NOTES

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A Short Proof of the Simple Continued Fraction Expansion of *e*

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1. INTRODUCTION. In [**3**], Euler analyzed the Ricatti equation to prove that the number e has the continued fraction expansion

$$e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \dots] = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cdots}}}}}.$$

The pattern becomes more elegant if one replaces the initial 2 with 1, 0, 1, which yields the equivalent continued fraction

$$[1, 0, 1, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, \dots] \tag{1}$$

because

$$1 + \cfrac{1}{0 + \cfrac{1}{1 + \cdots}} = 2 + \cdots .$$

One of the most interesting proofs is due to Hermite; it arose as a byproduct of his proof of the transcendence of e in [**5**]. (See [**6**] for an exposition by Olds.) The purpose of this note is to present an especially short and direct variant of Hermite's proof and to explain some of the motivation behind it.

Consider any continued fraction $[a_0, a_1, a_2, \dots]$. Its ith convergent is defined to be the continued fraction $[a_0, a_1, \dots, a_i]$. One of the most fundamental facts about continued fractions is that the ith convergent equals p_i/q_i, where p_i and q_i can be calculated recursively using

$$p_n = a_n p_{n-1} + p_{n-2}, \qquad q_n = a_n q_{n-1} + q_{n-2},$$

starting from the initial conditions $p_0 = a_0$, $q_0 = 1$, $p_1 = a_0 a_1 + 1$, and $q_1 = a_1$. For a proof see [**4**, p. 130] or any introduction to continued fractions.

2. PROOF OF THE EXPANSION. Let $[a_0, a_1, a_2, \dots]$ be the continued fraction (1). In other words, $a_{3i+1} = 2i$ and $a_{3i} = a_{3i+2} = 1$, so p_i and q_i are as follows:

i	0	1	2	3	4	5	6	7	8
p_i	1	1	2	3	8	11	19	87	106
q_i	1	0	1	1	3	4	7	32	39

(Note that $q_1 = 0$ so p_1/q_1 is undefined, but that will not be a problem.) Then p_i and q_i satisfy the recurrence relations

$$\begin{aligned} p_{3n} &= p_{3n-1} + p_{3n-2}, & q_{3n} &= q_{3n-1} + q_{3n-2}, \\ p_{3n+1} &= 2np_{3n} + p_{3n-1}, & q_{3n+1} &= 2nq_{3n} + q_{3n-1}, \\ p_{3n+2} &= p_{3n+1} + p_{3n}, & q_{3n+2} &= q_{3n+1} + q_{3n}. \end{aligned}$$

To verify that the continued fraction (1) equals e, we must prove that

$$\lim_{i\to\infty} \frac{p_i}{q_i} = e.$$

Define the integrals

$$\begin{aligned} A_n &= \int_0^1 \frac{x^n(x-1)^n}{n!} e^x\, dx, \\ B_n &= \int_0^1 \frac{x^{n+1}(x-1)^n}{n!} e^x\, dx, \\ C_n &= \int_0^1 \frac{x^n(x-1)^{n+1}}{n!} e^x\, dx. \end{aligned}$$

Proposition 1. *For $n \ge 0$, $A_n = q_{3n}e - p_{3n}$, $B_n = p_{3n+1} - q_{3n+1}e$, and $C_n = p_{3n+2} - q_{3n+2}e$.*

Proof. In light of the recurrence relations cited earlier, we need only verify the initial conditions $A_0 = e - 1$, $B_0 = 1$, and $C_0 = 2 - e$ (which are easy to check) and prove the three recurrence relations

$$A_n = -B_{n-1} - C_{n-1}, \tag{2}$$

$$B_n = -2nA_n + C_{n-1}, \tag{3}$$

$$C_n = B_n - A_n. \tag{4}$$

Of course, (4) is trivial. To prove (2) (i.e., $A_n + B_{n-1} + C_{n-1} = 0$) integrate both sides of

$$\frac{x^n(x-1)^n}{n!}e^x + \frac{x^n(x-1)^{n-1}}{(n-1)!}e^x + \frac{x^{n-1}(x-1)^n}{(n-1)!}e^x = \frac{d}{dx}\left(\frac{x^n(x-1)^n}{n!}e^x\right),$$

which follows immediately from the product rule for derivatives. To prove (3) (i.e., $B_n + 2nA_n - C_{n-1} = 0$) integrate both sides of

$$\frac{x^{n+1}(x-1)^n}{n!}e^x + 2n\frac{x^n(x-1)^n}{n!}e^x - \frac{x^{n-1}(x-1)^n}{(n-1)!}e^x = \frac{d}{dx}\left(\frac{x^n(x-1)^{n+1}}{n!}e^x\right),$$

which follows from the product rule and some additional manipulation. This completes the proof. ■

The recurrences (2) and (3) can also be proved by integration by parts.

