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Lower estimates for the expected Betti numbers of random real hypersurfaces

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Abstract

We estimate from below the expected Betti numbers of real hypersurfaces taken at random in a smooth real projective n-dimensional manifold. These random hypersurfaces are chosen in the linear system of a large dth power of a real ample line bundle equipped with a Hermitian metric of positive curvature. As for the upper bounds that we recently established, these lower bounds read as a product of a factor which only depends on the dimension n of the manifold with the Kähler volume of its real locus $\mathbb{R}X$ and $\sqrt{d}^n$. Actually, any closed affine real algebraic hypersurface appears with positive probability as part of such random real hypersurfaces in any ball of $\mathbb{R}X$ of radius $O(1/\sqrt{d})$.

1. *Introduction*

What is the topology of a real hypersurface taken at random in a smooth real projective manifold? When the latter is the projective line, this question reduces to: how many real roots does a random real polynomial in one variable have? This question was answered by Kac [**8**] in 1943 and for a different measure, by Kostlan [**9**] and Shub and Smale [**13**] in the early 1990s. In our recent paper [**5**] (see also [**6**]), we did bound from above the expected Betti numbers of such random real hypersurfaces in smooth real projective manifolds. Our purpose now is to bound these Betti numbers from below, see Corollary 1.3.

Let us first recall our framework. We denote by X a smooth complex projective manifold of positive dimension n defined over the reals, by $c_X : X \to X$ the induced Galois antiholomorphic involution and by $\mathbb{R}X = \mathrm{Fix}(c_X)$ the real locus of X which we implicitly assume to be non-empty. We then consider an ample line bundle L over X, also defined over the reals. It comes thus equipped with an antiholomorphic involution $c_L : L \to L$ which turns the bundle projection map $\pi : L \to X$ into a $\mathbb{Z}/2\mathbb{Z}$-equivariant one, so that $c_X \circ \pi = \pi \circ c_L$. We equip L in addition with a real Hermitian metric h, thus invariant under c_L, which has a positive curvature form ω, locally defined by $\omega = (1/2i\pi)\partial\bar{\partial} \log h(e, e)$ for any non-vanishing local holomorphic section e of L. This metric induces a Kähler metric $g_h = \omega(\cdot, i\cdot)$ on X, which reduces to a Riemannian metric g_h on $\mathbb{R}X$. It then induces an L^2-scalar product on every space of global holomorphic real sections of tensor products L^d of L, $d > 0$, which are denoted by

$$\mathbb{R}H^0(X, L^d) = \{\sigma \in H^0(X, L^d) \,|\, c_L \circ \sigma = \sigma \circ c_X\}.$$

This L^2-scalar product is defined by the formula

$$\forall(\sigma, \tau) \in \mathbb{R}H^0(X, L^d), \quad \langle \sigma, \tau \rangle = \int_X h^d(\sigma, \tau)(x)\, dx, \tag{1.1}$$

where dx denotes any volume form of X. For instance, dx can be chosen to be the normalized volume form $dV_h = \omega^n / \int_X \omega^n$. This L^2-scalar product finally induces a Gaussian probability

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measure $\mu_{\mathbb{R}}$ on $\mathbb{R}H^0(X, L^d)$ whose density with respect to the Lebesgue one at $\sigma \in \mathbb{R}H^0(X, L^d)$ reads $(1/\sqrt{\pi}^{N_d})\, e^{-\|\sigma\|^2}$, where $N_d = \dim \mathbb{R}H^0(X, L^d)$. It is with respect to this probability measure that we consider random real hypersurfaces (as in the works [**9**, **13**]) and our results hold for large values of d.

Let us recall the estimates from above that we recently established in [**5**]. For every $d > 0$, we denote by $\mathbb{R}\Delta_d$ the discriminant locus of sections $\sigma \in \mathbb{R}H^0(X, L^d)$ which do not vanish transversally. For every $\sigma \in \mathbb{R}H^0(X, L^d) \setminus \mathbb{R}\Delta_d$, we denote by $\mathbb{R}C_\sigma = \sigma^{-1}(0) \cap \mathbb{R}X$ its smooth real vanishing locus. Then, for every $i \in \{0, \dots, n-1\}$, we denote by $m_i(\mathbb{R}C_\sigma)$ the ith Morse number of $\mathbb{R}C_\sigma$ and by $b_i(\mathbb{R}C_\sigma)$ its ith Betti number with real coefficients. These are defined as the infinimum over all Morse functions f on $\mathbb{R}C_\sigma$ of the number of critical points of index i of f and as $b_i(\mathbb{R}C_\sigma) = \dim H_i(\mathbb{R}C_\sigma; \mathbb{R})$, respectively. It follows from Morse theory that $b_i(\mathbb{R}C_\sigma) \leqslant m_i(\mathbb{R}C_\sigma)$ and we set

$$E(b_i) = \int_{\mathbb{R}H^0(X,L^d)\setminus\mathbb{R}\Delta_d} b_i(\mathbb{R}C_\sigma)\, d\mu_{\mathbb{R}}(\sigma),$$

and $E(m_i) = \int_{\mathbb{R}H^0(X,L^d)\setminus\mathbb{R}\Delta_d} m_i(\mathbb{R}C_\sigma)\, d\mu_{\mathbb{R}}(\sigma)$. Then, we proved the following theorem.

THEOREM 1.1 [**5**, Theorem 1]. *Let X be a smooth real projective manifold of positive dimension n and (L, h) be a real holomorphic Hermitian line bundle of positive curvature over X. Then, for every $i \in \{0, \dots, n-1\}$,*

$$\limsup_{d\to\infty} \frac{1}{\sqrt{d}^n} E(m_i) \leqslant \frac{1}{\sqrt{\pi}} e_{\mathbb{R}}(i, n-1-i)\, \mathrm{Vol}_h(\mathbb{R}X).$$

In Theorem 1.1, $\mathrm{Vol}_h(\mathbb{R}X)$ denotes the total Riemannian volume of the real locus $\mathbb{R}X$ for the Kähler metric g_h, while $e_{\mathbb{R}}(i, n-1-i)$ is a constant which only depends on i and the dimension of X. The latter originates from random symmetric matrices and is defined as

$$e_{\mathbb{R}}(i, n-1-i) = \int_{\mathrm{Sym}(i,n-1-i,\mathbb{R})} |\det A|\, d\mu_{\mathbb{R}}(A),$$

where $\mathrm{Sym}(i, n-1-i, \mathbb{R})$ denotes the open cone of non-degenerated real symmetric matrices of size $n-1$ and signature $(i, n-1-i)$, while $d\mu_{\mathbb{R}}$ is the restriction to this cone of the classical Gaussian probability measure of the space of symmetric matrices, see [**5**]. In particular, for fixed $i \geqslant 0$, there exists $c_i > 0$ such that for large values of n,

$$e_{\mathbb{R}}(i, n-1-i) \leqslant \exp(-c_i n^2), \tag{1.2}$$

as follows from some large deviation estimates established in [**1**], see [**5**, Theorem 1.6].

Our aim now is to get similar asymptotic estimates from below for the expected Betti numbers of random real hypersurfaces linearly equivalent to L^d, see Corollary 1.3. These estimates will follow from our main result, Theorem 1.2, which we now formulate.

Let Σ be a closed hypersurface of $\mathbb{R}^n$, that is, a smooth compact hypersurface of $\mathbb{R}^n$ which has no boundary and which we do not assume to be connected. For every $\sigma \in \mathbb{R}H^0(X, L^d) \setminus \mathbb{R}\Delta_d$, we denote by $N_\Sigma(\sigma)$ the maximal number of disjoint open subsets of $\mathbb{R}X$ having the property that each such open subset U' contains a hypersurface Σ' such that $\Sigma' \subset \mathbb{R}C_\sigma$ and (U', Σ') is diffeomorphic to $(\mathbb{R}^n, \Sigma)$. We then set

$$E(N_\Sigma) = \int_{\mathbb{R}H^0(X,L^d)\setminus\mathbb{R}\Delta_d} N_\Sigma(\sigma)\, d\mu_{\mathbb{R}}(\sigma),$$

and we associate to Σ, in fact to its isotopy class in $\mathbb{R}^n$, a positive constant c_Σ out of the amount of transversality of a real polynomial P in n variables which vanishes along a hypersurface isotopic to Σ, see (2.6). Our main result is the following theorem.

