On the Casson Invariant

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Abstract

The Casson invariant is a topological invariant of closed oriented 3-manifolds. It is an integer that counts the $SU(2)$-representations of the fundamental group of these manifolds in a sense introduced by Casson in 1985. Its first properties allowed Casson to solve famous problems in 3-dimensional topology.

The Casson invariant can also be independently defined in a combinatorial way as a function of Alexander polynomials of framed links presenting the 3-manifolds. It has numerous interesting properties: It behaves nicely under most topological mutations such as orientation reversal, connected sum, surgery, regluing along surfaces... This makes it easy to compute and to use. The Casson invariant contains the Rohlin invariant of $\frac{\mathbf{Z}}{2\mathbf{Z}}$-homology spheres, that is the signature of any smooth spin 4-manifold bounded by such a sphere. It is also explicitly related to quantum invariants and is the first finite type invariant in the sense of Ohtsuki. (This Ohtsuki notion of finite type invariants for 3-manifolds is analogous to the Vassiliev notion for knots.)

This talk will be a general presentation of the Casson invariant where its newest properties and developments will be emphasized.

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1 Introduction

The Casson invariant is the simplest topological invariant of 3-manifolds among the invariants introduced after 1984. It interacts with many domains of mathematics. Casson defined it using the differential topology of $SU(2)$-representation spaces. It can be extended in the spirit of this definition using symplectic geometry [W, CLM, C]. It can also be defined using infinite-dimensional analysis and gauge theory [Ta]: it is the Euler characteristic of the Floer homology [BD]. Its natural connections with the structure of the mapping class group have been investigated in a series of articles of S. Morita [Mo2] who interpreted it as a certain secondary invariant associated with the first characteristic class of surface bundles. Its relationship with quantum invariants [Mu], links it to quantum physics; and the Dedekind sums which often show up in the study of its topological properties show that it even interacts with arithmetic.

Here, with the aim of being as elementary as possible, we will present a combinatorial definition of the Casson invariant, and describe most of its topological properties.

2 Surgery presentations of 3-manifolds

Here, all the manifolds are compact and oriented (unless otherwise mentioned). Boundaries are oriented with the "outward normal first" convention. We usually work with topological manifolds, but since any topological manifold of dimension less or equal than 3 has exactly one C^∞-structure (see [Ku]), we will sometimes make incursions in the smooth category. Manifolds are always considered up to oriented homeomorphism and embeddings and homeomorphisms are considered up to ambient isotopy.

Given a 3-manifold M, a knot K of M -that is an embedding of the circle S^1 in the interior of M-, and a parallel μ of K -that is a closed curve on the boundary of a tubular neighborhood $T(K)$ of K which runs parallel to K-, we can define the *manifold* $\chi(M;(K,\mu))$ *obtained from* M *by surgery on* K *with respect to* μ by the following construction: remove the interior of $T(K)$ from M and replace it by another solid torus $D^2 \times S^1$ glued along the boundary $\partial T(K)$ of $T(K)$ by a homeomorphism from $\partial T(K)$ to $\partial D^2 \times S^1$ which maps μ to $\partial D^2 \times \{1\}$.

$$\chi(M;(K,\mu)) = \overline{M \setminus T(K)} \cup_{\partial T(K) \sim \partial D^2 \times S^1} D^2 \times S^1$$

Note that $\chi(M;(K,\mu))$ is well-defined. Indeed, it is obtained by first gluing a thickened disk to $\overline{M \setminus T(K)}$ along an annulus around μ in $\partial T(K)$ and by next filling in the resulting sphere S^2 in the boundary by a standard 3-ball B^3.

(K,μ) is called a *framed knot*. A collection of disjoint framed knots is called a framed link. Surgery on framed links is the natural generalization of surgery on framed knots. The surgery is performed on each component of the link. A first motivation for taking this operation in consideration is the following theorem which is proved in a very elegant way in [Ro].

Theorem 2.1 (Lickorish [Li], Wallace [W] 1960) *Any closed (i.e. compact, connected, without boundary) 3-manifold can be obtained from the standard 3-sphere S^3 by surgery on a framed link.*

We now define linking numbers to help parametrizing surgeries. Let J and K be two disjoint knots in a 3-manifold M which are rationnally null-homologous (i.e. null in $H_1(M;\mathbf{Q})$). Then there is a closed surface Σ embedded in $\overline{M \setminus T(K)}$ whose boundary lies in $\partial T(K)$ and is homologous to $d[K]$ in $T(K)$ for some nonzero $d \in \mathbf{Z}$; and the *linking number* of J and K, $lk(J,K)$, is unambiguously defined as the algebraic intersection number of J and Σ divided out by d. It is symmetric.

For $R = \mathbf{Z}, \mathbf{Q}$ or $\mathbf{Z}/2\mathbf{Z}$, a 3-manifold with the same R-homology $H_*(.;R)$ as S^3 is called an R-sphere. When M is a $\mathbf{Q}$-sphere, the isotopy class in $\partial T(K)$ of the *characteristic curve* μ of the surgery is specified by the linking number of μ and K in M and the framed knot (K,μ) is also denoted by $(K, lk(K,\mu))$. In particular, a framed link in S^3 is a link of S^3 each component of which is equipped by an integer.

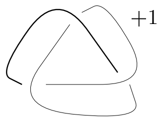

Figure 1: A surgery presentation of the Poincaré sphere (see [R])

A second motivation for studying surgeries is the Kirby calculus which relates two surgery presentations of the same 3-manifold. Following is the Fenn and Rourke version of the Kirby calculus:

Theorem 2.2 (Fenn-Rourke [FR], Kirby [K] 1978) *Any two framed links of S^3 presenting the same 3-manifold can be obtained from each other by a finite number of FR-moves, w.r.t. the following description of FR-moves.*

Let $\mathbf{L}$ be a framed link in S^3 such that a component $\mathbf{U}$ of $\mathbf{L}$ is a trivial knot U equipped with a parallel μ_U satisfying $lk(U,\mu_U) = \varepsilon = \pm 1$. Consider a cylinder $I \times D^2$ embedded in $\overline{S^3 \setminus T(U)}$ so that $I \times S^1$ is embedded in $\partial T(U)$. Let τ be the homeomorphism of $\overline{S^3 \setminus T(U)}$ which is the identity outside the cylinder, and which twists the cylinder around its axis so that μ_U is mapped to the meridian of U. Clearly, $\tau(\mathbf{L} \setminus \mathbf{U})$ presents the same 3-manifold as $\mathbf{L}$ does (where we think of framed links as links equipped with curves to give a meaning to $\tau(\mathbf{L} \setminus \mathbf{U})$). We define a FR-move as the operation described above which transforms $\mathbf{L}$ into $\tau(\mathbf{L} \setminus \mathbf{U})$ or its inverse. It is easy to see that such a move does not change the presented manifold.

