NOTES ON AUTOMORPHISM GROUPS
OF PROJECTIVE VARIETIES

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ABSTRACT. These are extended notes for my lectures at the School and Workshop on Varieties and Group Actions (Warsaw, September 23–29, 2018). They present old and new results on automorphism groups of normal projective varieties over an algebraically closed field.

CONTENTS

1. INTRODUCTION

Let X be a projective variety over an algebraically closed field k. It is known that the automorphism group, $\mathrm{Aut}(X)$, has a structure of locally algebraic group (see [Gro61, Ram64, MO67]). This yields an exact sequence

$$(1.0.1) \qquad 1 \longrightarrow \mathrm{Aut}^0(X) \longrightarrow \mathrm{Aut}(X) \longrightarrow \pi_0 \mathrm{Aut}(X) \longrightarrow 1,$$

where $\mathrm{Aut}^0(X)$ is (the group of k-rational points of) a smooth connected algebraic group, and $\pi_0 \mathrm{Aut}(X)$ is a discrete group.

To analyze the structure of $\mathrm{Aut}(X)$, one may start by considering the connected automorphism group $\mathrm{Aut}^0(X)$ and the group of components $\pi_0 \mathrm{Aut}(X)$ separately. It turns out that there is no restriction on the former: every smooth connected algebraic group is the connected automorphism group of some normal projective variety X (see [Bri14, Thm. 1]). In characteristic 0, we may

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further take X to be smooth by using equivariant resolution of singularities (see e.g. [Kol07, Chap. 3]).

By constrast, very little is known on the structure of the group of components. Every finite group G can be obtained in this way, as G is the full automorphism group of some smooth projective curve (see the main result of [MV83]). But the group of components is generally infinite, and it is unclear how infinite it can be.

The long-standing question whether this group is finitely generated has been recently answered in the negative by Lesieutre. He constructed an example of a smooth projective variety of dimension 6 having a discrete, non-finitely generated automorphism group (see [Les16]). His construction has been further extended in all dimensions at least 2 by Dinh and Oguiso, see [DO17]. The former result is obtained over an arbitrary field of characteristic 0, while the latter is over the complex numbers; the case of positive characteristics seems to be still open.

On the positive side, it is a folklore result that $\pi_0 \operatorname{Aut}(X)$ acts on the Néron-Severi lattice (the group of line bundles up to numerical equivalence) with a finite kernel. It follows for example that the group of components is finite or countable (this is easily seen directly), and commensurable with a torsion-free, residually finite group. Also, $\pi_0 \operatorname{Aut}(X)$ is known to be finitely presented for some interesting classes of projective varieties, including abelian varieties (see [Bor62]) and complex hyperkähler manifolds (see [CF18, Thm. 1.5]).

In these notes, we first present some fundamental results on automorphism groups of projective varieties. To cover the case of positive characteristics, one is lead to considering group schemes (as the automorphism group scheme is generally not smooth) and singular varieties (as resolution of singularities is unknown in this case, a fortiori equivariant desingularization which is a basic tool for studying automorphism groups in characteristic 0; normality turns out to be a handy substitute for smoothness in this setting.

We survey the construction of the automorphism group scheme and its action on the Picard scheme, from which one can derive interesting information. For example, the numerical equivalence class of any ample line bundle has a finite stabilizer under the action of $\pi_0 \operatorname{Aut}(X)$ (Theorem 2.10). Also, the isomorphism class of any such line bundle has an algebraic stabilizer under the action of $\operatorname{Aut}(X)$ (Theorem 2.17). Both results, which are again folklore, have a number of remarkable consequences, e.g., the automorphism group of any Fano variety is a linear algebraic group.

We then present three new results, all of which generalize recent work. The first one goes in the positive direction for *almost homogeneous* varieties, i.e., those on which a smooth connected algebraic group acts with an open orbit.

Theorem 1. *Let X be a normal projective variety, almost homogeneous under a linear algebraic group. Then $\mathrm{Aut}(X)$ is a linear algebraic group as well.*

This was first proved by Fu and Zhang in the setting of compact Kähler manifolds (see [FZ13, Thm. 1.2]). The main point of their proof is to show that the anticanonical line bundle is big. This relies on Lie-theoretical methods, in particular the $\mathfrak{g}$-anticanonical fibration of [HO84, I.2.7], also known as the Tits fibration. But this approach does not extend to positive characteristics, already when X is homogeneous under a semi-simple algebraic group: then any big line bundle on X is ample, but X is generally not Fano (see [HL93]).

To prove Theorem 1, we construct a normal projective variety X' equipped with a birational morphism $f : X' \to X$ such that the action of $\mathrm{Aut}(X)$ on X lifts to an action on X', which fixes a big line bundle. For this, we use a characteristic-free version of the Tits fibration (Lemma 3.2). We also check that the image of $\mathrm{Aut}(X)$ in $\mathrm{Aut}(X')$ is closed (Lemma 3.1).

Our second result goes in the negative direction, as it yields many examples of algebraic groups which cannot be obtained as the automorphism group of a normal projective variety. To state it, we introduce some notation.

Let G be a smooth connected algebraic group. By Chevalley's structure theorem (see [Con02] for a modern proof), there is a unique exact sequence of algebraic groups

$$(1.0.2) \qquad 1 \longrightarrow G_{\mathrm{aff}} \longrightarrow G \longrightarrow A \longrightarrow 1,$$

where G_{aff} is a smooth connected affine (or equivalently linear) algebraic group and A is an abelian variety. We denote by $\mathrm{Aut}_{\mathrm{gp}}^{G_{\mathrm{aff}}}(G)$ the group of automorphisms of the algebraic group G which fix G_{aff} pointwise.

Theorem 2. *With the above notation, assume that the group $\mathrm{Aut}_{\mathrm{gp}}^{G_{\mathrm{aff}}}(G)$ is infinite. If $G \subset \mathrm{Aut}(X)$ for some normal projective variety X, then G has infinite index in $\mathrm{Aut}(X)$.*

It is easy to show that $\mathrm{Aut}_{\mathrm{gp}}^{G_{\mathrm{aff}}}(G)$ is an arithmetic group, and to construct classes of examples for which this group is infinite, see Remarks 4.3.

If G is an abelian variety, then $\mathrm{Aut}_{\mathrm{gp}}^{G_{\mathrm{aff}}}(G)$ is just its group of automorphisms which fix the origin. In this case, Theorem 2 is due (in essence) to Lombardo and Maffei, see [LM18, Thm. 2.1]. They also obtain a converse over the complex numbers: given an abelian variety G with finite automorphism group, they construct a smooth projective variety X such that $\mathrm{Aut}(X) = G$ (see [LM18, Thm. 3.9]).

Like that of [LM18, Thm. 2.1], the proof of Theorem 2 is based on the structure of X as a homogeneous fibration over an abelian variety, quotient of A by a finite subgroup scheme. This allows us to construct an action on X of a subgroup of finite index of $\mathrm{Aut}_{\mathrm{gp}}^{G_{\mathrm{aff}}}(G)$, which normalizes G and intersects this group trivially.

When X is almost homogeneous under G, the Albanese morphism provides such a homogeneous fibration (see [Bri10, Thm. 3]). A finer analysis of its automorphisms leads to our third main result.

Theorem 3. *Let X be a normal projective variety, almost homogeneous under a smooth connected algebraic group G. Then $\pi_0 \operatorname{Aut}(X)$ is an arithmetic group. In positive characteristics, $\pi_0 \operatorname{Aut}(X)$ is commensurable with $\operatorname{Aut}_{\mathrm{gp}}^{G_{\mathrm{aff}}}(G)$.*

(The second assertion does not hold in characteristic 0, see Remark 5.5). These results leave open the question whether every *linear* algebraic group is the automorphism group of a normal projective variety. Further open questions are discussed in the recent survey [Can18], in the setting of smooth complex projective varieties; most of them address dynamical aspects of automorphisms, which are not considered in the present notes but play an important rôle in many developments.

Notation and conventions.

Throughout these notes, we fix an algebraically closed field k of arbitrary characteristic, denoted by $\operatorname{char}(k)$. By a *scheme*, we mean a separated scheme over k, unless otherwise stated; a *subscheme* is a locally closed k-subscheme. Morphisms and products of schemes are understood to be over k as well.

Given two schemes X and S, we use the (standard) notation $X(S)$ for the set of S-valued points of X. If $S = \operatorname{Spec}(R)$ for some algebra R, then we also denote $X(S)$ by $X(R)$.

A *variety* is an integral scheme of finite type.

An *algebraic group* is a group scheme of finite type; a *locally algebraic group* is a group scheme, locally of finite type. With this convention, the "linear algebraic groups" of the classical literature are exactly the smooth affine algebraic groups.

We use [Har77] as a general reference for algebraic geometry, and [FGA05] for more advanced developments on Hilbert and Picard schemes. We refer to [DG70] for some fundamental facts on (locally) algebraic groups, and to [BSU13] for structure results about smooth connected algebraic groups.

2. Some basic constructions and results

2.1. The automorphism group. Let X be a scheme.

Definition 2.1. A *family* of automorphisms of X parameterized by a scheme S is an automorphism of the S-scheme $X \times S$.

Any such automorphism g satisfies $g(x, s) = (f(x, s), s)$ for all schematic points $x \in X$, $s \in S$, where $f : X \times S \to S$ is a morphism such that the map $f_s : X \to X$, $s \mapsto f(x, s)$ is an automorphism for any $s \in S(k)$ (but the latter condition does not guarantee that g is an automorphism).

The families of automorphisms of X parameterized by a fixed scheme S form a group, that we denote by $\mathrm{Aut}(X \times S/S)$. Given such a family $g = f \times \mathrm{id}_S$ and a morphism of schemes $u : S' \to S$, the morphism

$$g' : X \times S' \longrightarrow S', \quad (x, s') \longmapsto (f(x, u(s')), s')$$

is a family of automorphisms of X parameterized by S'. Moreover, the pull-back map

$$u^* : \mathrm{Aut}(X \times S/S) \longrightarrow \mathrm{Aut}(X \times S'/S'), \quad g \longmapsto g'$$

is a group homomorphism. For any morphism of schemes $v : S'' \to S'$, we have $(u \circ v)^* = v^* \circ u^*$, and id_S^* is the identity.

Definition 2.2. The *automorphism functor* is the contravariant functor from schemes to groups

$$\mathbf{Aut}_X : S \longmapsto \mathrm{Aut}(X \times S/S), \quad u \longmapsto u^*.$$

Theorem 2.3. *If X is proper, then the group functor $\mathbf{Aut}_X$ is represented by a locally algebraic group.*

Proof. We sketch the argument under the stronger assumption that X is *projective*. The idea (due to Grothendieck) is to deduce the result from the existence of the Hilbert scheme via a graph construction.

More specifically, one associates with each $g \in \mathrm{Aut}(X \times S/S)$ its *graph* Γ_g; this is the image of the morphism

$$X \times S \longrightarrow X \times S \times_S X \times S, \quad (x, s) \longmapsto (x, s, g(x, s), s)$$

(a closed immersion). Let $g = f \times \mathrm{id}_S$ and identify $X \times S \times_S X \times S$ with $X \times X \times S$; then Γ_g is identified with the image Γ_f of the closed immersion

$$\mathrm{id}_X \times f : X \times S \longrightarrow X \times X \times S, \quad (x, s) \longmapsto (x, f(x, s)).$$

In particular, the graph of the identity automorphism is identified with the diagonal, $\mathrm{diag}(X) \times S$.

The projection $\Gamma_f \to S$ is flat, since it is identified with the projection $X \times S \to S$ via the above isomorphism $X \times S \xrightarrow{\cong} \Gamma_f$. Moreover, the assignment $g \mapsto \Gamma_g$ extends to a morphism of functors

$$\Gamma : \mathbf{Aut}_X \longrightarrow \mathbf{Hilb}_{X \times X},$$

where the right-hand side denotes the *Hilbert functor* which takes every scheme S to the set of closed subschemes of $X \times X \times S$ which are flat over S. The Hilbert functor is represented by a scheme $\mathrm{Hilb}_{X \times X}$, the disjoint union of open and closed projective schemes $\mathrm{Hilb}_{X \times X}^P$ obtained by prescribing the Hilbert polynomial P (see [Gro61], and [FGA05, Chap. 5] for a recent exposition of this existence result). In particular, $\mathrm{Hilb}_{X \times X}$ is locally of finite type.

To complete the argument, one shows that Γ identifies $\mathbf{Aut}_X$ with the functor of points of an open subscheme of $\mathrm{Hilb}_{X \times X}$. This can be done in two steps: first, one considers the functor $\mathbf{End}_X$ of endomorphisms of X, and shows that it is represented by an open subscheme End_X of $\mathrm{Hilb}_{X \times X}$ (via the graph construction again); then one shows that $\mathbf{Aut}_X$ is represented by an open subscheme of End_X. See [FGA05, 5.6.2] for details.

The general case, where X is only assumed proper, is due to Matsumura and Oort via a very different method; they obtain an axiomatic characterization of group schemes among group functors, and they check that their axioms are satisfied for the automorphism functor (see [MO67]). $\square$

We assume from now on that X is *proper*, and we denote by Aut_X its automorphism group scheme; then we have $\mathrm{Aut}_X(S) = \mathrm{Aut}(X \times S/S)$ for any scheme S.

Since Aut_X is a locally algebraic group, it has a largest closed connected normal subgroup scheme: the *neutral component*, that we denote by Aut_X^0. Moreover, Aut_X^0 is an algebraic group, and the quotient group scheme

$$\pi_0(\mathrm{Aut}_X) = \mathrm{Aut}_X/\mathrm{Aut}_X^0$$

exists and is discrete (see [DG70, II.5.1] for these results). We say that $\pi_0(\mathrm{Aut}_X)$ is the *group of components* of Aut_X.

If $\mathrm{char}(k) = 0$, then Aut_X is smooth by a result of Cartier (see [DG70, II.6.1]). This fails in positive characteristics, see e.g. Example 2.5 (i). But the largest reduced subscheme $\mathrm{Aut}_{X,\mathrm{red}}$ of Aut_X is a closed smooth subgroup scheme; it represents the functor from reduced schemes to groups given by $S \mapsto \mathrm{Aut}(X \times S/S)$ (see [DG70, II.5.2.3]).

In particular, we obtain the exact sequence (1.0.1), where

$$\mathrm{Aut}^0(X) := \mathrm{Aut}_X^0(k) = \mathrm{Aut}_{X,\mathrm{red}}^0(k) \quad \text{and} \quad \pi_0 \mathrm{Aut}(X) := \pi_0(\mathrm{Aut}_X)(k).$$

Identifying $\mathrm{Aut}_{X,\mathrm{red}}^0$ with its group of k-rational points, we will view $\mathrm{Aut}^0(X)$ as a smooth connected algebraic group.

The scheme Aut_X is the disjoint union of the cosets $g\,\mathrm{Aut}_X^0$, where $g \in \mathrm{Aut}(X)$, which form its connected components; these cosets are indexed by $\pi_0 \mathrm{Aut}(X)$. As a consequence, Aut_X is algebraic if and only if so is $\mathrm{Aut}(X)$. Moreover, Aut_X is linear if and only if so is $\mathrm{Aut}(X)$ (indeed, a scheme of finite type Y is affine if and only if Y_{red} is affine; see [Har77, Ex. III.3.1]).

