

## Riemann surfaces and totally real tori

Julien Duval and Damien Gayet

**Abstract.** Given a totally real torus unknotted in the unit sphere  $S^3$  of  $\mathbb{C}^2$ , we prove the following alternative: either the torus is rationally convex and there exists a filling of the torus by holomorphic discs, or its rational hull contains a holomorphic annulus or a pair of holomorphic discs.

**Mathematics Subject Classification (2010).** 32E20, 58J32.

**Keywords.** Totally real torus, filling by holomorphic discs, rational convexity.

### Introduction

In this paper we address the following question: given a totally real torus in  $\mathbb{C}^2$ , does there always exist a compact Riemann surface in  $\mathbb{C}^2$  with boundary in (or simply *attached to*) the torus?

Recall that (closed connected) surfaces in  $\mathbb{C}^2$  are *totally real* if they are never tangent to a complex line. The only orientable ones are tori. Special cases are *Lagrangian* tori, those on which the standard Kähler form of  $\mathbb{C}^2$  vanishes.

Our question is motivated by geometric function theory (see [15] for background). Given a compact set  $K$  in  $\mathbb{C}^2$ , its *polynomial hull*  $\hat{K}$  is defined as

$$\hat{K} = \{z \text{ in } \mathbb{C}^2 \mid |P(z)| \leq \|P\|_K \text{ for every polynomial } P\}.$$

The set  $K$  is *polynomially convex* if  $\hat{K} = K$ . In this case  $K$  satisfies Runge theorem. Note that any compact Riemann surface attached to  $K$  is contained in  $\hat{K}$ . It is therefore tempting to explain the presence of a non trivial hull by Riemann surfaces, at least for nice sets like orientable surfaces (they are not polynomially convex for homological reasons). But quite often a complex tangency of a surface locally gives birth to small holomorphic discs attached to it. Thus the very first global problem arises with totally real orientable surfaces, namely tori.

Note that, in the definitions above, instead of polynomials we could as well work with rational functions without poles on  $K$ . This gives rise to the notions of *rational hull* and *rational convexity*. Again an obstruction to rational convexity is the presence

of a compact Riemann surface  $C$  attached to  $K$  with the additional restriction that  $\partial C$  bounds in  $K$ .

Here is a bit of history around our question. In 1985 Gromov [10] gave a positive answer for Lagrangian tori, constructing holomorphic discs attached to them. In 1996 by the same method Alexander [1] exhibited for every totally real torus a proper holomorphic disc with all its boundary except one point in the torus. Later on [2] he gave examples of totally real tori without holomorphic discs with full boundary in them, but still admitting holomorphic annuli attached to them.

In the present work we focus<sup>1</sup> on tori in the unit sphere  $S^3$  of  $\mathbb{C}^2$ . They are *unknotted* if they are isotopic to the standard torus in  $S^3$ . We prove the following

**Theorem.** *Let  $T$  be a totally real torus unknotted in  $S^3$ . Then either  $T$  is rationally convex and bounds a solid torus foliated by holomorphic discs in the unit ball  $B$ , or its rational hull contains a holomorphic annulus or a pair of holomorphic discs attached to  $T$ .*

The solid torus is called a *filling* of  $T$  which is said in this case *fillable*. The standard torus is an example of the first situation, while the second is illustrated by the following

**Example** (compare with [2]). Consider the conjugate Hopf fibration

$$\pi : S^3 \subset \mathbb{C}^2 \rightarrow S^2 \subset \mathbb{C} \times \mathbb{R}, \quad (z, w) \mapsto (2zw, |z|^2 - |w|^2).$$

Remark that the fibers of  $\pi$  are circles. Denote by  $T_\gamma$  the preimage by  $\pi$  of an embedded closed curve  $\gamma$  in  $S^2$ . Then  $T_\gamma$  is an unknotted torus in  $S^3$ , totally real if the projection of  $\gamma$  on  $\mathbb{C}$  is immersed. Choose this projection as a figure eight which avoids the origin. It follows (see [2]) that every compact Riemann surface with boundary in  $T_\gamma$  is in a fiber of the polynomial  $p(z, w) = 2zw$ . But  $T_\gamma$  does not separate  $p^{-1}(a)$  except if  $a$  is the double point of the figure eight. We then get only one holomorphic annulus attached to  $T$ . If on the other hand the figure eight intersects itself at the origin we get instead a pair of holomorphic discs attached to  $T$ .