According to the above theorem, in order to define an invariant of closed 3-manifolds, it suffices to find a function of surgery presentations invariant under FR-moves. For lack of good candidates, this process had not been used before 1988. Since, with the invasion of quantum invariants, there is a lot of 3-manifolds invariants which have been proved to be invariant using this simple principle ([RT], [W],...) but for most of them a topological interpretation is still to be found. Here, I propose first to introduce such an invariant function and next to give the topological interpretation of the invariant of 3-manifolds it yields: a generalization of the Casson invariant.

3 A combinatorial definition of the Casson invariant.

In order to introduce our invariant function $\mathbf{F}$, we need some notation. Let $\mathbf{L} = (\mathbf{K}_i)_{i \in N}$ be a framed link in a $\mathbf{Q}$-sphere M, $\mathbf{K}_i = (K_i, \mu_i) = (K_i, lk(\mu_i, K_i))$. $N = \{1, \ldots, n\}$ is the set of indices of the components of $\mathbf{L}$. For a subset I of N, $\mathbf{L}_I = (\mathbf{K}_i)_{i \in I}$. $E(\mathbf{L}) = [\ell_{ij} = lk(\mu_i, K_j)]_{i,j=1,\ldots,n}$ denotes the symmetric *linking matrix* of $\mathbf{L}$. $b^-(\mathbf{L})$ (resp. $b^+(\mathbf{L})$) is the number of negative (resp. positive) eigenvalues of $E(\mathbf{L})$. signature$(E(\mathbf{L})) = b^+(\mathbf{L}) - b^-(\mathbf{L})$. For a $\mathbf{Z}$-module A, $|A|$ denotes the order of A, that is its cardinality if A is finite and 0 otherwise. Note that

$$|H_1(\chi(M; \mathbf{L}))| = (-1)^{b^-(\mathbf{L})} det(E(\mathbf{L}))|H_1(M)|$$

(Unless otherwise mentioned, the homology coefficients are the integers.)

Now, we can set:

$$\mathbf{F}_M(\mathbf{L}) = (-1)^{b^-(\mathbf{L})} \sum_{I \subset N, I \neq \emptyset} det(E(\mathbf{L}_{N \setminus I}))\alpha(\mathbf{L}_I)$$

$$+|H_1(\chi(M; \mathbf{L}))| \frac{\text{signature}(E(\mathbf{L}))}{8}$$

with

$$\alpha(\mathbf{L}_I) = |H_1(M)| \left(\tilde{\zeta}(L_I) + \frac{(-1)^{\sharp I}}{24} L_8(\mathbf{L}_I) \right)$$

where $L_8(\mathbf{L})$ and $\tilde{\zeta}(L)$ are described below.

$L_8(\mathbf{L})$ is the following homogeneous polynomial in the coefficients of the linking matrix. Let G be a graph whose vertices are indexed by N; for an edge e of G whose ends are indexed by i and j, we set $lk(\mathbf{L}; e) = \ell_{ij}$. Next we define $lk(\mathbf{L}; G)$ as the product running over all edges e of G of the $lk(\mathbf{L}; e)$. Now, $L_8(\mathbf{L})$ is the sum of the $lk(\mathbf{L}; G)$ running over all graphs G whose vertices are the elements of N and whose underlying spaces have the form of a figure eight made of two oriented distinguished circles (North and South) with one common vertex.

The coefficient $\tilde{\zeta}$ can be defined from the several-variable Alexander polynomial Δ (as defined and normalized in [Ha] and [BL2] or in Section 6 below) for several component links and from the classical Alexander polynomial Δ of knots which is the order of the H_1 of the infinite cyclic covering of $M \setminus K$, viewed as a natural $\mathbf{Z}[t, t^{-1}]$-module, normalized in such a way that $\Delta(1) > 0$ and Δ is symmetric.

$$\tilde{\zeta}(L) = \begin{cases} (-1)^{n-1} \frac{\partial^n \Delta}{\partial t_1 \ldots \partial t_n}(L)(1, \ldots, 1) & \text{if } n > 1 \\ \frac{O_M(K_1)}{2|H_1(M)|} \Delta''(K_1)(1) + \frac{(-1)}{24}\left(1 + \frac{1}{O_M(K_1)^2}\right) & \text{if } n = 1 \end{cases}$$

where $O_M(K_1) = |H_1(M)| / |\text{Torsion}(H_1(M \setminus K_1))|$ is the order of the class of K_1 in $|H_1(M)|$. It is one if M is a $\mathbf{Z}$-sphere.

We can now state the theorem:

Theorem 3.1 ([L3]) *There exists a rational topological invariant λ of closed 3-manifolds such that for any framed link $\mathbf{L}$ in S^3,*

$$\lambda(\chi(S^3; \mathbf{L})) = \mathbf{F}_{S^3}(\mathbf{L})$$

The so-defined λ-invariant satisfies the more general surgery formula:

Property 1 *For any framed link $\mathbf{H}$ in a $\mathbf{Q}$-sphere M,*

$$\lambda(\chi(M; \mathbf{H})) = \frac{|H_1(\chi(M; \mathbf{H}))|}{|H_1(M)|} \lambda(M) + \mathbf{F}_M(\mathbf{H})$$

The principle of the proof of the theorem is very simple. According to the Fenn and Rourke version of the Kirby theorem, it suffices to show the invariance of $\mathbf{F}$ under a FR-move; and the function $\mathbf{F}$ is a function of homological invariants of the exterior of the framed link whose variation under a homeomorphism of this exterior can be followed (with some combinatorial efforts). See [L3].