Theorem 1.2. *Let X be a smooth real projective manifold of positive dimension n and (L, h) be a real holomorphic Hermitian line bundle of positive curvature over X. Let Σ be a closed hypersurface of $\mathbb{R}^n$, which does not need to be connected. Then,*

$$\liminf_{d\to\infty} \frac{1}{\sqrt{d}^n} E(N_\Sigma) \geqslant c_\Sigma \operatorname{Vol}_h(\mathbb{R}X).$$

In particular, when Σ is connected, Theorem 1.2 provides a lower bound for the expected number of connected components that are diffeomorphic to Σ in the real vanishing locus of a random section $\sigma \in \mathbb{R}H^0(X, L^d)$. As in Theorem 1.1, the constant c_Σ does not depend on the choice of the triple (X, L, h), it only depends on Σ.

Let us now denote, for every positive integer n, by $\mathcal{H}_n$ the set of diffeomorphism classes of smooth closed connected hypersurfaces in $\mathbb{R}^n$. For every $i \in \{0, \dots, n-1\}$ and every $[\Sigma] \in \mathcal{H}_n$, we denote by $b_i(\Sigma) = \dim H_i(\Sigma; \mathbb{R})$ its ith Betti number with real coefficients and by $m_i(\Sigma)$ its ith Morse number. Then, we set $[c_\Sigma] = \sup_{\Sigma\in[\Sigma]} c_\Sigma$.

Corollary 1.3. *Let X be a smooth real projective manifold of positive dimension n and (L, h) be a real holomorphic Hermitian line bundle of positive curvature over X. Then, for every $i \in \{0, \dots, n-1\}$,*

$$\liminf_{d\to\infty} \frac{1}{\sqrt{d}^n} E(b_i) \geqslant \left(\sum_{[\Sigma]\in\mathcal{H}_n} c_{[\Sigma]} b_i(\Sigma) \right) \operatorname{Vol}_h(\mathbb{R}X),$$

and likewise $\liminf_{d\to\infty}(1/\sqrt{d}^n)E(m_i) \geqslant (\sum_{[\Sigma]\in\mathcal{H}_n} c_{[\Sigma]} m_i(\Sigma)) \operatorname{Vol}_h(\mathbb{R}X)$. *In particular, for every* $i \in \{0, \dots, n-1\}$,

$$\liminf_{d\to\infty} \frac{1}{\sqrt{d}^n} E(b_i) \geqslant \exp(-e^{5n+69}) \operatorname{Vol}_h(\mathbb{R}X).$$

The last part of Corollary 1.3 follows from the fact that for every $i \in \{0, \dots, n-1\}$, $\mathbb{R}^n$ contains the product of spheres $S^i \times S^{n-1-i}$ as a hypersurface, while $c_{S^i\times S^{n-1-i}} \geqslant \exp(-e^{5n+69})$, see Proposition 2.7. This double exponential decay has to be compared with (1.2) and is not an optimal bound. For instance, it follows from [**5**, Theorem 1] that when $n = 1$, $(1/\sqrt{d})E(b_0)$ converges to $(1/\sqrt{\pi}) \operatorname{Length}_h(\mathbb{R}X)$. The results given by Theorems 1.1 and 1.2 raise the following question: does the quotient $E(b_i)/\operatorname{Vol}_h(\mathbb{R}X)\sqrt{d}^n$ or likewise $E(m_i)/\operatorname{Vol}_h(\mathbb{R}X)\sqrt{d}^n$ have a limit in general, which only depends on $i \in \{0, \dots, n-1\}$ and the dimension n of X, but not on the triple (X, L, h)? This holds true for $n = 1$, see [**5**, Theorem 1] or also [**5**, Theorem 2] for similar results on the number of critical points of given index.

Note that another natural Gaussian probability measure could have been chosen on $\mathbb{R}H^0(X, L^d)$, induced by an L^2-product defined by integration over $\mathbb{R}X$ instead of the integration over X (see [**6**, § 3.1.1] for a discussion on our choice and other possible ones). This is the measure considered by Nazarov and Sodin in their study of random spherical harmonics in dimension 2, see [**12**], and more recently by Lerario and Lundberg in higher dimensions, see [**10**]. The upper and lower estimates they obtain for the number of connected components for these spherical harmonics are in d^n instead of $\sqrt{d}^n$. These estimates are also established for homogeneous polynomials on unit spheres in [**10**]. Note that such a behaviour was previously guessed through computational experiments by Raffalli, while Sarnak and Wigman informed us that they were able to prove the upper estimates in $\mathbb{R}P^2$.

In order to prove Theorem 1.2, we follow the same probability approach as Nazarov and Sodin (see [**12**, § 6.1] or also [**10**, § 2.2]) which we combine with the L^2-estimates of Hörmander, see § 3.1. The latter make it possible asymptotically to produce, for every smooth closed hypersurface Σ of $\mathbb{R}^n$ contained in a ball of radius R and every ball B_d of $\mathbb{R}X$ of radius $R/\sqrt{d}$, a section $\sigma \in \mathbb{R}H^0(X, L^d)$ which vanishes transversally in B_d along a hypersurface diffeomorphic to Σ, the transversality being quantitative in the sense of Donaldson, see [**3**, Definition 7] and Proposition 3.4. We then bound from above the expected C^1-norm of sections of $\mathbb{R}H^0(X, L^d)$ in such a ball B_d and deduce from Markov's inequality that a random section in $\mathbb{R}H^0(X, L^d)$ vanishes with positive probability in B_d along a hypersurface diffeomorphic to Σ, see Proposition 3.8. The result follows from the fact that there are more or less $\mathrm{Vol}_h(\mathbb{R}X)\sqrt{d}^n$ disjoint such balls in $\mathbb{R}X$. Recall that the construction in [**4**] of real Donaldson hypersurfaces with many spheres in their real locus was carried out in a similar manner.

The first part of the paper is devoted to preliminaries on closed affine real algebraic hypersurfaces and the second one to the proofs of Theorem 1.2 and Corollary 1.3.

2. *Closed affine real algebraic hypersurfaces*

This paragraph is devoted to preliminaries. We first introduce two real functions which play a role in the proof of Theorem 1.2. Then, we associate a positive constant c_Σ to any isotopy class of smooth closed hypersurface Σ of $\mathbb{R}^n$, see (2.6), using a notion of regular pair given by Definition 2.3. Throughout this paper, by closed manifold we mean smooth compact manifold without boundary. Finally, we estimate from below this constant in the case of product of spheres, see § 2.3.

2.1. *Two real functions*

We introduce here two real functions f_τ and g_R whose maximum and minimum turn out to play a role in the proof of Theorem 1.2. For every $\tau > 0$, we set

$$f_\tau : a \in [\sqrt{\tau}, +\infty[\longmapsto \frac{1}{\sqrt{\pi}}\left(1 - \frac{\tau}{a^2}\right)\int_a^{+\infty} e^{-t^2}\,dt,$$

so that $f_\tau(\sqrt{\tau}) = \lim_{a\to\infty} f_\tau(a) = 0$. We set

$$m_\tau = \sup_{[\sqrt{\tau}, +\infty[} f_\tau. \tag{2.1}$$

In particular, for every positive τ,

$$m_\tau \geqslant f_\tau(\sqrt{\tau+1}) \geqslant \frac{1}{\sqrt{\pi}(\tau+1)} e^{-(\sqrt{\tau+1}+1)^2}. \tag{2.2}$$

The estimate (2.2) is chosen in the light of the following Lemma 2.1.

Lemma 2.1. *For every positive τ, the function f_τ reaches its maximum on the interval $[\sqrt{\tau}, \sqrt{\tau+1}]$.*

Proof. For every positive τ and every $a \geqslant \sqrt{\tau}$, $f'_\tau(a) = (1/\sqrt{\pi})((2\tau/a^3)\int_a^{+\infty} e^{-t^2}\,dt - (1 - \tau/a^2)e^{-a^2})$, so that if f_τ reaches its maximum at the point $c \in [\sqrt{\tau}, +\infty[$, $(2\tau/c^3)\int_c^{+\infty} e^{-t}\,dt = (1-\tau/c^2)\,e^{-c^2}$. Now $\int_c^{+\infty} e^{-t^2}\,dt \leqslant e^{-c^2}/2c$, so that $(1-\tau/c^2) \leqslant \tau/c^4$ and $c^2 - \tau \leqslant \tau/c^2 \leqslant 1$. Hence, $c \leqslant \sqrt{1+\tau}$. □

Likewise, for every positive R and every positive integer n, we set

$$g_R : s \in \mathbb{R}_+^* \longmapsto \frac{(R+s)^{2n}}{s^{2n}} e^{\pi(R+s)^2},$$

so that $\lim_{s\to 0} g_R(s) = \lim_{s\to+\infty} g_R(s) = +\infty$. We set

$$\rho_R = \inf_{\mathbb{R}_+^*} g_R. \tag{2.3}$$

In particular, for every positive R,

$$\rho_R \leqslant g_R(R) = 4^n \exp(4\pi R^2). \tag{2.4}$$

2.2. *Real polynomials and transversality*

We introduce here the notion of regular pair, see Definition 2.3, and the constant c_Σ associated to any isotopy class of smooth closed hypersurface Σ of $\mathbb{R}^n$, see (2.6).

