Géométrie algébrique/*Algebraic Geometry*

Enriques diagrams, resolutions and toric clusters

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Abstract – We study morphisms obtained by blowing up the points in a constellation $\mathcal{C}$ of infinitely near nonsingular points, the associated Enriques diagrams, and singularities defined by r-tuples of non-degenerate functions with respect to an integer weighted constellation, *i.e.* a cluster. Non-degenerate r-tuples exist for clusters coming from complete ideals supported in $\mathcal{C}$ in characteristic 0. We describe the semigroup of these ideals and, if $\mathcal{C}$ is toric, we compute it.

Diagrammes d'Enriques, résolutions et amas toriques

Résumé – *Nous étudions les morphismes obtenus en éclatant les points d'une constellation $\mathcal{C}$ de points infiniment voisins non singuliers, les diagrammes d'Enriques associés et les singularités définies par un r-uple de fonctions non dégénéré relativement à une constellation $\mathbb{Z}$-pondérée, ou amas. Nous montrons l'existence de r-uples non-dégénérés relativement à un amas provenant d'un idéal complet à support dans $\mathcal{C}$ en caractéristique 0. Nous décrivons le semi-groupe formé par ces idéaux et nous le calculons lorsque $\mathcal{C}$ est torique.*

Version française abrégée – Soient X une variété algébrique non singulière de dimension $d \geqq 2$, définie sur un corps algébriquement clos K, et O un point fermé de X. Dans la suite, une *constellation* de points infiniment voisins de O signifiera un ensemble fini $\mathcal{C} = \{P_0 = O, P_1, ..., P_n\}$ où, $\sigma_i : X_{i+1} \to X_i$ désignant l'éclatement de X_i de centre P_i, on a $X_0 = X$ et P_{i+1} est un point de X_{i+1} se projetant sur P_0 dans X_0, $0 \leqq i \leqq n$. On identifiera deux constellations pour lesquelles les morphismes $\sigma = \sigma_{\mathcal{C}} := \sigma_0 \circ ... \circ \sigma_n$ de $X(\mathcal{C}) := X_{n+1}$ dans X sont X-isomorphes. La relation $P_j \geqq P_i$, si P_j se projette sur P_i dans X_i, est une relation d'ordre partielle sur les points de $\mathcal{C}$.

Pour chaque $P = P_i \in \mathcal{C}$, on note B_P (ou B_i) le diviseur $\sigma_i^{-1}(P)$ sur X_{i+1} et par E_P (ou E_i) ses transformées strictes successives, en particulier sur $X(\mathcal{C})$. On dit que P_j est *proche* de P_i si $P_j \in E_i$ et on note $P_j \to P_i$ (ou $j \to i$).

Le *diagramme d'Enriques* $\Gamma_{\mathcal{C}}$ de $\mathcal{C}$ est l'arbre avec racine muni de la relation binaire $(\rightsquigarrow)$ ainsi défini : les sommets correspondent aux points P de $\mathcal{C}$, les arêtes aux couples (P, Q) tels que P soit un point de E_Q en dehors du lieu exceptionnel du morphisme $E_Q \to B_Q$, la racine correspond au point O et la relation $(\rightsquigarrow)$ à la relation de proximité $(\to)$.

On caractérise d'abord les arbres finis avec racine munis d'une relation binaire sur l'ensemble de leurs sommets qui sont le diagramme d'Enriques d'une constellation. *Voir* theorem 1 de la version anglaise.

On désigne par *amas* une constellation pondérée $\mathcal{A} = (\mathcal{C}, \underline{m})$ où $\underline{m} = (m_0, ..., m_n) \in \mathbb{Z}^{n+1}$. On lui associe le diviseur à support exceptionnel sur $X(\mathcal{C})$, $D_{\mathcal{A}} = \sum m_i E_i^*$ où E_i^* est le transformé total de B_i sur $X(\mathcal{C})$ et l'idéal $I_{\mathcal{A}}$ de $R := \mathcal{O}_{X, O}$ germe en O de $\sigma_* \mathcal{O}_{X(\mathcal{C})}(-D_{\mathcal{A}})$.

