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Rational convexity of non-generic immersed Lagrangian submanifolds

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Abstract We prove that an immersed Lagrangian submanifold in $\mathbf{C}^n$ with quadratic self-tangencies is rationally convex. This generalizes former results for the embedded and the immersed transversal cases.

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0 Introduction

Let K a compact set in $\mathbf{C}^n$. It is *rationally convex* if its complement can be filled out by global complex hypersurfaces. According to the Oka–Weil theorem, holomorphic functions nearby such a rationally convex compact set K can be approximated on K by rational functions.

A classical obstruction to the rational convexity of K is the presence in $\mathbf{C}^n$ of a Riemann surface C with boundary in K, such that ∂C bounds in K. This obstruction disappears when K is Lagrangian for a global Kähler form, i.e. when K is a submanifold of real dimension n on which this form vanishes. Indeed N. Sibony and the first author proved in 1995 [3] that compact embedded Lagrangian submanifolds for a Kähler form in $\mathbf{C}^n$ are rationally convex.

It is natural to try and extend this result to immersed Lagrangian submanifolds, which are much more abundant. The second author obtained in 2000 [4] the

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rational convexity of generic compact immersed Lagrangian submanifolds, those with transversal self-intersections.

But more complicated self-intersections show up generically when looking at deformations of Lagrangian submanifolds, namely *quadratic self-tangencies*: double points where the two branches are transversal except for one direction along which they get a quadratic contact. Our result is:

Theorem *Let ω a Kähler form on $\mathbf{C}^n$, and L a compact immersed Lagrangian submanifold for ω with quadratic self-tangencies. Then L is rationally convex.*

As for the former cases, this statement has its counterpart in a symplectic setting in the spirit of Donaldson's work on symplectic hypersurfaces [2] (see [1] for the symplectic version of the embedded case). But for simplicity we will stick to our holomorphic context.

The second author would like to thank K. Cieliebak for asking this question and the french Agence Nationale de la Recherche for its support. His motivation stemmed from his work on the Floer homology for pairs of Lagrangian submanifolds.

Before entering the proof of our theorem, we will review briefly the embedded and the immersed transversal cases, with a new argument for the later.

1 The embedded case

Let L be a compact embedded submanifold of dimension n in $\mathbf{C}^n$, Lagrangian for a Kähler form ω. Choose a potential ϕ of this form, i.e. a global function such that $dd^c\phi = \omega$. By hypothesis $d^c\phi$ is a closed form on L. By perturbing ω, one can always arrange rational periods for this closed form, or even periods in $2\pi\mathbf{Z}$ after multiplying ω by some constant. Therefore $d^c\phi = d\psi$ where ψ is a function on L with values in $\mathbf{R}/2\pi\mathbf{Z}$, and $e^{\phi+i\psi}$ is a well defined complex-valued function on L. It can be extended as a global function h on $\mathbf{C}^n$, $\overline{\partial}$-flat on L (i.e. $|\overline{\partial}h| = O(d^m)$ for any m where d stands for the distance to L) and vanishing outside a neighborhood of L. By construction the first jets of $|h|$ and e^ϕ coincide along L. Actually it is not hard to check that $|h|_\phi \leq e^{-d^2}$ if d is suitably normalized. Here $|h|_\phi = |h|e^{-\phi}$.

Our model for a global hypersurface avoiding L will be $(h^k = 0)$. It needs only be corrected by means of a $\overline{\partial}$-equation with L^2-estimates.