The proof of the general surgery formula rests on the same remark. Take a surgery presentation $\mathbf{L}$ of the $\mathbf{Q}$-sphere M. By transversality, we may assume that $\mathbf{H}$ is disjoint of the link $\hat{L}$ made of the cores of the new solid tori glued by the surgery. Then the surgery presentation $\mathbf{H}$ can be seen in S^3. (Again we think of it as a link equipped with characteristic curves) and the equality to be shown is:

$$\mathbf{F}_{S^3}((\mathbf{H} \subset S^3) \cup \mathbf{L}) = \frac{|H_1(\chi(M; \mathbf{H}))|}{|H_1(M)|} \mathbf{F}_{S^3}(\mathbf{L}) + \mathbf{F}_M(\mathbf{H})$$

where both sides are functions of homological invariants of

$$S^3 \setminus (H \cup L) = M \setminus (H \cup \hat{L})$$

equipped with the surgery curves.

Because of the form of the surgery formula, it is easy to compare the λ-invariant with the Rohlin invariant for $\mathbf{Z}/2\mathbf{Z}$-spheres. Before stating the comparison property, let us give a definition of this invariant discovered in 1952. A spin structure on a smooth manifold of dimension greater or equal than 3 is a homotopy class of trivializations of its tangent bundle over its 2-skeleton. (See [Mi2] for other definitions.) The Rohlin invariant σ of a $\mathbf{Z}/2\mathbf{Z}$-sphere M is the signature mod 16 of (the intersection form on the H_2 of) a smooth spin (i.e. equipped with a spin structure) 4-manifold bounded by M.

Property 2 *For any* $\mathbf{Z}/2\mathbf{Z}$*-sphere* M*,*

$$\sigma(M) = 8|H_1(M)|\lambda(M) \bmod 16$$

To a surgery presentation $(\mathbf{L} \subset S^3)$ of a 3-manifold M, we may associate the following natural 4-manifold $W_{\mathbf{L}}$ bounded by M: $W_{\mathbf{L}}$ is constructed from the standard 4-dimensional ball B^4 by gluing 2-handles $D^2 \times D^2$ to each component of the tubular neighborhood of L, $T(L) \subset S^3 = \partial B^4$, with respect to the trivialization given by the characteristic curves (which allows to identify a component of $T(L)$ to $(D^2 \times S^1 \subset D^2 \times D^2)$). $W_{\mathbf{L}}$ is next smoothed in a standard way. The linking matrix $E(\mathbf{L})$ is the matrix of the intersection form on $H_2(W_{\mathbf{L}})$ w.r.t. the basis of $H_2(W_{\mathbf{L}})$ associated to its handle decomposition above. A necessary and sufficient condition for $W_{\mathbf{L}}$ to be spin is that the diagonal of $E(\mathbf{L})$ is even (see [GM1, p.43]), and this can always be realized by FR moves (see [Ka]). In this case, it is easy to check that when $\det(\mathrm{E}(\mathbf{L}))$ is odd (that is when M is a $\mathbf{Z}/2\mathbf{Z}$-sphere), $8|det(E(\mathbf{L}))|\mathbf{F}_{S^3}(\mathbf{L}) - \mathrm{signature}(E(\mathbf{L}))$ belongs to $16\mathbf{Z}$. This proves the congruence with the Rohlin invariant stated above; and a few classical easy arguments show that this also gives a proof of the original Rohlin theorem asserting that the signature of a closed smooth spin 4-manifold is divisible by 16 (this Rohlin theorem yields the well-definedness of the Rohlin invariant as a direct corollary) (see [L3, Sec. 6.3]).

The following properties of the λ-invariant can also be checked very easily:

Property 3 *For any closed 3-manifold* M*, the* λ*-invariant of the manifold* $-M$ *obtained from* M *by orientation reversal satisfies:*

$$\lambda(-M) = (-1)^{\beta_1(M)+1}\lambda(M)$$

where $\beta_1(M)$ *is the first Betti number of* M*.*

Property 4 *For any two closed 3-manifolds* M_1 *and* M_2*, the* λ*-invariant of their connected sum* $M_1 \sharp M_2 \stackrel{def}{=} \overline{M_1 \setminus B^3} \cup_{S^2} \overline{M_2 \setminus B^3}$ *satisfies*

$$\lambda(M_1 \sharp M_2) = |H_1(M_2)|\lambda(M_1) + |H_1(M_1)|\lambda(M_2)$$

But the main property of λ is that it can be expressed in terms of previously known invariants:

Property 5 *Let M be a closed 3-manifold.*

- *If $\beta_1(M) \geq 4$, then*

$$\lambda(M) = 0$$

- *If $\beta_1(M) = 3$, let (a,b,c) be a basis of $H^1(M)$ and let $\cup$ denote the cup product, then*

$$\lambda(M) = |\mathrm{Torsion}(H_1(M))|(a \cup b \cup c)([M])^2$$

- *If $\beta_1(M) = 2$, let $([F],[G])$ be a basis of $H_2(M)$, represent it by two closed surfaces F and G embedded in general position in M, call γ their oriented intersection, call γ' the parallel of γ w.r.t. the trivialization of the normal bundle of γ induced by F and G.*

$$\lambda(M) = -|\mathrm{Torsion}(H_1(M))| lk(\gamma, \gamma')$$

- *If $\beta_1(M) = 1$, let $\Delta(M)$ be the Alexander polynomial of M, that is (again) the order of the H_1 of the infinite cyclic covering of M, viewed as a natural $\mathbf{Z}[t,t^{-1}]$-module, normalized in such a way that $\Delta(M)(1) > 0$ and $\Delta(M)(t) = \Delta(M)(t^{-1})$.*

$$\lambda(M) = \frac{\Delta''(M)(1)}{2} - \frac{|\mathrm{Torsion}(H_1(M))|}{12}$$

- *If $\beta_1(M) = 0$ (i.e. if M is a $\mathbf{Q}$-sphere), then $\lambda(M)$ is the Casson-Walker invariant of M. More precisely, if M is a $\mathbf{Z}$-sphere, $\lambda(M)$ is the Casson invariant of M as normalized in [AM, GM2], and in general, if λ_W denotes the normalization of the Walker invariant used in [W],*

$$\lambda(M) = \frac{|H_1(M)|}{2}\lambda_W(M)$$

It is now time to describe the Casson invariant of $\mathbf{Z}$-spheres as introduced by Casson in 1985.

4 The Casson invariant after Casson

Let M be a $\mathbf{Z}$-sphere, A. Casson defined $\lambda(M)$ as an algebraic number of conjugacy classes of irreducible $SU(2)$-representations of $\pi_1(M)$ in the following way. (Details can be found in [AM] or [GM2]).