On définit la notion d'*élément* (resp. de *r-uple*, $1 \leqq r < d$, *d'éléments*) de R *propre* (resp. *non dégénéré*) relativement à $\mathcal{A}$. On montre que la singularité définie par un tel r-uple admet $\sigma_{\mathcal{C}}$ pour désingularisation plongée. De plus, si $r = d - 2$, *i.e.* s'il s'agit d'une surface, sa désingularisation minimale est obtenue en éclatant les points P_i de $\mathcal{C}$ tels que $m_i \neq 0, 1$. L'anneau R ne contient pas nécessairement d'élément propre, *a fortiori* de r-uple d'éléments non dégénéré. Cependant si $\mathcal{A}$ est *idéaliste*, *i.e.* s'il existe un idéal I primaire pour Max R tel que $I\mathcal{O}_{X(\mathcal{C})} = \mathcal{O}_{X(\mathcal{C})}(-D_{\mathcal{A}})$, un élément général de $I_{\mathcal{A}}$ est propre relativement à $\mathcal{A}$

et si la caractéristique de K est 0, un r-uple d'éléments généraux de $I_{\mathcal{A}}$ est non dégénéré relativement à $\mathcal{A}$.

Dans toute la suite, on fixe une constellation $\mathcal{C}$. Les groupes N^1 et N_1 respectivement quotient de $\operatorname{Pic} X(\mathcal{C})$ et du groupe des 1-cycles relatifs de $X(\mathcal{C})$ par la relation d'équivalence numérique relative, sont des groupes abéliens libres mis en dualité par la forme d'intersection $(D, C) \to D \cdot C$. On vérifie que $N^1 = \mathbb{E} := \oplus \mathbb{Z} E_P$.

Les idéaux complets de R primaires pour $\operatorname{Max} R$ (resp. les amas idéalistes) à support dans $\mathcal{C}$ forment un semi-groupe $\mathcal{F}_{\mathcal{C}}$ (resp. $\mathcal{G}_{\mathcal{C}}$ que nous appelons la galaxie de $\mathcal{C}$). La correspondance $\mathcal{A} \mapsto I_{\mathcal{A}}$ (resp. $\mathcal{A} \mapsto D_{\mathcal{A}}$) est un isomorphisme de $\mathcal{G}_{\mathcal{C}}$ sur $\mathcal{F}_{\mathcal{C}}$ (resp. le semi-groupe $\mathbb{E}^{\#}$ de $\mathbb{E}$ formé par les diviseurs engendrés par leurs sections globales au voisinage de $\sigma^{-1}(O)$ ou σ-engendrés). Soit $\langle E_0^*, \ldots, E_n^* \rangle$ le semi-groupe engendré par les E_i^* et soit NE (resp. $NE^{\vee}$) le cône engendré par les images des courbes effectives contractées par σ dans $N_1 \bigotimes_{\mathbb{Z}} \mathbb{Q}$ (resp. son dual dans $\mathbb{E} \bigotimes_{\mathbb{Z}} \mathbb{Q}$). On a les inclusions

$$\mathbb{E}^{\#} \subset -NE^{\vee} \cap \mathbb{E} \subset \langle E_0^*, \ldots, E_n^* \rangle .$$

Si $\mathcal{C}$ est *torique*, *i.e.* si $X = K^d$ muni de l'action naturelle du tore $T = (K^*)^d$, et si chaque P_i, $0 \leqq i \leqq n$, est une orbite fermée de X_i, l'égalité $\mathbb{E}^{\#} = -NE^{\vee} \cap \mathbb{E}$ est une conséquence immédiate de la caractérisation numérique des diviseurs T-stables engendrés par leurs sections globales [8]. Vu l'isomorphisme ci-dessus entre $\mathcal{G}_{\mathcal{C}}$ et $\mathbb{E}^{\#}$, on a $\mathcal{A} \in \mathcal{G}_{\mathcal{C}}$ si et seulement si pour tout $Q \in \mathcal{C}$ et toute courbe T-stable l de B_Q, on a $m_Q \geqq \sum_{P \in \mathcal{C} \cap \bar{l}} m_P$ où, par convention, $P \in \bar{l}$ signifie que P est dans l'adhérence de la transformée stricte de l. Désignant par « *courbe maximale* » l'adhérence de l dans $X(\mathcal{C})$ si $\mathcal{C} \cap \bar{l}$ est maximal pour l variant dans B_Q, il en résulte aussi que les vecteurs extrémaux de NE sont les classes dans N_1 des courbes maximales fournies par tous les $Q \in \mathcal{C}$.