Theorem 1. $e = [1, 0, 1, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, \dots]$.

Proof. Clearly A_n, B_n, and C_n tend to 0 as $n \to \infty$. It follows from Proposition 1 that

$$\lim_{i\to\infty} q_i e - p_i = 0.$$

Because $q_i \geq 1$ when $i \geq 2$, we see that

$$e = \lim_{i\to\infty} \frac{p_i}{q_i} = [1, 0, 1, 1, 2, 1, 1, 4, 1, \dots],$$

as desired. ■

3. MOTIVATION. The most surprising aspect of this proof is the integral formulas, which have no apparent motivation. The difficulty is that the machinery that led to them has been removed from the final proof. Hermite wrote down the integrals while studying Padé approximants, a context in which it is easier to see how one might think of them.

Padé approximants are certain rational function approximations to a power series. They are named after Hermite's student Padé, in whose 1892 thesis [**7**] they were studied systematically. However, for the special case of the exponential function they are implicit in Hermite's 1873 paper [**5**] on the transcendence of e.

We will focus on the power series

$$e^z = \sum_{k\geq 0} \frac{z^k}{k!},$$

since it is the relevant one for our purposes. A *Padé approximant* to e^z of type (m, n) is a rational function $p(z)/q(z)$ with $p(z)$ and $q(z)$ polynomials such that $\deg p(z) \leq m$, $\deg q(z) \leq n$, and

$$\frac{p(z)}{q(z)} = e^z + O\left(z^{m+n+1}\right)$$

as $z \to 0$. In other words, the first $m + n + 1$ coefficients in the Taylor series of $p(z)/q(z)$ agree with those for e^z. One cannot expect more agreement than that: $p(z)$ has $m + 1$ coefficients and $q(z)$ has $n + 1$, so there are $m + n + 2$ degrees of freedom in toto, one of which is lost because $p(z)$ and $q(z)$ can be scaled by the same factor without changing their ratio. Thus, one expects to be able to match $m + n + 1$ coefficients. (Of course this argument is not rigorous.)

It is easy to see that there can be only one Padé approximant of type (m, n): if $r(z)/s(z)$ is another, then

$$\frac{p(z)s(z) - q(z)r(z)}{q(z)s(z)} = \frac{p(z)}{q(z)} - \frac{r(z)}{s(z)} = O\left(z^{m+n+1}\right).$$

Because $p(z)s(z) - q(z)r(z)$ vanishes to order $m + n + 1$ at $z = 0$ but has degree at most $m + n$ (assuming $\deg r(z) \le m$ and $\deg s(z) \le n$), it must vanish identically. Thus, $p(z)/q(z) = r(z)/s(z)$.

We have no need to deal with the existence of Padé approximants here, because it will follow from a later argument. One can compute the Padé approximants of e^z by solving simultaneous linear equations to determine the coefficients of the numerator and denominator, after normalizing so the constant term of the denominator is 1.

The usefulness of Padé approximants lies in the fact that they provide a powerful way to approximate a power series. Those of type $(m, 0)$ are simply the partial sums of the series, and the others are equally natural approximations. If one is interested in approximating e, it makes sense to plug $z = 1$ into the Padé approximants for e^z and see what happens.

Let $r_{m,n}(z)$ denote the Padé approximant of type (m, n) for e^z. Computing continued fractions reveals that

$$\begin{aligned} r_{1,1}(1) &= [2, 1], \\ r_{1,2}(1) &= [2, 1, 2], \\ r_{2,1}(1) &= [2, 1, 2, 1], \\ r_{2,2}(1) &= [2, 1, 2, 1, 1], \\ r_{2,3}(1) &= [2, 1, 2, 1, 1, 4], \\ r_{3,2}(1) &= [2, 1, 2, 1, 1, 4, 1], \\ r_{3,3}(1) &= [2, 1, 2, 1, 1, 4, 1, 1], \end{aligned}$$

etc. In other words, when we set $z = 1$, the Padé approximants of types (n, n), $(n, n + 1)$, and $(n + 1, n)$ appear to give the convergents to the continued fraction of e. There is no reason to think Hermite approached the problem quite this way, but his paper [**5**] does include some numerical calculations of approximations to e, and it is plausible that his strategy was informed by patterns he observed in the numbers.

