Quantum Walks in Electric Fields

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Outline

Definition of e.m. fields in continuous space-time

Consider a Schrödinger operator

$$H = \frac{p^2}{2m} + V.$$

Introduce electromagnetic interaction via minimal coupling

$$p \mapsto p - eA, \quad i\hbar\partial_t \mapsto i\hbar\partial_t - e\phi$$
$$H \mapsto \tilde{H} = \frac{(p-eA)^2}{2m} + V + e\phi$$
$$\Rightarrow \quad [\tilde{H}, p_k + eA_k] = 0.$$

Hence, as a consequence,

$$T(a)T(b) = e^{iF_{\mu\nu}a^\mu b^\nu}\, T(b)T(a)$$

for $T(a) = e^{i(p_\mu + eA_\mu)a^\mu}$ the Magnetic Translation Operators.

Definition of e.m. fields in Discrete Time Settings

Dynamical map given by a unitary

- $\Rightarrow$ no notion of generator of (finite) time translations
- $\Rightarrow$ no notion of minimal coupling

BUT: we can nevertheless find a notion of electromagnetic fields by imposing obstructions on the commutators of translations for $T_\mu := T(a_\mu)$

$$T_\mu T_\nu = e^{iF_{\mu\nu}} T_\nu p T_\mu, \qquad \mu, \nu = 0, \dots, s \tag{1}$$

and we define as in the continuous case:

$$\begin{aligned} F_{\mu\mu} &= 0 \\ (F_{0,\mu})_{\mu=0,\dots,s} &=: E \\ (F_{\mu,\nu})_{\mu,\nu=1,\dots,s} &=: B \end{aligned}$$

Classification of 1D translationally invariant walks

To electrify a walk W with $W\,T_1 = T_1 W$ for $s = 1$, find an operator F such that $T_0 = F \cdot W$ and

$$T_0 T_1 = e^{iE} T_1 T_0. \qquad (2)$$

T_1 acts on wave functions as

$$(T_1\psi)(x) = \psi(x+1).$$

Hence an operator we may consider is $F = e^{iEQ}$ for $E \in \mathbb{R}$ which assures (2) as

$$\begin{aligned}\left(F^{-1} T_1^{-1} F T_1 \psi\right)(x) &= e^{-iEx} e^{iE(x+1)} \psi(x) \\ &= e^{iE} \psi(x) \qquad \forall \psi.\end{aligned}$$

Hence, to electrify a translationally invariant walk W we DEFINE

$$\boxed{W_E := e^{iEQ} W.} \qquad (3)$$

Discrete Space - Almost Mathieu Operator

Consider $\mathcal{A}_\theta = \{(u^k v^l)_{k,l} | vu = e^{2\pi i\theta} uv\}$ represented on discrete space $\mathcal{H} = \ell^2(\mathbb{Z})$ as

$$u \mapsto U : \psi(n) \mapsto \psi(n+1)$$
$$v \mapsto V : \psi(n) \mapsto e^{i2\pi\theta n}\psi(n)$$

Take the self adjoint element $\mathcal{A}_\theta \ni H = u + \lambda v + (u + \lambda v)^*$.

The Almost Mathieu operator is defined as

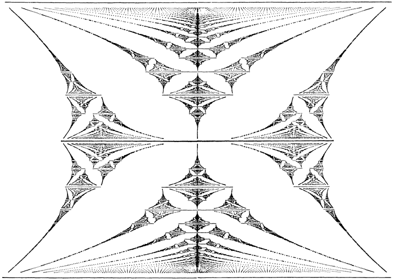
$$(H_{\lambda,\alpha,\theta}\psi)(n) = \psi(n+1) + \psi(n-1) + 2\lambda\cos(n\theta - \alpha)\,\psi(n)$$

Hofstadter '76:
"The problem of Bloch electrons in magnetic fields is a very peculiar problem, because it is one of the very few places in physics where the difference between rational numbers and irrational numbers makes itself felt."

Discrete Space - Almost Mathieu Operator

$\theta \in 2\pi\mathbb{Q}$:

Hofstadter '76: for $\theta \in 2\pi\mathbb{Q}$ the spectrum of $H_{\lambda=1,\alpha,\theta}$ consists of Denominator(θ) bands as



Discrete Space - Almost Mathieu Operator

$\theta \notin 2\pi\mathbb{Q}$:

Hofstadter '76: "...From this algorithm, the nature of the spectrum at an "irrational" field can be deduced; it is seen to be an uncountable but measure-zero set of points (a Cantor set)"

Conjecture (B.Simon)

The spectrum of $H_{\lambda,\alpha,\theta}$ is a Cantor set for all $\lambda > 0$ and $\theta \in \mathbb{R}\backslash 2\pi\mathbb{Q}$.