The proof of the theorem relies on the technique of filling spheres by holomorphic discs due to Bedford and Klingenberg [5] and Kruzhilin [12] (see also Eliashberg [8]). This is where the restriction to  $S^3$  enters. The spheres come into the picture as approximations of a lift of the torus in a suitable covering. More precisely take a totally real unknotted torus  $T$  in  $S^3$ . It divides  $S^3$  in two solid tori. In the same manner its hull  $\hat{T}$  separates the unit ball  $B$  in two pseudoconvex components. At least one of them has a universal covering which unwinds the corresponding solid torus. Push  $T$  slightly in this good component, building a sequence of tori  $T_n$  converging

<sup>1</sup>following [13] which by the way seems uncorrect (see our example)

toward  $T$ . We therefore get as a lift of  $T_n$  a periodic cylinder sitting in a pseudoconvex boundary. Approximate it by a sphere  $S_n$  containing say  $2n$  periods of the cylinder.

We are now in position to apply the technique of filling. It provides a sequence of balls bounded by  $S_n$  and foliated by holomorphic discs. Single out one of these discs passing through the equator of  $S_n$  and call  $\Delta_n$  its projection downstairs. The alternative reads as follows: either the area of  $\Delta_n$  remains bounded, or not.

In the former case (the *rationally convex case*) we check that the tori  $T_n$  are fillable for large  $n$ , and that their fillings converge in some sense to a filling of  $T$ . This relies on Gromov compactness theorem.

In the latter (the *non rationally convex case*) we rather look at the limit of  $\Delta_n$  in terms of currents. Define  $U$  as the limit of the normalized currents of integration on  $\Delta_n$ . Then  $U$  is a positive current such that  $dU$  bounds in  $T$ . Therefore the support of  $U$  is contained in the rational hull of  $T$ . Moreover a dividing process of  $U$  shows that it can be written as an integral of currents of integration over Riemann surfaces. Finally we apply Ahlfors theory of covering surfaces to prove that these Riemann surfaces are holomorphic discs or annuli.

Before entering the details of the proof, we collect some background. In the sequel a limit of a sequence often occurs up to extracting a subsequence, even if not explicitly mentioned. Pseudoconvex domains are also sometimes confused with their closure.

## 1. Background

**a) Filling spheres.** Recall the central result of [12] (see also [5]).

**Theorem.** *Let  $\Omega$  be a bounded strictly pseudoconvex domain in  $\mathbb{C}^2$  and  $S$  a sphere in  $\partial\Omega$ . Suppose that the complex tangencies of  $S$  are elliptic or hyperbolic points. Then  $S$  bounds a unique ball  $\Sigma$  in  $\Omega$  foliated by holomorphic discs.*

This ball  $\Sigma$  is called the *filling* of  $S$ . The *complex tangencies* of  $S$  are the points where  $S$  is tangent to a complex line. Being of elliptic or hyperbolic type (see [5], [12] for the definition) is a generic condition. It can be achieved by a small perturbation localized near the complex points.

The picture looks as follows. Take a sphere in  $\mathbb{R}^3$  endowed with its height function, which is Morse if the sphere is generic. Elliptic points correspond to local maxima and minima of the height, while hyperbolic points translate in saddle points. By Morse theory we have  $e - h = 2$  where  $e$  and  $h$  are respectively the number of elliptic and hyperbolic points. The filling corresponds to the ball bounded by the sphere foliated by the level sets of the height. Therefore all the holomorphic discs of the filling are smooth up to the boundary except those touching a hyperbolic point which have corners.