As any closed 3-manifold, M can be decomposed into two handlebodies A and B glued along a genus g surface $\Sigma = \partial A = -\partial B$. (A handlebody is a regular neighborhood of a wedge of circles in a 3-manifold.) Such a decomposition $M = A \cup_\Sigma B$ is called a Heegaard splitting of M.

For a topological space X, call $R(X)$ the space of $SU(2)$-representations of the discrete group $\pi_1(X)$ equipped with the compact open topology. The subspace of $R(X)$ consisting of irreducible representations is an open set in $R(X)$ denoted by $\tilde{R}(X)$. When $\pi_1(X)$ is a free group of rank g, for example, when X=A or B, $R(X)$ has a natural smooth structure which makes it diffeomorphic to $SU(2)^g \cong (S^3)^g$. $\tilde{R}(\Sigma)$ also has a natural smooth structure. Namely, call Σ_* the surface obtained from Σ by removing an open disk, choose a basepoint of Σ_* on $\partial\Sigma_*$ and call $(\partial : R(\Sigma_*) \to S^3)$ the evaluation of a representation of $R(\Sigma_*)$ at $\partial\Sigma_*$. The restriction of ∂ to $\tilde{R}(\Sigma_*)$ is a submersion. Thus, $\tilde{R}(\Sigma) = \tilde{R}(\Sigma_*) \cap \partial^{-1}(1)$ becomes a natural smooth $(6g-3)$-submanifold of $\tilde{R}(\Sigma_*)$. Let $X = A, B, \Sigma$ or Σ_*, the free smooth action of $SO(3) = SU(2)/\{-1,1\}$ by right conjugation on $\tilde{R}(X)$ identifies $\tilde{R}(X)$ with the total space of a principal $SO(3)$-bundle whose base is a smooth open manifold denoted by $\hat{R}(X)$. $\hat{R}(X)$ is the space of conjugacy classes of irreducible $SU(2)$-representations of $\pi_1(X)$.

The inclusions of Σ_* into A and B identify $R(A)$ and $R(B)$ with submanifolds of $R(\Sigma_*)$, and the Van Kampen theorem identifies $R(M)$ with $R(A) \cap R(B)$.

Since M is a $\mathbf{Z}$-sphere, the only reducible $SU(2)$-representation of $\pi_1(M)$ is the trivial one, ρ_0, and it can be shown that $R(A)$ and $R(B)$ intersect transversally at ρ_0. Thus, $\tilde{R}(A) \cap \tilde{R}(B)$ and hence $\hat{R}(A) \cap \hat{R}(B)$ are compact. Therefore, an isotopy with compact support perturbing the inclusion of $\hat{R}(A)$ into $\hat{R}(\Sigma)$ can make $\hat{R}(A)$ transverse to $\hat{R}(B)$ inside $\hat{R}(\Sigma)$. Now, since $\hat{R}(A)$ and $\hat{R}(B)$ are of complementary dimension in $\hat{R}(\Sigma)$, their intersection is a finite number of points which can be given signs (+1) or (-1) once $\hat{R}(A)$, $\hat{R}(B)$ and $\hat{R}(\Sigma)$ are oriented. The sum of these signs is denoted by $< \hat{R}(A), \hat{R}(B) >_{\hat{R}(\Sigma)}$. It is, up to sign, twice the Casson invariant.

In order to suppress the sign indetermination we must specify orientations. $SU(2)$, $R(A)$, $R(B)$ and $R(\Sigma_*)$ are oriented arbitrarily. $SO(3)$ is oriented by the double covering $SU(2) \to SO(3)$. $\tilde{R}(\Sigma)$ is oriented as the fiber of ∂ with the convention $(base \oplus fiber)$. Once $R(X)$ is oriented, $\hat{R}(X)$ is oriented as the base of a $SO(3)$-bundle with the convention $(base \oplus fiber)$. It can be shown that the (classical) algebraic intersection number $< R(A), R(B) >_{R(\Sigma_*)}$ is ± 1.

Now, with all the notations above, we can state Casson's original definition of λ.

$$\lambda(M) = \frac{(-1)^g}{2} \frac{< \hat{R}(A), \hat{R}(B) >_{\hat{R}(\Sigma)}}{< R(A), R(B) >_{R(\Sigma_*)}}$$

Casson proved the invariance of λ using the Reidemeister-Singer theorem which asserts that two Heegaard splittings of the same manifold become isomorphic after a finite number of stabilizations (that are connected sums with the genus one Heegaard splitting of S^3), and following the transformation of the above definition under a stabilization.

Casson's theorem was:

Theorem 4.1 (Casson, 1985) *There exists an integral topological invariant λ of* **Z***-spheres such that:*

1. *If the trivial representation is the only representation of $\pi_1(M)$ into $SU(2)$, then $\lambda(M) = 0$.*
2. $\lambda(-M) = -\lambda(M)$.
3. $\lambda(M_1 \sharp M_2) = \lambda(M_1) + \lambda(M_2)$.
4. *For any knot K in a homology sphere M, for any $\varepsilon = \pm 1$,*

$$\lambda(\chi(M;(K,\varepsilon))) = \lambda(M) + \frac{\varepsilon}{2}\Delta(K)''(1).$$

5. $\sigma(M) = 8\lambda(M) \bmod 16$.

The immediate corollaries of this theorem, *'The Rohlin invariant of a homotopy sphere is null.'* and *'The Rohlin invariant of an amphicheral* **Z***-sphere is null.'* answered two long-unsolved questions in low-dimensional topology and allowed Casson to show the existence of a topological 4-manifold which cannot be triangulated (see [AM]) as a simplicial complex.

Note that the first assertion of the theorem is a direct corollary of Casson's definition of λ. The second and third assertions can also be proved very easily from this definition. Since any **Z**-sphere can be obtained from S^3 by a sequence of surgeries on knots framed by ± 1 (see [GM2]), and because an analogous surgery formula was known for the Rohlin μ-invariant $\mu = \frac{\sigma}{8}$, the fifth assertion is a direct consequence of the surgery formula. Thus, the only difficulty in the proof of Casson's theorem (in addition to inventing this definition...) is to prove the surgery formula from the definition above.

To prove the surgery formula Casson proved the following lemma. A *boundary link* is a link whose components bound disjoint surfaces in the ambient manifold; T denotes the trefoil knot in S^3 pictured in Figure 1.