Proved by Avila and Jitomirskaya in 2005 (preceded by various partial results).

Analog in Quantum walks? Open question...

Electrification of 1D translationally invariant walks

$W_E := e^{iEQ}W$ leaves us as in the continuous case with two options:

- $E \in 2\pi\mathbb{Q}$
- $E \notin 2\pi\mathbb{Q}$

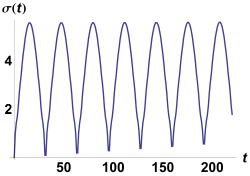
Hint for distinct behaviour:
Deelectrify W_E by either spatial or temporal regrouping in case $E \in 2\pi\mathbb{Q}$.

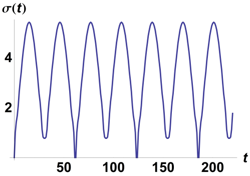
Outline

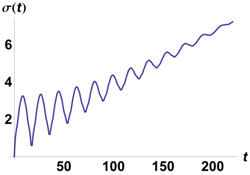
Numerical Results

Choose $E = 2\pi \frac{p}{q} \in 2\pi\mathbb{Q}$.

Denominator even (32) odd (31)







$E \in \mathbb{Q}$: Generic behaviour and spectral considerations

Take walk W on $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$ and electrify:

$$W_E = e^{iEQ} W.$$

$\mathbb{Q} \in E = 2\pi \frac{p}{q} \Rightarrow \rho := e^{iE}$ primitive q^{th} root of unity $\Rightarrow W_E^q$ translation inv.
$\Rightarrow$ Fourier-transform makes sense:

$$\begin{aligned}
(W_E^q \psi)(p) &= W(p+E) \cdot (W_E^{q-1}\psi)(p+E) &(4)\\
&= \mathcal{T}\left[\prod_{j=1}^{q} W(p+jE)\right] \psi(p) &(5)\\
&=: \tilde{\mathrm{W}}_{\mathrm{q}}(\mathrm{p})\, \psi(p). &(6)
\end{aligned}$$

This matrix may be written as

$$\tilde{\mathrm{W}}_{\mathrm{q}}(\mathrm{p}) = R \cdot W(p) \cdot R^2 \cdot W(p) \dots R^q \cdot W(p), \quad R = e^{iE\sigma_3} \tag{7}$$

The revival theorem

Theorem

Let $\tilde{W}_q$ be the temporal regrouped and hence translational invariant electric walk with $E = 2\pi\frac{p}{q}$. Then

$$\|W_E^{2q} + \mathbb{1}\|_{op} = 2|a|^q \qquad q \text{ odd}$$

$$\|W_E^{q} + (-1)^{\frac{q}{2}}\mathbb{1}\|_{op} = 2|a|^{\frac{q}{2}} \qquad q \text{ even}$$

where $SU(2) \ni C = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$.

This bound is exponentially good in q iff $|a| < 1$.

A trace formula

To determine the spectrum, solve

$$\det\left[\tilde{\mathrm{W}}_{\mathbf{q}}(\mathrm{p}) - \lambda\mathbb{1}\right] = \det\tilde{\mathrm{W}}_{\mathbf{q}}(\mathrm{p}) - \lambda\,\mathrm{Tr}\,\tilde{\mathrm{W}}_{\mathbf{q}}(\mathrm{p}) + \lambda^2 \stackrel{!}{=} 0 \tag{8}$$

$\det\tilde{\mathrm{W}}_{\mathbf{q}}(\mathrm{p})$ is simply given by $\det W(p)^q = 1$, and for the trace:

Lemma

Let $\tilde{\mathrm{W}}_{\mathbf{q}}(\mathrm{p})$ be given by (7) and $R^q = 1$ s.t. $R^k \neq 1\,\forall k < q$. Then $Tr\,\tilde{\mathrm{W}}_{\mathbf{q}}(\mathrm{p})$ is given by

$$Tr\,\tilde{\mathrm{W}}_{\mathbf{q}}(\mathrm{p}) = \begin{cases} a^q + d^q & q \text{ odd} \\ -(a^q + d^q) + (-1)^{\frac{q}{2}+1} 2\left((\det W(p))^{\frac{q}{2}} - (ad)^{\frac{q}{2}}\right) & q \text{ even} \end{cases} \tag{9}$$

Outline

Continued Fraction Expansions of $E \in \mathbb{R}\backslash 2\pi\mathbb{Q}$

Consider $E \in 2\pi\mathbb{R}\backslash\mathbb{Q}$. Represent E as

$$E = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots}} =: [a_0; a_1, a_2, \dots]$$