LEMMA 2.2. *If* $P = \sum_{(i_1,\dots,i_n)\in\mathbb{N}^n} a_{i_1,\dots,i_n} z_1^{i_1}\cdots z_n^{i_n} \in \mathbb{R}[X_1,\dots,X_n]$, *then*

$$\|P\|_{L^2}^2 = \int_{\mathbb{C}^n} |P(z)|^2\, e^{-\pi\|z\|^2}\, dz = \sum_{(i_1,\dots,i_n)\in\mathbb{N}^n} |a_{i_1,\dots,i_n}|^2 \frac{i_1!\cdots i_n!}{\pi^{i_1+\cdots+i_n}}.$$

Proof. We note that $\|P\|_{L^2}^2 = \sum_{I,J\in\mathbb{N}^n} a_I \bar{a}_J \int_{\mathbb{C}^n} z^I \bar{z}^J\, e^{-\pi\|z\|^2}\, dz$. But for every $k\neq 0$, $\int_{\mathbb{C}} z^k\, e^{-\pi|z|^2}\, dz = \int_{\mathbb{C}} \bar{z}^k\, e^{-\pi|z|^2}\, dz = 0$, whereas for every non-negative k,

$$\int_{\mathbb{C}} |z|^{2k}\, e^{-\pi|z|^2}\, dz = 2\pi\int_0^{+\infty} r^{2k+1}\, e^{-\pi r^2}\, dr = \pi\int_0^{+\infty} t^k\, e^{-\pi t}\, dt = \frac{1}{\pi^k}\Gamma(k+1) = \frac{k!}{\pi^k}.$$

The result follows then from Fubini's Theorem. □

DEFINITION 2.3. Let U be a bounded open subset of $\mathbb{R}^n$ and $P\in\mathbb{R}[X_1,\dots,X_n]$, $n>0$. The pair (U,P) is said to be regular if and only if

(i) zero is a regular value of the restriction of P to U;
(ii) the vanishing locus of P in U is compact.

DEFINITION 2.4. For every regular pair (U,P) given by Definition 2.3, we denote by $\mathcal{T}_{(U,P)}$ the set of $(\delta,\epsilon)\in(\mathbb{R}_+^*)^2$ such that

(i) there exists a compact subset K of U satisfying $\inf_{U\setminus K}|P|>\delta$;
(ii) for every $y\in U$, $|P(y)|<\delta \Rightarrow \|d_{|y}P\|>\epsilon$, where $\|d_{|y}P\|^2 = \sum_{i=1}^n |\partial P/\partial x_i|^2$.

We then set for every regular pair (U,P), $R_{(U,P)} = \max(1, \sup_{y\in U}\|y\|)$ and

$$\tau_{(U,P)} = 24\rho_{R_{(U,P)}}\|P\|_{L^2}^2 \inf_{(\delta,\epsilon)\in\mathcal{T}_{(U,P)}}\left(\frac{1}{\delta^2}+\frac{\pi n}{\epsilon^2}\right)\in\mathbb{R}_+^*, \tag{2.5}$$

where $\rho_{R_{(U,P)}}$ is defined by (2.3).

Now, let Σ be a closed hypersurface of $\mathbb{R}^n$, not necessarily connected. We denote by $\mathcal{I}_\Sigma$ the set of regular pairs (U,P) given by Definition 2.3, such that the vanishing locus of P in U contains a subset isotopic to Σ in $\mathbb{R}^n$. It follows from Nash's Theorem for hypersurfaces in $\mathbb{R}^n$ that $\mathcal{I}_\Sigma$ is non-empty, see [**11**, Theorem 1]. We then set

$$c_\Sigma = \sup_{(U,P)\in\mathcal{I}_\Sigma}\left(\frac{m_{\tau_{(U,P)}}}{2^n\,\mathrm{Vol}(B(R_{(U,P)}))}\right), \tag{2.6}$$

where $m_{\tau_{(U,P)}}$ is defined by (2.1) and $\mathrm{Vol}(B(R_{(U,P)}))$ denotes the volume of the Euclidean ball of radius $R_{(U,P)}$ in $\mathbb{R}^n$. From (2.2), follows that for every $(U,P) \in \mathcal{I}_\Sigma$,

$$c_\Sigma \geqslant \frac{\lfloor n/2 \rfloor! \exp(-(\sqrt{\tau_{(U,P)}+1}+1)^2)}{2^{n+1}\pi^{\lfloor n/2 \rfloor} R^n_{(U,P)}(1+\tau_{(U,P)})\sqrt{\pi}}, \tag{2.7}$$

since the volume of the ball of radius $R_{(U,P)}$ in $\mathbb{R}^n$ is bounded from below by $(2\pi^{\lfloor n/2 \rfloor}/\lfloor n/2 \rfloor!)R^n_{(U,P)}$ for every $n > 0$. For large values of $\tau_{(U,P)}$, as the ones which appear in the examples given in § 2.3, we deduce from (2.7) that

$$c_\Sigma \geqslant \exp(-2\tau_{(U,P)}). \tag{2.8}$$

2.3. *Examples*

2.3.1. *The spheres*

Proposition 2.5. *For every positive integer n, $c_{S^{n-1}} \geqslant \exp(-e^{4n+55})$.*

For every $n > 0$, we set $P_S(x_1, \dots, x_n) = \sum_{j=1}^n x_j^2 - 1$ and $U_S = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_{j=1}^n x_j^2 < 4\}$. The pair (U_S, P_S) is regular in the sense of Definition 2.3 and $P_S^{-1}(0) \subset U_S$ is isotopic in $\mathbb{R}^n$ to the unit sphere S^{n-1}.

Lemma 2.6. *For every $n > 0$ and every $0 < \delta < 1$, $(\delta, 2\sqrt{1-\delta}) \in \mathcal{T}_{(U_S,P_S)}$.*

Proof. For every $x \in \mathbb{R}^n$ and $\delta > 0$,

$$\begin{aligned} |P_S(x)| < \delta &\Longleftrightarrow 1-\delta < \|x\|^2 < 1+\delta \\ &\Longrightarrow \|d_{|x}P_S\|^2 = 4\|x\|^2 > 4(1-\delta). \end{aligned}$$

Moreover, when $0 < \delta < 1$, $K_\delta = \{x \in U \mid 1-\delta \leqslant \|x\|^2 \leqslant 1+\delta\}$ is compact in U_S. We deduce that $(\delta, \epsilon) \in \mathcal{T}_{(U_S,P_S)}$ for $\epsilon^2 = 4(1-\delta)$. □

Proof of Proposition 2.5. For every positive integer n, $R^2_{(U_S,P_S)} = 4$, while from Lemma 2.2, $\|P_S\|^2_{L^2} = 1 + 2n/\pi^2 \leqslant n+1$. From (2.5) and Lemma 2.6, we deduce

$$\begin{aligned} \tau_{(U_S,P_S)} &\leqslant 24\rho_{R_{(U_S,P_S)}}(n+1)\left(1+\frac{\pi n}{4}\right) \\ &\leqslant \exp(n\ln 4 + 16\pi + \ln(96) + 2\ln(n)) \quad \text{by (2.4)} \\ &\leqslant \exp(4n+54). \end{aligned}$$

The estimate $c_{S^{n-1}} \geqslant \exp(-e^{4n+55})$ follows then from (2.8). □

2.3.2. *Products of spheres*

Proposition 2.7. *For every positive integer n and every $0 \leqslant i \leqslant n-1$, $c_{S^i \times S^{n-i-1}} \geqslant \exp(-e^{5n+69})$.*

For every $n > 0$ and every $0 \leqslant i \leqslant n-1$, we set

$$Q_i((x_1, \dots, x_{i+1}), (y_1, \dots, y_{n-i-1})) = \left(\sum_{j=1}^{i+1} x_j^2 - 2\right)^2 + \sum_{j=1}^{n-i-1} y_j^2 - 1,$$

and $U_{Q_i} = \{(x,y) \in \mathbb{R}^{i+1} \times \mathbb{R}^{n-1-i} \mid \|x\|^2 + \|y\|^2 < 5\}$. The pair (U_{Q_i}, Q_i) is regular in the sense of Definition 2.3 and $Q_i^{-1}(0) \subset V_{Q_i}$ is isotopic in $\mathbb{R}^n$ to the product $S^i \times S^{n-i-1}$ of the unit spheres in $\mathbb{R}^{i+1}$ and $\mathbb{R}^{n-i}$.