Pour $Q \in \mathcal{C}$, on désigne par D_Q le diviseur correspondant à l'idéal $*$ simple spécial $\mathcal{P}_Q$ introduit par Lipman dans [7]. On calcule l'exposant $r_Q \in \mathbb{Z}$ de $\mathcal{P}_Q$ dans la « factorisation » unique de $I \in \mathcal{F}_{\mathcal{C}}$ en fonction des poids de l'amas $\mathcal{A}$ correspondant à I. Précisément, $r_Q = m_Q - \sum_{P \twoheadrightarrow Q} m_P$ où $P \twoheadrightarrow Q$ signifie que $P \in \mathcal{C} \cap \bar{l}$ pour un $l \subset B_Q$ comme ci-dessus. La nouvelle relation $(\twoheadrightarrow)$ entre points de $\mathcal{C}$ est appelée la relation de *proximité linéaire. Voir* theorem 6 de la version anglaise. On y trouvera aussi un codage des constellations toriques et une traduction des relations de proximité et de proximité linéaire dans ce code.

Toujours si $\mathcal{C}$ est torique, il en résulte que les conditions suivantes sont équivalentes : (i) $-NE^{\vee}$ (ou NE) est un cône régulier, (ii) les vecteurs extrémaux de $NE^{\vee}$ sont les D_Q, $Q \in \mathcal{C}$, (iii) $-NE^{\vee}$ est un cône simplicial, (iv) les exposants $r_Q \in \mathbb{N}$ pour tout $I \in \mathcal{F}_{\mathcal{C}}$, et une caractérisation des constellations vérifiant ces conditions. Il en est ainsi si $\mathcal{C}$ est une chaîne torique (*i.e.* totalement ordonnée par $\leqq$).

Let X be a smooth algebraic variety of dimension $d \geqq 2$ over an algebraically closed field K, $O \in X$ a closed point. By a *constellation* (of infinitely near points) at O, we mean a finite set $\mathcal{C} = \{P_0 = O, P_1, \ldots, P_n\}$ where, if $\sigma_i : X_{i+1} \to X_i$ is the blowing-up of X_i with center P_i, one has $X_0 = X$ and P_{i+1} is a closed point in X_{i+1} whose image in X_0 is P_0. Consider the morphism $\sigma : X(\mathcal{C}) \to X$, where $X(\mathcal{C}) := X_{n+1}$ and $\sigma = \sigma_{\mathcal{C}} := \sigma_0 \circ \sigma_1 \circ \ldots \circ \sigma_n$ and identify constellations whenever the corresponding morphisms σ are X-isomorphic. The set $\mathcal{C}$ is partially ordered by the relation $P_j \geqq P_i$ if

P_i is the image at X_i of P_j. For $P = P_i \in \mathcal{C}$, denote by B_P (or B_i) the divisor $\sigma_i^{-1}(P)$ and by E_P (or E_i) the strict transform of B_P at anyone of the X_j with $j \geqq i+1$, in particular at $X(\mathcal{C})$ (and B_P). A point $P_j \in \mathcal{C}$ is said to be *proximate* to P_i, $P_j \to P_i$ (or $j \to i$) in symbol, if $P_j \in E_i$. For a subset $\mathcal{J}$ of $\mathcal{C}$, set $E_{\mathcal{J}} = \bigcap_{P \in \mathcal{J}} E_P \subset X(\mathcal{C})$. If $E_{\mathcal{J}} \neq \varnothing$, then $\mathcal{J} = \{P_{i_1} < \ldots < P_{i_k}\}$ with $i_l \to i_h$ for every h, l with $1 \leqq h < l \leqq k$, and we set $B_{\mathcal{J}} = E_{i_1} \cap \ldots \cap E_{i_{k-1}} \cap B_{i_k}$ and $\sigma_{\mathcal{J}} = \sigma_{i_k+1} \circ \ldots \circ \sigma_n|_{E_{\mathcal{J}}} : E_{\mathcal{J}} \to B_{\mathcal{J}}$. Order and proximity are represented in the Enriques diagram [4], *i.e.* the rooted tree $\Gamma_{\mathcal{C}}$ whose vertices p correspond one to one to points $P \in \mathcal{C}$, the edges to pairs (P, Q) with $P \in E_Q$, out of the exceptional locus of the morphism $E_Q \to B_Q$, the root corresponds to O, and which is provided with a binary relation $p \rightsquigarrow q$ corresponding to pairs with $P \to Q$. The rooted tree order $\gtrsim$ corresponds to the order $\geqq$; write $l(p)$ for the number of edges in the path joining the root to p.