Also, note that every subgroup scheme of Aut_X is closed, as follows from [DG70, II.5.1].

Remarks 2.4. (i) Recall that an *action* of a group scheme G on X is a morphism $\alpha : G \times X \to X$ satisfying the following conditions : (a) $\alpha(e, x) = x$ for any schematic point $x \in X$, where $e \in G(k)$ denotes the neutral element,

and (b) $\alpha(g_1, \alpha(g_2, x)) = \alpha(g_1 g_2, x)$ for any schematic points $g_1, g_2 \in G$ and $x \in X$.

There is a bijective correspondence between the G-actions on X and the homomorphisms $G \to \operatorname{Aut}_X$. Indeed, every such action α defines an action of the group $G(S)$ on $X \times S$ by S-automorphisms, for any scheme S: to any morphism $u : S \to G$, one associates the morphism

$$g(u) : X \times S \longrightarrow X \times S, \quad (x, s) \longmapsto (\alpha(u(s), x), s).$$

This yields a homomorphism $G(S) \to \operatorname{Aut}(X \times S/S)$ which is compatible with pull-backs, and hence leads to the desired homomorphism $f : G \to \operatorname{Aut}_X$. Conversely, every such homomorphism f defines an action

$$\alpha : G \times X \to X, \quad (g, x) \mapsto f(g)(x),$$

and the assignments $\alpha \mapsto f$, $f \mapsto \alpha$ are inverse to each other.

(ii) The group of components $\pi_0 \operatorname{Aut}(X)$ is finite or countable. Indeed, since X is a scheme of finite type, it may be defined over a subfield k' of k which is finitely generated over its prime field, and hence is finite or countable. Since the algebraic closure of k' is countable, we may assume that k is (algebraically closed and) countable. Let $g \in \operatorname{Aut}(X)$ and recall that g is uniquely determined by its graph, a closed subscheme Γ_g of $X \times X$. Consider an open affine subscheme $U \subset X \times X$, and the ideal $I \subset \mathcal{O}(U)$ of $\Gamma_g \cap U$. Since $\mathcal{O}(U)$ is a finitely generated k-algebra, it is a countable set and its set of ideals is finite or countable. As $X \times X$ is covered by finitely many affine open subschemes, it follows that $\operatorname{Aut}(X)$ is finite or countable as well. This yields our assertion.

When X is projective, this assertion also follows by viewing Aut_X as an open subscheme of $\operatorname{Hilb}_{X \times X}$, since the latter is a finite or countable union of projective schemes (there are at most countably many Hilbert polynomials).

(iii) The Lie algebra of the group scheme Aut_X consists of the derivations of the structure sheaf $\mathcal{O}_X$; equivalently,

(2.1.1) $$\operatorname{Lie}(\operatorname{Aut}_X) = H^0(X, \mathcal{T}_X),$$

where $\mathcal{T}_X = \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$ denotes the tangent sheaf.

Consider indeed the ring of dual numbers, $k[\varepsilon] = k[t]/(t^2)$, and its spectrum S. Then $\operatorname{Lie}(\operatorname{Aut}_X)$ is the kernel of the homomorphism

$$\operatorname{Aut}(X \times S/S) = \operatorname{Aut}_X(k[\varepsilon]) \longrightarrow \operatorname{Aut}_X(k) = \operatorname{Aut}(X)$$

induced by the algebra homomorphism $k[\varepsilon] \to k$ (see [DG70, II.4.1]). Viewing $X \times S$ as the ringed space with the same underlying topological space as X and the structure sheaf $\mathcal{O}_X \otimes k[\varepsilon]$, this kernel consists of the pairs $(\operatorname{id}, \operatorname{id}_{\mathcal{O}_X} + \varepsilon D)$, where D is a derivation of $\mathcal{O}_X$.

When X is projective, we may recover the isomorphism (2.1.1) by identifying $\operatorname{Lie}(\operatorname{Aut}_X)$ with the Zariski tangent space of $\operatorname{Hilb}_{X \times X}$ at its k-rational

point $\mathrm{diag}(X)$ (the graph of id_X). By [FGA05, 6.4.10], we have

$$T_{\mathrm{diag}(X)}\mathrm{Hilb}_{X\times X} = \mathrm{Hom}(\mathcal{I}_{\mathrm{diag}(X)}/\mathcal{I}^2_{\mathrm{diag}(X)}, \mathcal{O}_{\mathrm{diag}(X)}),$$

where $\mathcal{I}_{\mathrm{diag}(X)} \subset \mathcal{O}_{X\times X}$ denotes the ideal sheaf of the diagonal. Since we have $\mathcal{I}_{\mathrm{diag}(X)}/\mathcal{I}^2_{\mathrm{diag}(X)} = \Omega^1_X$, this yields $T_{\mathrm{diag}(X)}\mathrm{Hilb}_{X\times X} = H^0(X, \mathcal{T}_X)$.

(iv) Consider the normalization map $\eta : \tilde{X} \to X$. It is easy to see that the action of $\mathrm{Aut}(X)$ on X lifts to a unique action on $\tilde{X}$. But this does not necessarily extend to an action of Aut_X in view of Example 2.5 (iii).

Examples 2.5. (i) Let X be a *finite scheme*, i.e., $X = \mathrm{Spec}(A)$ where A is a commutative k-algebra of finite dimension as a k-vector space. Then Aut_X is a closed subgroup scheme of the general linear group $\mathrm{GL}(A)$, where A is viewed as a k-vector space again. Thus, Aut_X is linear.

If X is reduced (equivalently, A is a product of finitely many copies of k), then Aut_X is finite and reduced as well. But Aut_X can be very large for an arbitrary finite scheme X: if $A = k \oplus I$ where I is an ideal of square 0 and of dimension n as a k-vector space, then Aut_A contains $\mathrm{GL}(I) \simeq \mathrm{GL}_n$.

Also, Aut_X is not necessarily smooth when $\mathrm{char}(k) = p > 0$, e.g. when $A = k[t]/(t^p)$. Indeed, for any scheme S, the group $\mathrm{Aut}_A(S)$ is identified with the automorphism group of the sheaf of $\mathcal{O}_S$-algebras $\mathcal{O}_S[x]$. Denoting by t the image of x in A, we may view $\mathrm{Aut}_A(S)$ as the group of maps

$$t \longmapsto a_0 + a_1 t + \cdots + a_{p-1} t^{p-1},$$

where $a_0 \in \mathcal{O}(S)$, $a_0^p = 0$, $a_1 \in \mathcal{O}(S)$ is invertible, and $a_2, \ldots, a_{p-1} \in \mathcal{O}(S)$. Note that X is a closed subgroup scheme of the additive group $\mathbb{G}_a$; this finite subgroup scheme of order p is denoted by α_p. Moreover, the maps $t \to a_0 + t$ are just the translations by points of α_p. Also, the maps $t \mapsto a_1 t + \cdots + a_{p-1} t^{p-1}$ define a smooth connected linear algebraic group G (the stabilizer of the origin in Aut_A), and Aut_A is the semi-direct product of its closed normal subgroup scheme α_p with G. In particular, Aut_A is non-reduced, and $\mathrm{Aut}_{A,\mathrm{red}} = G$ is not a normal subgroup scheme of Aut_A.

(ii) Let X be a *curve*; by this, we mean a complete variety of dimension 1. Then X is projective (see [Har77, Ex. III.5.8]). Moreover, Aut_X is an algebraic group, as follows e.g. from Corollary 2.12 below.

If X is smooth of genus $g \geq 2$, then $H^0(X, \mathcal{T}_X) = 0$ by the Riemann-Roch theorem. In view of Remark 2.4 (iii), it follows that Aut_X is finite and constant. Also, the automorphism group schemes of smooth curves of genus ≤ 1 are easily described; they turn out to be smooth.

This does not extend to singular curves in characteristic $p > 0$. Consider indeed the plane projective curve X with equation $x^{p+1} - y^p z = 0$. Then one may check that $\mathrm{Aut}(X)$ is the multiplicative group $\mathbb{G}_m$, acting via

$$t \cdot [x : y : z] := [t^p x : t^{p+1} y : z].$$

But X has an additional action of $\alpha_p \times \alpha_p$, via

$$(\lambda, \mu) \cdot [x : y : z] := [x : \lambda x + y + \mu z : z].$$

The normalization map of X is given by

$$\eta : \mathbb{P}^1 \longrightarrow X, \quad [u : v] \longmapsto [u^p v, u^{p+1}, v^{p+1}].$$

One may also check that the above action of $\alpha_p \times \alpha_p$ does not lift to an action on $\mathbb{P}^1$. Such a phenomenon was first observed by Seidenberg in a commutative algebra setting (see [Sei66, §5]).

(iii) Let X be an *abelian variety*, i.e., a smooth connected proper algebraic group. Then X is a projective variety and a commutative algebraic group; its group law will be denoted additively, with neutral element $0 \in X(k)$. One may check that Aut_X is again the semi-direct product of X (acting on itself by translation) with the stabilizer of the origin, $\mathrm{Aut}_{X,0}$; the latter is a constant group scheme, which consists of the automorphisms of the algebraic group X. As a consequence, Aut_X is smooth and $\pi_0(\mathrm{Aut}_X) \simeq \mathrm{Aut}_{X,0}$. This group (identified with its group of k-rational points, $\mathrm{Aut}(X, 0)$) consists of the invertible elements in the endomorphism ring $\mathrm{End}(X, 0)$. By [Mum08, §19], the additive group of $\mathrm{End}(X, 0)$ is free of finite rank; it follows that $\mathrm{Aut}(X, 0)$ is finitely presented (see [Bor62]). Note that this group may be infinite, e.g. when $X = Y \times Y$ for some nonzero abelian variety Y; then $\mathrm{Aut}(X, 0)$ contains $\mathrm{GL}_2(\mathbb{Z})$ acting by linear combinations of entries of $x = (y_1, y_2)$.

(iv) Let X be a complete *toric variety*, i.e., X is normal and equipped with an action of a torus T having a dense open orbit and trivial stabilizer. If X is smooth, then the automorphism group scheme Aut_X is smooth and linear, as shown by Demazure via an explicit description of this group scheme in terms of the fan of X (see [Dem70, §4]). For an arbitrary X, a description of the automorphism group $\mathrm{Aut}(X)$ à la Demazure has been obtained by several authors (see [Cox95, Büh96, BG99]). In particular, $\mathrm{Aut}(X)$, or equivalently Aut_X, is still linear in that setting; this also follows from Theorem 1. But the smoothness of Aut_X seems to be an open question.

2.2. **The Picard variety.** Let X be a scheme. The *Picard group* $\mathrm{Pic}(X)$ is the set of isomorphism classes of line bundles (or invertible sheaves) over X. The tensor product of line bundles yields a structure of commutative group on $\mathrm{Pic}(X)$, with neutral element the class of the structure sheaf $\mathcal{O}_X$. The inverse of the class of a line bundle L is the class of the dual line bundle $L^\vee$. Every morphism of schemes $f : X' \to X$ defines a pull-back homomorphism $f^* : \mathrm{Pic}(X) \to \mathrm{Pic}(X')$.

Like for automorphisms, we may define a family of line bundles over X, parameterized by a scheme S, to be a line bundle over $X \times S$. The isomorphism

classes of such families yield a contravariant functor from schemes to commutative groups, $S \mapsto \mathrm{Pic}(X \times S)$ (defined on morphisms via pull-back). But this functor has no chances to be representable, as it incorporates information on S via the pull-back map

$$\mathrm{pr}_2^* : \mathrm{Pic}(S) \longrightarrow \mathrm{Pic}(X \times S)$$

associated with the projection $\mathrm{pr}_2 : X \times S \to S$. (In fact, $S \mapsto \mathrm{Pic}(S)$ is not representable). If X has a k-rational point x, then pr_2 has a section $s \mapsto (x, s)$, and hence pr_2^* is injective. This motivates the following:

Definition 2.6. The *Picard functor* is the commutative group functor

$$\mathbf{Pic}_X : S \longmapsto \mathrm{Pic}(X \times S)/\mathrm{pr}_2^* \mathrm{Pic}(S).$$

Theorem 2.7. *Let X be a proper scheme. If $\mathcal{O}(X) = k$, then $\mathbf{Pic}_X$ is represented by a (commutative) locally algebraic group.*

This result is due to Murre (see [Mur64, II.15]) via an axiomatic characterization of representable group functors, analogous to that of [MO67]; it applies in particular to all complete varieties. For projective varieties, there is an alternative proof (due to Grothendieck again) via Hilbert schemes of divisors, see [Gro62] and [FGA05, Chap. 9] for a recent exposition. Specifically, the above theorem follows from [FGA05, 9.2.5, 9.4.8] when X is a projective variety.

Under the assumptions of Theorem 2.7, we have an exact sequence analogous to (1.0.1):

$$(2.2.1) \qquad 0 \longrightarrow \mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}(X) \longrightarrow \pi_0 \mathrm{Pic}(X) \longrightarrow 0,$$

where $\mathrm{Pic}^0(X)$ is (the group of k-rational points of) a smooth connected commutative algebraic group: the *Picard variety* of X, parameterizing line bundles up to *algebraic equivalence.* The group of components $\pi_0 \mathrm{Pic}(X)$ is finitely generated (see [SGA6, XIII.5.1]); it is called the *Néron-Severi group* and denoted by $\mathrm{NS}(X)$.

We now define an action of the automorphism group scheme Aut_X on the *Picard scheme* Pic_X. For any scheme S, the group $\mathbf{Aut}_X(S) = \mathrm{Aut}(X \times S/S)$ acts on $\mathrm{Pic}(X \times S)$ by group automorphisms via pull-back. This action fixes pointwise $\mathrm{pr}_2^* \mathrm{Pic}(S)$, and hence induces an action on $\mathrm{Pic}(X \times S)/\mathrm{pr}_2^* \mathrm{Pic}(S) = \mathbf{Pic}_X(S)$ by group automorphisms again. Moreover, for any morphism of schemes $u : S' \to S$, we have a commutative diagram

$$
\begin{array}{ccc}
\mathbf{Aut}_X(S) \times \mathbf{Pic}_X(S) & \longrightarrow & \mathbf{Pic}_X(S) \\
\downarrow & & \downarrow \\
\mathbf{Aut}_X(S') \times \mathbf{Pic}_X(S') & \longrightarrow & \mathbf{Pic}_X(S'),
\end{array}
$$

where the horizontal arrows are the actions, and the vertical arrows are pull-backs. This yields the desired action in view of Yoneda's lemma.

The above action is a scheme-theoretic version of the natural action of $\mathrm{Aut}(X) = \mathrm{Aut}_X(k)$ on $\mathrm{Pic}(X) = \mathrm{Pic}_X(k)$. It follows that $\mathrm{Aut}(X)$ acts algebraically on $\mathrm{Pic}^0(X)$ and $\mathrm{NS}(X)$.

Lemma 2.8. *Let X be a projective variety. Then $\mathrm{Aut}^0(X)$ acts trivially on $\mathrm{NS}(X)$. If X is normal, then $\mathrm{Pic}^0(X)$ is an abelian variety, on which $\mathrm{Aut}^0(X)$ acts trivially as well.*

Proof. Let L be a line bundle over X. Then the *polarization map*

$$(2.2.2) \qquad \tau_L : \mathrm{Aut}(X) \longrightarrow \mathrm{Pic}(X), \quad g \longmapsto g^*(L) \otimes L^{-1}$$

takes the identity to the trivial bundle, and hence $\mathrm{Aut}^0(X)$ to $\mathrm{Pic}^0(X)$. This yields the first assertion.