It is not clear how to prove this numerical pattern directly from the definitions. However, Hermite found an ingenious way to derive the Padé approximants from integrals, which can be used to prove it. In fact, it will follow easily from Proposition 1.

It is helpful to reformulate the definition as follows. For a Padé approximant of type (m, n), we are looking for polynomials $p(z)$ and $q(z)$ of degrees at most m and n, respectively, such that $q(z)e^z - p(z) = O\left(z^{m+n+1}\right)$ as $z \to 0$. In other words, the function

$$z \mapsto \frac{q(z)e^z - p(z)}{z^{m+n+1}}$$

must be holomorphic. (Here that is equivalent to being bounded for z near 0. No complex analysis is needed in this article.)

One way to recognize a function as being holomorphic is to write it as a suitable integral. For example, because

$$\frac{(z-1)e^z + 1}{z^2} = \int_0^1 x e^{zx}\, dx,$$

it is clear that $z \mapsto ((z-1)e^z + 1)/z^2$ is holomorphic. Of course, that is unnecessary for such a simple function, but Hermite realized that this technique was quite powerful.

It is not clear how he thought of it, but everyone who knows calculus has integrated an exponential times a polynomial, and one can imagine he simply remembered that the answer has exactly the form we seek.

Lemma 1. *Let $r(x)$ be a polynomial of degree k. Then there are polynomials $q(z)$ and $p(z)$ of degree at most k such that*

$$\int_0^1 r(x)e^{zx}\,dx = \frac{q(z)e^z - p(z)}{z^{k+1}}.$$

Specifically,

$$q(z) = r(1)z^k - r'(1)z^{k-1} + r''(1)z^{k-2} - \cdots$$

and

$$p(z) = r(0)z^k - r'(0)z^{k-1} + r''(0)z^{k-2} - \cdots.$$

Proof. Integration by parts implies that

$$\int_0^1 r(x)e^{zx}\,dx = \frac{r(1)e^z - r(0)}{z} - \frac{1}{z}\int_0^1 r'(x)e^{zx}\,dx,$$

from which the desired result follows by induction. ■

To get a Padé approximant $p(z)/q(z)$ of type (m, n), we want polynomials $p(z)$ and $q(z)$ of degrees m and n, respectively, such that

$$z \mapsto \frac{q(z)e^z - p(z)}{z^{m+n+1}}$$

is holomorphic. That suggests we should take $k = m + n$ in the lemma. However, if $r(x)$ is not chosen carefully, then the degrees of $q(z)$ and $p(z)$ will be too high. To choose $r(x)$, we examine the explicit formulas

$$q(z) = r(1)z^{m+n} - r'(1)z^{m+n-1} + r''(1)z^{m+n-2} - \cdots$$

and

$$p(z) = r(0)z^{m+n} - r'(0)z^{m+n-1} + r''(0)z^{m+n-2} - \cdots.$$

The condition that $\deg q(z) \le n$ simply means $r(x)$ has a root of order m at $x = 1$, and similarly $\deg p(z) \le m$ means $r(x)$ has a root of order n at $x = 0$. Since $\deg r(x) = m + n$, our only choice (up to a constant factor) is to take

$$r(x) = x^n(x-1)^m,$$

and that polynomial works. Thus,

$$\int_0^1 x^n(x-1)^m e^{zx}\,dx = \frac{q(z)e^z - p(z)}{z^{m+n+1}},$$

where $p(z)/q(z)$ is the Padé approximant of type (m, n) to e^z.

Setting $z = 1$ recovers the integrals used in the proof of the continued fraction expansion of e, except for the factor of $1/n!$, which simply makes the answer prettier. Fundamentally, the reason why the factorial appears is that

$$\frac{d^n}{dx^n}\left(\frac{x^n(x-1)^n}{n!}\right)$$

has integral coefficients. Note also that up to a change of variables, this expression is the Rodrigues formula for the Legendre polynomial of degree n [**1**, p. 99].

Natural generalizations of these integrals play a fundamental role in Hermite's proof of the transcendence of e. See [**2**, p. 4] for an especially short version of the proof or chapter 20 of [**8**] for a more leisurely account (although the integrals used there are slightly different from those in this paper).

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A Proof of the Continued Fraction Expansion of $e^{1/M}$

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1. INTRODUCTION. This paper gives another proof for the remarkable simple continued fraction

$$e^{1/M} = 1 + \cfrac{1}{M-1+\cfrac{1}{1+\cfrac{1}{3M-1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{5M-1+\cfrac{1}{1+\cdots}}}}}}}.$$