Lemma 4.2 *Let ν be a rational invariant of* **Z***-spheres such that: For any 2-component boundary link* $\mathbf{L}$ *in a* **Z***-sphere M whose components are framed by ± 1:*

$$\sum_{I \subset \{1,2\}} \nu(\chi(M;\mathbf{L}_I)) = 0$$

Then

$$\nu(\chi(M;(K,\varepsilon))) = \nu(M) + \frac{\varepsilon}{2}\Delta(K)''(1)(\nu(\chi(S^3;(T,1))) - \nu(S^3))$$

Then he computed $\lambda(\chi(S^3;(T,1))) = 1$ and proved that λ satisfied the hypothesis of the lemma from his definition. In fact, it is possible [GM2] to compute the Casson invariant of Seifert fibered $\mathbf{Z}$-spheres with 3 exceptional fibers, and the variation of the Casson invariant under a surgery along a knot bounding an unknotted genus one Seifert surface directly from Casson's definition. Both computations give $\lambda(\chi(S^3;(T,1)))$.

Remark 4.3 In [Lin], X. S. Lin proved that the signature of a knot can also be obtained by counting some $SU(2)$-representations of the π_1 of its exterior 'à la Casson'. Like Casson's comparison of his representation number with the Rohlin invariant, Lin's proof that his representation number coincides with the signature is not direct. In both cases, it would be interesting to have a more direct identification.

Note also that Lemma 4.2 provides a nice characterization of the Casson invariant. In the same spirit, it can be shown [L5]:

Property 6 *Any two $\mathbf{Z}$-spheres which have the same Casson invariant can be obtained from one another by a sequence of surgeries on knots with trivial Alexander polynomial framed by ± 1.*

In 1988, K. Walker [W] used the stratified symplectic structure of the representation spaces [Go] to give a complete generalization of Casson's work to $\mathbf{Q}$-spheres. In this case, reducible representations can not be ignored, and basic differential topology does not suffice anymore to provide a powerful generalization. (A weaker generalization of the Casson invariant to $\mathbf{Q}$-spheres had been proposed by S. Boyer and A. Nicas [BN].). Furthermore, K. Walker gave a very nice proof based on Kirby calculus that his one-component surgery formula gives a consistent definition of his invariant λ_W.

Next, S. Cappell, R. Lee and E. Miller [CLM] generalized Walker's definition to other Lie groups like $SU(n)$, but they have not yet found interesting properties for their invariants. C. Curtis [C] studied the $SO(3)$, $U(2)$, $Spin(4)$ and $SO(4)$-invariants more precisely and proved that they are functions of the Walker $SU(2)$-invariant.

Of course, the combinatorial extension of the Casson invariant described in Section 3 is also a development of Casson's work. Indeed, without Walker's generalization of the Casson theorem above, and Boyer-Lines's work [BL1] the author would not have been able to find the general surgery formula of Property 1. In their work independent from Walker's, S. Boyer and D. Lines gave a combinatorial definition of the restriction of the Walker invariant λ_W to homology lens spaces, they proved a two-component surgery formula formula for the Casson invariant, they exhibited the first part $\mathbf{F}_1$ of the surgery function $\mathbf{F}$, the combination of the coefficients $\tilde{\zeta}$, and they proved that $(\lambda(\chi(S^3;.)) - \mathbf{F}_1)$ is invariant under link homotopy. It must also be mentioned that the surgery formula for algebraically split links, that are links whose components do not algebraically link each other, is due to Hoste [Ho] to close this section about the Casson work and some of its developments.

5 Further topological properties of the Casson invariant

Since the Alexander-Conway polynomial is a well-understood knot invariant, it is easy to apply the surgery formula satisfied by λ in order to compute the λ-invariant of any manifold presented by surgery [L2, L1], in order to study the behaviour of λ under other topological mutations as in [D, Ki, Wo] or in Properties 7, 8 and 9 described below, or in order to compare λ with other invariants as H. Murakami did to prove that the Walker invariant is equal to a function of the Reshetikhin and Turaev invariants that he appropriately defined [Mu].

Remark 5.1 In [O1], T. Ohtsuki generalized Murakami's work and renormalized the Reshetikhin and Turaev invariants into an invariant series of $\mathbf{Q}$-spheres whose first coefficients are $|H_1(.)|$ and λ. It would be interesting to know whether the other coefficients of this series are related to Casson-type invariants. To study his series, Ohtsuki [O2] defined the notion of finite type invariant for $\mathbf{Z}$-spheres. This notion is analogous to the notion of Vassiliev invariants of knots. Say that a rational invariant ν of $\mathbf{Z}$-spheres is of AS-type (resp. of B-type) less or equal than n if for any $(n+1)$-component algebraically split (resp. boundary link) L in a $\mathbf{Z}$-sphere M whose components are framed by ± 1:

$$\sum_{I \subset \{1,\ldots,n+1\}} \nu(\chi(M;\mathbf{L}_I)) = 0$$

Note that Casson's lemma (4.2) proves that the B-type 1 invariants are exactly the degree 1 polynomials in λ while the Hoste surgery formula [Ho] shows that λ is of AS-type 3. In fact, it is proved that the AS-type is always a multiple of 3, and it is conjectured (proved ?) that the two mentioned notions of finite type invariants coincide and that the AS-type is three times the B-type. It is not hard to see that for any integer n, a degree n polynomial in λ is an invariant of B-type n and of AS-type $3n$. Thus, the polynomials in λ are nice prototypes for finite type invariants. But, T. T. Q. Le proved [Le] that they are not the only ones. It would be interesting to place the $SU(n)$-invariants of Cappell, Lee and Miller among these finite type invariants.

It is worth mentioning the existence of some variants of the surgery formula (Property 1), that have not yet been mentioned to avoid introducing too many notations. Note that in the surgery definition, we do not need the characteristic curve μ of the surgery to be parallel to the knot K. Any non-separating simple closed curve of $\partial T(K)$ can play the role of the characteristic curve, and the surgery defined by such a curve is (sometimes) called a rational surgery. The surgery formula extends naturally to rational surgeries. For surgeries starting from $\mathbf{Z}$-spheres the surgery function $\mathbf{F}$ can be expressed only in terms of linking numbers and one-variable Alexander-Conway polynomials. (See [L3].)

There are also some formulae for the Casson invariant of p-fold branched cyclic coverings. For a link L in a $\mathbf{Z}$-sphere M, let $R_p(M;L)$ be the p-fold cyclic covering of M branched along L, obtained from the covering of the exterior of L associated with the 'linking number with L modulo p' by filling it in by solid tori whose meridians are sent to old meridians of L.