Knowing $(a_i)_{i\in\mathbb{N}}$ one defines finite approximations

$$\frac{p_n}{q_n} := [a_0; a_1, \dots, a_n], \qquad \text{where} \quad \begin{cases} \frac{p_n}{q_n} > E & n \text{ odd} \\ \frac{p_n}{q_n} < E & n \text{ even} \end{cases}$$

which approximate E as

$$\left| E - \frac{p_n}{q_n} \right| < \frac{1}{a_{n+1} q_n^2}.$$

Estimates for Revivals for $E \in \mathbb{R}\backslash 2\pi\mathbb{Q}$

Consider $\psi \in \mathcal{H}$ with supp $\psi = [-L, L]$. Then

$$\begin{aligned} &\|W_E\psi - W_{E'}\psi\| \leq L|E - E'| \\ \Rightarrow \quad &\|W_E^t\psi - W_{E'}^t\psi\| \leq \frac{t}{2}(t + 2L - 1)|E - E'| \end{aligned}$$

hence with $E' = 2\pi\frac{p_n}{q_n}$ and the Revival Theorem

$$\begin{aligned} \|W_E^{2q_n}\psi + \psi\| &\leq \frac{4\pi}{a_{n+1}} + \mathcal{O}\left(Lq_n^{-1}\right) \qquad q_n \text{ odd} \\ \|W_E^{q_n}\psi + (-1)^{\frac{q_n}{2}}\psi\| &\leq \frac{\pi}{a_{n+1}} + \mathcal{O}\left(Lq_n^{-1}\right) \qquad q_n \text{ even} \end{aligned}$$

Spectral Considerations for $E \in \mathbb{R}\backslash 2\pi\mathbb{Q}$

Hence

$$\lim_{n\to\infty} a_n = \infty \quad \Longrightarrow \quad \text{inf. seq. of sharper and sharper revivals}$$

$$\Rightarrow \quad \sigma_{\mathrm{ac}} = \varnothing$$

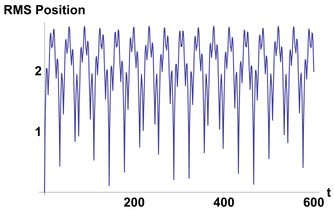
and

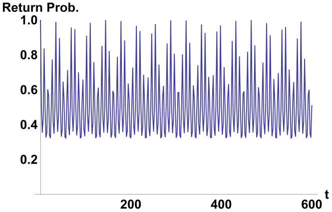
$$\lim_{n\to\infty} a_{n+1} - a_n = \infty \text{ sufficiently fast} \qquad \Rightarrow \quad \sigma_{\mathrm{pp}}(W_E) = \varnothing$$

$$\Rightarrow \quad \boxed{\sigma(W_E) = \sigma_{\mathrm{sc}}(W_E)}$$

Bounded CFE

In the case of $\lim_{n\to\infty} a_n < \infty$ (e.g. E =Golden Ratio $= [0;1,1,1,\ldots]$)



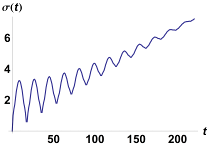
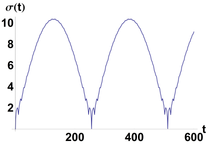
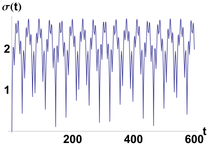


Conjecture

Let $E = [a_0; a_1, \ldots]$ with $\lim_{n\to\infty} a_n < \infty$. Then the system shows Anderson localization, i.e.

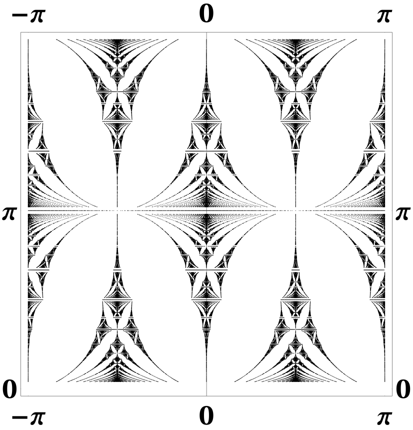
$$\sigma(W_E) \equiv \sigma_{pp}(W_E)$$

Summary

E	rational	irrat., CFE unbounded	irrat., CFE bounded
			
$\sigma(W_E)$	absolutely continuous	singular continuous	pure point
	proved	proved	conjectured

Outlook

Next step: obviously magnetic fields - as an appetizer:



Acknowledgements

Thanks for your attention!

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