THEOREM 1. – *Let* Γ *be a finite rooted tree provided with a binary relation* $\rightsquigarrow$ *on the vertex set. Then* $\Gamma = \Gamma_{\mathcal{C}}$ *for some constellation* $\mathcal{C}$ *if and only if the following conditions hold:* (i) *if* $p \rightsquigarrow q$, *then* $p \gtrsim q$ *and* $p \neq q$, (ii) *if* $p \gtrsim q$ *and* $l(p) = l(q)+1$, *then* $p \rightsquigarrow q$, (iii) *if* $r \gtrsim p \gtrsim q$ *with* $p \neq q$ *and* $r \rightsquigarrow q$, *then* $p \rightsquigarrow q$.

Let $\mathbb{E}$ be the free group $\oplus \mathbb{Z}\, E_P$. If E_P^* denotes the total transform of B_P at $X(\mathcal{C})$, one has $E_Q = E_Q^* - \sum_{P \to Q} E_P^*$, so (E_P^*) is also a basis of $\mathbb{E}$. By a *cluster*, we mean a weighted constellation $\mathcal{A} = (\mathcal{C}, \underline{m})$ where $\underline{m} = (m_P) = (m_0, \ldots, m_n) \in \mathbb{Z}^{n+1}$. Associated to $\mathcal{A}$, one has the divisor $D_{\mathcal{A}} = \sum m_i\, \mathbb{E}_i^* \in \mathbb{E}$; thus the data $\mathcal{A}$ is equivalent to $(\mathcal{C}, D_{\mathcal{A}})$. Given an effective divisor $\mathfrak{h}$ on X, for any j, the divisor

$$\tilde{\mathfrak{h}}_j = (\sigma_1 \circ \ldots \circ \sigma_j)^*(\mathfrak{h}) - \sum_{0 \leqq i < j} m_i\, E_i^*$$

is called the *virtual transform* of $\mathfrak{h}$ on X_j. Then $\mathfrak{h}$ is said to *pass* (resp. to *pass properly*) through $\mathcal{A}$ if for any $\mathcal{J} = \{P_{i_1} < \ldots < P_{i_k}\}$ with $k = 1$ (resp. $1 \leqq k \leqq d$) such that $E_{\mathcal{J}} \neq \varnothing$, the multiplicity at P_{i_k} of the inverse image $\mathfrak{h}_{\mathcal{J}}$ of $\tilde{\mathfrak{h}}_{i_k}$ on $E_{i_1} \cap \ldots \cap E_{i_{k-1}} \subset X_{i_k}$ is $\geqq$ (resp. $=$) to $m_{\mathcal{J}} := m_{i_k}$. Above notions are local at O; thus, if f is a local equation for $\mathfrak{h}$ and $\mathfrak{h}$ passes properly through $\mathcal{A}$, we say that f is *proper with respect to* $\mathcal{A}$ and the projective tangent cone to $\mathfrak{h}_{\mathcal{J}}$ at Q_{i_k} is denoted by $\mathfrak{h}_{\mathcal{J}}(f)$ (this is a divisor of degree $m_{\mathcal{J}}$ on $B_{\mathcal{J}} \simeq \mathbb{P}^{d-k}$). An *r-tuple* $f_1, \ldots, f_r$ of elements of $R := \mathcal{O}_{X,O}$ with $1 \leqq r < d$ is said to be *non-degenerate with respect to* $\mathcal{A}$ if $f_1, \ldots, f_r$ are proper with respect to $\mathcal{A}$ and for any $\mathcal{J}$ the scheme $\mathfrak{h}_{\mathcal{J}}(f_1) \cap \ldots \cap \mathfrak{h}_{\mathcal{J}}(f_r)$ is nonsingular of codimension r on $B_{\mathcal{J}} \setminus \mathcal{C}$. Finally, for a cluster $\mathcal{A}$, set $\mathcal{C}'(\mathcal{A}) = \{P_i \in \mathcal{C} \mid m_i \neq 0, 1\}$.