Another way to describe the complex points of  $S$  is via its *characteristic foliation*. This is the foliation generated by the characteristic line field  $T_{\mathbb{C}}\partial\Omega \cap TS$  where  $T_{\mathbb{C}}\partial\Omega$  is the complex part of  $T\partial\Omega$ . It is singular precisely at the complex points of  $S$ , elliptic points corresponding to foci and hyperbolic to saddle points. The characteristic foliation gives a control on the discs of the filling, in the sense that their boundaries are always transversal to it. This comes from Hopf lemma which asserts that a holomorphic disc contained in  $\Omega$  is transversal to  $\partial\Omega$ .

Here are further properties of the filling. First every compact Riemann surface in  $\Omega$  attached to  $S$  is contained in  $\Sigma$ . Next  $\Sigma$  is the envelope of holomorphy of  $S$ . Hence  $\Sigma$  is contained in any pseudoconvex domain containing  $S$ . Finally if we divide out the sphere  $S$  into two half spheres by an equator, at least one of them can be partially filled in the sense of [8]: the surface swept by the boundaries of the discs in  $\Sigma$  contained in the half sphere reaches the equator.

In the sequel we will apply this technique of filling to a sphere in  $\partial\tilde{\Omega}$  where  $\tilde{\Omega}$  is the universal covering of a pseudoconvex domain  $\Omega$  which is strictly pseudoconvex where the sphere projects down. The reader can check that all the arguments of [5], [12] apply mutatis mutandis.

**b) Geometric function theory.** We will use the following facts concerning polynomial convexity (see [15] for this paragraph). Let  $K$  be a compact set in  $S^3$  separating the sphere in finitely many components, then its polynomial hull  $\hat{K}$  divides  $B$  in the same number of components. Moreover by Rossi local maximum principle these components are pseudoconvex domains. We will also rely on the theorem by Alexander describing the polynomial hull of a curve of finite length (with finitely many components): it is a Riemann surface attached to the curve.

We move on to rational convexity. The rational hull  $r(K)$  of a compact set  $K$  in  $\mathbb{C}^2$  is geometrically defined as the set of points  $z$  such that any algebraic curve passing through  $z$  meets  $K$ . If  $K \subset P$  where  $P$  is a rational polyhedron, then the algebraic curves can be replaced by analytic curves in  $P$ . The usual obstruction to rational convexity is the presence of a compact Riemann surface with boundary in  $K$  with the additional restriction that this boundary bounds in  $K$ . In our theorem (second situation) the holomorphic annulus or the pair of holomorphic discs will satisfy this condition and therefore be part of  $r(T)$ . As for the first situation we have the following

**Lemma.** *A fillable totally real torus in  $S^3$  is rationally convex.*

*Proof.* Call  $T$  the torus and  $\Theta$  its filling. We first prove that  $\Theta$  is rationally convex. By Rossi local maximum principle and the Runge property of  $B$  it is enough to construct through any point near  $\Theta$  in the ball  $B$  an analytic curve in  $B$  (smooth up to  $S^3$ ) avoiding  $\Theta$ . We produce them by stability of the filling of  $T$  (see [4] for a similar situation). Foliate a neighborhood of  $T$  in  $S^3$  by tori, then the fillings of these tori

foliate a neighborhood of  $\Theta$  in  $B$ . Therefore the corresponding holomorphic discs fill out this neighborhood and avoid  $T$  if they are not in  $\Theta$ . At this stage  $r(T) \subset \Theta$ .

We now prove that  $r(T) = T$ . According to the first step  $\Theta$  is a decreasing limit of rational polyhedrons. It is then enough to construct through any point  $z$  of  $\Theta \setminus T$  an analytic curve in a neighborhood of  $\Theta$  avoiding  $T$ . Take through  $z$  a real closed curve in  $\Theta \setminus T$  transversal to the holomorphic discs, parametrized by the unit circle. Extend this parametrization as a smooth map  $f$  from a thin round annulus in such a way that  $\bar{\partial}f$  vanishes to infinite order along the unit circle. By solving an adequate  $\bar{\partial}$ -equation perturb now  $f$  into a holomorphic map. This map parametrizes a thin holomorphic annulus still passing through  $z$  and intersecting  $\Theta$  near the initial curve, hence avoiding  $T$ .

