, \mathcal{O}_Z(p))$ is trivial and hence x^p yields a G-invariant trivialization of $\mathcal{O}_Z(p)$, in agreement with the construction of Example 1.12.

2.5 Picard groups of torsors

Definition 2.15. Let G be an algebraic group, X a G-variety, and $f : X \to Y$ a morphism of varieties. We say that f is a *G-torsor* if it satisfies the following three conditions:

(i) f is G-invariant.

(ii) f is faithfully flat.

(iii) The morphism $F : G \times X \to X \times_Y X$, $(g,x) \mapsto (x, g \cdot x)$ is an isomorphism.

Note that for any G-invariant morphism f, the image of F is contained in the fibered product $X \times_Y X$ (the scheme-theoretic preimage of the diagonal under the morphism $f \times f : X \times X \to Y \times Y$). Also, condition (iii) implies that given $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$, there exists a unique $g \in G$ such that $x_2 = g \cdot x_1$; in particular, G acts freely on X with quotient Y.

Proposition 2.16. *Let G be an algebraic group, and $f : X \to Y$ a G-torsor.*
(i) *The pull-back by f yields isomorphisms* $\mathrm{U}(Y) \cong \mathrm{U}(X)^G$ *and* $\mathrm{Pic}(Y) \cong \mathrm{Pic}_G(X)$.
(ii) *If G is connected and X (or equivalently Y) is irreducible, then there is an exact sequence*

$$(7) \quad 0 \to \mathrm{U}(Y) \longrightarrow \mathrm{U}(X) \longrightarrow \widehat{G} \xrightarrow{\gamma} \mathrm{Pic}(Y) \longrightarrow \mathrm{Pic}(X) \xrightarrow{\psi} \mathrm{Pic}(G \times X)/p_2^*\mathrm{Pic}(X),$$

where γ is the characteristic homomorphism that assigns to any $\chi \in \widehat{G}$, the class of the associated line bundle on Y.

Proof. (i) Since f is G-equivariant, the pull-back of any line bundle on Y is equipped with a G-linearization. Conversely, a G-linearization of a line bundle L on X is exactly a descent datum for L. So the assertion on Picard groups follows from descent theory (see [SGA1, Exp. VIII, Cor. 1.3, Prop. 1.10]). The assertion on units is proved similarly.

(ii) follows from (i) combined with Theorem 2.10 and Proposition 2.11. □

We now obtain a cohomological interpretation of most of the exact sequence (7). For this, we need the following:

Lemma 2.17. *With the notation and assumptions of Proposition 2.16 (ii), there is an exact sequence of sheaves*

$$0 \longrightarrow \mathcal{O}_Y^* \longrightarrow f_*(\mathcal{O}_X^*) \longrightarrow \widehat{G} \longrightarrow 0, \tag{8}$$

where $\widehat{G}$ is viewed as a constant sheaf on Y.

Proof. Let $V \subset Y$ be an open subset, and $U := f^{-1}(V) \subset X$. Then U is an irreducible G-variety, and the restriction $f : U \to V$ is a G-torsor. By Lemma 2.5 and Proposition 2.16 (i), we have an exact sequence

$$0 \longrightarrow \mathcal{O}(V)^* \longrightarrow \mathcal{O}(U)^* \longrightarrow \widehat{G}.$$

This yields the complex of sheaves (8), and its left exactness. To check its right exactness, it suffices to show the following assertion: for any $\chi \in \widehat{G}$ and $y \in Y$, there exist an open neighborhood V of y and a function $f \in \mathcal{O}(f^{-1}(V))^*$ such that $\chi(f) = \chi$.

Denote by T the torus with character group $\widehat{G}$. Since $\widehat{T} = \widehat{G}$, we have a surjective homomorphism of algebraic groups $\pi : G \to T$. Since T is affine, we may form the fiber bundle associated with the G-torsor f and the G-variety T; this yields a T-torsor $\varphi : Z \to Y$ (see [MFK94, Prop. 7.1]). Moreover, $f : X \to Y$ factors as $\psi \circ \varphi$, where $\psi : X \to Z$ is a torsor under the scheme-theoretic kernel of π. (One can show that this kernel is a connected algebraic group; we will not need this fact). Thus, it suffices to show the claim with f replaced by φ. But the latter is a T-torsor, and hence locally trivial for the Zariski topology. Moreover, the claim holds obviously for any trivial torsor. $\square$