When X is normal, $\mathrm{Pic}^0(X)$ is projective in view of [FGA05, 9.5.4], and hence is an abelian variety. As a consequence, the automorphism group of the algebraic group $\mathrm{Pic}^0(X)$ is discrete. It follows that the connected algebraic group $\mathrm{Aut}^0(X)$ acts trivially on this group. $\qquad\square$

Remarks 2.9. (i) Given a line bundle L on a normal projective variety X, the image of the polarization map (2.2.2) is fixed pointwise by $\mathrm{Aut}^0(X)$ in view of the above lemma. Thus, we have

$$g_1^* g_2^*(L) \otimes g_1^*(L^\vee) \otimes g_2^*(L^\vee) \otimes L \simeq \mathcal{O}_X$$

for any $g_1, g_2 \in \mathrm{Aut}^0(X)$ (this is the *theorem of the square*). Equivalently, the polarization map restricts to a group homomorphism $\mathrm{Aut}^0(X) \to \mathrm{Pic}^0(X)$.

(ii) If X is an abelian variety, then Pic_X^0 is the dual abelian variety (see [Mum08, §13]). Also, $\mathrm{Aut}^0(X) = X$ by Example 2.5 (iii). In this case, Lemma 2.8 implies that every algebraically trivial line bundle is translation-invariant; the converse holds by [Mum08, §13] again.

Still considering a projective variety X, we obtain an action of $\pi_0 \mathrm{Aut}(X)$ on $\mathrm{NS}(X) = \mathrm{Pic}(X)/\mathrm{Pic}^0(X)$ in view of Lemma 2.8. Thus, $\pi_0 \mathrm{Aut}(X)$ acts on the quotient of $\mathrm{NS}(X)$ by its torsion subgroup, and this quotient is a free abelian group of finite rank: the *Picard number* $\rho(X)$. We denote this "Néron-Severi lattice" by $\mathrm{N}^1(X)$; it may also be seen as the group of line bundles up to *numerical equivalence* (see [FGA05, 9.6.3]), where two line bundles L, M on X are said to be numerically equivalent if the intersection number $L \cdot C$ equals $M \cdot C$ for any curve $C \subset X$. The class of a line bundle L in $\mathrm{N}^1(X)$ is denoted by $[L]_{\mathrm{num}}$.

We may now state the following result, analogous to a theorem of Fujiki and Lieberman on automorphism groups of compact Kähler manifolds (see [Fuj78, Thm. 4.8], [Lie78, Prop. 2.2]):

Theorem 2.10. *Let X be a projective variety, and L an ample line bundle over X. Then the stabilizer of $[L]_{\mathrm{num}}$ under the action of $\pi_0 \operatorname{Aut}(X)$ is finite.*

Proof. Denote by G the stabilizer of $[L]_{\mathrm{num}}$ under the action of $\operatorname{Aut}(X)$. By Lemma 2.8, G contains $\operatorname{Aut}^0(X)$, and hence is closed in $\operatorname{Aut}(X)$. Let $g \in G$, with graph $\Gamma_g \subset X \times X$. Then the Hilbert polynomial of Γ_g, relative to the ample line bundle $L \boxtimes L$ on $X \times X$, is given by

$$n \in \mathbb{Z} \longmapsto \chi(\Gamma_g, L^{\otimes n} \boxtimes L^{\otimes n}) = \chi(X, L^{\otimes n} \otimes g^*(L^{\otimes n})).$$

Since $L^{\otimes n} \otimes g^*(L^{\otimes -n})$ is numerically trivial, we have

$$\chi(X, L^{\otimes n} \otimes g^*(L^{\otimes n})) = \chi(X, L^{\otimes 2n})$$

in view of [FGA05, 9.6.3]. So Γ_g is a k-rational point of the Hilbert scheme $\operatorname{Hilb}^P(X \times X)$, where $P(n) := \chi(X, L^{\otimes 2n})$ is independent of g. As a consequence, the graphs of all elements of G belong to a subscheme of finite type of $\operatorname{Hilb}_{X \times X}$. Thus, G has finitely many components. $\square$

As a direct but noteworthy consequence of the above proposition, we obtain:

Corollary 2.11. *Let X be a projective variety. Then the kernel of the action of $\pi_0 \operatorname{Aut}(X)$ on $\mathrm{N}^1(X)$ is finite.*

To state our next result, recall that the *ample cone* of a projective variety X consists of the positive real multiples of the classes of ample line bundles, in the finite-dimensional real vector space $\mathrm{N}^1(X) \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$. This is an open convex cone, with closure the *nef cone* $\operatorname{Nef}(X)$. Both cones are clearly stable under the action of $\pi_0 \operatorname{Aut}(X)$.

Corollary 2.12. *Let X be a projective variety. If the convex cone $\operatorname{Nef}(X)$ is rational polyhedral, then $\operatorname{Aut}(X)$ is an algebraic group.*

Proof. It suffices to show that $\pi_0 \operatorname{Aut}(X)$ is finite. in view of Theorem 2.10, it suffices in turn to check that $\pi_0 \operatorname{Aut}(X)$ fixes an ample class in $\mathrm{N}^1(X)$. By assumption, $\operatorname{Nef}(X)$ is generated by finitely many boundary rays, which are in turn generated by primitive vectors $e_1, \ldots, e_n \in \mathrm{N}^1(X)$. Thus, the action of $\pi_0 \operatorname{Aut}(X)$ on this lattice permutes $e_1, \ldots, e_n$, and hence fixes $e := e_1 + \cdots + e_n$. Moreover, e lies in the interior of the nef cone; therefore, e is ample. $\square$

The above corollary fails if the nef cone is only assumed to be polyhedral. Indeed, there are examples of (smooth projective) K3 surfaces S with Picard number 2, such that the two boundary rays of $\operatorname{Nef}(S)$ are irrational and $\operatorname{Aut}(S)$ is an infinite discrete group; see [We88].

2.3. **The lifting group.** Throughout this subsection, we consider a projective variety X and a line bundle $\pi : L \to X$. As $\mathcal{O}(X) = k$, the Picard scheme Pic_X exists and is equipped with an action of Aut_X. The stabilizer of the class $[L] \in \mathrm{Pic}(X) = \mathrm{Pic}_X(k)$ is a closed subgroup scheme of Aut_X, that we denote by $\mathrm{Aut}_{X,[L]}$; its reduced subgroup is denoted by $\mathrm{Aut}(X, [L])$.

Note that $\mathrm{Aut}_{X,[L]}$ does not come with an action on the variety L, as it only fixes the isomorphism class of this line bundle. We will construct a central extension of $\mathrm{Aut}_{X,[L]}$ by the multiplicative group $\mathbb{G}_m$, which acts on L by lifting the action of $\mathrm{Aut}_{X,[L]}$ on X. For this, we introduce some additional notation.

The automorphism group of L, viewed as a line bundle over X, is the group of global units $\mathcal{O}(X)^* = k^*$ acting by multiplication on fibers. More generally, for any scheme S, the automorphism group of the pull-back line bundle $\pi \times \mathrm{id}_S : L \times S \to X \times S$ is identified with $\mathcal{O}(S)^*$, since X is proper and $\mathcal{O}(X) = k$. Thus, the automorphism functor of the line bundle L is represented by $\mathbb{G}_m$. Its fixed point subscheme (for its natural action on L) is the zero section L_0, and the restriction

$$\pi^\times : L^\times := L \setminus L_0 \longrightarrow X$$

is a $\mathbb{G}_m$-torsor.

We denote by $\mathbf{Aut}_L^{\mathbb{G}_m}$ the centralizer of $\mathbb{G}_m$ in the automorphism group functor $\mathbf{Aut}_L$: for any scheme S, the group $\mathbf{Aut}_L^{\mathbb{G}_m}(S)$ consists of those $\gamma \in \mathrm{Aut}(L \times S/S)$ such that for any morphism of schemes $u : S' \to S$ and any $f \in \mathcal{O}(S')^* = \mathbb{G}_m(S')$, the S'-automorphism $u^*(\gamma)$ of $L \times S'$ commutes with the action of f by multiplication. Note that $\mathbb{G}_m$ (viewed as a group functor) is identified with a central subgroup functor of $\mathbf{Aut}_L^{\mathbb{G}_m}$.

Proposition 2.13. *With the above notation and assumptions, the group functor $\mathbf{Aut}_L^{\mathbb{G}_m}$ is represented by a locally algebraic group, $\mathrm{Aut}_L^{\mathbb{G}_m}$. Moreover, there is an exact sequence of locally algebraic groups*

(2.3.1) $$1 \longrightarrow \mathbb{G}_m \longrightarrow \mathrm{Aut}_L^{\mathbb{G}_m} \xrightarrow{\pi_*} \mathrm{Aut}_{X,[L]} \longrightarrow 1,$$

and π_* (viewed as a morphism of schemes) has a section.

Proof. We argue as in the proof of [BSU13, Thm. 6.2.1]. Consider the projective completion of L,

$$\bar{\pi} : \bar{L} := \mathbb{P}(L \oplus \mathcal{O}_X) \longrightarrow X.$$

This is a projective line bundle over X, equipped with the section at infinity $\bar{L}_\infty$. Given a scheme S, every $\gamma \in \mathrm{Aut}^{\mathbb{G}_m}(L \times S/S)$ extends uniquely to an S-automorphism of $\bar{L} \times S$ which normalizes $\bar{L}_\infty \times S$ and commutes with the action of $\mathbb{G}_{m,S}$. This identifies $\mathbf{Aut}_L^{\mathbb{G}_m}$ with the subgroup functor of $\mathbf{Aut}_{\bar{L}}$ which normalizes $\bar{L}_\infty$ and centralizes $\mathbb{G}_m$. Since $\bar{L}$ is a projective variety, its

automorphism functor is represented by a locally algebraic group (Theorem 2.3). It follows that so is $\mathbf{Aut}_L^{\mathbb{G}_m}$ by using [DG70, II.3.6].

We now construct a homomorphism

$$\pi_* : \mathrm{Aut}_L^{\mathbb{G}_m} \longrightarrow \mathrm{Aut}_X.$$

Given S as above, the action of $\mathrm{Aut}_L^{\mathbb{G}_m}(S)$ on $L \times S$ restricts to an action on $L^\times \times S$. For any $\gamma \in \mathrm{Aut}_L^{\mathbb{G}_m}(S)$, the composition

$$L^\times \times S \xrightarrow{\gamma} L^\times \times S \xrightarrow{\pi^\times \times \mathrm{id}} X \times S$$

is invariant under $\mathbb{G}_{m,S}$. Since $\pi^\times \times \mathrm{id}_S$ is a $\mathbb{G}_{m,S}$-torsor, it is a categorical quotient. Thus, there is a unique morphism of S-schemes $g : X \times S \to X \times S$ such that the diagram

$$
\begin{array}{ccc}
L^\times \times S & \xrightarrow{\gamma} & L^\times \times S \\
{\scriptstyle \pi^\times \times \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \pi^\times \times \mathrm{id}} \\
X \times S & \xrightarrow{g} & X \times S
\end{array}
$$

commutes. We then set $g := \pi_*(\gamma)$. Clearly, we have $\pi_*(\gamma_1)\pi_*(\gamma_2) = \pi_*(\gamma_1 \gamma_2)$ for any $\gamma_1, \gamma_2 \in \mathrm{Aut}_L^{\mathbb{G}_m}(S)$, and $\pi_*(\mathrm{id}_{L^\times \times S}) = \mathrm{id}_{X \times S}$; also, π_* is compatible with pull-backs. Thus, π_* yields a homomorphism of group functors, and hence the desired homomorphism.

By construction, the kernel of π_* is the subgroup scheme of automorphisms of the line bundle L, i.e., the multiplicative group $\mathbb{G}_m$. To complete the proof, it suffices to show that the image of $\pi_*(S) : \mathrm{Aut}_L^{\mathbb{G}_m}(S) \to \mathrm{Aut}_X(S)$ equals $\mathrm{Aut}_{X,[L]}(S)$ for any scheme S. Let again $\gamma \in \mathrm{Aut}_L^{\mathbb{G}_m}(S)$ and $g = \pi_*(\gamma)$. The diagram

$$
\begin{array}{ccc}
L \times S & \xrightarrow{\gamma} & L \times S \\
{\scriptstyle \pi \times \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \pi \times \mathrm{id}} \\
X \times S & \xrightarrow{g} & X \times S
\end{array}
$$

also commutes, since $L^\times \times S$ is schematically dense in $L \times S$. It follows that γ defines an isomorphism

$$\eta : L \times S \xrightarrow{\cong} g^*(L \times S)$$

of schemes over $X \times S$, which commutes with the $\mathbb{G}_m$-action. Thus, η is an isomorphism of line bundles over $X \times S$; hence $g \in \mathrm{Aut}_{X,L}(S)$. Conversely, if $g \in \mathrm{Aut}_{X,[L]}(S)$, then we have an isomorphism of line bundles η as above, and hence an S-automorphism of the scheme $L \times S$ which lifts g and commutes with the $\mathbb{G}_{m,S}$-action. $\qquad \square$

Remarks 2.14. (i) As seen in the proof of Proposition 2.13, we may view $\mathbf{Aut}_L^{\mathbb{G}_m}$ as the group functor of pairs (g, γ), where g is an automorphism of X, and $\gamma : g^*(L) \to L$ is an isomorphism of line bundles. The group law is given by $(g_1, \gamma_1)(g_2, \gamma_2) = (g_1 g_2, \gamma_2 \, g_2^*(\gamma_1))$. As a consequence, we obtain an isomorphism of group schemes

$$\mathrm{Aut}_L^{\mathbb{G}_m} \longrightarrow \mathrm{Aut}_{L^\vee}^{\mathbb{G}_m}, \quad (g, \gamma) \longmapsto (g, (\gamma^\vee)^{-1}),$$

which induces the automorphism $t \mapsto t^{-1}$ of the central $\mathbb{G}_m$, and the identity of $\mathrm{Aut}_{X,[L]} = \mathrm{Aut}_{X,[L^\vee]}$.

(ii) Let G be a group scheme acting on X and fixing the isomorphism class of L. The pull-back of the exact sequence (2.3.1) under the corresponding homomorphism $G \to \mathrm{Aut}_{X,[L]}$ is a central extension of group schemes

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \mathcal{G}(L) \xrightarrow{f} G \longrightarrow 1,$$

where f has a section as a morphism of schemes. Note that $\mathcal{G}(L)$ is (locally) algebraic if and only if so is G.

A G-*linearization* of L is a section of f as a homomorphism of group schemes, i.e., an action of G on L which lifts the action on X and commutes with the $\mathbb{G}_m$-action. In view of (i), such a linearization is given by isomorphisms

$$\gamma_g : g^*(L) \xrightarrow{\simeq} L$$

for all schematic points g of G, satisfying the cocycle condition

$$\gamma_{g_1 g_2} = \gamma_{g_2} \, g_2^*(\gamma_{g_1})$$

(this is the original definition of a G-linearization, see [MFK93, 3.1.6]). Also, note that any two G-linearizations of L differ by a homomorphism $G \to \mathbb{G}_m$, i.e., a character of G.