LEMMA 2.8. *For every positive integer n and every $0 \leqslant i \leqslant n-1$,*

$$\left(\frac{1}{2\sqrt{n}}, 2\sqrt{1-\frac{1}{2\sqrt{n}}}\right) \in \mathcal{T}_{(U_{Q_i}, Q_i)}.$$

Proof. For every $(x,y) \in \mathbb{R}^{i+1} \times \mathbb{R}^{n-i-1}$ and every $0 < \delta < \frac{1}{2}$,

$$\begin{aligned}|Q_i(x,y)| < \delta &\Longleftrightarrow 1-\delta < (\|x\|^2-2)^2 + \|y\|^2 < 1+\delta \\ &\Longrightarrow \|d_{|(x,y)}Q_i\|^2 = 4\|y\|^2 + 16\|x\|^2(\|x\|^2-2)^2,\end{aligned}$$

with $\|x\|^2 > 2 - \sqrt{1+\delta} > \frac{1}{2}$. Thus, $\|d_{|(x,y)}Q_i\|^2 > 4((\|x\|^2-2)^2 + \|y\|^2) > 4(1-\delta)$ and we deduce the result by choosing $\delta = 1/2\sqrt{n}$. □

Proof of Proposition 2.7. For every positive integer n and every $0 \leqslant i \leqslant n-1$, $R^2_{(U_{Q_i}, Q_i)} = 5$, $\rho_{R_{(U_{Q_i}, Q_i)}} \leqslant 4^n \exp(20\pi)$ by (2.4), while from Lemma 2.2,

$$\|Q_i\|^2_{L^2} = 9 + \frac{2}{\pi^2}(n-i-1) + \frac{32}{\pi^2}(i+1) + \frac{24}{\pi^4}(i+1) + \frac{16}{\pi^4}\begin{pmatrix} i+1 \\ 2 \end{pmatrix} \leqslant 13n^2.$$

We deduce from (2.5) and Lemma 2.8 the upper estimate $\tau_{(U_{Q_i}, Q_i)} \leqslant 1560 n^3 4^n e^{20\pi} \leqslant e^{5n+68}$ since $\ln n \leqslant n-1$. We then deduce from (2.8) the lower estimate $c_{S^i \times S^{n-i-1}} \geqslant \exp(-2e^{5n+68})$. □

REMARK 2.9. The lower estimates given by Propositions 2.5 and 2.7 are far from being optimal.

3. *Lower bounds for the Betti numbers of random real algebraic hypersurfaces*

We first realize the affine real algebraic hypersurfaces in every smooth real projective manifold at the scale $1/\sqrt{d}$ thanks to Hörmander L^2-estimates, see Proposition 3.4. We then follow the approach of Nazarov and Sodin (see [**12**] or also [**10**]) by first estimating the expected local C^1-norm of sections, see Proposition 3.7, and then deducing a positive probability of presence of such affine real algebraic hypersurfaces in the vanishing locus of random sections in any ball of radius $O(1/\sqrt{d})$, see Proposition 3.8. Theorem 1.2 and Corollary 1.3 follow.

3.1. *Hörmander sections*

DEFINITION 3.1. Let (X, c_X) be a smooth real projective manifold of positive dimension n and (L, h, c_L) be a real holomorphic Hermitian line bundle of positive curvature over X. For every x in $\mathbb{R}X$, let us call an *h-trivialization* of L in the neighbourhood of x the following data:

(i) a local holomorphic chart $\psi_x : (W_x, x) \subset X \to (V_x, 0) \subset \mathbb{C}^n$ such that
 (a) $\psi_x \circ c_X = \mathrm{conj} \circ \psi_x$, where $\mathrm{conj} : (y_1, \ldots, y_n) \in \mathbb{C}^n \mapsto (\bar{y}_1, \ldots, \bar{y}_n) \in \mathbb{C}^n$;
 (b) $d_{|x}\psi_x : (T_x X, g_h) \to \mathbb{C}^n$ is an isometry;

(ii) a non-vanishing holomorphic section e of L defined over W_x and such that
 (a) $c_L \circ e \circ c_X = e$;
 (b) $\phi = -\log h(e,e)$ vanishes at x and is positive everywhere else;

(c) there exists a positive constant α_1 such that on V_x.

$$|\phi \circ \psi_x^{-1}(y) - \pi\|y\|^2| \leqslant \alpha_1 \|y\|^3. \tag{3.1}$$

Definition 3.2. Let X be a smooth real projective manifold of positive dimension n and (L, h) be a real holomorphic Hermitian line bundle of positive curvature over X. A *field of h-trivializations* on $\mathbb{R}X$ is the data at every real point x of an h-trivialization in the neighbourhood of x such that the open subset V_x of $\mathbb{C}^n$ given by Definition 3.1 does not depend on $x \in \mathbb{R}X$, and such that the composition $\phi \circ \psi_x^{-1}$ gets uniformly bounded from below by a positive constant on this open set $V = V_x$, while the constant α_1 can be chosen not to depend on $x \in \mathbb{R}X$.

Lemma 3.3. *Any smooth real projective manifold of positive dimension equipped with a real holomorphic Hermitian line bundle of positive curvature admits a field of h-trivializations.*

Proof. Let (X, c_X) be a smooth real projective manifold of positive dimension n and (L, h, c_L) be a real holomorphic Hermitian line bundle with positive curvature ω over X. Let x be a real point of X and let us first prove the existence of an h-trivialization of L near x. The existence of the local chart $\psi_x : (W_x, x) \to (V_x, 0)$ satisfying the first condition of Definition 3.1 is given by definition. Now, restricting W_x if necessary and averaging a local holomorphic section, we get a local holomorphic section $\tilde{e}$ of $L_{|W_x}$ which does not vanish and satisfies $c_L \circ \tilde{e} = \tilde{e} \circ c_X$. The plurisubharmonic function $\tilde{\phi} = -\log h(\tilde{e}, \tilde{e})$ takes real values. Its composition $\tilde{\phi} \circ \psi_x^{-1}$ reads $\tilde{\phi} \circ \psi_x^{-1} = \Re\phi_1 + \phi_2$, where $\phi_1 \in \mathbb{C}[y_1, \ldots, y_n]$ is a degree 2 polynomial and $\phi_2(y) = \pi\|y\|^2 + O(\|y\|^3)$, since the Hermitian part of the second derivative of $\tilde{\phi}$ at x is πg_h by definition. We then set, following [7], $e = \exp(\phi_1 \circ \psi_x)\tilde{e}$ which satisfies the second condition of Definition 3.1 after restricting the open subset W_x if necessary.

The larger the higher order derivatives of ϕ_2 are, the smaller W_x has to be chosen. However, these higher order derivatives are the same as the ones of $\tilde{\phi} \circ \psi_x^{-1}$ since they are not affected by ϕ_1. Now, we can cover $\mathbb{R}X$ with the supports of finitely many real sections $\tilde{e}_1, \ldots, \tilde{e}_k$. The derivative of these sections is uniformly bounded over $\mathbb{R}X$. We can thus choose an h-trivialization near every point x of $\mathbb{R}X$ in such a way that the open subset V_x of $\mathbb{C}^n$ does not depend on $x \in \mathbb{R}X$. Restricting V_x if necessary, this ensures the existence of a field of h-trivialization on $\mathbb{R}X$. □

For every positive d and every $\sigma \in \mathbb{R}H^0(X, L^d)$, $\|\sigma\|_{L^2(h)}$ denotes the L^2-norm for the normalized volume form induced by the Kähler form ω, that is,

$$\|\sigma\|^2_{L^2(h)} = \int_X \|\sigma\|^2_{h^d}\, dV_h,$$

where $dV_h = \omega^n / \int_X \omega^n$. Moreover, if the restriction of σ to W_x reads $\sigma = f_\sigma\, e^d$ for some holomorphic $f_\sigma : W_x \to \mathbb{C}$, we set $|\sigma| = |f_\sigma|$, so that $\|\sigma\|^2_{h_d} = |\sigma|^2 \exp(-d\phi)$ on W_x and for every z in W_x,

$$|d_{|z}\sigma| = |d_{|y}(f_\sigma \circ \psi_x^{-1})| \quad \text{where } y = \psi_x(z). \tag{3.2}$$

We also denote, for every small enough $R > 0$, by $B(x, R) \subset W_x$ the ball centred at x and of radius R for the flat metric of V pulled back by ψ_x, so that $B(x, R) = \psi_x^{-1}(B(0, R))$. We finally denote by $\delta_L = \int_X c_1(L)^n$ the degree of the bundle L.