Next, taking the long exact sequence of cohomology associated with (7) and using the isomorphism $\mathrm{Pic}(Y) \cong H^1(Y, \mathcal{O}_Y^*)$, we obtain the exact sequence

$$0 \to \mathcal{O}(Y)^* \longrightarrow \mathcal{O}(X)^* \xrightarrow{\chi} \widehat{G} \longrightarrow \mathrm{Pic}(Y) \longrightarrow H^1(Y, f_*(\mathcal{O}_X^*)).$$

Moreover, the Leray spectral sequence associated with the map $f : X \to Y$ and the sheaf $\mathcal{O}_X^*$ yields an injective map $H^1(Y, f_*(\mathcal{O}_X^*)) \to H^1(X, \mathcal{O}_X^*) \cong \mathrm{Pic}(X)$. Thus, we obtain the exact sequence

$$0 \to \mathcal{O}(Y)^* \longrightarrow \mathcal{O}(X)^* \xrightarrow{\chi} \widehat{G} \longrightarrow \mathrm{Pic}(Y) \longrightarrow \mathrm{Pic}(X)$$

which gives back part of (7). But we do not know how to recover the obstruction map $\psi : \mathrm{Pic}(X) \to \mathrm{Pic}(G \times X)/p_2^*\mathrm{Pic}(X)$ via this cohomological approach.

3 Normal G-varieties

3.1 Geometry of linear algebraic groups

Throughout this section, G denotes a connected linear algebraic group.

Proposition 3.1. *The variety G is rational, and its Picard group is finite.*

Proof. Choose a Borel subgroup $B \subset G$. Then the B-torsor $G \to G/B$ admits local sections by [Bo91, Cor. 15.8]. Thus, G is birationally isomorphic to $B \times G/B$. Moreover, the variety B is isomorphic to a product of copies of $\mathbb{A}^1$ and $\mathbb{A}^1 \setminus \{0\}$; also, G/B is rational in view of the Bruhat decomposition (see [Bo91, Thm. 14.12]). Thus, G is rational as well.

To show the finiteness of the Picard group, we use the exact sequence

$$0 \longrightarrow \widehat{G} \longrightarrow \widehat{B} \xrightarrow{\gamma} \mathrm{Pic}(G/B) \longrightarrow \mathrm{Pic}(G) \longrightarrow \mathrm{Pic}(G \times B)/p_2^*\mathrm{Pic}(G)$$

which follows from Proposition 2.16 in view of the isomorphisms $\mathrm{U}(G) \cong \widehat{G}$, $\mathrm{U}(B) \cong \widehat{B}$ (Lemma 2.4) and $\mathrm{U}(G/B) = 0$ (as G/B is a complete irreducible variety). Since G is smooth, the pull-back map $\mathrm{Pic}(G) \to \mathrm{Pic}(G \times \mathbb{A}^1)$ is an isomorphism, and $\mathrm{Pic}(G \times \mathbb{A}^1) \to \mathrm{Pic}(G \times (\mathbb{A}^1 \setminus \{0\}))$ is surjective (see [Ha74, Chap. II, §6]). It follows that $p_2^* : \mathrm{Pic}(G) \to \mathrm{Pic}(G \times B)$ is surjective as well (actually, this is an isomorphism; we will not use this fact). So it suffices to show that the characteristic homomorphism $\gamma : \widehat{B} \to \mathrm{Pic}(G/B)$ has finite cokernel. Denote by $R(G)$ the radical of G, i.e., the largest closed connected normal solvable subgroup; then the quotient $G' := G/R(G)$ is semisimple. Also, $R(G) \subset B$ and hence B is the pull-back of a unique Borel subgroup $B' \subset G'$; moreover, $G/B \cong G'/B'$ and γ factors through the analogous map $\gamma' : \widehat{B'} \to \mathrm{Pic}(G'/B')$. So we may assume that G is semisimple. Then $\widehat{G} = \{0\}$; moreover, the rank of $\widehat{B}$ is the rank of G, say r. Thus, it suffices to show that the group $\mathrm{Pic}(G/B)$ is generated by r elements. But this follows from the Bruhat decomposition again, since G/B contains the open Bruhat cell isomorphic to an affine space, and its complement is the union of r irreducible divisors. $\square$

Proposition 3.1 does not extend to arbitrary fields, as shown by the following:

Example 3.2. Let K be an imperfect field; then $\mathrm{char}(K) = p > 0$ and there exists $a \in K$ such that $a \notin K^p$. Consider the closed subscheme $G \subset \mathbb{A}^2$ defined by the equation $y^p = x + ax^p$. Over the extension $L := K(a^{\frac{1}{p}})$, this equation may be written as $(y - a^{\frac{1}{p}}x)^p = x$; thus, the base change G_L is isomorphic to the affine line $\mathbb{A}^1_L$. In particular, G is a geometrically irreducible variety. Also, G is a subgroup scheme of $\mathbb{A}^2$ viewed as $\mathbb{G}_a \times \mathbb{G}_a$, and hence G is a connected linear algebraic group (a form of $\mathbb{G}_a$). We claim that the variety G is not rational if $p \neq 2$; if in addition the group $G(K)$ is infinite (e.g., if K is separably closed), then $\mathrm{Pic}(G)$ is infinite as well.

The closure of $G \subset \mathbb{A}^2$ in the projective plane $\mathbb{P}^2$ is the curve C with homogeneous equation $y^p = xz^{p-1} + ax^p$. The complement $C \setminus G$ consists of a single point P_∞, with homogeneous coordinates $[1 : a^{\frac{1}{p}} : 0]$; in particular, the residue field of P_∞ is L. Note that P_∞ is a regular point of C, since one can check that the maximal ideal of the local ring $\mathcal{O}_{C,P_\infty}$ is generated by $\frac{z}{x}$; as a consequence, the variety C is regular (but not smooth, since the base change C_L has homogeneous equation $(y - a^{\frac{1}{p}}x)^p = xz^{p-1}$ and hence is singular at P_∞).

To show that G is not rational, it suffices to check that the arithmetic genus $p_a(C) := \dim H^1(C, \mathcal{O}_C)$ is nonzero. For this, note that the exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-p) \longrightarrow \mathcal{O}_{\mathbb{P}^2} \longrightarrow \mathcal{O}_C \longrightarrow 0$$

yields an isomorphism

$$H^1(C, \mathcal{O}_C) \cong H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-p))$$

and moreover,

$$H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-p)) \cong H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(p-3))^\vee \cong K[x,y,z]_{p-3}^\vee.$$

Thus, $p_a(C) = \frac{(p-1)(p-2)}{2}$ is indeed nonzero.

We now turn to the Picard group of G. As C is regular, $\mathrm{Pic}(C) = \mathrm{Cl}(C)$ and we have a right exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathrm{Cl}(C) \longrightarrow \mathrm{Cl}(G) \longrightarrow 0,$$

where $1 \in \mathbb{Z}$ is sent to the class $[P_\infty]$. This class is not torsion (since $\deg(P_\infty) = p$) and hence the above sequence is exact. Also, since C has points of degree 1 (for example, the origin, $[0:0:1] =: P_0$), we have an exact sequence

$$0 \longrightarrow \mathrm{Cl}^0(C) \longrightarrow \mathrm{Cl}(C) \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0,$$

where $\mathrm{Cl}^0(X)$ denotes the group of divisor classes of degree 0. Combining both exact sequences readily yields an exact sequence

$$0 \longrightarrow \mathrm{Cl}^0(C) \longrightarrow \mathrm{Cl}(G) \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow 0,$$

which is split by the subgroup of $\mathrm{Cl}(G)$ generated by $[P_0]$.

To check the claim, it suffices to show that $\mathrm{Cl}^0(C)$ is infinite. Consider the map $f : G(K) \to \mathrm{Cl}^0(C)$ that sends any point P to the class $[P] - [P_0]$. Then f is injective: with an obvious notation, if $[Q] - [P_0]$ is linearly equivalent to $[P] - [P_0]$, then there exists $f \in K(C)$ such that $\mathrm{div}(f) = [Q] - [P]$. If in addition $Q \neq P$, then f yields a birational morphism $C \to \mathbb{P}^1$, which contradicts the fact that C is not rational.

For more details on this example and further developments, see [Ru70] and [KMT74] (especially §6.7).