Next, consider the *section ring*

$$R = R(X, L) = \bigoplus_{n=0}^{\infty} H^0(X, L^{\otimes n}).$$

This is a graded algebra, not necessarily finitely generated. Since $R = \mathcal{O}(L^\vee)$, we have a canonical morphism

$$\alpha : L^\vee \longrightarrow \mathrm{Spec}(R)$$

(the *affinization morphism*); clearly, α is dominant. The group scheme $\mathrm{Aut}_L^{\mathbb{G}_m}$ acts naturally on $L^\vee$ (Remark 2.14 (i)); this induces an action on $\mathrm{Spec}(R)$ such that α is equivariant (see Lemma 2.15 below). Also, note that $\alpha_*(\mathcal{O}_{L^\vee}) = \mathcal{O}_{\mathrm{Spec}(R)}$, as follows e.g. from [Har77, Prop. II.5.8].

Denote by 0 the k-rational point of $\mathrm{Spec}(R)$ that corresponds to the maximal homogeneous ideal of R, and let $Y := \mathrm{Proj}(R)$. Then 0 is fixed by the action of $\mathbb{G}_m$ on $\mathrm{Spec}(R)$, and the natural morphism

$$q : \mathrm{Spec}(R) \setminus \{0\} \longrightarrow Y$$

satisfies

$$q_*(\mathcal{O}_{\mathrm{Spec}(R)\setminus\{0\}})^{\mathbb{G}_m} = \mathcal{O}_Y.$$

As a consequence, q is the categorical quotient under the $\mathbb{G}_m$-action. Thus, q is equivariant for the action of $\mathrm{Aut}_L^{\mathbb{G}_m}$, which acts on Y through its quotient $\mathrm{Aut}_{X,[L]}$.

The scheme-theoretic fiber $\alpha^{-1}(0)$ is a closed subscheme of $L^\vee$, stable under $\mathrm{Aut}_L^{\mathbb{G}_m}$ and containing the zero section $L_0^\vee$. Thus, the complement $\alpha^{-1}(0) \setminus L_0^\vee$ is a $\mathbb{G}_m$-torsor over a closed subscheme of X called the *stable base locus* $\mathrm{B}(L)$; the action of $\mathrm{Aut}_{X,[L]}$ normalizes $\mathrm{B}(L)$. This yields a commutative square

(2.3.2)
$$
\begin{array}{ccc}
L^\vee \setminus \alpha^{-1}(0) & \xrightarrow{\ \alpha\ } & \mathrm{Spec}(R) \setminus \{0\} \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle q} \\
X \setminus \mathrm{B}(L) & \xrightarrow{\ \ \ f\ \ \ } & Y,
\end{array}
$$

where $f = f_L$ is $\mathrm{Aut}_{X,[L]}$-equivariant. Since α is dominant, so is f. Moreover, we have

$$f_*(\mathcal{O}_{X\setminus\mathrm{B}(L)}) = \mathcal{O}_Y.$$

Lemma 2.15. *Let G be a locally algebraic group, and X a scheme of finite type.*

  (i) *The canonical map*

$$f_{G,X} : \mathcal{O}(G) \otimes \mathcal{O}(X) \longrightarrow \mathcal{O}(G \times X), \quad (\varphi, \psi) \longmapsto ((g,x) \mapsto \varphi(g)\,\psi(x))$$

     is an isomorphism.

  (ii) *The linear representation of G in $\mathcal{O}(X)$ via $(g \cdot f)(x) = f(g^{-1} \cdot x)$ yields an action of G on $\mathrm{Spec}\,\mathcal{O}(X)$ such that the affinization morphism*

$$\alpha : X \longrightarrow \mathrm{Spec}\,\mathcal{O}(X)$$

     is equivariant.

Proof. (i) Recall that G is the disjoint union of its connected components $g_i G^0$ for a family $(g_i)_{i\in I}$ of $G(k)$, and these components are of finite type. Thus, each $f_{g_i G^0, X}$ is an isomorphism in view of [DG70, I.2.2.6]. The assertion follows from this in view of the decompositions $\mathcal{O}(G) = \prod_{i\in I} \mathcal{O}(g_i G^0)$, $\mathcal{O}(G \times X) = \prod_{i\in I} \mathcal{O}(g_i G^0 \times X)$, and $f_{g,x} = \prod_{i\in I} f_{g_i G^0, X}$.

(ii) This follows from (i) by a standard argument, see e.g. [DG70, II.1.1.5]. $\qquad\square$

Remarks 2.16. (i) The above morphism $f : X \setminus \mathrm{B}(L) \to Y$ may be defined directly as follows: let $s \in H^0(X, L^{\otimes n})$ be a homogeneous element of R. Then f defines an open affine subvariety $D_+(s) \subset \mathrm{Proj}\, R$, and an open subvariety $X_s \subset X$. Moreover, $\mathcal{O}(D_+(s))$ consists of the homogeneous elements of degree 0 in the localization $R[s^{-1}]$, i.e., $\mathcal{O}(D_+(s)) = \bigcup_{m \geq 1} H^0(X, L^{\otimes mn}) s^{-m}$. This yields an injective algebra homomorphism $\mathcal{O}(D_+(s)) \to \mathcal{O}(X_s)$, or equivalently, a dominant morphism $X_s \to D_+(s)$; also, these homomorphisms are compatible when s runs over the homogeneous elements of R. But the equivariance property of f is not easily seen on this definition.

(ii) The group scheme $\mathrm{Aut}^L_{\mathbb{G}_m}$ acts on $\mathrm{Spec}(R)$ by commuting with the action of $\mathbb{G}_m$, and hence acts on each homogeneous component $H^0(X, L^{\otimes n})$ via a linear representation. Thus, the projectivization $\mathbb{P}H^0(X, L^{\otimes n})$ (the space of hyperplanes in $H^0(X, L^{\otimes n})$) may be viewed as a projective representation of $\mathrm{Aut}^{\mathbb{G}_m}_L / \mathbb{G}_m = \mathrm{Aut}_{X,[L]}$. Also, we have a rational map

$$f_n : X \dashrightarrow \mathbb{P}H^0(X, L^{\otimes n})$$

defined on the complement of the base locus $\mathrm{Bs}(L^{\otimes n})$ of the complete linear system associated with L^n. Moreover, $\mathrm{Bs}(L^{\otimes n})$ is stable under $\mathrm{Aut}_{X,[L]}$, and the morphism $X \setminus \mathrm{Bs}(L^{\otimes n}) \to \mathbb{P}H^0(X, L^{\otimes n})$ is equivariant under this group scheme. As a topological space, the base locus $\mathrm{B}(L)$ is the intersection of the $\mathrm{Bs}(L^{\otimes n})$, and $\mathrm{Bs}(L^{\otimes m}) \subset \mathrm{Bs}(L^{\otimes n})$ whenever m divides n. Thus, $\mathrm{B}(L) = \mathrm{Bs}(L^{\otimes n})$ (as topological spaces again) for any sufficiently divisible n. Moreover, the morphism f may be viewed as the filtered inverse limit of the f_n, where the positive integers are ordered by divisibility.

(iii) If L is ample, then the algebra R is finitely generated; moreover, α is proper and restricts to an isomorphism $L^\vee \setminus L_0^\vee \xrightarrow{\simeq} \mathrm{Spec}(R) \setminus \{0\}$. As a consequence, $\mathrm{B}(L) = \emptyset$ and f yields an isomorphism $X \xrightarrow{\simeq} Y$.

Conversely, if α restricts to an isomorphism as above, then L is ample by Grauert's criterion (see [EGA, II.8.8, II.8.9] for these results).

More generally, R is finitely generated whenever L is *semi-ample*, i.e., $L^{\otimes n}$ is generated by its global sections for some $n \geq 1$; equivalently, $\mathrm{B}(L) = \emptyset$. Then we obtain a morphism $f : X \to Y$, where Y is a projective variety and $f_*(\mathcal{O}_X) = \mathcal{O}_Y$; moreover, $L^{\otimes n} = f^*(M)$ for some ample line bundle M on Y and some $n \geq 1$.

(iv) If X is smooth, then its canonical line bundle ω_X is equipped with a natural $\mathrm{Aut}(X)$-linearization. (Indeed, the action of $\mathrm{Aut}(X)$ on X linduces an action on differential forms of top degree, which commutes with the action of $\mathbb{G}_m$). So we obtain an action of $\mathrm{Aut}(X)$ on all the objects constructed above, and on those associated with the anti-canonical line bundle $\omega_X^\vee$.

Theorem 2.17. *Let X be a projective variety, and L an ample line bundle over X. Then $\mathrm{Aut}(X, [L])$ is a linear algebraic group.*

Proof. Let $G := \mathrm{Aut}(X, [L])$. Since G is a closed subgroup of $\mathrm{Aut}(X, [L^{\otimes n}])$ for any $n \geq 1$, we may assume that L is very ample. Then we have a G-equivariant immersion

$$f_1 : X \longrightarrow \mathbb{P}H^0(X, L^{\otimes n}) =: \mathbb{P}^N.$$

Thus, G is isomorphic with the (reduced) normalizer of X in $\mathrm{Aut}(\mathbb{P}^N)$. This normalizer is a closed subgroup of PGL_{N+1}, a linear algebraic group. $\qquad\square$

We now present several consequences of Theorem 2.17. To state the first one, recall that a smooth projective variety X is called *Fano*, if its anti-canonical bundle $\omega_X^\vee$ is ample. In view of Remark 2.16 (iv), we obtain:

Corollary 2.18. *Let X be a Fano variety. Then $\mathrm{Aut}(X)$ is a linear algebraic group.*

Next, we obtain criteria for the *connected* automorphism group to be linear:

Corollary 2.19. *Let X be a projective variety. If $\mathrm{Pic}^0(X)$ is trivial, then $\mathrm{Aut}^0(X)$ is linear.*

Proof. By Lemma 2.8 and the assumption, $\mathrm{Aut}^0(X)$ acts trivially on $\mathrm{Pic}(X)$. In particular, $\mathrm{Aut}^0(X)$ fixes the class of some ample line bundle. So the assertion follows from Theorem 2.17. $\qquad\square$

Recall that every variety X has a universal morphism to an abelian variety: the *Albanese variety*, $\mathrm{Alb}(X)$ (see [Ser60, Thm. 5]). If X is normal and projective, then $\mathrm{Alb}(X)$ is the dual abelian variety of $\mathrm{Pic}^0(X)$, see [FGA05, 9.5.25]. In view of Corollary 2.19, this yields:

Corollary 2.20. *Let X be a normal projective variety. If $\mathrm{Alb}(X)$ is trivial, then $\mathrm{Aut}^0(X)$ is linear.*

This applies in particular to any unirational variety X, since every rational map from a projective space to an abelian variety is constant. Note that the group of components $\pi_0 \mathrm{Aut}(X)$ may be infinite in this setting; this already holds for many smooth projective rational surfaces (see e.g. [CD12]).

2.4. **Automorphisms of fibrations.** Let $f : X \to Y$ be a morphism of schemes. We define an *automorphism of f* as a pair (g, h), where $g \in \mathrm{Aut}(X)$, $h \in \mathrm{Aut}(Y)$ and the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\
X & \xrightarrow{\ f\ } & Y
\end{array}
$$

commutes. The automorphisms of f form a subgroup $\operatorname{Aut}(f) \subset \operatorname{Aut}(X) \times \operatorname{Aut}(Y)$. Moreover, the first projection $\operatorname{Aut}(X) \times \operatorname{Aut}(Y) \to \operatorname{Aut}(X)$ yields an exact sequence of (abstract) groups

$$1 \longrightarrow \operatorname{Aut}(X/Y) \longrightarrow \operatorname{Aut}(f) \longrightarrow \operatorname{Aut}(X).$$

By considering $f \times \operatorname{id}_S : X \times S \to Y \times S$ for any scheme S, we obtain a group functor $\mathbf{Aut}_f$ which lies in an analogous exact sequence of group functors. If X and Y are proper, then $\mathbf{Aut}_f$ is represented by a closed subgroup scheme $\operatorname{Aut}_f \subset \operatorname{Aut}_X \times \operatorname{Aut}_Y$, and we still have an exact sequence of group schemes

$$(2.4.1) \qquad 1 \longrightarrow \operatorname{Aut}_{X/Y} \longrightarrow \operatorname{Aut}_f \longrightarrow \operatorname{Aut}_X.$$

To obtain more information on Aut_f under appropriate additional assumptions, we need the following descent result for actions of locally algebraic groups:

Lemma 2.21. *Let $f : X \to Y$ be a proper morphism of schemes such that $f_*(\mathcal{O}_X) = \mathcal{O}_Y$. Let G be a locally algebraic group acting on X via $\alpha : G \times X \to X$. Then the following conditions are equivalent:*

(i) *There exists an action $\beta : G \times Y \to Y$ such that the diagram*

$$(2.4.2) \qquad \begin{array}{ccc} G \times X & \xrightarrow{\ \alpha\ } & X \\ {\scriptstyle \operatorname{id}_G \times f}\downarrow & & \downarrow{\scriptstyle f} \\ G \times Y & \xrightarrow{\ \beta\ } & Y \end{array}$$

commutes (i.e., f is equivariant).

(ii) *The diagonal action of G on $X \times X$ normalizes the closed subscheme $X \times_Y X$.*

(iii) *The composition $f \circ \alpha$ is constant on the fibers of $\operatorname{id}_G \times f$.*

Under these conditions, β is unique. Moreover, these hold if G is connected.

Proof. The implications (i)$\Rightarrow$(ii) and (ii)$\Rightarrow$(iii) are readily verified.

(iii)$\Rightarrow$(i) By [EGA, II.8.11], there exists a unique morphism β such that the diagram (2.4.2) commutes, provided that (a) $\operatorname{id}_G \times f$ is proper, and (b) $(\operatorname{id}_G \times f)_*(\mathcal{O}_{G \times X}) = \mathcal{O}_{G \times Y}$. But (a) holds as $\operatorname{id}_G \times f$ is obtained from f by the base change $\operatorname{pr}_2 : G \times Y \to Y$. To check (b), choose open affine coverings (U_i) of G and (V_j) of Y. Then the $U_i \times V_j$ form an open affine covering of $G \times Y$, and

$$(\operatorname{id}_G \times f)_*(\mathcal{O}_{G \times X})(U_i \times V_j) = \mathcal{O}(U_i \times f^{-1}(V_j)) = \mathcal{O}(U_i) \otimes \mathcal{O}(f^{-1}(V_j))$$

$$= \mathcal{O}(U_i) \otimes \mathcal{O}(V_j) = \mathcal{O}(U_i \times V_j),$$

where the second equality holds since $f^{-1}(V_j)$ is quasi-compact, and the third one, since $f_*(\mathcal{O}_X) = \mathcal{O}_Y$.

This completes the proof of the existence and uniqueness of the morphism β. That β is an action also follows from the uniqueness assertion in [EGA, II.8.11]. More specifically, the morphism $\beta \circ (e_G \times \mathrm{id}_Y) : Y \to Y$ is the identity, since its composition with f equals f. Also, denoting by $m : G \times G \to G$ the multiplication, the morphisms

$$\beta \circ (m \times \mathrm{id}_Y), \beta \circ (\mathrm{id}_G \times \beta) : G \times G \times Y \longrightarrow Y$$

are equal, since so are their compositions with $\mathrm{id}_{G \times G} \times f$ (note that

$$\mathrm{id}_{G \times G} \times f : G \times G \times X \to G \times G \times Y$$

satisfies (a) and (b) by the above argument).