Finally let us recall the analogue in terms of currents of the usual obstructions to polynomial or rational convexity [7]. Let  $K$  a compact set in  $\mathbb{C}^2$  and  $U$  a positive 1,1-current with compact support. If  $\text{supp}(dU) \subset K$  then  $\text{supp}(U) \subset \hat{K}$ . If moreover  $dU = dV$  where  $V$  is a current supported by  $K$ , then  $\text{supp}(U) \subset r(K)$ .

**c) Ahlfors currents.** They are the local version of the currents built from an entire curve in complex hyperbolicity. In our context a current  $U$  is an *Ahlfors current* if  $U = \lim \frac{[\Delta_n]}{a_n}$ , where  $[\Delta_n]$  are currents of integration over holomorphic discs  $\Delta_n$  of area  $a_n$  contained in  $B$  whose boundary sits mainly in  $S^3$ . Precisely, one has  $\text{length}(\partial\Delta_n \setminus S^3) = o(a_n)$ . Hence  $U$  is a positive 1,1-current with compact support such that  $\text{supp}(dU) \subset S^3$ . The following lemma (compare with [6]) will be important for the non rationally convex case.

**Lemma.** *Let  $U$  be an Ahlfors current whose support is an analytic curve in  $B$ . Then each irreducible component of this curve is a holomorphic disc or annulus.*

*Proof.* It relies on Ahlfors theory of covering surfaces [14] under the form of the following

**Isoperimetric inequality.** *Let  $E$  be a compact connected Riemann surface with boundary, of negative Euler characteristic. Then there is a constant  $c$  such that for any holomorphic disc  $f : D \rightarrow E$  we have  $\text{area}(f(D)) \leq c \text{length}(f(\partial D) \setminus \partial E)$ .*

Here area and length are computed by means of a given metric on  $E$ , taking into account multiplicities.

As in [6] we proceed by contradiction. Let  $C$  be a component of the analytic curve which is neither a disc nor an annulus. Then there exists a figure eight  $e$  in  $C$  such that any component of  $C \setminus e$  meets  $\partial C$ . In particular  $e$  is polynomially convex [15]. We may suppose moreover that  $e$  avoids the singularities of  $C$ . Thickening  $e$  slightly in  $C$  we get a disc with two holes  $E$ . Identify now a polynomially convex neighborhood  $V$  of  $e$  to  $E \times d$  where  $d$  is a small disc. Call  $\pi$  the projection of  $V$

on  $E$ . Recall that the current  $U$  comes from a sequence of discs  $(\Delta_n)$ . Shrinking  $d$  a bit we may suppose that  $\text{length}(\Delta_n \cap (E \times \partial d)) = o(a_n)$ . This uses the fact that  $U|_V$  does not charge  $V \setminus E$  and the coarea formula. Now, as  $V$  is polynomially convex,  $\Delta_n \cap V$  consists in a union of discs  $\delta_n$  by the maximum principle. By construction the boundaries of  $\delta_n$  sit mainly in  $\partial E \times d$ . We infer that  $\text{area}(\pi(\Delta_n \cap V)) = o(a_n)$  by applying the isoperimetric inequality to the maps  $\pi : \delta_n \rightarrow E$  and summing up. This contradicts the fact that  $U$  charges  $E$ .

**Remark.** Suppose we have an annulus  $A$  among the components of the analytic curve. Then the discs  $\Delta_n$  approximating  $U$  satisfy the following additional property: they cannot avoid (for large  $n$ ) a fixed analytic curve  $C$  in  $B$  meeting  $A$ . Indeed if not we could work out the previous argument in the complement of  $C$ , replacing everywhere the polynomial convexity by the convexity with respect to the algebra  $\mathcal{M}$  of meromorphic functions in  $B$  with poles on  $C$ . We would find a  $\mathcal{M}$ -convex figure eight in the punctured annulus  $A \setminus C$  and proceed as above to reach a contradiction.

We enter now the proof of the theorem.

## 2. The set up

Let  $T$  be an unknotted totally real torus  $T$  in  $S^3$ . It divides  $S^3$  into two solid tori  $\omega_i$  (diffeomorphic to  $S^1 \times D^2$ ) and its polynomial hull  $\hat{T}$  separates  $B$  into two pseudoconvex domains  $\Omega_i$  containing  $\omega_i$  in their closure (§1 b)).