THEOREM 2. – *Assume* $f_1, \ldots, f_r$ *is non-degenerate with respect to* $\mathcal{A}$, $1 \leqq r < d$, *and let* S *be the singularity defined by* $(f_1, \ldots, f_r)$. *Then*

(*a*) $\sigma_{\mathcal{C}}$ *is an embedded resolution of singularities for* S;

(*b*) *if* $r = d-2$, *the strict transform of* S *at* $X(\mathcal{C}'(\mathcal{A}))$ *is the minimal resolution of the surface singularity* S.

For any $Q \in \sigma^{-1}(O)$, let $\mathcal{J}$ be the maximal set with $Q \in E_{\mathcal{J}}$; then (*a*) follows as the strict transform of S intersects $E_{\mathcal{J}}$ transversally. Since the multiplicity at P_j of S is m_j^r, then $\mathcal{C}'(\mathcal{A})$ is a constellation. Now, a general hyperplane section of the surface S is an ordinary curve singularity, so if $m_0 > 1$ any desingularization of S factorizes through the blowing-up of O and (*b*) follows. ■

Non-degeneracy and theorem 2 (*a*) are analogous to [6] where the role of $\mathcal{A}$ is played by the Newton polyhedron $\mathcal{N}$ and proper means to have $\mathcal{N}$ as polyhedron.

A cluster $\mathcal{A} = (\mathcal{C}, \underline{m})$ is called *idealistic* if one has $\mathcal{O}_{X(\mathcal{C})}(-D_{\mathcal{A}}) = I\,\mathcal{O}_{X(\mathcal{C})}$ for some ideal I in R, or, equivalently, if $\mathcal{O}_{X(\mathcal{C})}(-D_{\mathcal{A}})$ is σ-generated. The ideals $I \subset R$ primary for Max R and such that $I\,\mathcal{O}_{X(\mathcal{C})}$ is invertible for some $\mathcal{C}$ are called *finitely supported* [7]. For such an I, if $I\,\mathcal{O}_{X(\mathcal{C})} = \mathcal{O}_{X(\mathcal{C})}(-D_I)$, then the cluster defined by D_I is idealistic, and the completion (*i.e.* the integral closure) $\overline{I}$ of I is recovered from D_I by $\overline{I} = \sigma_* \mathcal{O}_{X(\mathcal{C})}(-D_I)$. Thus, there is only one complete ideal associated to an idealistic cluster. For any $\mathcal{A}$, the ideal $I_{\mathcal{A}} = \sigma_* \mathcal{O}_{X(\mathcal{C})}(-D_{\mathcal{A}})$ consists of those germs f defining divisors which pass through $\mathcal{A}$ [2]. In general, $I_{\mathcal{A}}$ does not contain proper elements. The following Bertini type result allows us to apply theorem 2.

PROPOSITION 3. – *For an idealistic cluster one has:*

(a) a general element $f \in I_{\mathcal{A}}$ is proper with respect to $\mathcal{A}$;

(b) if char$(K) = 0$, *for any r, $1 \leqq r < d$, a general r-tuple of elements of $I_{\mathcal{A}}$ is non-degenerate with respect to $\mathcal{A}$.*

Fix a constellation $\mathcal{C}$ and set $Z = X(\mathcal{C})$. The idealistic clusters with support in $\mathcal{C}$ form an additive semigroup with the addition of weights, called the *galaxy* $\mathcal{G}_{\mathcal{C}}$ of $\mathcal{C}$. The correspondences $\mathcal{A} \to D_{\mathcal{A}}$, $\mathcal{A} \to I_{\mathcal{A}}$ give isomorphisms between $\mathcal{G}_{\mathcal{C}}$ and the respective semigroups $\mathbb{E}^{\#} \subset \mathbb{E}$ of σ-generated divisors and $\mathcal{F}_{\mathcal{C}}$ of complete ideals supported in $\mathcal{C}$ (with law $I * J = \overline{I.J}$). For each $P = P_i \in \mathcal{C}$, Lipman introduces the ideal $\mathcal{P}_P \in \mathcal{F}_{\mathcal{C}}$ corresponding to the cluster with minimum (for descending lexicographic order) $\underline{m}$ among those with $m_i = 1$ and $m_j = 0$ if $P_j \nleq P_i$. The divisors $D_P := D_{\mathcal{P}_P}$ are indecomposable in $\mathbb{E}^{\#}$ and they form a new basis for $\mathbb{E}$. Thus, every $I \in \mathcal{F}_{\mathcal{C}}$ has a unique « factorization » $I = \prod^{*}_{P} \mathcal{P}_P^{r_P}$ where $r_P \in \mathbb{Z}$. Now, denote by $Z_1(Z/X)$ the group of relative 1-cycles on Z (*i.e.* contracted by σ) and by $(D, C) \mapsto D \cdot C$ the intersection product $\mathrm{Pic}\, Z \times Z_1(Z/X) \to \mathbb{Z}$. The quotient $N^1(Z/X)$ of Pic Z by the left kernel, which is identified to $\mathbb{E}$, becomes the dual of the quotient $N_1(Z/X)$ of $Z_1(Z/X)$ by the right kernel. Let $NE = NE(Z/X)$ be the cone in $N_1(Z/X) \otimes \mathbb{Q}$ generated by the classes of effective curves, $NE^{\vee}$ its dual in $\mathbb{E} \otimes \mathbb{Q}$, and $\langle E_0^*, ..., E_n^* \rangle$ the semigroup generated by the E_i^*'s.