Finally, (iii) holds when G is connected, as follows from the rigidity lemma in [Mum08, §4]. □

Definition 2.22. A *fibration* is a morphism $f : X \to Y$, where X, Y are projective varieties and $f_*(\mathcal{O}_X) = \mathcal{O}_Y$.

Note that every fibration is a surjective morphism with connected fibres. Conversely, a surjective morphism of projective varieties $f : X \to Y$ with connected fibres is a fibration, if Y is normal and the function field $k(X)$ is separable over $k(Y)$ (this follows from the Stein factorization, see [Har77, Cor. III.11.5]). The latter assumption is satisfied when $\mathrm{char}(k) = 0$.

The fibrations of normal projective varieties are also known as *contractions* in the setting of the Minimal Model Program. Every birational morphism of such varieties is a fibration in view of Zariski's Main Theorem (see [Har77, Cor. III.11.4] and its proof).

Proposition 2.23. *Let $f : X \to Y$ be a fibration. Then the first projection* $\mathrm{pr}_1 : \mathrm{Aut}_f \to \mathrm{Aut}_X$ *is a closed immersion, and its image contains* Aut_X^0.

Proof. Applying Lemma 2.21 to $f \times \mathrm{id}_S$ for any scheme S, we see that pr_1 induces an isomorphism of Aut_f with the normalizer of $X \times_Y X$ in Aut_X; moreover, this normalizer contains Aut_X^0. □

With the assumptions of the above proposition, we obtain an exact sequence of group schemes

$$1 \longrightarrow \mathrm{Aut}_X^0 \longrightarrow \mathrm{Aut}_f \longrightarrow \pi_0 \mathrm{Aut}(f),$$

where $\pi_0 \mathrm{Aut}(f)$ is a subgroup of $\pi_0 \mathrm{Aut}(X)$. We will obtain a geometric interpretation of this subgroup. For this, we recall some basic objects of the Minimal Model Program, refering to [KM98, Chap. 2] for details.

Choose an ample line bundle M on Y, and set $L := f^*(M)$. With the notation of Remark 2.16 (iii), the line bundle L is semi-ample; moreover, the

natural map $R(Y, M) \to R(X, L)$ is an isomorphism by the projection for-mula. As $Y \simeq \operatorname{Proj} R(Y, M)$, it follows that f is identified with the canonical morphism

$$f_L : X \longrightarrow \operatorname{Proj} R(X, L).$$

Also, a curve $C \subset X$ is *contracted by* f (i.e., C is contained in a fiber of f) if and only if $0 = M \cdot f(C) = L \cdot C$.

Denote by $\mathrm{N}_1(X)$ the group of 1-dimensional cycles on X up to numerical equivalence; this is the dual lattice of the Néron-Severi lattice $\mathrm{N}^1(X)$. The numerical equivalence classes of (integral) curves span a convex cone $NE(X)$ in the finite-dimensional real vector space $\mathrm{N}_1(X)_{\mathbb{R}} := \mathrm{N}_1(X) \otimes_{\mathbb{Z}} \mathbb{R}$, with closure $\overline{NE}(X)$. Moreover, $[L]_{\mathrm{num}}$ is a linear form on $\mathrm{N}_1(X)_{\mathbb{R}}$ which is non-negative on $\overline{NE}(X)$. Denote by $F \subset \overline{NE}(X)$ the face defined by $[L]_{\mathrm{num}} = 0$. Then a curve $C \subset X$ is contracted by f if and only if its class in $\mathrm{N}^1(X)$ lies in F. As the fibers of f are geometrically connected, they are uniquely determined by F. In view of Lemma 2.21, it follows that $\operatorname{Aut}(f)$ is the stabilizer of F under the natural action of $\pi_0 \operatorname{Aut}(X)$ on $\mathrm{N}^1(X)$.

This can be reformulated by using the duality between the closed convex cones $\overline{NE}(X) \subset \mathrm{N}_1(X)_{\mathbb{R}}$ and $\operatorname{Nef}(X) \subset \mathrm{N}^1(X)_{\mathbb{R}}$: the face F corresponds to a face $F^{\perp} \subset \operatorname{Nef}(X)$, which may be identified with the image of $\operatorname{Nef}(Y)$ under the (injective) pull-back map $\mathrm{N}^1(Y)_{\mathbb{R}} \to \mathrm{N}^1(X)_{\mathbb{R}}$. Moreover, $\pi_0 \operatorname{Aut}(f)$ is the stabilizer of the face $F^{\perp}$ in $\pi_0 \operatorname{Aut}(X)$; it acts on this face via the homomorphism $\pi_0 \operatorname{Aut}(f) \to \pi_0 \operatorname{Aut}(Y)$ induced by the second projection $\operatorname{Aut}(f) \to \operatorname{Aut}(Y)$.

If the cone $\operatorname{Nef}(X)$ is rational polyhedral, then $\pi_0 \operatorname{Aut}(f)$ is the stabilizer of a single class $[L]_{\mathrm{num}}$, where $L = f^*(M)$ for some ample line bundle M on Y; equivalently, $[L]_{\mathrm{num}}$ lies in the relative interior of $F^{\perp}$ (this may be seen by arguing as in the proof of Corollary 2.12). But this generally fails: consider for example the projection

$$f : X := E \times E \times \mathbb{P}^1 \longrightarrow E \times E =: Y,$$

where E is an elliptic curve. Then $\operatorname{Aut}(f) = \operatorname{Aut}(X)$ has no non-zero fixed point in $\mathrm{N}^1(Y)_{\mathbb{R}} = \mathbb{R} \times \mathbb{R}$, since it acts there by an overgroup of $\mathrm{GL}_2(\mathbb{Z})$.

2.5. **Big line bundles.**

Definition 2.24. Let L be a line bundle on a projective variety X. Then L is *big* if the rational map $f_n : X \dashrightarrow \mathbb{P} H^0(X, L^{\otimes n})$ is birational onto its image for some $n \geq 1$.

See [KM98, Lem. 2.60] for further characterizations of big line bundles. We now record a bigness criterion, to be used in the proof of Theorem 1:

Lemma 2.25. *Consider an effective Cartier divisor D on a projective variety X and let $U := X \setminus \operatorname{Supp}(D)$. If U is affine, then $\mathcal{O}_X(D)$ is big.*

Proof. Denote by $s \in H^0(X, \mathcal{O}_X(D))$ the canonical section; then f is defined on the open subvariety $X_s \subset X$, the complement of the zero locus of s. Let $h_1, \ldots, h_r$ be generators of the k-algebra $\mathcal{O}(U)$. Then there exist positive integers $n_1, \ldots, n_r$ such that $h_i \, s^{n_i} \in H^0(X, \mathcal{O}_X(n_i D))$ for $i = 1, \ldots, r$. As a consequence, $h_i \, s^n \in H^0(X, \mathcal{O}_X(nD))$ for any $n \geq n_1, \ldots, n_r$. It follows that f restricts to a closed immersion $X_s \to \mathbb{P} H^0(X, \mathcal{O}_X(nD))_s$. $\qquad \square$

Next, we obtain a slight generalization of Theorem 2.10:

Proposition 2.26. *Let L a big and nef line bundle over a projective variety X. Then the stabilizer of $[L]_{\mathrm{num}}$ under the action of $\pi_0 \operatorname{Aut}(X)$ is finite.*

Proof. Arguing as in the proof of Theorem 2.10, it suffices to show that $G := \operatorname{Aut}(X, [L]_{\mathrm{num}})$ has finitely many components. For this, we adapt the arguments of [Lie78, Prop. 2.2] and [Zha09, Lem. 2.23].

By Kodaira's lemma, we have $L^{\otimes n} \simeq A \otimes E$ for some positive integer $n \geq 1$, some ample line bundle A and some effective line bundle E on X (see [KM98, Lem 2.60]). Since G is a closed subgroup of $\operatorname{Aut}(X, [L^{\otimes n}]_{\mathrm{num}})$, we may assume that $n = 1$.

Consider the ample line bundle $A \boxtimes A$ on $X \times X$. We will show that the degrees of the graphs $\Gamma_g \subset X \times X$, where $g \in G$, are bounded independently of g. These graphs form a family of normal irreducible algebraic cycles on $X \times X$, parameterized by G (a disjoint union of open and closed smooth varieties). This yields a morphism from G to the "Chow variety" $\operatorname{Chow}_{X \times X}$, which is an open immersion in view of [Kol99, I.3.17, I.6.3]. Since the cycles of any prescribed degree form a subscheme of finite type of $\operatorname{Chow}_{X \times X}$, it follows that G has finitely many components indeed.

Let $d := \dim(X)$ and denote by $L_1 \cdots L_d$ the intersection number of the line bundles $L_1, \ldots, L_d$ on X; also, we denote the line bundles additively. By [FGA05, Thm. 9.6.3], $L_1 \cdots L_d$ only depends on the numerical equivalence classes of $L_1, \ldots, L_d$. With this notation, the degree of Γ_g relative to $A \boxtimes A$ is the self-intersection number $(A + g^* A)^d$. We have

$$(A + g^* A)^d = (A + g^* A)^{d-1} \cdot (L + g^* L) - (A + g^* A)^{d-1} \cdot (E + g^* E).$$

We now use the fact that $L_1 \cdots L_{d-1} \cdot E \geq 0$ for any ample line bundles $L_1, \ldots, L_{d-1}$ (this is a very special case of [Ful, Ex. 12.1.7]), and hence for any nef line bundles $L_1, \ldots, L_{d-1}$ (since the nef cone is the closure of the ample cone). It follows that

$$(A + g^* A)^d \leq (A + g^* A)^{d-1} \cdot (L + g^* L)$$

$$= (A + g^* A)^{d-2} \cdot (L + g^* L)^2 - (A + g^* A)^{d-2} \cdot (L + g^* L) \cdot (E + g^* E).$$

Using again the above fact, this yields

$$(A + g^* A)^d \leq (A + g^* A)^{d-2} \cdot (L + g^* L)^2.$$

Proceeding inductively, we obtain

$$(A + g^*A)^d \leq (L + g^*L)^d.$$

Since g^*L is numerically equivalent to L, this yields the desired bound

$$(A + g^*A)^d \leq 2^d L^d.$$

$\square$

Finally, we also obtain a slight generalization of Theorem 2.17, which will also be used in the proof of Theorem 1:

Proposition 2.27. *Let L be a big line bundle on a normal projective variety X. Then $\mathrm{Aut}(X, [L])$ is a linear algebraic group.*

Proof. Since $\mathrm{Aut}(X, [L])$ is a closed subgroup of $\mathrm{Aut}(X, [L^{\otimes n}])$, we may assume that $f_1 : X \dashrightarrow \mathbb{P}\, H^0(X, L)$ is birational onto its closed image Y_1.

Consider the blowing-up of the base locus of L, as in [Har77, Ex. II.7.17.3], and its normalization $\tilde{X}$. Denote by

$$\pi : \tilde{X} \longrightarrow X$$

the resulting birational morphism, and let $\tilde{L} := \pi^*(L)$. Then $\tilde{L} = L' \otimes E$, where L' is a line bundle generated by its subspace of global sections $H^0(X, L)$, and E is an effective line bundle. Moreover, the action of $G := \mathrm{Aut}(X, [L])$ on X lifts to an action on $\tilde{X}$ which fixes both classes $[\tilde{L}]$ and $[L']$. We now claim that the image of the resulting homomorphism $\rho : G \to \mathrm{Aut}(\tilde{X}, [L'])$ is closed.

Since X is normal, we have $\pi_*(\mathcal{O}_{\tilde{X}}) = \mathcal{O}_X$ by Zariski's Main Theorem. Thus, we may view G as a subgroup scheme of the automorphism group of the fibration π. In view of Lemma 2.21 and of the isomorphism $L \simeq \pi_*(\tilde{L})$, the image of this subgroup in $\mathrm{Aut}(\tilde{X})$ is the intersection of the stabilizer of $[\tilde{L}]$ and the normalizer of the fibered product $\tilde{X} \times_X \tilde{X}$. In particular, this image is closed, proving our second claim.

By this claim, we may replace the pair (X, L) with $(\tilde{X}, L')$; equivalently, we may assume that L is generated by its global sections. Then $L = f^*(M)$, where $f = f_L : X \to Y$ is obtained from the Stein factorization of $f_1 : X \to Y_1$, and M is an ample line bundle on the normal projective variety Y; also, recall that f is G-equivariant. In particular, L is nef, and hence $\mathrm{Aut}(X, [L]_{\mathrm{num}})$ is an algebraic group (Proposition 2.26). Since G is a closed subgroup of $\mathrm{Aut}(X, [L]_{\mathrm{num}})$, it is algebraic as well. Thus, the image of the homomorphism $G \to \mathrm{Aut}\,\mathbb{P}H^0(X, L)$ is closed. As f is birational, the kernel of this homomorphism is trivial; this completes the proof. $\square$

As a consequence, if a projective variety X is *weak Fano* (that is, X is smooth and ω_X^{-1} is big and nef), then $\mathrm{Aut}(X)$ is a linear algebraic group.

3. Proof of Theorem 1

We will use the following preliminary result:

Lemma 3.1. *Let* $f : X \to Y$ *be a birational morphism, where* X *and* Y *are normal projective varieties. Assume that the action of* $\mathrm{Aut}(Y)$ *on* Y *lifts to an action on* X. *Then the corresponding homomorphism* $\rho : \mathrm{Aut}(Y) \to \mathrm{Aut}(X)$ *is a closed immersion.*

Proof. Note that ρ yields a section of the projection $\mathrm{pr}_2 : \mathrm{Aut}(f) \to \mathrm{Aut}(Y)$. Also, the scheme-theoretic kernels of ρ and π_2 are trivial, since f restricts to an isomorphism over a dense open subvariety of Y. It follows that pr_2 is an isomorphism, and ρ restricts to an isomorphism of $\mathrm{Aut}(Y)$ with the image of $\mathrm{Aut}(f)$ under the first projection pr_1. The latter image is closed by Proposition 2.23. $\qquad\square$

Next, we obtain a characteristic-free analogue of the Tits fibration:

Lemma 3.2. *Let* G *be a connected linear algebraic group, and* H *a subgroup scheme. For any* $n \geq 1$, *denote by* G_n *the* n-*th infinitesimal neighborhood of the neutral element* e *in* G.

  (i) *The union of the* G_n *(*$n \geq 1$*) is dense in* G.
  (ii) *For* $n \gg 0$, *we have* $N_G(H^0) = N_G(H^0 \cap G_n)$.
  (iii) *The canonical morphism* $f : G/H \to G/N_G(H^0)$ *is affine.*

Proof. (i) Denote by $\mathfrak{m} \subset \mathcal{O}(G)$ the maximal ideal of e; then

$$G_n = \mathrm{Spec}(\mathcal{O}(G)/\mathfrak{m}^n)$$

for all n. Thus, the assertion is equivalent to $\bigcap_{n \geq 1} \mathfrak{m}^n = 0$. This is proved in [Jan03, I.7.17]; we recall the argument for the reader's convenience. If G is smooth, then $\mathcal{O}(G)$ is a noetherian domain, hence the assertion follows from Nakayama's lemma. For an arbitrary G, we have an isomorphism of algebras

$$\mathcal{O}(G) \simeq \mathcal{O}(G_{\mathrm{red}}) \otimes A,$$

where A is a local k-algebra of finite dimension as a k-vector space (see [DG70, III.3.6.4]). Thus, $\mathfrak{m} = \mathfrak{m}_1 \otimes 1 + 1 \otimes \mathfrak{m}_2$, where $\mathfrak{m}_1$ (resp. $\mathfrak{m}_2$) denotes the maximal ideal of e in $\mathcal{O}(G_{\mathrm{red}})$ (resp. the maximal ideal of A). We may choose an integer $N \geq 1$ such that $\mathfrak{m}_2^N = 0$; then $\mathfrak{m}^n \subset \mathfrak{m}_1^{n-N} \otimes A$ for all $n \geq N$. Since G_{red} is smooth and connected, we have $\bigcap_{n \geq 1} \mathfrak{m}_1^n = 0$ by the above step; this yields the assertion.