Namely consider $f = h^k - v$ where v is a solution of $\overline{\partial}v = \overline{\partial}(h^k)$ satisfying $||v||_{2,k\phi} \leq ||\overline{\partial}(h^k)||_{2,k\phi}$ on a big ball (we tend from now on to neglect irrelevant constants). This translates in L^∞-estimates. Indeed we have (cf. [5])

$$|v(p)| \leq \epsilon||\overline{\partial}v||_{\infty,\epsilon} + \epsilon^{-n}||v||_{2,\epsilon}$$

where the norms are taken on $B(p,\epsilon)$. Then, introducing the weight, we get

$$|v|e^{-k\phi}(p) \leq e^{k\epsilon}(\epsilon||\overline{\partial}(h^k)e^{-k\phi}||_{\infty,\epsilon} + \epsilon^{-n}||ve^{-k\phi}||_{2,\epsilon}).$$

Taking now $\epsilon = \frac{1}{k^2}$ we end up with

$$||v||_{\infty,k\phi} \leq \frac{1}{k} + k^{2n}||v||_{2,k\phi} \leq \frac{1}{k} + k^{2n}||\overline{\partial}(h^k)||_{2,k\phi}.$$

This last norm is roughly estimated by $k^{2n+1}(\int_0^1 s^{2m}e^{-ks^2}ds)^{\frac{1}{2}}$ using the properties of h. It decays with k if m is large.

Choose now a point p outside L. We have $|h^k|_{k\phi}(p) = 0$ but $|h^k|_{k\phi} = 1$ on L. Hence $|f|_{k\phi}(p) \leq \frac{1}{2}\min_L |f|_{k\phi}$ for k big enough. Conversely, as in Hörmander [5] (see also Donaldson [2] for peak sections in a symplectic setting), one can construct a global holomorphic function g which peaks at p in weighted norm, i.e. satisfying $|g|_{k\phi}(p) \geq \max_L |g|_{k\phi}$. For this, take h to be the cut-off of the exponential of the holomorphic 2-jet of ϕ at p and proceed as above. Therefore $(f(p)g = g(p)f)$ is a global holomorphic hypersurface passing through p and avoiding L. See [3] for more details.

2 The immersed transversal case

The proof follows exactly the same scheme, the only differences appearing near the double points. Namely in a neighborhood of such a point p the function h^k will be replaced by $h_1^k + h_2^k$, each h_i being constructed as before for each branch L_i of L at p. Recall that $|h_i|_\phi = 1$ on L_i and that $|h_i|_\phi \leq e^{-d_i^2}$ where d_i is the distance to L_i. It remains to guarantee a lower bound on L for $|h_1^k+h_2^k|_{k\phi}$. Let us treat the case of L_1. On this branch we have $|h_1^k+h_2^k|_{k\phi} = |h_1^k|_{k\phi}|1+(\frac{h_2}{h_1})^k| = |1+(\frac{h_2}{h_1})^k|$. But by construction the first jets of h_1 and h_2 coincide at p. Hence $\frac{h_2}{h_1} = \frac{h_2e^{-\phi}}{h_1e^{-\phi}} = e^{-d_2^2+iO(d^2)}$ on L_1 where d is the distance to p. Remark now that d_2 and d are equivalent on L_1 by the transversality of the two branches. Therefore we get that $|h_1^k + h_2^k|_{k\phi} = |1 + e^{k(-d^2+iO(d^2))}|$, and this is bounded from below by the minimum of $|1 + e^z|$ on a cone directed by the negative real axis.

3 The immersed quadratic case

As before the difficulties appear near the double points. But we cannot rely on such a simple argument as in the transversal case : the two branches interact too closely along the quadratic tangency to prevent the vanishing of $|h_1^k + h_2^k|_{k\phi}$ on L. We will follow instead the method of [4]. Let, for simplicity, L be an immersed Lagrangian submanifold for a Kähler form ω with a single quadratic self-tangency at p.

Lemma *There exists a local perturbation of ω near p such that*

(i) *L remains Lagrangian for the new form*
(ii) *this form has a potential ϕ presenting a strict local minimum 0 at p*
(iii) *$d^c\phi$ vanishes along L near p.*

Assume this for a moment. The theorem follows with the same method as in the transversal case. We just replace $h_1^k + h_2^k$ by $\chi^k + h_1^k + h_2^k$ near p in order to separate the two branches. Here χ is a cut-off function such that $\chi = e^\epsilon$ ($\epsilon > 0$) near p. It is not hard to check that $|\chi^k + h_1^k + h_2^k|_{k\phi}$ is bounded from below on L. Indeed the first term dominates on $L \cap (\phi < \epsilon)$ and one of the other dominates on $L_i \cap (\phi > \epsilon)$. At the junction $L_i \cap (\phi = \epsilon)$ no cancellation appears : indeed h_i are real positive along L_i because of the vanishing of $d^c\phi$.