3.2 Linearization of powers of line bundles

Lemma 3.3. *Let X, Y be normal irreducible varieties, and assume that Y is rational. Then the map*

$$p_1^* \times p_2^* : \mathrm{Cl}(X) \times \mathrm{Cl}(Y) \longrightarrow \mathrm{Cl}(X \times Y)$$

is an isomorphism. Moreover, this map restricts to an isomorphism

$$\mathrm{Pic}(X) \times \mathrm{Pic}(Y) \cong \mathrm{Pic}(X \times Y).$$

Proof. By assumption, Y contains a nonempty open subset V which is isomorphic to an open subset of some affine space $\mathbb{A}^n$. Since the pull-back map $\mathrm{Cl}(X) \to \mathrm{Cl}(X \times \mathbb{A}^n)$ is an isomorphism (see [Ha74, Prop. II.6.6]) and the pull-back map $\mathrm{Cl}(X \times \mathbb{A}^n) \to \mathrm{Cl}(X \times V)$ is surjective, the map $p_1^* : \mathrm{Cl}(X) \to \mathrm{Cl}(X \times V)$ is surjective. Also, the kernel of the pull-back map $\mathrm{Cl}(X \times Y) \to \mathrm{Cl}(X \times V)$ is generated by the classes $[X \times E_i]$, where E_i denote the irrreducible components of $Y \setminus V$ of codimension 1 in Y. Since $[X \times E_i] = p_2^*([E_i])$, it follows that the map $p_1^* \times p_2^*$ is surjective.

To show that this map is injective, consider Weil divisors D on X and E on Y such that the class of $p_1^*(D)+p_2^*(E)$ is zero in $\mathrm{Cl}(X \times Y)$. In other words, there exists $f \in k(X\times Y)$ such that $\mathrm{div}(f) = p_1^*(D) + p_2^*(E)$. For a general point $y \in Y$, the rational function $f_y : x \mapsto f(x,y)$ is defined and satisfies $\mathrm{div}(f_y) = D$; thus, the class $[D]$ is zero in $\mathrm{Cl}(X)$. Likewise, $[E] = 0$ in $\mathrm{Cl}(Y)$; this completes the proof of the first assertion.

For the second assertion, it suffices to show the following claim: given Weil divisors D on X and E on Y such that $p_1^*(D) + p_2^*(E)$ is Cartier, D and E must be Cartier as well. But this claim follows by the argument for the injectivity of $p_1^* \times p_2^*$. □

Remark 3.4. The rationality assumption on Y (or X) cannot be omitted in Lemma 3.3. Consider indeed an elliptic curve C and the diagonal, $\mathrm{diag}(C) \subset C \times C$. There there exist no divisors D, E on C such that $\mathrm{diag}(C)$ is linearly equivalent to $p_1^*(D) + p_2^*(E)$. Otherwise, taking intersection numbers with $\{x\} \times C$ and $C \times \{y\}$ for $x, y \in C$, we obtain $\deg(D) = \deg(E) = 1$. Thus, the self-intersection number $\mathrm{diag}(C)^2$ equals 1. But $\mathrm{diag}(C)$ is the scheme-theoretic fiber of a point under the quotient morphism $C \times C \to (C\times C)/\mathrm{diag}(C)$. Therefore, the normal bundle to $\mathrm{diag}(C)$ in $C\times C$ is trivial, and hence $\mathrm{diag}(C)^2 = 0$, a contradiction.

Theorem 3.5. *Let X be a normal G-variety. Then every line bundle on X is G-invariant. Moreover, there exists a positive integer n such that $L^{\otimes n}$ admits a G-linearization; we may take for n the exponent of* $\mathrm{Pic}(G)$.

Proof. Since X is normal, it is the disjoint union of its irreducible component; moreover, any such component is G-stable, as G is connected. Thus, we may assume that X is irreducible.

By Lemma 3.3 applied to $G \times X$, we have $\alpha^*(L) \cong p_1^*(M) \otimes p_2^*(N)$ for some line bundles M on G and N on X. Pulling back to $\{e_G\} \times X$, we obtain that $L \cong N$; then pulling back to $\{g\} \times X$, we obtain that $g^*(L) \cong L$. Thus, L is G-invariant.

Let n be the exponent of $\mathrm{Pic}(G)$; then $M^{\otimes n}$ is trivial, hence an isomorphism $\alpha^*(L^{\otimes n}) \cong p_2^*(L^{\otimes n})$. In view of Lemma 2.9, it follows that $L^{\otimes n}$ is G-linearizable. □

Proposition 3.6. *Let $f : X \to Y$ be a G-torsor, where X (or equivalently Y) is a normal variety. Then there is an exact sequence*

$$\mathrm{Pic}(Y) \xrightarrow{f^*} \mathrm{Pic}(X) \xrightarrow{\alpha_x^*} \mathrm{Pic}(G),$$

where $\alpha_x : G \to X$ denotes the orbit map $g \mapsto g \cdot x$, and x is an arbitrary point of X.