  (ii) Denote by $I \subset \mathcal{O}(G)$ the ideal of H^0; then the ideal of $H^0 \cap G_n$ is $I + \mathfrak{m}^n$. Consider the action of G on itself by conjugation; this yields a linear representation of G on $\mathcal{O}(G)$ for which $N_G(H^0) = N_G(I)$ and $N_G(H^0 \cap G_n) = N_G(I + \mathfrak{m}^n)$. Given an algebra R, the group of R-points $N_G(I + \mathfrak{m}^n)(R)$ consists

of those $g \in G(R)$ such that

$$\text{Int}(g)((I + \mathfrak{m}^n) \otimes R \otimes S) = (I + \mathfrak{m}^n) \otimes R \otimes S$$

for any algebra S (see [DG70, II.1.3.5]). This is equivalent to

$$\text{Int}(g)(I \otimes R \otimes S) \subset (I + \mathfrak{m}^n) \otimes R \otimes S$$

for all such S. Since $\mathfrak{m}^n \subset \mathfrak{m}^{n-1}$, it follows that $N_G(I + \mathfrak{m}^n) \subset N_G(I + \mathfrak{m}^{n-1})$. This decreasing sequence of closed subschemes of G must stop, say at

$$K = N_G(I + \mathfrak{m}^{n_0}) = \bigcap_{n \geq 0} N_G(I + \mathfrak{m}^n).$$

Let $g \in K(R)$, then $J := \text{Int}(g)(I \otimes R \otimes S)$ is an ideal of $R \otimes S$. Denote by $\bar{J}$ its image in the quotient $(\mathcal{O}(G)/I) \otimes R \otimes S = \mathcal{O}(H^0) \otimes R \otimes S$. Then $\bar{J} \subset \bar{\mathfrak{m}}^n \otimes R \otimes S$ for all n, where $\bar{\mathfrak{m}}$ is the maximal ideal of $\mathcal{O}(H^0)$ associated with the neutral element. Since $\bigcap_{n \geq 1}(\bar{\mathfrak{m}}^n \otimes R \otimes S) = (\bigcap_{n \geq 1} \bar{\mathfrak{m}}^n) \otimes R \otimes S$ and $\bigcap_{n \geq 1} \bar{\mathfrak{m}}^n = 0$ by (i), we obtain $\bar{J} = 0$, that is, $\text{Int}(g)(I \otimes R \otimes S) \subset I \otimes R \otimes S$. Replacing g with g^{-1}, we see that $g \in N_G(I) = N_G(H^0)$. This shows that $K \subset N_G(H^0)$; the opposite inclusion follows from the fact that G normalizes each G_n.

(iii) We have a commutative triangle

$$\begin{array}{ccc} G/H^0 & & \\ \varphi \downarrow & \searrow^{\psi} & \\ G/H & \xrightarrow{f} & G/N_G(H^0), \end{array}$$

where φ is a torsor under H/H^0 (a finite constant group), and ψ is a torsor under $N_G(H^0)/H^0$ (a linear algebraic group). In particular, ψ is affine. Let U be an open affine subscheme of $G/N_G(H^0)$. Then $\psi^{-1}(U) \subset G/H^0$ is open, affine and stable under H/H^0. Hence $f^{-1}(U) = \varphi(\psi^{-1}(U))$ is affine. $\square$

Remarks 3.3. (i) The first infinitesimal neighborhood G_1 may be identified with the Lie algebra $\mathfrak{g}$ of G; thus, $G_1 \cap H^0 = H_1^0$ is identified with the Lie algebra $\mathfrak{h}$ of H.

If $\text{char}(k) = 0$, then $N_G(H^0) = N_G(\mathfrak{h})$, since every subgroup scheme of G is uniquely determined by its Lie subalgebra (see e.g. [DG70, II.6.2.1]). As a consequence, the morphism $f : G/H \to G/N_G(H^0)$ is the $\mathfrak{g}$-anticanonical fibration considered in [HO84, I.2.7].

By contrast, if $\text{char}(k) > 0$ then the natural morphism $G/H \to G/N_G(\mathfrak{h})$ is not necessarily affine (see e.g. [Bri10, Ex. 5.6]). In particular, the inclusion $N_G(H^0) \subset N_G(\mathfrak{h})$ may be strict.

(ii) If $\text{char}(k) = p > 0$, then G_{p^n} is the nth Frobenius kernel of G, as defined for example in [Jan03, I.9.4]; in particular, G_{p^n} is a normal infinitesimal subgroup

scheme of G. Then the above assertion (i) just means that the union of the iterated Frobenius kernels is dense in G.

We may now prove Theorem 1. Recall its assumptions: X is a normal projective variety on which a smooth connected linear algebraic group G acts with an open dense orbit. Since the variety G is rational, X is unirational and hence $\mathrm{Aut}^0(X)$ is linear (Corollary 2.20). We may thus assume that $G = \mathrm{Aut}^0(X)$. In particular, G is a normal subgroup of $\mathrm{Aut}(X)$.

Denote by $X_0 \subset X$ the open G-orbit; then X_0 is normalized by $\mathrm{Aut}(X)$. Choose $x_0 \in X_0(k)$ and denote by H its stabilizer in G. Then we have $X_0 = G \cdot x_0 \simeq G/H$ equivariantly for the G-action. We also have $X_0 = \mathrm{Aut}(X) \cdot x_0 \simeq \mathrm{Aut}(X)/\mathrm{Aut}(X, x_0)$ equivariantly for the $\mathrm{Aut}(X)$-action. As a consequence, $\mathrm{Aut}(X) = G \cdot \mathrm{Aut}(X, x_0)$.

Next, choose a positive integer n such that $N_G(H^0) = N_G(H^0 \cap G_n)$ (Lemma 3.2). The action of $\mathrm{Aut}(X)$ on G by conjugation normalizes G_n and induces a linear representation of $\mathrm{Aut}(X)$ in $V := \mathcal{O}(G_n)$, a finite-dimensional vector space. The ideal of $H^0 \cap G_n$ is a subspace $W_0 \subset V$, with stabilizer $N_G(H^0)$ in G. We consider W_0 as a k-rational point of the Grassmannian $\mathrm{Grass}(V)$ parameterizing linear subspaces of V of the appropriate dimension. The linear action of $\mathrm{Aut}(X)$ on V yields an action on $\mathrm{Grass}(V)$. The subgroup scheme $\mathrm{Aut}(X, x_0)$ fixes W_0, since it normalizes $\mathrm{Aut}(X, x_0)^0 = H^0$. Thus, we obtain

$$\mathrm{Aut}(X) \cdot W_0 = G \cdot \mathrm{Aut}(X, x_0) \cdot W_0 = G \cdot W_0 \simeq G/N_G(H^0).$$

As a consequence, the morphism $f : G/H \to G/N_G(H^0)$ yields an $\mathrm{Aut}(X)$-equivariant morphism

$$\tau : X_0 \longrightarrow Y,$$

where Y denotes the closure of $\mathrm{Aut}(X) \cdot W_0$ in $\mathrm{Grass}(V)$.

We may view τ as a rational map $X \dashrightarrow Y$. Let X' denote the normalization of the graph of this rational map, i.e., of the closure of X_0 embedded diagonally in $X \times Y$. Then X' is a normal projective variety equipped with an action of $\mathrm{Aut}(X)$ and with an equivariant morphism $f : X' \to X$ which restricts to an isomorphism above the open orbit X_0. By Lemma 3.1, the image of $\mathrm{Aut}(X)$ in $\mathrm{Aut}(X')$ is closed; thus, it suffices to show that $\mathrm{Aut}(X')$ is a linear algebraic group. So we may assume that τ extends to a morphism, that we will still denote by $\tau : X \to Y$.

Next, consider the boundary, $\partial X := X \setminus X_0$, that we view as a closed reduced subscheme of X; it is normalized by $\mathrm{Aut}(X)$. Thus, the action of $\mathrm{Aut}(X)$ on X lifts to an action on the blowing-up of X along ∂X, and on its normalization. Using Lemma 3.1 again, we may further assume that ∂X is the support of an effective Cartier divisor Δ, normalized by $\mathrm{Aut}(X)$; thus, the line bundle $\mathcal{O}_X(\Delta)$ is $\mathrm{Aut}(X)$-linearized.

We also have an $\mathrm{Aut}(X)$-linearized line bundle M on Y, the pull-back of $\mathcal{O}(1)$ under the Plücker embedding of $\mathrm{Grass}(V)$. Thus, there exists a positive integer m and a nonzero section $t \in H^0(Y, M^{\otimes m})$ which vanishes identically on the boundary $\partial Y := Y \setminus G \cdot W_0$. Then $L := \tau^*(M)$ is an $\mathrm{Aut}(X)$-linearized line bundle on X, equipped with a nonzero section $s := \tau^*(t)$ which vanishes identically on $\tau^{-1}(\partial Y) \subset \partial X$. Denote by D (resp. E) the divisor of zeroes of s (resp. t). Then $D + \Delta$ is an effective Cartier divisor on X, and we have

$$X \setminus \mathrm{Supp}(D + \Delta) = X_0 \setminus \mathrm{Supp}(D) = f^{-1}(G \cdot W_0 \setminus \mathrm{Supp}(E)).$$

Since $\partial Y \subset \mathrm{Supp}(E)$, we have $G \cdot W_0 \setminus \mathrm{Supp}(E) = Y \setminus \mathrm{Supp}(E)$. The latter is affine as M is ample. Since the morphism f is affine (Lemma 3.2), it follows that $X \setminus \mathrm{Supp}(D + \Delta)$ is affine as well. Hence $D + \Delta$ is big (Lemma 2.25). Also, $\mathcal{O}_X(D + \Delta) = L \otimes \mathcal{O}_X(\Delta)$ is $\mathrm{Aut}(X)$-linearized. In view of Proposition 2.27, we conclude that $\mathrm{Aut}(X)$ is a linear algebraic group.

4. Proof of Theorem 2

By [Bri10, Thm. 2], there exists a G-equivariant morphism

$$f : X \longrightarrow G/H$$

for some subgroup scheme $H \subset G$ such that $H \supset G_{\mathrm{aff}}$ and H/G_{aff} is finite; equivalently, H is affine and G/H is an abelian variety. Then the natural map $A = G/G_{\mathrm{aff}} \to G/H$ is an isogeny. Denote by Y the scheme-theoretic fiber of f at the origin of G/H; then Y is normalized by H, and the action map $G \times Y \to X$, $(g, y) \mapsto g \cdot y$ factors through an isomorphism

$$G \times^H Y \xrightarrow{\simeq} X,$$

where $G \times^H Y$ denotes the quotient of $G \times Y$ by the action of H via

$$h \cdot (g, y) := (gh^{-1}, h \cdot y).$$

This is the fiber bundle associated with the fppf H-torsor $G \to G/H$ and the H-scheme Y. The above isomorphism identifies f with the morphism $G \times^H Y \to G/H$ obtained from the projection $G \times Y \to G$.

We now obtain a reduction to the case where G is *anti-affine*, i.e., $\mathcal{O}(G) = k$. Recall that G has a largest anti-affine subgroup scheme G_{ant}; moreover, G_{ant} is smooth, connected and centralizes G (see [DG70, III.3.8]). We have the Rosenlicht decomposition $G = G_{\mathrm{ant}} \cdot G_{\mathrm{aff}}$; moreover, the scheme-theoretic intersection $G_{\mathrm{ant}} \cap G_{\mathrm{aff}}$ contains $(G_{\mathrm{ant}})_{\mathrm{aff}}$, and the quotient $(G_{\mathrm{ant}} \cap G_{\mathrm{aff}})/(G_{\mathrm{ant}})_{\mathrm{aff}}$ is finite (see [BSU13, Thm. 3.2.3]). As a consequence,

$$G = G_{\mathrm{ant}} \cdot H \simeq (G_{\mathrm{ant}} \times H)/(G_{\mathrm{ant}} \cap H) \quad \text{and} \quad G/H \simeq G_{\mathrm{ant}}/(G_{\mathrm{ant}} \cap H).$$

Thus, we obtain an isomorphism of schemes

$$(4.0.1) \qquad\qquad G_{\mathrm{ant}} \times^{G_{\mathrm{ant}} \cap H} Y \xrightarrow{\simeq} X,$$

and an isomorphism of abstract groups

$$\text{(4.0.2)} \qquad \text{Aut}_{\text{gp}}^H(G) \xrightarrow{\simeq} \text{Aut}_{\text{gp}}^{G_{\text{ant}} \cap H}(G_{\text{ant}}).$$

Next, we construct an action of the subgroup $\text{Aut}_{\text{gp}}^H(G) \subset \text{Aut}_{\text{gp}}^{G_{\text{aff}}}(G)$ on X. Let $\gamma \in \text{Aut}^H(G)$. Then $\gamma \times \text{id}$ is an automorphism of $G \times Y$, equivariant under the above action of H. Moreover, the quotient map $G \times Y \to G \times^H Y$ is a fppf H-torsor, and hence a categorical quotient. It follows (as in the proof of Proposition 2.13) that $\gamma \times \text{id}$ defines an automorphism δ of $G \times^H Y = X$, such that the diagram

$$
\begin{array}{ccc}
G \times Y & \xrightarrow{\gamma \times \text{id}} & G \times Y \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\
X & \xrightarrow{\quad \delta \quad} & X
\end{array}
$$

commutes. Clearly, the assignement $\gamma \mapsto \delta$ yields a homomorphism of abstract groups

$$\varphi : \text{Aut}_{\text{gp}}^H(G) \longrightarrow \text{Aut}(X).$$

We also have a natural homomorphism

$$\psi : \text{Aut}_{\text{gp}}^H(G) \longrightarrow \text{Aut}_{\text{gp}}(G/H) = \text{Aut}(G/H, 0).$$

By construction, f is equivariant under the action of $\text{Aut}_{\text{gp}}^H(G)$ on X via φ, and its action on G/H via ψ. Moreover, we have in $\text{Aut}(X)$

$$\varphi(\gamma)\, g\, \varphi(\gamma)^{-1} = \gamma(g)$$

for all $\gamma \in \text{Aut}_{\text{gp}}^H(G)$ and $g \in \text{Aut}(X)$. In particular, the image of the homomorphism φ normalizes G.

