Cubics of dimension seven and the Cayley plane

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(joint work with Atanas Iliev)

Consider a smooth hypersurface X in the complex projective space $\mathbb{P}^{n+1}$. Suppose that X is Fano, i.e., that its degree d is at most $n+1$. Denote its index by $\iota = n+2-d$. Kuznetsov proved in [2]:

Theorem 1. *The derived category $D(X)$ of coherent sheaves on X has a semi-orthogonal decomposition*

$$D(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{O}_X(1), \dots, \mathcal{O}_X(\iota - 1)\rangle,$$

where the full subcategory $\mathcal{A}_X$ is a Calabi-Yau category of dimension $n - \frac{2\iota}{d}$.

(When d does not divide 2ι, this means that $\mathcal{A}_X$ has a Serre functor, a suitable power of which is a shift).

An interesting example is the case of cubic fourfolds ($d = 3, n = 4$), for which $\mathcal{A}_X$ is in general a *noncommutative K3 surface*. In certain cases there is a genuine K3 surface attached to the cubic fourfold X. This is for example the case when X is defined as the Pfaffian of a 6×6 skew-symmetric matrix of linear forms in 6 variables – otherwise said, as a linear section $X = \mathrm{Pf} \cap \mathbb{P}^5_X$ of the Pfaffian hypersurface $\mathrm{Pf} \subset \mathbb{P}(\wedge^2\mathbb{C}^6)^\vee$. On the dual side, $\mathbb{P}(\wedge^2\mathbb{C}^6)$ contains the Grassmannian $G(2,6)$, whose intersection with $(\mathbb{P}^5_X)^\perp$ is a K3 surface S such that $\mathcal{A}_X \simeq D(S)$ [2]. When X deforms to a non Pfaffian cubic, $\mathcal{A}_X$ is still defined but S is no longer present.

The next case for which $\mathcal{A}_X$ has integral dimension is the case of cubic sevenfolds ($d = 3, n = 7$), for which $\mathcal{A}_X$ is a *noncommutative Calabi-Yau threefold*. A first important difference with the case of cubic fourfolds is that there can exist no genuine Calabi-Yau threefold Z such that $\mathcal{A}_X \simeq D(Z)$. One can nevertheless generalize the Pfaffian construction by replacing $G(2,6)$ by the next Severi variety [4]. The *Cayley plane* $\mathbb{OP}^2$ is a sixteen dimensional complex projective variety, homogeneous under the exceptional group E_6, with an equivariant embedding inside $\mathbb{P}^{26}$. Its dual variety $C = (\mathbb{OP}^2)^*$ is a cubic hypersurface that we call the Cartan cubic (its equation was first written down by E. Cartan in terms of tritangent planes to a cubic surface).

Proposition 1. *A general seven dimensional cubic X can be represented as a linear section of the Cartan cubic, in finitely many ways.*

It would be interesting to know the number of such representations. If it is equal to one, the cubic sevenfold has a canonical form of a quite unexpected type. If it is bigger than one, the noncommutative Calabi-Yau threefold $\mathcal{A}_X$ must be very symmetric.

Indeed, each realization of X as a linear section of the Cartan cubic endows it with a special rank nine vector bundle E, defined as follows. Since C is also the secant variety to the dual Cayley plane $(\mathbb{OP}^2)^\vee$, each point $x \in X$ defines an entry locus $Q_x \subset (\mathbb{OP}^2)^\vee$, which turns out to be an eight-dimensional quadric. (As observed by Freudenthal and Tits in the fifties, these quadrics must be considered as projective lines over the octonions, and the family of these lines have many of the characteristic properties of a plane projective geometry.) The polar hyperplane to x with respect to Q_x inside its linear span, is by definition $\mathbb{P}(E_x)$ and this defines the bundle E on X.

Theorem 2. *The vector bundle E is arithmetically Cohen-Macaulay and infinitesimally rigid. Moreover E and $E(1)$ are spherical objects in $\mathcal{A}_X$.*

This means in particular that the cohomology of $End(E)$ is that of a three-dimensional sphere. Seidel and Thomas [6] showed how to associate to a spherical object a *spherical twist*, which is in our case a self-equivalence of the category $\mathcal{A}_X$. Thus every representation of X as a section of the Cartan cubic produces nontrivial self-equivalences of the corresponding noncommutative Calabi-Yau. Note that on the contrary, $D(X)$ itself has no interesting self-equivalence since X is Fano.

On the dual side, to a general $X = (\mathbb{OP}^2)^* \cap \mathbb{P}^8_X$ we can associate the orthogonal section $Y = \mathbb{OP}^2 \cap (\mathbb{P}^8_X)^\perp$ of the Cayley plane. This is a Fano manifold of dimension seven and index three.

Proposition 2. *The two varieties X and Y are birationally equivalent.*

More precisely one can associate a birationality to any general point of Y, and this birationality can be constructed in very simple terms from the plane projective geometry supported by the Cayley plane.

Now, the derived category of $\mathbb{OP}^2$ was described in [5] in terms of a special rank ten vector bundle S, and it follows that $S_Y, \mathcal{O}_Y, S_Y(1), \mathcal{O}_Y(1), S_Y(2), \mathcal{O}_Y(2)$ is an exceptional collection in $D(Y)$. Consider the semiorthogonal decomposition

$$D(Y) = \langle S_Y, \mathcal{O}_Y, S_Y(1), \mathcal{O}_Y(1), S_Y(2), \mathcal{O}_Y(2), \mathcal{A}_Y \rangle.$$

Conjecture. *$\mathcal{A}_X$ and $\mathcal{A}_Y$ are equivalent.*

This statement would be very similar to the Pfaffian-Grassmannian derived equivalence of Borisov and Caldararu [1], except that we deal with noncommutative Calabi-Yau's attached to our Fano varieties. It would also be a new instance of Kuznetsov's homological projective duality [3].

References

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