Property 7 (Hoste [Ho]) *Let K be a knot in a $\mathbf{Z}$-sphere M. Let $D_\varepsilon K$ be the untwisted double of K with an ε-clasp, then*

$$\lambda(R_p(M;D_\varepsilon K)) = p\lambda(M) + \varepsilon p \Delta''(K)(1)$$

The following Mullins property relates the Walker invariant of 2-fold branched coverings to the Jones polynomial V and the oriented signature σ of links:

Property 8 (Mullins [Mul]) *Let L be a link in S^3 such that $R_2(S^3;L)$ is a $\mathbf{Q}$-sphere, then*

$$\lambda_W(R_2(S^3;L)) = \frac{\sigma(L)}{4} - \frac{V'(L)(-1)}{6V(L)(-1)}$$

To prove this formula, Mullins studied the variation of $\lambda_W(R_2(S^3;L))$ under a crossing change of L. Owing to the fact that the 2-fold branched covering of the ball of the crossing change is a solid torus, such a crossing change induces a surgery on $R_2(S^3;L)$.

For other p-fold cyclic branched coverings, a crossing change induces a handlebody replacement. This leads us to the following natural question. What can we say about $\lambda(A \cup_\Sigma B)$ for a $\mathbf{Q}$-sphere obtained by gluing two pieces A and B along a genus g surface Σ?

Our partial answer is the following property of λ [L4]. A $\mathbf{Q}$-*handlebody* is a 3-manifold with the same rational homology as a standard handlebody. For a 3-manifold A with boundary, the kernel $\mathcal{L}_A$ of the map from $H_1(\partial A;\mathbf{Q})$ to $H_1(A;\mathbf{Q})$ induced by the inclusion is called the *Lagrangian* of A.

Property 9 *Let A, A', B and B' be four $\mathbf{Q}$-handlebodies such that ∂A, $\partial A'$, $-\partial B$ and $-\partial B'$ are identified via orientation-preserving homeomorphisms with a genus g surface Σ. Assume that $\mathcal{L}_A = \mathcal{L}_{A'}$ and $\mathcal{L}_B = \mathcal{L}_{B'}$ and that $\mathcal{L}_A \cap \mathcal{L}_B = \{0\}$ inside $H_1(\Sigma;\mathbf{Q})$. Then*

$$\lambda_W(A \cup_\Sigma B) - \lambda_W(A' \cup_\Sigma B) - \lambda_W(A \cup_\Sigma B') + \lambda_W(A' \cup_\Sigma B') = R(A, A', B, B')$$

where $R(A, A', B, B')$, described below in general, is zero if $g \leq 2$.

Before describing $R(A,A',B,B')$ in general, note that, for $g=0$ and $A'=B'=B^3$, this property is nothing but the additivity of λ_W under connected sum. The genus one formula, when A' and B' are solid tori, is the splicing formula, shown by several authors [BN, FM] for the Casson invariant, and generalized by Fujita to the Walker invariant [F]. In this case, starting with $A\cup_\Sigma B$, there is a unique way of filling in A with a solid torus B' having the right Lagrangian, $A'\cup B$ and $A'\cup B'$ are similarly well-determined, and the Walker invariant of the lens space $A'\cup B'$ is a known Dedekind sum.

Now, let us decribe $R(A,A',B,B')$ under the hypotheses of Property 9. The isomorphism $\partial_{AA'}$ from $H_2(A\cup_\Sigma -A';\mathbf{Q})$ to $\mathcal{L}_A$ which maps the homology class of a surface S of $A\cup_\Sigma -A'$ (transverse to ∂A) to the class of $\partial(S\cap A)$ carries the algebraic intersection defined on $\bigwedge^3 H_2(A\cup_\Sigma -A';\mathbf{Q})$ to a form $\mathcal{I}_{AA'}$ defined on $\bigwedge^3\mathcal{L}_A$. Define $\mathcal{I}_{BB'}$ similarly. Let $(\alpha_1,\dots,\alpha_g)$ and $(\beta_1,\dots,\beta_g)$ be two bases for $\mathcal{L}_A$ and $\mathcal{L}_B$, respectively, that are dual for the intersection form $<,>_\Sigma$ on Σ ($<\alpha_i,\beta_j>_\Sigma=\delta_{ij}$). Then

$$R(A,A',B,B')=-4\sum_{\{i,j,k\}\subset\{1,\dots,g\}}\mathcal{I}_{AA'}(\alpha_i\wedge\alpha_j\wedge\alpha_k)\mathcal{I}_{BB'}(\beta_i\wedge\beta_j\wedge\beta_k)$$

Remark 5.2 Let $(\Sigma,\mathcal{L}_A)$ be a closed, connected surface equipped with a rational Lagrangian (as above). In [S], D. Sullivan proved that any integral form on $\bigwedge^3(H_1(\Sigma;\mathbf{Z})\cap\mathcal{L}_A)$ may be realized as a $\mathcal{I}_{AA'}$ for two standard handlebodies A and A' with boundary Σ and Lagrangian $\mathcal{L}_A$.

A splitting $A\cup_\Sigma B$ of a $\mathbf{Q}$-sphere induces the following function λ_{AB} on the Torelli group of Σ. The Torelli group is the group of the (isotopy classes of) homeomorphisms of Σ which induce the identity on $H_1(\Sigma)$. For a homeomorphism f of the Torelli group, $A\cup_f B$ denotes the manifold obtained by replacing the (underlying) identification $j_B:\Sigma\longrightarrow -\partial B$ by $j_B\circ f$.

$$\lambda_{AB}(f)=\frac{1}{2}(\lambda_W(A\cup_f B)-\lambda_W(A\cup_\Sigma B))$$

As a direct corollary of Property 9, we see that $\lambda_{AB}(g\circ f)-\lambda_{AB}(g)-\lambda_{AB}(f)$ is a function of the evaluations of the Johnson homomorphism at f and g (see [J, Second definition, p.170] for a definition of the Johnson homomorphism which is a homomorphism from the Torelli group to $\bigwedge^3 H^1(\Sigma)$). With completely different methods (based mainly on Johnson's study of the Torelli group), S. Morita proved this corollary for Heegaard splittings of $\mathbf{Z}$-spheres [Mo, Theorem 4.3], but he did not think that it extended to general embeddings [Mo, Remark 4.7].