**Lemma.** *For one of these domains the map  $H_1(\omega_i, \mathbb{Z}) \rightarrow H_1(\Omega_i, \mathbb{Z})$  is injective.*

*Proof.* If not, let  $\gamma_i$  be a generator of  $H_1(\omega_i, \mathbb{Z})$ . Note first that  $\gamma_1$  and  $\gamma_2$  are linked in  $S^3$ , and next that the linking number of two disjoint cycles in  $S^3$  can be computed as the intersection number of the chains they bound in  $B$ . Now by assumption  $n_i \gamma_i$  bounds a chain in  $\Omega_i$  for some integer  $n_i$ . But  $\Omega_1$  and  $\Omega_2$  being disjoint this shows that  $n_1 \gamma_1$  and  $n_2 \gamma_2$  (hence  $\gamma_1$  and  $\gamma_2$ ) are not linked in  $S^3$ . Contradiction.

Let us call simply  $\Omega$  this good side and  $\omega$  the corresponding solid torus. We push slightly  $T$  inside  $\omega$ , creating a sequence of tori  $T_n$  approximating  $T$ .

Consider the universal covering  $p: \tilde{\Omega} \rightarrow \Omega$ . Because  $\pi_1(\omega) \rightarrow \pi_1(\Omega)$  is injective, all the components of  $p^{-1}(\omega)$  are diffeomorphic to  $\mathbb{R} \times D^2$ . Fix one of them and call it  $\tilde{\omega}$ . Then  $T_n$  lifts to a cylinder  $\tilde{T}_n$  (diffeomorphic to  $\mathbb{R} \times S^1$ ) inside  $\tilde{\omega}$ . Let  $\tau$  be the automorphism of  $\tilde{\Omega}$  induced by the action of a generator of  $\pi_1(\omega)$ . It acts on  $\tilde{\omega}$  as a translation on the factor  $\mathbb{R}$  and  $\tilde{T}_n$  is invariant under this action.

Construct the sphere  $S_n$  approximating the cylinder  $\tilde{T}_n$  as follows. Pick a disc  $D$  in  $\omega$  (diffeomorphic to  $* \times D^2$ ). Its boundary is a *meridian* of  $T$ . Deform  $D$  slightly in  $D_n$  with boundary in  $T_n$ . Choose a lift  $\tilde{D}_n$  of  $D_n$  in  $\tilde{\omega}$ . The curves  $\tau^{\pm n}(\partial \tilde{D}_n)$

bound an annulus  $\tilde{A}_n$  in  $\tilde{T}_n$ . The sphere  $S_n$  is obtained by smoothing the sphere with corners  $\tau^{-n}(\tilde{D}_n) \cup \tilde{A}_n \cup \tau^n(\tilde{D}_n)$ . Note that the complex points of  $S_n$  can be made generic after a perturbation localized near the caps  $\tau^{\pm n}(\tilde{D}_n)$ . By construction  $S_n$  projects down to the interior of  $\omega$  where  $\Omega$  is strictly pseudoconvex. Hence the technics of §1 a) apply. Denote by  $\Sigma_n$  the filling of  $S_n$  in  $\tilde{\Omega}$ . Now the equator  $\partial\tilde{D}_n$  divides  $S_n$  into two half spheres  $S_n^\pm$ . At least one of them, say  $S_n^-$ , has a partial filling. This means that we may single out a disc  $\tilde{\Delta}_n$  of  $\Sigma_n$  touching the equator and whose boundary is entirely contained in  $S_n^-$ . Put  $\Delta_n = p(\tilde{\Delta}_n)$ .

The alternative reads as follows: either the area  $a_n$  of  $\Delta_n$  remains bounded or not. In the former case we will verify by Gromov compactness theorem that  $T$  is fillable. This is the *rationally convex case*. In the latter we will consider the Ahlfors current  $U = \lim \frac{[\Delta_n]}{a_n}$ . By construction its support will be in the rational hull of  $T$  and, after a detailed analysis, we will detect holomorphic annuli or discs in it. This is the *non rationally convex case*.