Proposition 3.4. *Let X be a smooth real projective manifold of positive dimension n and (L, h) be a real holomorphic Hermitian line bundle of positive curvature over X. Let $\mathcal{F}$ be a field of h-trivializations on $\mathbb{R}X$. Then, for every regular pair (U, P), every large enough integer d*

and every x in $\mathbb{R}X$, there exist $\sigma_{(U,P)} \in \mathbb{R}H^0(X, L^d)$ and an open subset $U_d \subset B(x, R_{(U,P)}/\sqrt{d})$ such that

(i) $\|\sigma_{(U,P)}\|_{L^2(h)}$ *converges to* $\|P\|_{L^2}/\sqrt{\delta_L}$ *as d grows to infinity;*
(ii) $(U_d, \sigma^{-1}_{(U,P)}(0) \cap U_d)$ *is diffeomorphic to* $(U, P^{-1}(0) \cap U) \subset \mathbb{R}^n$;
(iii) *for every* $(\delta, \epsilon) \in \mathcal{T}_{(U,P)}$, *there exists a compact subset* $K_d \subset U_d$ *such that*

$$\inf_{U_d \setminus K_d} |\sigma_{(U,P)}| > \frac{\delta}{2}\sqrt{d}^n;$$

while for every y in U_d,

$$|\sigma_{(U,P)}(y)| < \frac{\delta}{2}\sqrt{d}^n \Longrightarrow |d_{|y}\sigma_{(U,P)}| > \frac{\epsilon}{2}\sqrt{d}^{n+1}. \tag{3.3}$$

Recall that the norm of the derivative is given by (3.2), and note that the quantitative transversality condition (3.3) is the one used by Donaldson [**3**].

Under the hypotheses of Proposition 3.4, let $x \in \mathbb{R}X$. We set $U_d = \psi_x^{-1}((1/\sqrt{d})U) \subset B(x, R_{(U,P)}/\sqrt{d})$ for every sufficiently large d so that $(1/\sqrt{d})U \subset V$. Let $\chi : \mathbb{C}^n \to [0,1]$ be a smooth function with compact support in V which equals one in a neighbourhood of the origin. Then, let σ be the global smooth section of L^d defined by $\sigma_{|(X \setminus W_x)} = 0$ and

$$\sigma_{|W_x} = (\chi \circ \psi_x) P(\sqrt{d}\psi_x) e^d.$$

From the L^2-estimates of Hörmander, see [**7**, **14**, **15**], there exists a global section τ of L^d such that $\bar\partial\tau = \bar\partial\sigma$ and $\|\tau\|_{L^2(h)} \leqslant \|\bar\partial\sigma\|_{L^2(h)}$ for d large enough. This section can be chosen orthogonal to the holomorphic sections and is then unique, in particular real, so that $c_{L^d} \circ \tau \circ c_X = \tau$. Moreover, we obtain the following lemma.

Lemma 3.5. *There exist positive constants c_1 and c_2, which do not depend on $x \in \mathbb{R}X$ and satisfy $\|\tau\|_{L^2(h)} \leqslant c_1\, e^{-c_2 d}$ as well as $|\tau|_{C^1(U_d)} \leqslant c_1\, e^{-c_2 d}$, where the C^1-norm is defined by (3.2).*

Proof. The L^2-estimates of Hörmander (see [**15**, Proposition 1.1], for example) read for large enough d

$$\|\tau\|^2_{L^2(h)} \leqslant \int_X \|\bar\partial\sigma\|^2\, dV_h = \int_{W_x} |\psi_x^*\bar\partial\chi|^2 |P(\sqrt{d}\psi_x)|^2\, e^{-d\phi}\, dV_h \tag{3.4}$$

$$\leqslant d^{\deg(P)} \sup_{y \in V} \left(|\bar\partial\chi|^2_{|y} \left| \frac{P(\sqrt{d}y)}{\sqrt{d}^{\deg P}} \right|^2 \right) e^{-d\tilde c_2}, \tag{3.5}$$

where $\tilde c_2 = \inf_{y \in \mathrm{supp}(\bar\partial\chi)} (\phi \circ \psi_x^{-1}(y)) > 0$ by Definition 3.2, so that for d large enough there exist positive constants c_1, c_2 not depending on x such that $\|\tau\|_{L^2(h)} \leqslant c_1 \exp(-2c_2 d)$. Now, since $\tau \circ \psi_x^{-1}$ is holomorphic on $\chi^{-1}(1)$, the mean value inequality for plurisubharmonic functions implies that for every z in U_d,

$$\begin{aligned}
|\tau(z)|^2 &\leqslant \frac{1}{\mathrm{Vol}(B(1/\sqrt{d}))} \int_{B(\psi_x(z), 1/\sqrt{d})} |\tau \circ \psi_x^{-1}(y)|^2\, dy \\
&\leqslant \frac{1}{\mathrm{Vol}(B(1/\sqrt{d}))} \int_{B(z, 1/\sqrt{d})} \|\tau\|^2_{h^d}\, e^{d\phi} \psi_x^*\, dy \\
&\leqslant \frac{1}{\mathrm{Vol}(B(1/\sqrt{d}))} \sup_{B(z, 1/\sqrt{d})} (e^{d\phi} |\det d_{|z}\psi_x|) \|\tau\|^2_{L^2(h)},
\end{aligned}$$

where the determinant $|\det d_{|z}\psi_x|$ is computed with respect to the volume forms dV_h and dy. Since the coefficient in the front of $\|\tau\|^2_{L^2(h)}$ in the right-hand side of the inequality has a polynomial growth, it gets bounded from above by $\exp(2c_2 d)$ for d large enough and we deduce that $|\tau| \leqslant e^{-c_2 d}$ on U_d. The estimate for $|d\tau|$ is proved along the same lines. $\square$

Proof of Proposition 3.4. We set $\sigma_{(U,P)} = \sqrt{d}^n(\sigma - \tau)$ and $K_d = \psi_x^{-1}((1/\sqrt{d})K)$, see Definition 2.4. The section $\sigma_{(U,P)}$ is global and holomorphic. Lemma 3.5 shows that on U_d, $\sigma_{(U,P)}$ is a small perturbation of $\sqrt{d}^n\sigma$. In particular,

$$\|\sigma_{(U,P)}\|^2_{L^2(h)} \underset{d\to\infty}{\sim} d^n \int_{\chi^{-1}(1)} |P(\sqrt{d}y)|^2\, e^{-d\pi\|y\|^2}\frac{dy}{\delta_L} \underset{d\to\infty}{\longrightarrow} \frac{\|P\|^2_{L^2}}{\delta_L}. \tag{3.6}$$

Moreover, for every pair $(\delta, \epsilon) \in \mathcal{T}_{(U,P)}$ and every $z \in U_d \setminus K_d$,

$$\left|\frac{1}{\sqrt{d}^n}\sigma_{(U,P)}(z)\right| = |\sigma(z) - \tau(z)| \tag{3.7}$$
$$\geqslant |\sigma(z)| - \sup_{B(x,R_{(U,P)}/\sqrt{d})} |\tau| \tag{3.8}$$
$$\geqslant |P(\sqrt{d}\psi_x(z))| - \sup_{B(x,R_{(U,P)}/\sqrt{d})} |\tau| \tag{3.9}$$
$$> \delta - \sup_{B(x,R_{(U,P)}/\sqrt{d})} |\tau| \quad \text{from Definition 2.4.} \tag{3.10}$$

Thus, by Lemma 3.5, if d is large enough, then $\inf_{U_d\setminus K_d} |\sigma_{(U,P)}| > (\delta/2)\sqrt{d}^n$ whenever $x \in \mathbb{R}X$. Moreover, for every $z \in U_d$,

$$\begin{aligned} |\sigma_{(U,P)}(z)| < \frac{\delta}{2}\sqrt{d}^n &\Longrightarrow |\sigma(z) - \tau(z)| < \frac{\delta}{2} \\ &\Longrightarrow |\sigma(z)| < \frac{\delta}{2} + |\tau(z)| \\ &\Longrightarrow |P(\sqrt{d}\psi_x(z))| < \delta, \end{aligned}$$

for d large enough, whatever $x \in \mathbb{R}X$ is. Thus,

$$\begin{aligned} |\sigma_{(U,P)}(z)| < \frac{\delta}{2}\sqrt{d}^n &\Longrightarrow |d_{|\sqrt{d}\psi_x(z)}P| > \epsilon \\ &\Longrightarrow |d_{|z}\sigma| > \epsilon\sqrt{d} \quad \text{using notation (3.2)} \\ &\Longrightarrow |d_{|z}\sigma_{(U,P)}| > \frac{\epsilon}{2}\sqrt{d}^{n+1}, \end{aligned}$$

for d large enough by Lemma 3.5. Finally, Lemma 3.5 together with Lemma 3.6 imply that $(\sigma^{-1}_{(U,P)}(0) \cap U_d)$ is isotopic to $(\sigma^{-1}(0) \cap U_d)$ and so diffeomorphic to $(P^{-1}(0) \cap U)$ when d is large enough. $\square$