The above corollary also proves that, when $A\cup B$ is a $\mathbf{Z}$-sphere the function μ_{AB} induced by the Rohlin μ-invariant $\mu=\frac{\sigma}{8}$ defines a homomorphism from the Torelli group to $\mathbf{Z}/2\mathbf{Z}$. These homomorphisms were first studied by J. Birman and R. Craggs [BC], they are the so-called Birman-Craggs homomorphisms.

It is worth mentioning that the best natural generalization of Property 9 that may be expected for the generalized Casson invariant of Section 3 is true [L4]. This generalized Casson invariant also admits a homogeneous definition via Kontsevich integrals [LMMO]. Both of these properties together with the homogeneous surgery formula enhance the naturality of the generalization of λ proposed in Section 3.

To prove Property 9, we first find a sequence of simple surgeries on links transforming A into A' and staying among the $\mathbf{Q}$-handlebodies with Lagrangian $\mathcal{L}_A$. Then we apply the surgery formula of [L3, BL1] to these surgeries and we analyse how the involved formulae depend on B when B varies among the $\mathbf{Q}$-handlebodies with boundary $-\partial A$ and with fixed Lagrangian.

This analysis led us [L4] to construct a tautological generalization of Alexander polynomials to 3-manifolds with boundary which may be useful to prove other properties of the Casson invariant. We conclude this article with a brief presentation of this function called the Alexander function which will allow us to define the normalized several variable Alexander polynomial.

6 More about Alexander polynomials: the Alexander function

All the assertions of this section are proved in [L4, Section 3]. Here, A denotes a connected 3-manifold with non-empty boundary and with non-negative *genus* $g = g(A) = 1 - \chi(A)$. Λ_A denotes the group ring:

$$\Lambda_A = \mathbf{Z}\left[\frac{H_1(A)}{\mathrm{Torsion}(H_1(A))}\right]$$

Recall that $\Lambda_A = \bigoplus_{x \in \frac{H_1(A)}{\mathrm{Torsion}}} \mathbf{Z}\, exp(x)$ as a $\mathbf{Z}$-module, that its multiplication sends $(exp(x), exp(y))$ to $exp(x+y)$, and that the units of Λ_A are its elements of the form $\pm exp(x \in H_1(A)/\mathrm{Torsion})$.

The maximal free abelian covering of A is denoted by $\tilde{A}$ and the covering map from $\tilde{A}$ to A by p_A. We fix a basepoint $\star$ in A. The Λ_A-module $H_1(\tilde{A}, p_A^{-1}(\star); \mathbf{Z})$ is denoted by $\mathcal{H}_A$.

Definition 6.1 The *Alexander function* $\mathcal{A}_A$ of A is the Λ_A-morphism

$$\mathcal{A}_A : \bigwedge^{g} \mathcal{H}_A \longrightarrow \Lambda_A$$

which is defined up to a (global) multiplication by a unit of Λ_A as follows. Take a presentation of $\mathcal{H}_A$ over Λ_A with $(r+g)$ generators $\gamma_1, \ldots, \gamma_{r+g}$ and r relators $\rho_1, \ldots, \rho_r$ (which are Λ_A-linear combinations of the γ_i). Let $\hat{u} = u_1 \wedge \ldots \wedge u_g$ be an element of $\bigwedge^g \mathcal{H}_A$. Then $\mathcal{A}_A(\hat{u})$ is defined by the equality:

$$\mathcal{A}_A(\hat{u})\hat{\gamma} = \hat{\rho} \wedge \hat{u}$$

where $\hat{\rho} = \rho_1 \wedge \ldots \wedge \rho_r$, $\hat{\gamma} = \gamma_1 \wedge \ldots \wedge \gamma_{r+g}$, the u_i are represented as combinations of the γ_j, and the exterior products are to be taken in $\bigwedge^{r+g}\left(\bigoplus_{i=1}^{r+g} \Lambda_A \gamma_i\right)$.

Of course, $\mathcal{A}_A(\hat{u})$ is just the order of the Λ_A-module $\mathcal{H}_A/(\oplus \Lambda_A u_i)$. But, hopefully, some of the properties of $\mathcal{A}_A$ mentioned below will convince the reader that it may be interesting to work with a fixed normalization of $\mathcal{A}_A$.

Fix a preferred lift $\star_0$ of $\star$ in $\tilde{A}$. Let ∂ denote the boundary map from $\mathcal{H}_A$ to $H_0(p_A^{-1}(\star)) = \Lambda_A[\star_0] = \Lambda_A$. Once a normalization of $\mathcal{A}_A$ is fixed, $\mathcal{A}_A$ satisfies the easy property:

For any $v = (v_1, \ldots, v_g) \in \mathcal{H}_A^g$, for any $u \in \mathcal{H}_A$,

$$\sum_{i=1}^{g} \partial(v_i)\mathcal{A}_A(\hat{v}(\frac{u}{v_i})) = \mathcal{A}_A(\hat{v})\partial(u)$$

where $\hat{v} = v_1 \wedge \ldots \wedge v_g$ and $\hat{v}(\frac{u}{v_i}) = v_1 \wedge \ldots \wedge v_{i-1} \wedge u \wedge v_{i+1} \wedge \ldots \wedge v_g$.

This property shows that the next property of the Alexander function gives a consistent definition of the Reidemeister torsion τ (which yields the Alexander polynomial). If A is a link exterior, then for any element u of $\mathcal{H}_A$,

$$\mathcal{A}_A(u) = \partial(u)\tau(A)$$

If A is a several component link exterior, then $\tau(A)$ belongs to Λ_A, it is defined up to a multiplication by a unit of Λ_A.

In fact, a well-chosen multiplication by an element of the form $exp(\frac{1}{2}x)$ makes the Reidemeister torsion satisfy $\tau(A) = \pm\overline{\tau(A)}$ where the conjugation sends $exp(x)$ to $exp(-x)$ [Mi]. Thus, the Reidemeister torsion is an element defined up to sign in $\mathbf{Z}[\frac{1}{2}H_1(A)/\mathrm{Torsion}] \subset \mathbf{Z}[H_1(A;\mathbf{Q})]$. The choice of an orientation $\mathcal{O}$ of the vector space $H_1(A;\mathbf{R}) \oplus H_2(A;\mathbf{R})$ suppresses the sign indetermination and allows one to define $\tau(A,\mathcal{O}) \in \mathbf{Z}[H_1(A;\mathbf{Q})]$ unambiguously. (See [T, L3].) If A is the exterior of an n-component link L in a $\mathbf{Q}$-sphere M, such an orientation $\mathcal{O}_L$ is unambiguously defined by a basis of $H_1(A;\mathbf{R}) \oplus H_2(A;\mathbf{R})$ of the form $(m_1, \ldots, m_n, \partial T(K_1), \ldots, \partial T(K_{n-1}))$ where m_i and $\partial T(K_i)$ denote the oriented meridian and the boundary of the tubular neighborhood of the i^{th} component of L, respectively.