In any case we need to control the boundary of  $\Delta_n$ . We know by Hopf lemma that  $T\partial\Delta_n$  is transversal on  $T_n$  to the characteristic line field. Actually we have more. Denote by  $\omega_n$  the solid torus bounded by  $T_n$  in  $\omega$ . Perturb the ball  $B$  in a new strictly convex domain  $B_n$  by bumping slightly  $\omega_n$  out, keeping  $T_n$  still in  $\partial B_n$ . Note that by construction  $\partial\Delta_n \subset B_n$ , so  $\Delta_n \subset B_n$  by the maximum principle. But tilting the boundary of the domain along  $T_n$  translates in rotating the characteristic line field on  $T_n$ . We infer that  $T\partial\Delta_n$  avoids a full cone field on  $T_n$  bounded on one side by the original characteristic line field. As this can be done uniformly in  $n$ , we end up with  $T\phi_n(\partial\Delta_n \cap T_n)$  avoiding a cone field on  $T$ . Here  $\phi_n$  is a diffeomorphism between  $T_n$  and  $T$  close to identity. We may perturb slightly the characteristic line field of  $T$  to push it *inside* this cone field, still keeping its name. We summarize this discussion by saying that  $\gamma_n = \phi_n(\partial\Delta_n \cap T_n)$  is *uniformly transversal* to the characteristic foliation  $\mathcal{C}$  of  $T$ . This actually holds for any disc of  $\Sigma_n$ .

It follows that the length  $l_n$  of  $\gamma_n$  is controlled by  $a_n$ . For this construct a 1-form  $\beta$  on  $T$  whose kernel is the characteristic line field and extend it to  $\mathbb{C}^2$ . Then  $l_n \lesssim |\int_{\gamma_n} \beta| \lesssim |\int_{\Delta_n} d\beta| + |\int_{\partial\Delta_n \cap D_n} \beta|$  by the uniform transversality and Stokes theorem. Here  $\lesssim$  stands for an estimate up to a multiplicative constant. The first integral on the right is controlled by  $a_n$ . The second one is bounded. Indeed note that  $\partial\tilde{\Delta}_n$  bounds a disc  $\tilde{V}_n$  in  $S_n^-$ . Call  $V_n$  its projection downstairs. Then  $|\int_{\partial\Delta_n \cap D_n} \beta| \leq \int_{\partial D_n \cap V_n} |\beta| + \int_{D_n \cap V_n} |d\beta| \lesssim \text{length}(\partial D) + \text{area}(D)$  by Stokes theorem and the closeness of  $D_n$  and  $D$ . We end up with an estimate of the form  $l_n \leq C(1 + a_n)$ .

Conversely  $a_n$  is controlled by  $l_n$  in the same way. Indeed recall that  $a_n = \int_{\Delta_n} \omega$  where  $\omega$  is the standard Kähler form of  $\mathbb{C}^2$ . Write  $\lambda$  for a primitive of  $\omega$ . Then  $a_n = \int_{\partial\Delta_n} \lambda \lesssim l_n + \int_{\partial\Delta_n \cap D_n} \lambda$  by Stokes theorem and, as before, the last integral is bounded.

### The rationally convex case

In this case  $a_n$  remains bounded, and so is  $l_n$ .

We first check that  $T_n$  is fillable. By assumption  $\partial\tilde{\Delta}_n$  remains at bounded distance of the equator of  $S_n$ . This means that  $\tilde{\Delta}_n$  is attached to both  $S_n$  and  $\tau^{-1}(S_n)$ , hence belongs to their fillings (§1 a)). In other words both  $\tilde{\Delta}_n$  and  $\tau(\tilde{\Delta}_n)$  are part of  $\Sigma_n$ . The discs of  $\Sigma_n$  interpolating between them project down to the desired filling  $\Theta_n$  of  $T_n$ . Note that all the discs  $\Delta'_n$  of  $\Theta_n$  have bounded area. Indeed we have  $\int_{\Delta'_n} \omega \leq \int_{\Delta_n} \omega + \int_{T_n} |\omega| \lesssim a_n + \text{area}(T)$  by Stokes theorem.