Proof. Via the isomorphism $\mathrm{Pic}(G \times X)/p_2^*\mathrm{Pic}(X) \cong \mathrm{Pic}(G)$ (obtained in the proof of Theorem 3.5), the obstruction map $\psi : \mathrm{Pic}(X) \to \mathrm{Pic}(G \times X)/p_2^*\mathrm{Pic}(X)$ is identified with α_x^* for any $x \in X$. This observation combined with Proposition 2.11 implies the statement. □

Prominent examples of torsors are quotients of algebraic groups by closed subgroups. For these, Proposition 3.6 and Lemma 2.4 imply readily the following:

Corollary 3.7. *Let G be a connected algebraic group, and H a closed connected linear subgroup. Then there is an exact sequence*

$$0 \longrightarrow \mathrm{U}(G/H) \longrightarrow \widehat{G} \longrightarrow \widehat{H} \xrightarrow{\gamma} \mathrm{Pic}(G/H) \longrightarrow \mathrm{Pic}(G) \longrightarrow \mathrm{Pic}(H),$$

where γ denotes the characteristic homomorphism, and all other maps are pull-backs.

This result goes back to Raynaud (see [Ra70, Prop. VII.1.5]). Further developments may be found e.g. in [Sa81].

3.3 Local G-quasiprojectivity

Lemma 3.8. *Let X be a normal irreducible G-variety, and D a Weil divisor on X. Then $g^*(D)$ is linearly equivalent to D for any $g \in G$.*

Proof. As in the proof of Lemma 3.3, consider the regular locus X_{reg}, an open G-stable subset of X with complement of codimension at least 2. It suffices to show the statement for the pull-back $D_{X_{\mathrm{reg}}}$, and hence we may assume that X is smooth. Denote by L the line bundle on X associated with D. Then L is G-invariant by Theorem 3.5; this is equivalent to the desired statement. □

Lemma 3.9. *Let X be a normal G-variety, and D an effective Weil divisor on X. If* $\mathrm{Supp}(D)$ *contains no G-orbit, then D is a Cartier divisor, generated by global sections s_g, where $g \in G$, such that* $\mathrm{div}(s_g) = g^*(D)$. *If in addition $X \setminus$* $\mathrm{Supp}(D)$ *is affine, then D is ample.*

Proof. Let $U := X \setminus \mathrm{Supp}(D)$; this is an open subset of X, and the translates gU, where $g \in G$, cover X (since $\mathrm{Supp}(D)$ contains no G-orbit). Also, the pull-back D_U is trivial. For any $g \in G$, there exists $f = f_g \in k(X)^*$ such that $g^*(D) = D + \mathrm{div}(f)$, in view of Lemma 3.8. Thus, the pull-back $g^*(D)_U$ is trivial as well; equivalently, D_{gU} is trivial. It follows that D is Cartier. Moreover, for any g as above, there exists a section $s = s_g \in \Gamma(X, \mathcal{O}_X(D))$ with divisor $g^*(D)$. Since the supports of these divisors have no common point, D is generated by the corresponding global sections.

Next, assume that U is affine; then so is of course each translate gU. Note that $\mathcal{O}(gU)$ is the increasing union of the vector spaces $\Gamma(X, \mathcal{O}_X(nD))s_g^{-n}$, where n runs over the positive integers. Since the algebra $\mathcal{O}(gU)$ is finitely generated, there exists n and a finite-dimensional subspace $V \subset \Gamma(X, \mathcal{O}_X(nD))$ such that $s_g^n \in V$ and the subspace $V s_g^{-n}$ generates $\mathcal{O}(gU)$. As X is covered by finitely many translates $g_1U, \ldots, g_mU$, we may choose n and V so that each algebra $\mathcal{O}(g_iU)$ is generated by $V s_{g_i}^{-n}$. Then the rational map $f_V : X \dashrightarrow \mathbb{P}(V^\vee)$ (introduced in the proof of Proposition 1.8) is defined everywhere, and restricts to an immersion on each $g_iU = f_v^{-1}(H_i)$, where H_i denotes the hyperplane of $\mathbb{P}(V^\vee)$ corresponding to $s_{g_i}^n \in V$. Thus, f_V is an immersion, and $\mathcal{O}_X(nD) = f_v^*\mathcal{O}_{\mathbb{P}(V^\vee)}(1)$ is very ample. □

Theorem 3.10. *Let X be a normal G-variety. Then X admits a covering by G-quasi-projective open subsets.*

Proof. Let $x \in X$ and choose an affine open neighborhood U of x. Then $G \cdot U$ is a G-stable neighborhood of x containing U. We may thus replace X with $G \cdot U$, and it suffices to show that X is then G-quasiprojective.