PROPOSITION 4. – *One has* $\mathbb{E}^{\#} \subset -NE^{\vee} \cap \mathbb{E} \subset \langle E_0^*, ..., E_n^* \rangle$.

It follows from the fact that σ-generated implies σ-nef, *i.e.* for $D \in \mathbb{E}^{\#}$ one has $(-D) \cdot C \geqq 0$ for every effective exceptional curve C. ■

Next, to describe $NE^{\vee}$, we compute intersection numbers. For an irreducible closed subvariety $V \subset \cup E_i$ of dimension $s \geqq 1$, if $\mathcal{J}$ is the maximal set such that $V \subset E_{\mathcal{J}}$, then V is the strict transform by $\sigma_{\mathcal{J}}$ of $V_{\mathcal{J}} := \sigma_{\mathcal{J}}(V) \subset B_{\mathcal{J}}$. Let $\deg(V_{\mathcal{J}})$ be the degree of $V_{\mathcal{J}}$ in $B_{\mathcal{J}}$, and for each $j \to \mathcal{J}$, *i.e.* $j \to i$ for all $P_i \in \mathcal{J}$, denote by $e_j(V_{\mathcal{J}})$ the multiplicity at P_j of the strict transform of $V_{\mathcal{J}}$.

PROPOSITION 5. – *For any* $D = \sum m_j E_j^* \in \mathbb{E}$ *and V as above, one has*

$$(-D)^s \cdot V = \deg(V_{\mathcal{J}})\, m_{\mathcal{J}}^s - \sum_{j \to \mathcal{J}} e_j(V_{\mathcal{J}})\, m_j^s.$$

Note that if $-D$ is σ-nef, then one also has $(-D)^s \cdot V \geqq 0$ [5].

From now on, we deal with *toric constellations*. Let N be a lattice of dimension $d \geqq 2$ and fix an ordered basis $\mathcal{B} = \{v_a\}$ of N. Let $\Delta = \langle \mathcal{B} \rangle$ be the regular cone in $N \otimes \mathbb{Q}$ generated by $\mathcal{B}$, $X \simeq K^d$ the toric variety associated to the fan Σ_0 given by all the faces of Δ, P_0 the closed T-orbit in X, $(T \simeq (K^*)^d)$ and $\sigma_0 : X_1 \to X$ the blowing-up of P_0. The toric variety X_1 is associated to the fan Σ_1, the minimal subdivision of Σ_0 containing the ray through $u = \sum v_a$. For each a, let $\mathcal{B}_a$ be the ordered basis of N obtained by replacing v_a in $\mathcal{B}$ by u, and let $\Delta_a := \langle \mathcal{B}_a \rangle$. Then B_0 is the closure of the T-orbit defined by the ray through u, and closed orbits in X_1 correspond bijectively to cones Δ_a of Σ_1. A constellation $\mathcal{C} = \{P_0, P_1, ..., P_n\}$ at P_0 is toric if each P_i is a closed T-orbit of X_i.