We want now to prove that  $T$  is fillable as well. We rely on Gromov compactness theorem [10] (see also [11]). In our context it reads as follows: given a disc  $\Delta'_n$  in  $\Theta_n$ , then the sequence  $(\Delta'_n)$  converges (after extracting a subsequence) toward a finite bunch (with multiplicities) of holomorphic discs  $\Delta'$  attached to  $T$ . These discs do not present self-intersections or mutual intersections in the interior of  $B$ . This relies on two facts: intersections of distinct holomorphic curves persist under local deformation, and the convergence does not show accidents inside the ball. Actually an accident means an annulus component of  $\Delta'_n$  in a fixed small ball converging toward a pair of two discs (its modulus blows up). But all such local components are discs by the maximum principle. Moreover the discs  $\Delta'$ , if *simple*, are embedded inside  $B$  by a knot-theoretic argument [5]. We want to build the filling of  $T$  out of these limit discs. The problem is to exhibit sufficiently many such discs, embedded and disjoint in the closed ball. The difficulty takes place at their boundaries. We focus on them.

For a sequence  $(\Delta'_n)$  as above call  $\Gamma' = \cup\partial\Delta'$  the boundary of its limit. By Hopf lemma it is a finite union of immersed curves (with multiplicities). Denote by  $\text{Sing}(\Gamma')$  the set of multiple points of  $\Gamma'$ , i.e. its geometric singularities and its multiple components. Similarly put  $\Gamma$  for the boundary of the limit of the original sequence  $(\Delta_n)$  (after the same extraction). Our first observation is that  $\text{Sing}(\Gamma') \subset \Gamma$ . Indeed locally at least two strands of  $\partial\Delta'_n$  converge at a given point of  $\text{Sing}(\Gamma')$ : if  $\alpha$  is a short piece of the characteristic leaf through this point, it meets  $\phi_n(\partial\Delta'_n)$  at least twice. Here again  $\phi_n$  is a diffeomorphism between  $T_n$  and  $T$  close to identity. In other words  $\alpha$  runs from one boundary to the other in the cylinder obtained from  $T$  by cutting out  $\phi_n(\partial\Delta'_n)$ . As  $\phi_n(\partial\Delta_n)$  is parallel to these boundaries it always intersects  $\alpha$ , and so does  $\Gamma$ . Shrinking  $\alpha$  to the initial point concludes.

In particular at each point  $q \in \Gamma' \setminus \Gamma$  the convergence of  $(\Delta'_n)$  is *good*: there exists a unique simple disc  $\Delta'$  through  $q$  in the limit such that  $\Delta'_n$  converges toward  $\Delta'$  near  $q$ . Our second observation is that this disc does not really depend on  $(\Delta'_n)$ . If we consider another similar sequence  $(\Delta''_n)$  converging after the same extraction, such that  $q \in (\Gamma' \cap \Gamma'') \setminus \Gamma$ , then  $\Delta' = \Delta''$ . Indeed if not,  $\Delta'$  and  $\Delta''$  would be distinct. But intersections of distinct holomorphic discs attached to a totally real surface and on the same side of a strictly pseudoconvex boundary persist under local deformation. This can be seen by reflecting the discs through the surface to get (pseudo)holomorphic curves in a neighborhood of  $q$  and using the positivity of their intersections [16].

Therefore  $\Delta'_n$  and  $\Delta''_n$  would still intersect, contradicting their being part of the same filling  $\Theta_n$ .

According to our previous discussion we focus on  $T^* = T \setminus \Gamma$  where all the convergences are good. Pick a countable set  $Q$  dense in  $T^*$ . Denote by  $\Delta_{n,q}$  the disc of  $\Theta_n$  passing through  $\phi_n^{-1}(q)$  for  $q \in Q$ . By extracting once more we may suppose that all sequences  $(\Delta_{n,q})$  converge in Gromov sense. Hence there exists a unique simple disc  $\Delta_q$  through  $q$  in  $\lim_{n \rightarrow \infty} \Delta_{n,q}$ . We want to extend this construction to  $T^*$ .