Lemma 4.1. *With the above notation, ψ is injective. Moreover, $\text{Aut}_{\text{gp}}^H(G)$ is a subgroup of finite index of $\text{Aut}_{\text{gp}}^{G_{\text{aff}}}(G)$.*

Proof. To show both assertions, we may assume that G is anti-affine by using the isomorphism (4.0.2). Then G is commutative and hence its endomorphisms (as an algebraic group) form a ring, $\text{End}_{\text{gp}}(G)$. Let $\gamma \in \text{Aut}_{\text{gp}}^H(G)$ such that $\psi(\gamma) = \text{id}_{G/H}$; then $\gamma - \text{id}_G \in \text{End}_{\text{gp}}(G)$ restricts to the trivial endomorphism of H and takes G to H. Thus, $\gamma - \text{id}_G$ factors through a homomorphism $G/H \to H$; but every such homomorphism is trivial, since G/H is an abelian variety and H is affine. So $\gamma - \text{id}_G = 0$, proving the first assertion.

For the second assertion, we may replace H with any larger subgroup scheme K such that K/G_{aff} is finite. By [Bri15, Thm. 1.1], there exists a finite subgroup scheme $F \subset G$ such that $H = G_{\text{aff}} \cdot F$. Let n denote the order of F; then F is contained in the n-torsion subgroup scheme $G[n]$.

We now claim that $G[n]$ is finite for any integer $n > 0$. If $\text{char}(k) = p > 0$, then G is a semi-abelian variety (see [BSU13, Prop. 5.4.1]) and the claim

follows readily. If $\text{char}(k) = 0$, then G is an extension of a semi-abelian variety by a vector group U (see [BSU13, §5.2]. Since the multiplication map n_U is an isomorphism, we have $G[n] \simeq (G/U)[n]$; this completes the proof of the claim.

By this claim, we may replace H with the larger subgroup scheme $G_{\text{aff}} \cdot G[n]$ for some integer $n > 0$. Then the restriction map $\rho : \text{Aut}_{\text{gp}}(G) \to \text{Aut}_{\text{gp}}(G[n])$ yields an exact sequence

$$1 \longrightarrow \text{Aut}_{\text{gp}}^H(G) \longrightarrow \text{Aut}_{\text{gp}}^{G_{\text{aff}}}(G) \longrightarrow \text{Aut}_{\text{gp}}(G[n]).$$

Moreover, the image of ρ is finite. Indeed, it is contained in the image of the analogous map $\text{End}_{\text{gp}}(G) \to \text{End}_{\text{gp}}(G[n])$, and the latter image is a finitely generated abelian group (since so is $\text{End}_{\text{gp}}(G)$ in view of [BSU13, Lem. 5.1.3]) and is n-torsion (since so is $\text{End}_{\text{gp}}(G[n])$). This completes the proof. $\qquad\square$

Lemma 4.2. *With the above notation, φ is injective. Moreover, its image is the subgroup of $\text{Aut}(X)$ which normalizes G and centralizes Y; this subgroup intersects G trivially.*

Proof. Let $\gamma \in \text{Aut}_{\text{gp}}^H(G)$ such that $\varphi(\gamma) = \text{id}_X$. In view of the equivariance of f, it follows that $\psi(\gamma) = \text{id}_{G/H}$. Thus, $\gamma = \text{id}_G$ by Lemma 4.1. So φ is injective; we will therefore identify $\text{Aut}_{\text{gp}}^H(G)$ with its image in $\text{Aut}(X)$.

As already noticed, $\text{Aut}_{\text{gp}}^H(G)$ normalizes G; it also centralizes Y by construction. Conversely, let $u \in \text{Aut}(X)$ normalizing G and centralizing Y. Since the normalizer of Y in G equals H, the commutator $uhu^{-1}h^{-1}$ centralizes Y for any schematic point $h \in H$. Also, $uhu^{-1}h^{-1} \in G$. But in view of (4.0.1), we have $X = G_{\text{ant}} \cdot Y$, where G_{ant} is central in G. It follows that $uhu^{-1}h^{-1}$ centralizes X. Hence u centralizes H, and acts on G by conjugation via some $\gamma \in \text{Aut}_{\text{gp}}^H(G)$. For any schematic points $g \in G$, $y \in Y$, we have $u(g \cdot y) = ugu^{-1}u(y) = \gamma(g)u(y) = \gamma(g)y$, that is, $u = \varphi(\gamma)$.

It remains to show that $\text{Aut}_{\text{gp}}^H(G)$ intersects G trivially. Let $\gamma \in \text{Aut}_{\text{gp}}^H(G)$ such that $\varphi(\gamma) \in G$. Then γ acts on G/H by a translation, and fixes the origin. So $\psi(\gamma) = \text{id}_{G/H}$, and $\gamma = \text{id}_G$ by using Lemma 4.1 again. $\qquad\square$

Now Theorem 2 follows by combining Lemmas 4.1 and 4.2.

Remarks 4.3. (i) With the above notation, $\text{Aut}_{\text{gp}}^H(G)$ is the group of integer points of a linear algebraic group defined over the field of rational numbers. Indeed, we may reduce to the case where G is anti-affine, as in the beginning of the proof of Lemma 4.1. Then $\text{Aut}_{\text{gp}}^H(G)$ is the group of units of the ring $\text{End}_{\text{gp}}^H(G) =: R$; moreover, the additive group of R is free of finite rank (as follows from [BSU13, Lem. 5.1.3]). So the group of units of the finite-dimensional $\mathbb{Q}$-algebra $R_{\mathbb{Q}} := R \otimes_{\mathbb{Z}} \mathbb{Q}$ is a closed subgroup of $\text{GL}(R_{\mathbb{Q}})$ (via the regular representation), and its group of integer points relative to the lattice $R \subset R_{\mathbb{Q}}$ is just $\text{Aut}_{\text{gp}}^H(G)$.

In other terms, $\mathrm{Aut}^H_{\mathrm{gp}}(G)$ is an *arithmetic group*; it follows e.g. that this group is finitely presented (see [Bor62]).

(ii) The commensurability class of $\mathrm{Aut}^{G_{\mathrm{aff}}}_{\mathrm{gp}}(G)$ is an isogeny invariant. Consider indeed an isogeny $u : G \to G'$, i.e., u is a faithfully flat homomorphism and its kernel F is finite. Then u induces isogenies $G_{\mathrm{aff}} \to G'_{\mathrm{aff}}$ and $G_{\mathrm{ant}} \to G'_{\mathrm{ant}}$. In view of Lemma 4.1, we may thus assume that G and G' are anti-affine. Then F is contained in the n-torsion subgroup scheme $G[n]$ for some integer $n \geq 1$, and this subgroup scheme is finite (as seen in the proof of Lemma 4.1). As a consequence, there exists an isogeny $v : G' \to G$ such that $v \circ u = n_G$ (the multiplication map by n in G). Then we have a natural homomorphism

$$u_* : \mathrm{Aut}^{G_{\mathrm{aff}} \cdot F}_{\mathrm{gp}}(G) \longrightarrow \mathrm{Aut}^{G'_{\mathrm{aff}}}_{\mathrm{gp}}(G')$$

which lies in a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Aut}^{G_{\mathrm{aff}} \cdot G[n]}_{\mathrm{gp}}(G) \xrightarrow{\;u_*\;} \mathrm{Aut}^{G'_{\mathrm{aff}} \cdot \mathrm{Ker}(v)}_{\mathrm{gp}}(G') \xrightarrow{\;v_*\;} \mathrm{Aut}^{G_{\mathrm{aff}}}_{\mathrm{gp}}(G) \\
\downarrow \qquad\qquad\qquad \downarrow \qquad\qquad\qquad \\
\mathrm{Aut}^{G_{\mathrm{aff}} \cdot F}_{\mathrm{gp}}(G) \xrightarrow{\;u_*\;} \mathrm{Aut}^{G'_{\mathrm{aff}}}_{\mathrm{gp}}(G'),
\end{array}
$$

where all arrows are injective, and the images of the vertical arrows have finite index (see Lemma 4.1 again). Moreover, the image of the homomorphism $v_* \circ u_* = (v \circ u)_* = (n_G)_*$ has finite index as well, since this homomorphism is identified with the inclusion of $\mathrm{Aut}^{G_{\mathrm{aff}} \cdot G[n]}_{\mathrm{gp}}(G)$ in $\mathrm{Aut}^{G_{\mathrm{aff}}}_{\mathrm{gp}}(G)$. This yields our assertion.

(iii) There are many examples of smooth connected algebraic groups G such that $\mathrm{Aut}^{G_{\mathrm{aff}}}_{\mathrm{gp}}(G)$ is infinite. The easiest ones are of the form $A \times H$, where A is an abelian variety such that $\mathrm{Aut}(A, 0)$ is infinite, and H is a smooth connected linear algebraic group. To construct further examples, let G be a smooth connnected algebraic group, $\alpha : G \to A$ the quotient homomorphism by G_{aff}, and $h : A \to B$ a non-zero homomorphism to an abelian variety. Then $G' := G \times B$ is a smooth connected algebraic group as well, and the assignment $(g, b) \mapsto (g, b + h(\alpha(g)))$ defines an automorphism of G' of infinite order, which fixes pointwise $G'_{\mathrm{aff}} = G_{\mathrm{aff}}$.

5. Proof of Theorem 3

By [Bri10, Thm. 3], the Albanese morphism of X is of the form

$$f : X \longrightarrow G/H,$$

where H is a linear subgroup scheme of G containing G_{aff}. Thus, we have as in Section 4

$$X \simeq G \times^H Y \simeq G_{\mathrm{ant}} \times^K Y,$$

where $K := G_{\mathrm{ant}} \cap H$, and Y denotes the fiber of f at the origin of G/H. Then Y is a closed subscheme of X, normalized by H.

Lemma 5.1.   (i) G_{ant} *is the largest anti-affine subgroup of* $\mathrm{Aut}(X)$. *In particular, G_{ant} is normal in* $\mathrm{Aut}(X)$.
  (ii) *f is a fibration.*
 (iii) *If K is smooth, then Y is a normal projective variety, almost homogeneous under H_{red}^0.*

Proof. (i) Let $J \subset \mathrm{Aut}(X)$ be an anti-affine subgroup scheme. Then J is a central subgroup scheme of $\mathrm{Aut}^0(X)$, and hence so is $J \cdot G_{\mathrm{ant}}$. Since we have $\mathcal{O}(J \cdot G_{\mathrm{ant}}) \subset \mathcal{O}(J \times G_{\mathrm{ant}})$, we see that $J \cdot G_{\mathrm{ant}}$ is anti-affine as well. To prove the assertion, we may thus assume that $G_{\mathrm{ant}} \subset J$. Then $G' := G_{\mathrm{aff}} \cdot J$ is a smooth connected subgroup scheme of $\mathrm{Aut}^0(X)$ containing G as a normal subgroup scheme. Since X is almost homogeneous under G', we obtain $G/H = G'/H'$ for some linear subgroup scheme H' of G'. Thus, $G' = G \cdot H$, and $G'/G \simeq H'/H$ is linear. But $G'/G = J/J \cap G_{\mathrm{aff}}$ is anti-affine. Hence $G' = G$, and $J = G_{\mathrm{ant}}$.

(ii) Consider the Stein factorization $f = g \circ h$, where $g : X' \to G/H$ is finite, and $h : X \to X'$ is a fibration. By Lemma 2.21, the G-action on X descends to a unique action on X' such that g is equivariant. As a consequence, X' is a normal projective variety, almost homogeneous under this action, and h is equivariant as well. Let $H' \subset G$ denote the stabilizer of a point of the open G'-orbit in X. Then $H' \subset H$ and the homogeneous space H/H' is finite. It follows that $H' \supset G_{\mathrm{aff}}$, i.e., G/H' is an abelian variety. Thus, so is X'; then $X' = X$ and $h = \mathrm{id}$ by the universal property of the Albanese variety.

(iii) Consider the largest reduced subscheme $Y_{\mathrm{red}} \subset Y$. Since K is smooth, it normalizes Y_{red}. Thus, $G_{\mathrm{ant}} \times^K Y_{\mathrm{red}}$ may be viewed as a closed subscheme of $G_{\mathrm{ant}} \times^K Y = X$, with the same k-rational points (since $Y_{\mathrm{red}}(k) = Y(k)$). It follows that $X = G_{\mathrm{ant}} \times^K Y_{\mathrm{red}}$, i.e., Y is reduced.

Next, let $\eta : \tilde{Y} \to Y$ denote the normalization map. Then the action of K lifts uniquely to an action on $\tilde{Y}$. Moreover, the resulting morphism $G_{\mathrm{ant}} \times^K \tilde{Y} \to X$ is finite and birational, hence an isomorphism. It follows that Y is normal. Since Y is connected and closed in X, it is a projective variety.

It remains to show that Y is almost homogenous under $H_{\mathrm{red}}^0 =: H'$ (a smooth connected linear algebraic group). Since the homogeneous space H/H' is finite, so is the natural map $G/H' \to G/H$; as a consequence, this map is an isogeny and G/H' is an abelian variety. We have a cartesian square

$$
\begin{array}{ccc}
X' := G \times^{H'} Y & \longrightarrow & G/H' \\
\downarrow & & \downarrow \\
X = G \times^H Y & \longrightarrow & G/H.
\end{array}
$$

As a consequence, X' is a normal projective variety, almost homogeneous under G. Its open G-orbit intersects Y along an open subvariety, which is unique orbit of H'.   □

We denote by $\mathrm{Aut}(X, Y)$ the normalizer of Y in $\mathrm{Aut}(X)$; then $\mathrm{Aut}(X, Y)$ is the stabilizer of the origin for the action of $\mathrm{Aut}(X)$ on $\mathrm{Alb}(X) = G/H$. Likewise, we denote by $\mathrm{Aut}_{\mathrm{gp}}(G_{\mathrm{ant}}, K)$ (resp. $\mathrm{Aut}(Y, K)$) the normalizer of K in $\mathrm{Aut}_{\mathrm{gp}}(G_{\mathrm{ant}})$ (resp. $\mathrm{Aut}(Y)$).

Lemma 5.2.   (i) *There is an exact sequence*

(5.0.3) $\pi_0(K) \longrightarrow \pi_0 \mathrm{Aut}(X) \longrightarrow \pi_0 \mathrm{Aut}(X, Y) \longrightarrow 0.$

   (ii) *We have a closed immersion*

(5.0.4) $\iota : \mathrm{Aut}(X, Y) \longrightarrow \mathrm{Aut}_{\mathrm{gp}}(G_{\mathrm{ant}}, K) \times \mathrm{Aut}(Y, K)$

      with image consisting of the pairs (γ, v) such that $\gamma|_K = \mathrm{Int}(v)|_K$.

Proof. (i) Since the normal subgroup scheme G_{ant} of $\mathrm{Aut}(X)$ acts transitively on $G/H = \mathrm{Aut}(X)/\mathrm{Aut}(X, Y)$, we have $\mathrm{Aut}(X) = G_{\mathrm{ant}} \cdot \mathrm{Aut}(X, Y)$. Moreover, $G \cap \mathrm{Aut}(X, Y) = H$, hence $G_{\mathrm{ant}} \cap \mathrm{Aut}(X, Y) = K$. Thus, we obtain $\mathrm{Aut}^0(X) = G_{\mathrm{ant}} \cdot \mathrm{Aut}^0(X, Y)$ and

$$\pi_0 \mathrm{Aut}(X) = \mathrm{Aut}(X)/\mathrm{Aut}^0(X) = \mathrm{Aut}(X, Y)/(G_{\mathrm{ant}} \cdot \mathrm{Aut}^0(X, Y) \cap \mathrm{Aut}(X, Y))$$

$$= \mathrm{Aut}(X, Y)/(G_{\mathrm{ant}} \cap \mathrm{Aut}(X, Y)) \cdot \mathrm{Aut}^0(X, Y) = \mathrm{Aut}(X, Y)/K \cdot \mathrm{Aut}^0(X, Y).$$

This yields readily the desired exact sequence.

