

# Graphic surgery formulae for finite type invariants of 3-manifolds

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With  $\{i, j\} \subset \{1, 2, 3, 4\}$ , associate

$$p_{ij}: C_4(M) = (M \setminus \infty)^4 \setminus \text{all diagonals} \rightarrow C_2(M)$$

$$(x_1, x_2, x_3, x_4) \mapsto (x_i, x_j)$$

and set  $\omega_{ij} = p_{ij}^*(\omega)$ .

## Theorem 2 (M. Kontsevich, G. Kuperberg, D. Thurston)

$\forall \omega$  as in Theorem 1,

$$\lambda_2(M) = \int_{C_4(M)} \frac{2}{16} \omega \text{ (diagram of two adjacent squares)} + \frac{1}{24} \omega \text{ (diagram of a triangle with a vertex inside)}$$

is a **non trivial degree 2** invariant of  $M$ .

$$\omega_1 \text{ (diagram of two adjacent squares)} = \omega_{12} \wedge \omega_{13}^2 \wedge \omega_{24}^2 \wedge \omega_{34}$$

$$\omega \text{ (diagram of a triangle with a vertex inside)} = -\omega_{12} \wedge \omega_{13} \wedge \omega_{14} \wedge \omega_{23} \wedge \omega_{24} \wedge \omega_{34}$$

## Setting

- $M$  is a  $\mathbb{Q}$ -sphere (closed, 3-dimensional such that  $H_*(M; \mathbb{Q}) = H_*(S^3; \mathbb{Q})$ )
- $\infty \in M$ .

## Theorem 1 (G. Kuperberg, D. Thurston)

$\forall \omega$  closed 2-form on  $C_2(M) = (M \setminus \infty)^2 \setminus \text{diagonal}$

- such that  $\forall J \sqcup K: S^1 \sqcup S^1 \rightarrow M$  inducing  $J \times K: S^1 \times S^1 \rightarrow C_2(M)$ ,

$$\int_{J \times K} \omega = \text{lk}(J, K)$$

- canonically fixed on  $\partial \overline{C_2(M)}$ , antisymmetric,

$$\lambda_{\text{Casson-Walker}}(M) = \frac{2}{12} \int_{C_2(M)} \omega^3.$$

$[\omega] \in H^2(\overline{C_2(M)})$  is Poincaré dual to  $F^4 \in H_4(\overline{C_2(M)}, \partial \overline{C_2(M)})$ .

$$\int_{X^2 \subset C_2(M)} \omega = \langle F^4, X^2 \rangle_{C_2(M)}.$$

$$\int_{C_4(M)} \omega \text{ (diagram of two adjacent squares)} \text{ and } \int_{C_4(M)} \omega \text{ (diagram of a triangle with a vertex inside)}$$

are algebraic intersections of 6 codimension 2 manifolds  $F_{ij}$  in

$$C_4(M) \subset M^4.$$

Thus,  $\lambda_2(M)$  is a mixture of

- linking numbers** associated to **edges**
- triple intersections** in  $M$  associated to **vertices**

associated to some combination of trivalent diagrams.

For any  $n$ , there are (degree  $n$ ) invariants that are similar combinations of such **configuration space integrals** over  $C_{2n}(M)$  associated to trivalent vertices with  $2n$  vertices, and that distinguish  $\mathbb{Z}$ -spheres that are not distinguished by the invariants of degree less than  $n$ .

### Notation (surgery)

Let  $K$  be a knot in  $M$ .

$$M(K; p/q) = M \setminus \text{Int}(T(K)) \cup_{\partial T(K)} D^2 \times S^1$$

$$(pm(K) + ql(K)) \sim S^1 \times *$$

If  $K = \partial \Sigma$ ,

$$F^4(M(K; p/q)) = F^4(M) - \frac{q}{2p} (\Sigma \times \Sigma^+ + \Sigma^+ \times \Sigma)$$

in  $C_2(M) \cap C_2(M(K; p/q))$

up to minor corrections.

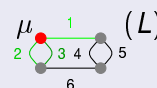
$$\left( lk_{M(K; p/q)}(U, V) = lk_M(U, V) - \left( \frac{q}{p} \right) lk_M(K, U) lk_M(K, V). \right)$$

For  $i \in \{1, \dots, 6\}$ , let  $\Sigma_i$  be a surface with boundary a knot  $\partial \Sigma_i = K_i$ , such that  $\Sigma_i \cap K_j = \emptyset$  if  $i \neq j$ .  
 Set  $\mu_{ijk}(L = (K_1, \dots, K_6)) = -\langle \Sigma_i, \Sigma_j, \Sigma_k \rangle_M$ .  
 For  $\sigma \in \mathfrak{S}_6$ ,  $\mu_{ijk}^\sigma = \mu_{\sigma(i)\sigma(j)\sigma(k)}$ .  
 Let  $q_1, \dots, q_6 \in \mathbb{Z} \setminus \{0\}$ . For  $I \subset \{1, \dots, 6\}$ ,  $M_I = M(K_i; 1/q_i)_{i \in I}$ .