Lemma 3.6. *Let U be an open subset of $\mathbb{R}^n$, $f : U \to \mathbb{R}$ be a function of class C^1 and $(\delta, \epsilon) \in (\mathbb{R}^*_+)^2$ be such that*

(i) *there exists a compact subset K of U such that $\inf_{U\setminus K} |f| > \delta$;*
(ii) *for every y in U, $|f(y)| < \delta \Rightarrow |df_{|y}| > \epsilon$.*

Then, for every function $g : U \to \mathbb{R}$ of class C^1 such that $\sup_U |g| < \delta$ and $\sup_U |dg| < \epsilon$, zero is a regular value of $f + g$ and $(f + g)^{-1}(0)$ is compact and isotopic to $f^{-1}(0)$ in U.

Proof. For every $t \in [0,1]$ and every $y \in U$,

$$f + tg(y) = 0 \Longrightarrow |f(y)| = |tg(y)| < \delta.$$

The point y is then contained in K and $|df_{|y}| > \epsilon$. Hence, we have

$$|d_{|y}(f + tg)| \geqslant |d_{|y}f| - |td_{|y}g| > 0,$$

so that 0 is a regular value of $f + tg$. The hypersurface $\Sigma_t = (f + tg)^{-1}(0)$ is smooth and included in K by the implicit function theorem. It produces an isotopy between $f^{-1}(0)$ and $(f + g)^{-1}(0)$. $\square$

3.2. *The expected local C^1-norm of sections*

The following Proposition 3.7 computes the expected local C^1-norm of sections. It is inspired by an analogous result of Nazarov and Sodin, see [**12**] (or also [**10**]). Recall that we denote by δ_L the degree $\int_X c_1(L)^n$ of the line bundle L over X, that $|\cdot|$ denotes the modulus evaluated in the charts given by h-trivializations, see (3.2), and that the constant ρ_R is defined by (2.3). Finally, we denote by v the density of dV_h with respect to the volume form dx chosen in (1.1) to define the L^2-product, so that $dV_h = v(x)\,dx$.

PROPOSITION 3.7. *Let X be a smooth real projective manifold of positive dimension n and (L, h) be a real holomorphic Hermitian line bundle of positive curvature over X. We equip $\mathbb{R}X$ with a field of h-trivializations. Then, for every positive R,*

$$\limsup_{d\to\infty} \sup_{x\in\mathbb{R}X} \frac{1}{d^n} E\left(\sup_{B(x,R/\sqrt{d})} \frac{|\sigma|^2}{v(x)}\right) \leqslant 6\delta_L\rho_R$$

and

$$\limsup_{d\to\infty} \sup_{x\in\mathbb{R}X} \frac{1}{d^{n+1}} E\left(\sup_{B(x,R/\sqrt{d})} \frac{|d\sigma|^2}{v(x)}\right) \leqslant 6\pi n\delta_L\rho_R.$$

Proof. Let $R > 0$, $x \in \mathbb{R}X$ and W_x be a neighbourhood given by the h-trivialization. When d is large enough, $B(x, R/\sqrt{d}) \subset W_x$ and $\phi \circ \psi_x^{-1}(y) = \pi\|y\|^2 + o(\|y\|^2)$. We deduce from the mean value inequality that for every $s \in \mathbb{R}_+$ and $\sigma \in \mathbb{R}H^0(X, L^d)$,

$$\begin{aligned}\forall z \in B\left(x, \frac{R}{\sqrt{d}}\right), \quad |\sigma(z)|^2 &\leqslant \frac{1}{\mathrm{Vol}(B(s/\sqrt{d}))} \int_{B(z,s/\sqrt{d})} |\sigma|^2 \psi_x^* \, dy \\ &\leqslant \frac{1}{\mathrm{Vol}(B(s/\sqrt{d}))} \int_{B(x,(R+s)/\sqrt{d})} |\sigma|^2 \psi_x^* \, dy.\end{aligned}$$

Thus, $\sup_{B(x,R/\sqrt{d})} |\sigma|^2 \leqslant (1/\mathrm{Vol}(B(s/\sqrt{d}))) \int_{B(x,(R+s)/\sqrt{d})} |\sigma|^2 \psi_x^* \, dy$ and after exchange of the integrals,

$$E\left(\sup_{B(x,R/\sqrt{d})} |\sigma|^2\right) \leqslant \frac{1}{\mathrm{Vol}(B(s/\sqrt{d}))} \int_{B(x,(R+s)/\sqrt{d})} E(|\sigma|^2) \psi_x^* \, dy.$$

Then, let $z \in B(x, (R+s)/\sqrt{d}) \cap \mathbb{R}X$ and $\sigma_0 \in \mathbb{R}H^0(X, L^d)$ be the Bergman section at z. Its norm equals 1 and it is orthogonal to the space of sections vanishing at z. Assume for the moment that the volume form dx chosen to define the L^2-scalar product equals dV_h, so that $v = 1$. Then, from [**15**, Lemma 2.2] (see also [**2**, **6**]),

$$\|\sigma_0(z)\|_{h^d}^2 \underset{d\to\infty}{\sim} \delta_L d^n.$$

But $\|\sigma_0(z)\|^2_{h^d} = |\sigma_0(z)|^2 \, e^{-d\phi(z)}$, from which we deduce

$$|\sigma_0(z)|^2 \leqslant \delta_L d^n \, e^{\pi(R+s)^2} + o(d^n),$$

where the $o(d^n)$ term can be chosen not to depend on $x \in \mathbb{R}X$. As a consequence,

$$\begin{aligned} E(|\sigma(z)|^2) &= \int_{\mathbb{R}H^0(X,L^d)} |\sigma(z)|^2 \, d\mu_{\mathbb{R}}(\sigma) = |\sigma_0(z)|^2 \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} a^2 \, e^{-a^2} \, da \\ &\leqslant \frac{1}{2} \delta_L d^n \, e^{\pi(R+s)^2} + o(d^n). \end{aligned}$$

When $z \in B(x, (R+s)/\sqrt{d}) \setminus \mathbb{R}X$, the space of real sections vanishing at z gets of real co-dimension 2 in $\mathbb{R}H^0(X, L^d)$. Let $\langle \theta_1, \theta_2 \rangle$ be an orthonormal basis of its orthogonal complement. From [**15**, Lemmas 1.2 and 2.1], we deduce as before that for every $j \in \{1, 2\}$,

$$\limsup_{d\to\infty} \frac{1}{d^n} |\theta_j(z)|^2 \leqslant 2\delta_L \, e^{\pi(R+s)^2},$$

an upper bound which does not depend on z. We deduce that

$$\begin{aligned} E(|\sigma(z)|^2) &= \int_{\mathbb{R}^2} |(a_1\theta_1(z) + a_2\theta_2(z))|^2 \, e^{-a_1^2 - a_2^2} \frac{1}{\pi} \, da_1 \, da_2 \\ &\leqslant 2\delta_L d^n \, e^{\pi(R+s)^2} (1 + o(1)) \int_{\mathbb{R}^2} (a_1^2 + a_2^2 + 2|a_1||a_2|) \, e^{-a_1^2 - a_2^2} \frac{1}{\pi} \, da_1 \, da_2 \\ &\leqslant 6\delta_L d^n \, e^{\pi(R+s)^2} (1 + o(1)). \end{aligned}$$

Finally,

$$\begin{aligned} \sup_{x \in \mathbb{R}X} E\left(\sup_{B(x,R/\sqrt{d})} |\sigma|^2 \right) &\leqslant \frac{6\delta_L d^n}{\mathrm{Vol}(B(s/\sqrt{d}))} \int_{B(0,(R+s)/\sqrt{d})} e^{\pi(R+s)^2} \, dy + o(d^n) \\ &\leqslant 6\delta_L d^n \, e^{\pi(R+s)^2} \frac{(R+s)^{2n}}{s^{2n}} + o(d^n). \end{aligned}$$