Note that for a general A, a basis $\mathcal{M} = \{m_1, \ldots, m_n\}$ of $H_1(A;\mathbf{Q})$ induces the natural ring inclusion $\psi_{\mathcal{M}}$ from $\mathbf{Z}[H_1(A;\mathbf{Q})]$ into the ring $\mathbf{Q}[[x_1, \ldots, x_n]]$ of formal series in the x_i: $\psi_{\mathcal{M}}(exp(m_i)) = exp(x_i)$.

In particular, if A is the exterior of a several component link L in a $\mathbf{Q}$-sphere M, then we use the natural basis $\mathcal{M}$ of the meridians of L to define the Alexander series

$$\mathcal{D}(L) = \psi_{\mathcal{M}}(\tau(M \setminus \overset{\circ}{T}(L), \mathcal{O}_L))$$

which is equivalent to the several variable Alexander polynomial

$$\Delta(L)\,(t_1 = exp(x_1), \dots, t_n = exp(x_n)) = (-1)^{n-1}\frac{\mathcal{D}(L)}{|H_1(M)|}$$

In general, a morphism $\psi_{\mathcal{M}}$ allows us to define the *order* of an element of Λ_A as the order of its image under $\psi_{\mathcal{M}}$. It does not depend on $\mathcal{M}$. Similarly, we will speak of the *low degree parts* of the elements of Λ_A. Indeed, the information required to compute the coefficients $\tilde{\zeta}$ of the surgery formula (Property 1) is contained in the low-degree parts of Alexander functions images. Thus, it is worth noting that the degree 0 part of $\mathcal{A}_A(\hat{u})$ is $\varepsilon(\mathcal{A}_A(\hat{u})) = |H_1(A)/(\oplus \mathbf{Z} p_{A*}(u_i))|$, and that the order of $\mathcal{A}_A(\hat{u})$ is greater or equal than the dimension of $H_1(A;\mathbf{Q})/(\oplus \mathbf{Q} p_{A*}(u_i))$. The Alexander function also satisfies the following interesting property which relates the low degree parts of some of its images to algebraic intersections.

Proposition 6.2 *For any (A, ℓ, m), where A is a $\mathbf{Q}$-handlebody whose boundary is equipped with two systems of curves $\ell = (\ell_1, \dots, \ell_g)$ and $m = (m_1, \dots, m_g)$ as in Figure 2, such that the homology classes of the ℓ_i generate $\mathcal{L}_A$,*

$$\mathcal{A}_A(\hat{m}(\frac{\ell_j}{m_k})) = \varepsilon(\mathcal{A}_A(\hat{m}))\sum_{i=1}^{g} \mathcal{I}_{A\Sigma_\ell}(\ell_i \wedge \ell_j \wedge \ell_k)(exp(m_i) - 1) + O(2)$$

where $O(2)$ makes up for an element of Λ_A of order greater or equal than 2, and Σ_ℓ is the standard handlebody with boundary ∂A where the ℓ_i bound disks.

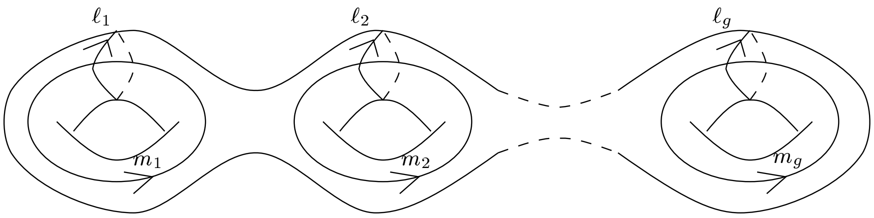

Figure 2: Two systems of curves on ∂A

Recall that $\mathcal{I}$ is defined in Section 5. Though ℓ_i denotes the curve ℓ_i, its homology class, and the class of a lifting of the curve ℓ_i (joined to the basepoint) in $\mathcal{H}_A$, depending on the context, the statement is unambiguous.

It is also worth observing the natural good behaviour of Alexander functions under the two operations: (1) Adding a 2-handle to A, (2) Performing a connected sum along the boundary of two 3-manifolds A and B. A lot of properties of Alexander polynomials can be derived from this natural behaviour. More generally, if A is a submanifold of the interior of a 3-manifold B, in order to compute $\mathcal{A}_B$, it is enough to know $\overline{B \setminus A}$, $\mathcal{A}_A$ and the inclusion from ∂A into $\overline{B \setminus A}$.

Let us use these remarks to be more specific about the sign determination of the Alexander series.

Let $L = (K_i)_{i \in \{1,\ldots,g\}}$ be a link in a $\mathbf{Q}$-sphere M. $g \geq 2$. Consider a regular neighborhood of a graph made of the K_i and paths joining them to the basepoint. This is a handlebody which is a connected sum along boundaries of the $T(K_i)$. Removing the interior of this handlebody from M yields a $\mathbf{Q}$-handlebody A whose boundary is equipped with the meridians m_i and some longitudes ℓ_i of the K_i which sit there as in Figure 2. We let δ_i denote the boundary of the genus one subsurface of ∂A with connected boundary containing m_i and ℓ_i.

Then, up to units of the form $exp(x \in \frac{1}{2}H_1(A)/\text{Torsion})$, for any $j,k \in \{1,\ldots,g\}$,

$$\mathcal{D}(L) = sign(\varepsilon(\mathcal{A}_A(\hat{m})))\psi_L\left(\frac{\mathcal{A}_A(\hat{\delta}(\frac{m_j}{\delta_k}))}{\partial(m_j)}\right)$$

Now, the definition of the coefficient $\tilde{\zeta}$ is complete and we know enough about the surgery formula. Thus, we can apply it together with the helpful formalism introduced above, and we are hopefully ready to discover more properties for the Casson invariant.

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