   (ii) Let $u \in \mathrm{Aut}(X, Y)$, and v its restriction to Y. Since u normalizes G_{ant}, we have $u(g \cdot y) = \mathrm{Int}(u)(g) \, g \cdot y = \mathrm{Int}(u)(g) \cdot v(y)$ for all schematic points $g \in G_{\mathrm{ant}}$ and $y \in Y$. Moreover, u normalizes K, hence $\mathrm{Int}(u) \in \mathrm{Aut}_{\mathrm{gp}}(G_{\mathrm{ant}}, K)$. Since $g \cdot y = gh^{-1} \cdot h \cdot y$ for any schematic point $h \in H$, we obtain $v(h \cdot y) = \mathrm{Int}(u)(h) \cdot v(y)$, i.e., $\mathrm{Int}(u) = \mathrm{Int}(v)$ on K. Thus, u is uniquely determined by the pair $(\gamma := \mathrm{Int}(u), v)$, and this pair satisfies the assertion. Conversely, any such pair yields an automorphism u of X normalizing Y, via $u(g \cdot y) := \gamma(g) \cdot v(y)$.   □

Lemma 5.3. *The projection $\rho : \mathrm{Aut}(X, Y) \to \mathrm{Aut}_{\mathrm{gp}}(G_{\mathrm{ant}}, K)$ induces an exact sequence*

$$\pi_0 \mathrm{Aut}^K(Y) \longrightarrow \pi_0 \mathrm{Aut}(X, Y) \longrightarrow I \longrightarrow 1,$$

where I denotes the subgroup of $\mathrm{Aut}_{\mathrm{gp}}(G_{\mathrm{ant}}, K)$ consisting of those γ such that $\gamma|_K = \mathrm{Int}(v)|_K$ for some $v \in \mathrm{Aut}(Y, K)$.

Proof. By Lemma 5.2 (ii), we have an exact sequence

$$1 \longrightarrow \mathrm{Aut}^K(Y) \longrightarrow \mathrm{Aut}(X, Y) \xrightarrow{\ \rho\ } I \longrightarrow 1.$$

Moreover, the connected algebraic group $\mathrm{Aut}(X,Y)^0$ centralizes G_{ant} in view of [BSU13, Lem. 5.1.3]; equivalently, $\mathrm{Aut}(X,Y)^0 \subset \mathrm{Ker}(\rho)$. This readily yields the assertion. $\qquad\square$

We now assume that K is *smooth*; this holds e.g. if $\mathrm{char}(k) = 0$. Then $\mathrm{Aut}(Y)$ is a linear algebraic group by Theorem 1 and Lemma 5.1 (iii). Thus, so is the subgroup scheme $\mathrm{Aut}^K(Y)$, and hence $\pi_0\,\mathrm{Aut}^K(Y)$ is finite. In view of Lemmas 5.2 and 5.3, it follows that $\pi_0\,\mathrm{Aut}(X)$ *is commensurable with* I.

To analyze the group I, we consider the homomorphism

$$\pi : I \longrightarrow \mathrm{Aut}_{\mathrm{gp}}(K), \quad \gamma \longmapsto \gamma|_K$$

with kernel $\mathrm{Aut}_{\mathrm{gp}}^K(G_{\mathrm{ant}})$. Since K is a commutative linear algebraic group, it has a unique decomposition

$$K \simeq D \times U,$$

where D is diagonalizable and U is unipotent. Thus, we have

$$\mathrm{Aut}_{\mathrm{gp}}(G_{\mathrm{ant}}, K) = \mathrm{Aut}_{\mathrm{gp}}(G_{\mathrm{ant}}, D) \cap \mathrm{Aut}_{\mathrm{gp}}(G_{\mathrm{ant}}, U)$$

and

$$\mathrm{Aut}_{\mathrm{gp}}(Y, K) = \mathrm{Aut}_{\mathrm{gp}}(Y, D) \cap \mathrm{Aut}_{\mathrm{gp}}(Y, U).$$

Moreover,

$$\mathrm{Aut}_{\mathrm{gp}}(K) \simeq \mathrm{Aut}_{\mathrm{gp}}(D) \times \mathrm{Aut}_{\mathrm{gp}}(U).$$

Under the latter identification, the image of π is contained in the product of the images of the natural homomorphisms

$$\pi_D : \mathrm{Aut}(Y, D) \longrightarrow \mathrm{Aut}_{\mathrm{gp}}(D), \quad \pi_U : \mathrm{Aut}(Y, U) \longrightarrow \mathrm{Aut}_{\mathrm{gp}}(U).$$

The kernel of π_D (resp. π_U) contains $\mathrm{Aut}^D(Y)$ (resp. $\mathrm{Aut}^U(Y)$; also, the quotient $\mathrm{Aut}(Y, D)/\mathrm{Aut}^D(Y)$ is finite in view of the rigidity of diagonalizable group schemes(see [DG70, II.5.5.10]). Thus, the image of π_D is finite as well.

If $\mathrm{char}(k) > 0$, then G_{ant} is a semi-abelian variety (see [BSU13, Prop. 5.4.1]) and hence U is finite. In view of the assumption that K is smooth, U is smooth as well, hence $\mathrm{Aut}_{\mathrm{gp}}(U)$ is finite. Thus, so is the image of π. Therefore, $\pi_0\,\mathrm{Aut}(X)$ is commensurable with $\mathrm{Aut}_{\mathrm{gp}}^K(G)$, and hence with $\mathrm{Aut}_{\mathrm{gp}}^{G_{\mathrm{aff}}}(G)$ by Lemma 4.1. This completes the proof of Theorem 3 in positive characteristics, under the additional assumption that K is smooth.

To handle the case where $\mathrm{char}(k) > 0$ and K is no longer assumed smooth, we consider the nth Frobenius kernel $I_n \subset G_{\mathrm{ant}}$, where n is a positive integer. Then $I_n \cap K$ is the nth Frobenius kernel of K; thus, the image of K in G_{ant}/I_n is smooth for $n \gg 0$ (see [DG70, III.3.6.10]). Also, $\mathrm{Aut}(X)$ normalizes I_n (since it normalizes G_{ant}), and hence acts on the quotient X/I_n. The latter is a normal projective variety, almost homogeneous under G/I_n (as follows from the results in [Bri17, §2.4]). Moreover, $(G/I_n)_{\mathrm{ant}} = G_{\mathrm{ant}}/I_n$ and we have

$$\mathrm{Alb}(X/I_n) \simeq \mathrm{Alb}(X)/I_n \simeq G_{\mathrm{ant}}/I_n K \simeq (G_{\mathrm{ant}}/I_n)/(K/I_n \cap K),$$

where $K/I_n \cap K$ is smooth for $n \gg 0$.

We now claim that the homomorphism $\mathrm{Aut}(X) \to \mathrm{Aut}(X/I_n)$ is bijective on k-rational points. Indeed, every $v \in \mathrm{Aut}(X/I_n)(k)$ extends to a unique automorphism of the function field $k(X)$, since this field is a purely inseparable extension of $k(X/I_n)$. As X is the normalization of X/I_n in $k(X)$, this implies the claim.

It follows from this claim that the induced map $\pi_0 \mathrm{Aut}(X) \to \pi_0 \mathrm{Aut}(X/I_n)$ is an isomorphism. Likewise, every algebraic group automorphism of G_{ant} induces an automorphism of G_{ant}/I_n and the resulting map $\mathrm{Aut}_{\mathrm{gp}}(G_{\mathrm{ant}}) \to \mathrm{Aut}_{\mathrm{gp}}(G_{\mathrm{ant}}/I_n)$ is an isomorphism, which restricts to an isomorphism

$$\mathrm{Aut}_{\mathrm{gp}}^{K}(G_{\mathrm{ant}}) \xrightarrow{\simeq} \mathrm{Aut}_{\mathrm{gp}}^{K/I_n \cap K}(G_{\mathrm{ant}}/I_n).$$

This yields a reduction to the case where K is smooth, and hence completes the proof of Theorem 3 in positive characteristics.

It remains to treat the case where $\mathrm{char}(k) = 0$. Then the exact sequence

$$0 \longrightarrow D \times U \longrightarrow G_{\mathrm{ant}} \longrightarrow A \longrightarrow 0$$

is classified by a pair of injective homomorphisms

$$X^*(D) \longrightarrow \widehat{A}(k), \quad U^\vee \longrightarrow H^1(A, \mathcal{O}_A) = \mathrm{Lie}(\widehat{A}),$$

where $X^*(D)$ denotes the character group of D, and $\widehat{A}$ stands for the dual abelian variety of A (see [BSU13, §5.5]). This defines a finitely generated subgroup $\Lambda \subset \widehat{A}(k)$ and a subspace $V \subset \mathrm{Lie}(\widehat{A})$. Moreover, $\mathrm{Aut}_{\mathrm{gp}}(G_{\mathrm{ant}})$ stabilizes U, and is identified with the subgroup of $\mathrm{Aut}_{\mathrm{gp}}(\widehat{A})$ which stabilizes Λ and V. This identifies I with the subgroup of $\mathrm{Aut}_{\mathrm{gp}}(\widehat{A}, \Lambda, V)$ consisting of those γ such that $\gamma|_V \in \mathrm{Aut}(Y, K)|_V$, where $\mathrm{Aut}(Y, K)$ acts on V via the dual of its representation in U. Note that $\mathrm{Aut}(Y, K)|_V$ is an algebraic subgroup of $\mathrm{GL}(V)$. Thus, the proof of Theorem 3 will be completed by the following result due to Gaël Rémond:

Lemma 5.4. *Assume that $\mathrm{char}(k) = 0$. Let A be an abelian variety. Let Λ be a finitely generated subgroup of $A(k)$. Let V be a vector subspace of $\mathrm{Lie}(A)$. Let G be an algebraic subgroup of $\mathrm{GL}(V)$. Let*

$$\Gamma := \{\gamma \in \mathrm{Aut}_{\mathrm{gp}}(A, \Lambda, V) \mid \gamma|_V \in G\}.$$

Then Γ is an arithmetic group.

Proof. By the Lefschetz principle, we may assume that k is a subfield of $\mathbb{C}$.

As k is algebraically closed, there is no difference between the automorphisms of A and those of its extension to $\mathbb{C}$. Thus, we may assume that $k = \mathbb{C}$.

We denote $W := \mathrm{Lie}(A)$, $L \subset W$ its period lattice, and L' the subgroup of W containing L such that $\Lambda = L'/L$; then L' is a free abelian group of finite rank. Let

$$\Gamma' := \mathrm{Aut}(L) \times \mathrm{Aut}(L') \subset G' := \mathrm{Aut}(L \otimes \mathbb{Q}) \times \mathrm{Aut}(L' \otimes \mathbb{Q}).$$

If we choose bases of L and L' of rank r and s say then this inclusion reads $\mathrm{GL}_r(\mathbb{Z}) \times \mathrm{GL}_s(\mathbb{Z}) \subset \mathrm{GL}_r(\mathbb{Q}) \times \mathrm{GL}_s(\mathbb{Q})$. We see G' as the group of $\mathbb{Q}$-points of the algebraic group $\mathrm{GL}_r \times \mathrm{GL}_s$ over $\mathbb{Q}$. To show that Γ is arithmetic, it suffices to show that it is isomorphic with the intersection in G' of Γ' with the $\mathbb{Q}$-points of some algebraic subgroup G'' of $\mathrm{GL}_r \times \mathrm{GL}_s$ defined over $\mathbb{Q}$.

Now it is enough to ensure that $G''(\mathbb{Q})$ is the set of pairs $(\varphi, \psi) \in G'$ satisfying the following conditions (where $\varphi_{\mathbb{R}}$ stands for the extension $\varphi \otimes \mathrm{id}_{\mathbb{R}}$ of φ to $W = L \otimes \mathbb{R}$) :

 (1) $\varphi_{\mathbb{R}}$ is a $\mathbb{C}$-linear endomorphism of W,
 (2) $\varphi_{\mathbb{R}}(L' \otimes \mathbb{Q}) \subset L' \otimes \mathbb{Q}$,
 (3) $\varphi_{\mathbb{R}}|_{L' \otimes \mathbb{Q}} = \psi$,
 (4) $\varphi_{\mathbb{R}}(V) \subset V$,
 (5) $\varphi_{\mathbb{R}}|_V \in G$.

Indeed, if $(\varphi, \psi) \in \Gamma'$ satisfies these five conditions then $\varphi \in \mathrm{Aut}(L)$ induces an automorphism of A thanks to (1), it stabilizes $\Lambda \otimes \mathbb{Q}$ because of (2) and then Λ itself by (3), since $\psi \in \mathrm{Aut}(L')$. With (4) it stabilizes V and (5) yields that it lies in Γ.

We are thus reduced to showing that these five conditions define an algebraic subgroup of $\mathrm{GL}_r \times \mathrm{GL}_s$ (over $\mathbb{Q}$). But the subset of $\mathrm{M}_r(\mathbb{Q}) \times \mathrm{M}_s(\mathbb{Q})$ consisting of pairs (φ, ψ) satisfying (1), (2) and (3) is a sub-$\mathbb{Q}$-algebra, so its group of invertible elements comes indeed from an algebraic subgroup of $\mathrm{GL}_r \times \mathrm{GL}_s$ over $\mathbb{Q}$.

On the other hand, (4) and (5) clearly define an algebraic subgroup of $\mathrm{GL}_r \times \mathrm{GL}_s$ over $\mathbb{R}$. But as we are only interested in $\mathbb{Q}$-points, we may replace this algebraic subgroup by the (Zariski) closure of its intersection with the $\mathbb{Q}$-points $\mathrm{GL}_r(\mathbb{Q}) \times \mathrm{GL}_s(\mathbb{Q})$. This closure is an algebraic subgroup of $\mathrm{GL}_r \times \mathrm{GL}_s$ over $\mathbb{Q}$ and the $\mathbb{Q}$-points are the same. $\square$

Remark 5.5. Assume that $\mathrm{char}(k) = 0$. Then by the above arguments, $\pi_0 \mathrm{Aut}(X)$ is commensurable with $\mathrm{Aut}_{\mathrm{gp}}^{G_{\mathrm{aff}}}(G)$ whenever K is *diagonalizable*, e.g., when G is a semi-abelian variety. But this fails in general. Consider indeed a non-zero abelian variety A and its universal vector extension,

$$0 \longrightarrow U \longrightarrow G \longrightarrow A \longrightarrow 0.$$

Then G is a smooth connected algebraic group, and $G_{\mathrm{aff}} = U \simeq H^1(A, \mathcal{O}_A)^{\vee}$ is a vector group of the same dimension as A. Moreover, G is anti-affine (see [BSU13, Prop. 5.4.2]). Let $Y := \mathbb{P}(U \oplus k)$ be the projective completion of

U. Then the action of U on itself by translation extends to an action on Y, and $X := G \times^U Y$ is a smooth projective equivariant completion of G by a projective space bundle over A. One may check that $\mathrm{Aut}^U_{\mathrm{gp}}(G)$ is trivial, and $\mathrm{Aut}(X) \simeq \mathrm{Aut}(A)$; in particular, the group $\pi_0 \mathrm{Aut}(X) \simeq \mathrm{Aut}(A, 0)$ is not necessarily finite.

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