Choosing $s \in \mathbb{R}_+^*$ such that $g_R(s) = \rho_R$, see (2.3), we deduce that

$$\limsup_{d\to\infty} \sup_{x \in \mathbb{R}X} \frac{1}{d^n} E\left(\sup_{B(x,R/\sqrt{d})} |\sigma|^2 \right) \leqslant 6\rho_R \delta_L.$$

Likewise, we deduce from the mean value inequality that for every $s \in \mathbb{R}_+$, $j \in \{1, \ldots, n\}$ and $z \in B(x, R/\sqrt{d})$,

$$\left| \frac{\partial \sigma}{\partial y_j} \right|^2 (z) \leqslant \frac{1}{\mathrm{Vol}(B(s/\sqrt{d}))} \int_{B(0,(s+R)/\sqrt{d})} \left| \frac{\partial(\sigma \circ \psi_x^{-1})}{\partial y_j} \right|^2 (y) \, dy,$$

from which follows after summation over $j \in \{1, \ldots, n\}$ that

$$E\left(\sup_{B(x,R/\sqrt{d})} |d\sigma|^2 \right) \leqslant \frac{1}{\mathrm{Vol}(B(s/\sqrt{d}))} \int_{B(0,(s+R)/\sqrt{d})} E(|d_{|\psi_x^{-1}(y)} \sigma|^2) \, dy.$$

Let $z \in B(x, (R+s)/\sqrt{d}) \cap \mathbb{R}X$ and for every $j \in \{1, \ldots, n\}$, $\sigma_j \in \mathbb{R}H^0(X, L^d)$ be the normed section orthogonal to the hyperplane of sections σ such that $\partial\sigma/\partial y_j|z = 0$. Still assuming that $dx = dV_h$, we know from [**15**, Lemma 2.1], see also [**6**, Lemma 2.2.3], that

$$\left\| \frac{\partial \sigma_j}{\partial y_j}(z) \right\|^2_{h^d} \underset{d\to\infty}{\sim} \pi\delta_L d^{n+1},$$

so that again

$$\left| \frac{\partial \sigma_j}{\partial y_j}(z) \right|^2 \leqslant \pi\delta_L \, e^{\pi(R+s)^2} d^{n+1} + o(d^{n+1}),$$

with an $o(d^{n+1})$ term which does not depend on $x \in \mathbb{R}X$. We deduce that

$$\begin{aligned} E(|d\sigma_{|z}|^2) &= \sum_{j=1}^n E\left(\left|\frac{\partial \sigma}{\partial y_j}(z)\right|^2\right) \\ &= \sum_{j=1}^n \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} a^2 \left|\frac{\partial \sigma_j}{\partial y_j}(z)\right|^2 e^{-a^2}\, da \\ &\leqslant \frac{n}{2}\pi\delta_L\, e^{\pi(R+s)^2} d^{n+1} + o(d^{n+1}). \end{aligned}$$

When $z \in B(x, (R+s)/\sqrt{d}) \setminus \mathbb{R}X$, we get in the same way as before that $E(|d\sigma_{|z}|^2) \leqslant 6n\pi\delta_L\, e^{\pi(R+s)^2} d^{n+1} + o(d^{n+1})$. Finally, $\limsup_{d\to\infty} \sup_{x\in\mathbb{R}X} (1/d^{n+1}) E(\sup_{B(x,R/\sqrt{d})} |d\sigma|^2) \leqslant 6n\pi\delta_L g_R(s)$. By choosing s such that $g_R(s) = \rho_R$, see (2.3), we obtain the result in the case where $v = 1$ on X.

In general, the Bergman section at x for the L^2-product (1.1) associated to the volume form dx is equivalent to the Bergman section σ_0 at x for dV_h times $\sqrt{v(x)}$, because σ_0 has its L^2-norm concentrated on the ball $B(x, \log d/\sqrt{d})$. The same holds true for every σ_j, $j \in \{1, \dots, n\}$, and the result follows by replacing δ_L with $v(x)\delta_L$. □

3.3. *Probability of the local presence of closed affine hypersurfaces*

Following the approach of Nazarov and Sodin (see [**12**] or also [**10**]), we deduce the following Proposition 3.8 from Propositions 3.4 and 3.7. It estimates from below the probability of presence, in a ball of radius inversely proportional to $\sqrt{d}$, of a given closed affine real algebraic hypersurface in the vanishing locus of sections of high tensor powers of an ample real line bundle.

Let (X, c_X) be a smooth real projective manifold of positive dimension n and (L, h, c_L) be a real holomorphic Hermitian line bundle of positive curvature over X. Let (U, P) be a regular pair given by Definition 2.3 and $\Sigma = P^{-1}(0) \subset U$. Then, for every $x \in \mathbb{R}X$, we set $B_d = B(x, R_{(U,P)}/\sqrt{d})$ and denote by $\mathrm{Prob}_{x,\Sigma}(L^d)$ the probability that $\sigma \in \mathbb{R}H^0(X, L^d)$ has the property that $\sigma^{-1}(0) \cap B_d$ contains a hypersurface Σ' such that the pair (B_d, Σ') be diffeomorphic to $(\mathbb{R}^n, \Sigma)$. That is,

$$\mathrm{Prob}_{x,\Sigma}(L^d) = \mu_{\mathbb{R}}\{\sigma \in \mathbb{R}H^0(X, L^d) \,|\, \sigma^{-1}(0) \cap B_d \supset \Sigma' \text{ and } (B_d, \Sigma') \sim (\mathbb{R}^n, \Sigma)\}.$$

We then set $\mathrm{Prob}_{\Sigma}(L^d) = \inf_{x\in\mathbb{R}X} \mathrm{Prob}_{x,\Sigma}(L^d)$.

Proposition 3.8. *Let X be a smooth real projective manifold of positive dimension n and L be a real holomorphic Hermitian line bundle of positive curvature over X. Let (U, P) be a regular pair given by Definition 2.3 and $\Sigma = P^{-1}(0) \subset U$. Then,*

$$\liminf_{d\to\infty} \mathrm{Prob}_{\Sigma}(L^d) \geqslant m_{\tau_{(U,P)}},$$

see (2.1).

Proof. Let $x \in \mathbb{R}X$ and let us choose an h-trivialization of (L, h) given by Definition 3.1. By Proposition 3.4, there exist, for every d large enough, a compact K_d, an open set U_d and a section $\sigma_{(U,P)} \in \mathbb{R}H^0(X, L^d)$ such that

$$K_d \subset U_d \subset B_d \subset W_x,$$

and $(U_d, \sigma^{-1}(0) \cap U_d)$ be diffeomorphic to (U, Σ). Moreover, for every $(\delta, \epsilon) \in \mathcal{T}_{(U,P)}$,

$$\inf_{U_d \setminus K_d} |\sigma_{(U,P)}| > \frac{\delta}{2}\sqrt{d}^{\,n}, \tag{3.11}$$

and for every z in U_d,

$$|\sigma_{(U,P)}(z)| < \frac{\delta}{2}\sqrt{d}^n \Rightarrow |d_{|z}\sigma_{(U,P)}| > \frac{\epsilon}{2}\sqrt{d}^{n+1}. \tag{3.12}$$

The moduli $|\sigma_{(U,P)}|$ and $|d\sigma_{(U,P)}|$ are computed here in the h-trivialization of L^d, see (3.2). Denote by $\sigma_P^\perp$ the hyperplane orthogonal to $\sigma_{(U,P)}$ in $\mathbb{R}H^0(X, L^d)$ and by s_P the orthogonal symmetry of $\mathbb{R}H^0(X, L^d)$ which fixes $\sigma_P^\perp$. Then, the average value of $\sup_{B_d} |\theta|^2$ on $\sigma_P^\perp$ satisfies

$$\begin{aligned}
E\left(\sup_{B_d} |\theta|^2\right) &= \int_{\sigma_P^\perp} \sup_{B_d} |\theta|^2 \, d\mu_{\mathbb{R}}(\theta) \\
&= \int_{\mathbb{R}H^0(X,L^d)} \sup_{B_d} \left|\frac{\sigma + s_P(\sigma)}{2}\right|^2 d\mu_{\mathbb{R}}(\sigma) \\
&\leqslant \frac{1}{2}\int_{\mathbb{R}H^0(X,L^d)} \sup_{B_d} |\sigma|^2 \, d\mu_{\mathbb{R}}(\sigma) + \frac{1}{2}\int_{\mathbb{R}H^0(X,L^d)} \sup_{B_d} |s_P(\sigma)|^2 \, d\mu_{\mathbb{R}}(\sigma) \\
&\leqslant E\left(\sup_{B_d} |\sigma|^2\right) \\
&\leqslant 6\delta_L \rho_{R_{(U,P)}} v(x) d^n + o(d^n),
\end{aligned}$$

from Proposition 3.7, where the $o(d^n)$ term does not depend on $x \in \mathbb{R}X$.