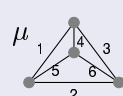
### Theorem 1

$$\sum_{I \subset \{1, 2, \dots, 6\}} (-1)^{|I|} \lambda_2(M_I) = \left( \prod_{i=1}^6 q_i \right) \left( 2\mu_{\text{cylinder}}(L) + \mu_{\text{triangle}}(L) \right)$$

where

$$\mu_{\text{cylinder}}(L) = \frac{1}{16} \sum_{\sigma \in \mathfrak{S}_6} \mu_{123}^\sigma \mu_{145}^\sigma \mu_{263}^\sigma \mu_{465}^\sigma$$


and

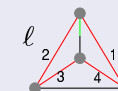
$$\mu_{\text{triangle}}(L) = \frac{1}{24} \sum_{\sigma \in \mathfrak{S}_6} \mu_{143}^\sigma \mu_{251}^\sigma \mu_{236}^\sigma \mu_{456}^\sigma$$


Let  $K_1, K_2, K_3, K_4$  bound  $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4$  in  $M$  such that  $\Sigma_i \cap K_j = \emptyset$  when  $i \neq j$ .  
 Assume  $\mu_{ijk}(L_4 = (K_1, \dots, K_4)) = 0 \quad \forall \{i, j, k\} \subset \{1, 2, 3, 4\}$ .  
 For  $I \subset \{1, 2, 3, 4\}$ ,  $M_I = M(K_i; 1/q_i)_{i \in I}$ .

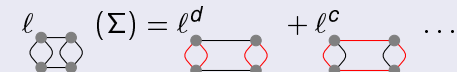
### Theorem 2

$$\sum_{I \subset \{1, 2, 3, 4\}} (-1)^{|I|} \lambda_2(M_I) = \left( \prod_{i=1}^4 q_i \right) \left( 2\ell_{\text{cylinder}}(\Sigma) + \ell_{\text{triangle}}(\Sigma) \right)$$

where

$$\ell_{\text{triangle}}(\Sigma) = \frac{3}{24} \sum_{\sigma \in \mathfrak{S}_4} lk(\Sigma_{\sigma(1)} \cap \Sigma_{\sigma(2)}, \Sigma_{\sigma(3)} \cap \Sigma_{\sigma(4)})$$


and

$$\ell_{\text{cylinder}}(\Sigma) = \ell^d + \ell^c + \dots$$


$$\ell^d = \frac{1}{16} \sum_{\sigma \in \mathfrak{S}_4} lk(\Sigma_{\sigma(1)} \cap \Sigma_{\sigma(2)}, \Sigma_{\sigma(3)} \cap \Sigma_{\sigma(4)})^2$$

and

$$\ell^c = \frac{4}{16} \sum_{\sigma \in \mathfrak{S}_4} lk^\sigma(\Sigma_1 \cap \Sigma_2, \Sigma_2 \cap \Sigma_3) lk^\sigma(\Sigma_3 \cap \Sigma_4, \Sigma_4 \cap \Sigma_1)$$

- Theorems 1 and 2 generalize to any degree  $n$  invariant (with  $3n-$  and  $2n-$  component links, respectively).
- They generalize for  $p/q$ -surgeries on null-homologous knots in  $\mathbb{Q}$ -spheres.
- For  $\mathbb{Z}$ -spheres, together with other results of math.GT/0703347, they generalize Garoufalidis-Goussarov-Polyak comparison of filtrations of the vector space generated by  $\mathbb{Z}$ -spheres.

### Questions

- Study filtrations of the space of  $\mathbb{Q}$ -spheres.
- Find **topological** surgery formulae for all configuration space invariants.

### Theorem (Le, Murakami, Ohtsuki, Garoufalidis ...)

- $\lambda_2(M_1 \# M_2) = \lambda_2(M_1) + \lambda_2(M_2) (\Rightarrow \lambda_2(S^3) = 0)$
- $\lambda_2(M(K; 1/q)) - \lambda_2(M) = q^2 \lambda_2''(K) + qv_3(K)$ .
- $\lambda_2''(K)$  is determined by the Alexander polynomial (Garoufalidis-Habegger).
- Any degree  $n$  invariant has a degree  $n$  polynomial variation, and an explicit graphic formula for the leading term is in math.GT/0703347.
- E. Auclair explicitly expressed  $v_3(K)$  (that is a coefficient of the Jones polynomial for  $K \subset S^3$ ) in terms of invariants of  $H_1(\widetilde{M} \setminus \Sigma; \mathbb{Q})$ .