4 Proof of the Lemma

For simplicity we present it in dimension 2. Take p to be 0 and denote by $z = x + iy$, $w = u + iv$ the coordinates and by d and δ the distances to 0 and to the x-axis respectively. These coordinates can be choosen in such way that ω is standard at 0, say $\omega = 3dd^c(|z|^2 + |w|^2) + O(d)$. We may suppose moreover that the tangent planes of the two branches at 0 generate the (x, u, v)-hyperplane and intersect along the x-axis. Each branch is then parameterized by $j : (x, t) \mapsto (x + ip, e^{i\theta}(t + iq)) + O(d^3)$, where θ is a specific angle and p, q are real quadratic polynomials.

We will construct the potential ϕ of the new form in two steps: firstly we choose a good potential ψ of ω such that $d^c\psi$ vanishes at a certain order at 0 along L, namely $d^c\psi = O(d^3 + \delta d)$; secondly we slightly perturb ψ in ϕ independently for each branch to achieve the vanishing of $d^c\phi$ along L.

We start with a global potential ψ of ω whose expansion near 0 is of the form $2x^2 + 4y^2 + 3u^2 + 3v^2 + O(d^3)$. The reason for this particular choice lies in the fact that $d^c(x^2 + 2y^2)$ vanishes on any parabola tangent to the x-axis at 0 in $\mathbf{C}$. It is not hard to check that $j^*d^c\psi = \alpha x^2 dx + \beta x^2 dt + O(d^3 + \delta d)$ where α is the same constant for the two branches, β being specific to each branch. Now we use the freedom to change the rest of order 3 in ψ by global pluriharmonic terms. Adding $\gamma \mathrm{Im}(z^3)$ for a suitable real constant γ (resp. $\mathrm{Im}(\mu z^2 w)$ for some complex number μ) allows us to kill α (resp. β for each branch).

We now perturb ψ to construct ϕ near L_1 such that $d^c\phi$ vanishes on L_1. Without loss of generality we can think of L_1 being flat, say after a rotation the (x, u)-plane. Indeed it is possible to straighten L_1 by means of a local diffeomorphism tangent to the identity at 0 and $\bar\partial$-flat on the branch. Write $j^*d^c\psi = a dx + b du$ on L_1 near 0 with $a, b = O(d^3 + \delta d)$. By cutting off a primitive of it, we can also assume the closed form $a dx + b du$ to be compactly supported on L_1. Remark that $j^*d^c(\psi - ay - bv) = 0$ near the origin. By construction this modification of ψ is supported in a cylinder based on a disk in L_1.

We then cut this perturbation of ψ off L_1 to avoid the interaction with L_2. For this, remark that the set $(y^2 + v^2 \leq \epsilon(u^2 + x^4))$ doesn't intersect L_2 except at 0 for a small ϵ. This is a consequence of the quadratic tangency between the two branches. Let $\phi = \psi - \chi(\frac{y^2+v^2}{u^2+x^4})(ay + bv)$ where χ is a smooth function with small support on $\mathbf{R}^+$ equal to 1 near 0. It can be checked that ϕ is a small C^2-perturbation of ψ because of our good control of a and b. Hence ϕ is strictly plurisubharmonic. Moreover ϕ coincides with ψ outside a neighborhood of 0. Summing up $d^c\phi$ vanishes on L_1 near 0, L

remains Lagrangian for the new form $dd^c\phi$, and this form coincides with ω outside a neighborhood of 0. Adding a similar term for L_2 concludes.

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