Let $D := X \setminus U$; this is a reduced Weil divisor on X which contains no G-orbit. Moreover, $X \setminus D = U$ is affine. By Lemma 3.9, it follows that D is ample. Let L be the line bundle associated with D; then some positive power of L is G-linearizable in view of Theorem 3.5. By Proposition 1.8, we conclude that X is G-quasiprojective. □

Corollary 3.11. *Let X be a normal variety equipped with an action of a torus T. Then X admits a covering by T-stable affine open subsets.*

Proof. In view of Theorem 3.10, we may assume that X is a T-stable subvariety of $\mathbb{P}(V)$ for some finite-dimensional T-module V. The dual module $V^\vee$ admits a basis consisting of T-eigenvectors, say $f_1, \ldots, f_n$. Thus, the complements of the hyperplanes $(f_i = 0) \subset \mathbb{P}(V)$ form a covering by T-stable affine open subsets. As a consequence, we may assume that X is a T-subvariety of an affine T-variety Z.

We now argue as in the proof of Corollary 1.17. Consider the closure $\bar{X}$ of X in Z, and the complement $Y := \bar{X} \setminus X$. Let $x \in X$; then there exists a T-eigenvector $f \in \mathcal{O}(X)$ such that f which vanishes identically on Y, and $f(x) \neq 0$. Then the open subset of $\overline{X}$ on which $f \neq 0$ is affine, T-stable, contains x, and is contained in X. □

References

[Bo91] A. Borel, *Linear algebraic groups. Second edition*, Grad. Texts in Math. **126**, Springer-Verlag, New York, 1991.

[Br94] K. Brown, *Cohomology of groups*, Grad. Texts Math. **87**, Springer-Verlag, New York, 1994.

[FI73] R. Fossum, B. Iversen, *On Picard groups of algebraic fibre spaces*, J. Pure Applied Algebra **3** (1973), 269–280.

[Ha74] R. Hartshorne, *Algebraic Geometry*, Grad. Texts in Math. **52**, Springer-Verlag, New York, 1977.

[KMT74] T. Kambayashi, M. Miyanishi, M. Takeuchi, *Unipotent algebraic groups*, Lecture Notes Math. **414**, Springer-Verlag, New York, 1974.

[KKLV89] F. Knop, H. Kraft, D. Luna, T. Vust: *Local properties of algebraic group actions*, in: Algebraische Transformationgruppen und Invariantentheorie, 63–75, DMV Sem. **13**, Birkhäuser, Basel, 1989.

[KKV89] F. Knop, H. Kraft, T. Vust: *The Picard group of a G-variety*, in: Algebraische Transformationgruppen und Invariantentheorie, 77–87, DMV Sem. **13**, Birkhäuser, Basel, 1989.

[Mu70] D. Mumford, *Abelian varieties*, Oxford University Press, Oxford, 1970.

[MFK94] D. Mumford, J. Fogarty, F. Kirwan, *Geometric invariant theory. Third edition*, Ergeb. Math. Grenzgeb. (2) **34**, Springer-Verlag, Berlin, 1994.

[Ra64] C. P. Ramanujam, *A note on automorphism groups of algebraic varieties*, Math. Ann. **156** (1964), no. 1, 25–33.

[Ra70] M. Raynaud, *Faisceaux amples sur les schémas en groupes et les espaces homogènes*, Lecture Notes Math. **119**, Springer-Verlag, New York, 1970.

[Ru70] P. Russell, *Forms of the affine line and its additive group*, Pacific J. Math. **32** (1970), no. 2, 527–539.

[SGA1] *Revêtements étales et groupe fondamental (SGA1)*, Séminaire de Géométrie Algébrique du Bois Marie 1960–1961, Documents Mathématiques **3**, Soc. Math. France, Paris, 2003.

[Sa81] J-J. Sansuc, *Groupe de Brauer et arithmétique des groupes algébriques linéaires sur un corps de nombres*, J. Reine Angew. Math. **327** (1981), 12–80.

[Su74] H. Sumihiro, *Equivariant completion*, J. Math. Kyoto Univ. **14** (1974), no. 1, 1–28.

[Su75] H. Sumihiro, *Equivariant completion. II*, J. Math. Kyoto Univ. **15** (1975), no. 3, 573–605.