One can code the edges of Γ_C by integers $a \in \{1, ..., d\}$, the code of an edge being a if it is obtained by replacing the a-th vector according to the above process. The coding determines the proximity relation and so the Enriques diagram, as $p \rightsquigarrow q$ if and only if the code sequence of the edges in the path joining q to p is of type $ab_1...b_t$ with $t \geqq 0$ and $b_i \neq a$, $\forall\, i$ [1]. A finer relation, called *linear proximity*, and denoted by $P \twoheadrightarrow Q$, is defined by: P is in the strict transform of the closure of some 1-dimensional T-orbit of B_Q, or, equivalently, in the above code sequence from q to p, one has either $t = 0$ or $b := b_1 = ... = b_t$ and $a \neq b$. Since a 1-dimensional T-orbit in B_Q is determined by two integers $a \neq b$, the set of points $P \in \mathcal{C}$ with $P \twoheadrightarrow Q$ on it, *i.e.* coded by either $ab...b$ or $ba...a$, will be denoted by $\overrightarrow{\mathcal{C}}_Q\{a, b\}$. For any Q, the strict transforms in Z of the closure of 1-dimensional orbits in B_Q with set $\overrightarrow{\mathcal{C}}_Q\{a, b\}$ maximal will be called *maximal curves.*

THEOREM 6. – *Let $\mathcal{C}$ be a toric constellation and $\mathcal{A} = (\mathcal{C}, \underline{m})$ a cluster. Then*

(i) *$\mathcal{A} \in \mathcal{G}_C$ if and only if $m_Q \geqq \sum_{P \in \overrightarrow{\mathcal{C}}_Q\{a,b\}} m_P$ for every Q and $a \neq b$.*

(ii) *If $\mathcal{A} \in \mathcal{G}_C$, then the exponents in the factorization of $I_\mathcal{A}$ are given by $r_Q = m_Q - \sum_{P \twoheadrightarrow Q} m_P$.*

(iii) $\mathbb{E}^\# = -NE^\vee \cap \mathbb{E}$.

(iv) *The extremal vectors of NE are the classes in $N_1(Z/X)$ of the maximal curves.*

(i) and (iii) follow from p. 47 [8] and proposition 5. By computing D_P from (i), one gets $\check{D}_Q = \check{E}^*_Q - \sum_{P \twoheadrightarrow Q} \check{E}^*_P$ for dual basis in $N_1(Z/X)$ and (ii) follows. By (iii), $NE = -E^{\#\vee}$, hence (iv). They are also given by $L^*_Q - \sum_{P \in \overrightarrow{\mathcal{C}}_Q\{a,b\}} L^*_P$ for $Q \in \mathcal{C}$ and $\overrightarrow{\mathcal{C}}_Q\{a, b\}$ maximal since $\check{E}^*_Q = -L^*_Q$, where L^*_Q denotes the class in $N_1(Z/X)$ of the total transform of a general line in B_Q. ■

Note that idealistic toric clusters $\mathcal{A}$ are just those coming from complete ideals generated by monomials. If $\underline{m}$ satisfies the inequalities (i), then the Newton polyhedron of $I_\mathcal{A}$ may be explicitely obtained from $\mathcal{A}$. This gives a constructive proof that these inequalities are sufficient for $\mathcal{A}$ to be idealistic [2].

COROLLARY 7. – *For a toric constellation $\mathcal{C}$, the following conditions are equivalent:*

(i) *The cone $-NE^\vee$ is regular,* i.e. *$\mathcal{G}_C$ is a free semigroup.*

(ii) *The set of extremal vectors of $-NE^\vee$ is $\{D_Q\}_{Q \in C}$.*

(iii) *The cone $-NE^\vee$ is simplicial,* i.e. *the set of extremals is $\mathbb{Q}$-free.*

(iv) *The exponents r_Q in the factorization of ideals in $\mathcal{F}_C$ are non negative.*

(v) *For each $Q \in \mathcal{C}$, there is only one maximal set $\overrightarrow{\mathcal{C}}_Q\{a, b\}$*

(vi) *The tree Γ_C is binary and if, for $q \in \Gamma_C$ and a, b, c with $a \neq b$, the points derived from q by codes a, b, ac are in Γ_C, then $b = c$.*

From (v) and (vi), it follows that the galaxy $\mathcal{G}_C$ of a toric chain $\mathcal{C}$ (*i.e.* $\mathcal{C}$ is totally ordered for $\geqq$) is a free semigroup (*see* [1]). Property (i) [resp. (iii)] when referred to $\mathcal{F}_C$, are usually called factoriality (resp. semifactoriality) (*see* [3]).

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