Likewise, from Proposition 3.7,

$$E\left(\sup_{B_d} |d\theta|^2\right) \leqslant 6\delta_L \pi n \rho_{R_{(U,P)}} v(x) d^{n+1} + o(d^{n+1}),$$

where the $o(d^{n+1})$ term does not depend on $x \in \mathbb{R}X$. From Markov's inequality follows that for every $M > 0$,

$$\mu_{\mathbb{R}}\left\{\theta \in \sigma_P^\perp \mid \sup_{B_d} |\theta|^2 \geqslant M^2 v(x) \frac{\delta^2 \delta_L}{4\|P\|^2_{L^2}} d^n\right\} \leqslant \frac{24\|P\|^2_{L^2}\rho_{R_{(U,P)}}}{M^2\delta^2} + o(1)$$

and

$$\mu_{\mathbb{R}}\left\{\theta \in \sigma_P^\perp \mid \sup_{B_d} |d\theta|^2 \geqslant M^2 v(x) \frac{\epsilon^2 \delta_L}{4\|P\|^2_{L^2}} d^{n+1}\right\} \leqslant \frac{24\pi n\|P\|^2_{L^2}\rho_{R_{(U,P)}}}{M^2\epsilon^2} + o(1),$$

where the $o(1)$ term does not depend on $x \in \mathbb{R}X$. As a consequence, setting

$$\mathcal{E}_{\sigma_P^\perp} = \left\{\theta \in \sigma_P^\perp \mid \sup_{B_d} |\theta|^2 < M^2 v(x) \frac{\delta^2 \delta_L}{4\|P\|^2_{L^2}} d^n \text{ and } \sup_{B_d} |d\theta|^2 < M^2 v(x) \frac{\epsilon^2 \delta_L}{4\|P\|^2_{L^2}} d^{n+1}\right\},$$

we have

$$\mu_{\mathbb{R}}(\mathcal{E}_{\sigma_P^\perp}) \geqslant 1 - 24\frac{\|P\|^2_{L^2}\rho_{R_{(U,P)}}}{M^2}\left(\frac{1}{\delta^2} + \frac{\pi n}{\epsilon^2}\right) - o(1),$$

where the $o(1)$ term does not depend on x. Choosing (δ, ϵ) which minimizes the function $(\delta, \epsilon) \mapsto (1/\delta^2 + \pi n/\epsilon^2)$, we deduce from (2.5) that $\mu_{\mathbb{R}}(\mathcal{E}_{\sigma_P^\perp}) \geqslant 1 - \tau_{(U,P)}/M^2 - o(1)$. Now, setting

$$\mathcal{F}_{M,(U,P)} = \left\{a\frac{\sigma_{(U,P)}}{\|\sigma_{(U,P)}\|_{L^2}} + \theta \in \mathbb{R}H^0(X, L^d) \mid a > M \text{ and } \theta \in \mathcal{E}_{\sigma_P^\perp}\right\},$$

where

$$\|\sigma_{(U,P)}\|_{L^2} \underset{d\to\infty}{\sim} \frac{\|p\|}{\sqrt{v(x)\delta_L}}, \tag{3.13}$$

by the first part of Proposition 3.4 and the fact that the mass of $\sigma_{(U,P)}$ concentrates on small balls $B(x, \log d/\sqrt{d})$. Take $\sigma \in \mathcal{F}_{M,(U,P)}$. From the estimates (3.11)–(3.13) and the definition of $\mathcal{E}_{\sigma_P^\perp}$, for d large enough, 0 is a regular value of σ and from Lemma 3.6, $\sigma^{-1}(0) \cap U_d$ is isotopic to $\sigma^{-1}_{(U,P)}(0) \cap U_d$, so that the pair $(B_d, \sigma^{-1}(0) \cap U_d)$ is diffeomorphic to $(B(0, R_{(U,P)}), \Sigma)$. The result follows from the fact that

$$\begin{aligned}\mathrm{Prob}_{x,\Sigma}(L^d) \geqslant \mu_{\mathbb{R}}(\mathcal{F}_{M,(U,P)}) &\geqslant \left(\frac{1}{\sqrt{\pi}} \int_M^{+\infty} e^{-t^2}\, dt\right) \left(1 - \frac{\tau_{(U,P)}}{M^2} - o(1)\right) \\ &= f_{\tau_{(U,P)}}(M) - o(1),\end{aligned}$$

see § 2.1. We choose $M \in [\sqrt{\tau_{(U,P)}}, +\infty[$ which maximizes $f_{\tau_{(U,P)}}$, see (2.1), and take the limit. □

Proof of Theorem 1.2. Let $(U, P) \in \mathcal{I}_\Sigma$, see § 2.2. For every $d > 0$, let Λ_d be a maximal subset of $\mathbb{R}X$ with the property that two distinct points of Λ_d are at distance greater than $2R_{(U,P)}/\sqrt{d}$. The balls centred at points of Λ_d and of radius $R_{(U,P)}/\sqrt{d}$ are disjoints, whereas the ones of radius $2R_{(U,P)}/\sqrt{d}$ cover $\mathbb{R}X$. For every $x \in \Lambda_d$ and every $\sigma \in \mathbb{R}H^0(X, L^d) \setminus \mathbb{R}\Delta_d$, we set $N_\Sigma(x, \sigma) = 1$ if the ball B_d contains a hypersurface Σ' such that $\Sigma' \subset \sigma^{-1}(0)$ and (B_d, Σ') is diffeomorphic to $(\mathbb{R}^n, \Sigma)$, whereas $N_\Sigma(x, \sigma) = 0$ otherwise. Recall that $N_\Sigma(\sigma)$ denotes the maximal number of disjoint open subsets of $\mathbb{R}X$ having the property that each such open subset U' contains a hypersurface Σ' such that $\Sigma' \subset \mathbb{R}C_\sigma$ and (U', Σ') be diffeomorphic to $(\mathbb{R}^n, \Sigma)$. Thus,

$$\begin{aligned}E(N_\Sigma) &\geqslant \int_{\mathbb{R}H^0(X,L^d)\setminus\mathbb{R}\Delta_d} \left(\sum_{x\in\Lambda_d} N_\Sigma(x, \sigma)\right) d\mu_{\mathbb{R}}(\sigma) \\ &= \sum_{x\in\Lambda_d} \mathrm{Prob}_{x,\Sigma}(L^d) \\ &\geqslant |\Lambda_d|\, \mathrm{Prob}_\Sigma(L^d),\end{aligned}$$

by Proposition 3.8. We deduce from the inclusion $\mathbb{R}X \subset \bigcup_{x\in\Lambda_d} B(x, 2R_{(U,P)}/\sqrt{d})$ that

$$\begin{aligned}\mathrm{Vol}_h(\mathbb{R}X) &\leqslant \sum_{x\in\Lambda_d} \mathrm{Vol}\left(B\left(x, \frac{2R_{(U,P)}}{\sqrt{d}}\right)\right) \\ &\leqslant 2^n |\Lambda_d|\, \mathrm{Vol}(B_d) + o\left(\frac{|\Lambda_d|}{\sqrt{d}^n}\right).\end{aligned}$$

From Proposition 3.8 follows then that

$$\liminf_{d\to\infty} \frac{1}{\sqrt{d}^n} E(N_\Sigma) \geqslant \frac{m_{\tau_{(U,P)}} \mathrm{Vol}_h(\mathbb{R}X)}{2^n\, \mathrm{Vol}(B_d)}.$$

This lower bound holds for every pair $(U, P) \in \mathcal{I}_\Sigma$ and we get the result by taking the supremum, see (2.6). □

Proof of Corollary 1.3. For every $d > 0$,

$$\begin{aligned}E(b_i) &= \int_{\mathbb{R}H^0(X,L^d)\setminus\mathbb{R}\Delta_d} b_i(\mathbb{R}C_\sigma, \mathbb{R})\, d\mu_{\mathbb{R}}(\sigma) \\ &\geqslant \int_{\mathbb{R}H^0(X,L^d)\setminus\mathbb{R}\Delta_d} \left(\sum_{\Sigma\in\mathcal{H}_n} N_\Sigma(\sigma) b_i(\Sigma)\right) d\mu_{\mathbb{R}}(\sigma) \\ &\geqslant \sum_{\Sigma\in\mathcal{H}_n} b_i(\Sigma) E(N_\Sigma).\end{aligned}$$

Hence, the first lower bound follows from Theorem 1.2, while the second one follows along the same lines. The last part of Corollary 1.3 is then a consequence of Proposition 2.7. □

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