Pick a point  $p$  in  $T^*$ . Then the component through  $p$  of  $\lim_{q \rightarrow p} \Delta_q$  (in Gromov sense) is well defined. Indeed any component through  $p$  in  $\lim_{q \rightarrow p} \Delta_q$  appears also as a limit of discs in  $\Theta_n$ : consider discs of the form  $\Delta_{n_k, q_k}$  for some sequence  $q_k$  going to  $p$  and  $n_k$  rapidly growing. Therefore by the observations above this component is unique and does not depend on any choice. We get a distribution of holomorphic discs  $\Delta_p$  ( $p \in T^*$ ) whose boundaries are embedded and disjoint (if distinct) in  $T^*$ . It turns out that the same holds in the whole  $T$ .

**Lemma.** *The curves  $\partial\Delta_p$  are embedded and disjoint (if distinct).*

*Proof.* We proceed by contradiction. Pick an intersection point  $s$  (necessarily in  $\Gamma$ ) of two different local branches  $\gamma', \gamma''$  of such curves. Note that  $\gamma' \cup \gamma''$  cuts out four components in  $T$  near  $s$ , two of which avoiding the characteristic leaf through  $s$ . Call  $C$  the union of these two components and put  $C^* = C \setminus \Gamma$ . Now for all  $p$  in  $C^*$  the curve  $\partial\Delta_p$  is canalized by  $\gamma'$  and  $\gamma''$  through  $s$ . Thus we get a whole family of holomorphic discs  $\Delta_p$  attached to  $T$  with a common point. On the other hand by the maximum principle these discs sit in  $\hat{T}$  and even in  $\partial\hat{T}$  as limits of discs in  $\Theta_n \subset \Omega$ . This will be the contradiction.

Let us make this precise. Recall first that we may associate to an immersed holomorphic disc  $\Delta$  attached to  $T$  an even integer, its *Maslov index*  $\mu(\Delta)$  (see [3] for background). This index is related to the dimension of the manifold of the holomorphic discs close to  $\Delta$  and attached to  $T$ . If  $\mu(\Delta) \leq 0$  this manifold is of dimension 0:  $\Delta$  does not have any deformation attached to  $T$ . If  $\mu(\Delta) > 0$  it is of positive dimension  $\mu(\Delta) - 1$ . Moreover if  $\mu(\Delta) = 2$  we get a small 1-parameter family of nearby *locally disjoint* discs attached to  $T$ . In particular they cannot pass through a common point. On the other hand if  $\mu(\Delta) > 2$  the (at least) 3-parameter family of nearby discs attached to  $T$  fills out a whole neighborhood of  $\Delta$  in  $B$ . This forbids  $\Delta$  to be in  $\partial\hat{T}$ . To conclude it remains to exhibit a genuine deformation among the family  $\Delta_p$  passing through  $s$ .

What we know already is that  $\Delta_p$  is the unique component through  $p$  in  $\lim_{q \rightarrow p} \Delta_q$  for  $p$  in  $C^*$ . We would like to really have  $\lim_{q \rightarrow p} \Delta_q = \Delta_p$ . This will be at least the case for  $\Delta_p$  big enough. For this recall that any holomorphic disc attached to  $T$  cannot be too small. This relies for instance on the existence of a basis of strictly pseudoconvex neighborhoods of  $T$ . Then according to Lelong theorem the area of

such a disc is bounded from below by some positive constant, say  $2\epsilon$ . Now pick  $p$  in  $C^*$  such that  $\text{area}(\Delta_p) \geq \sup_{C^*} \text{area}(\Delta_q) - \epsilon$ . As the area is preserved under the convergence in Gromov sense, we infer that there is no other component but  $\Delta_p$  in  $\lim_{q \rightarrow p} \Delta_q$ . This concludes.

At this stage we do have a whole smooth family of disjoint embedded holomorphic discs attached to  $T$  whose boundaries sweep out at least  $T^*$ . To achieve the filling it remains to close this family up on  $\Gamma$ . This goes along the same lines as before. The main point is that if  $p$  is in  $\Gamma$  then  $\lim_{q \rightarrow p} \Delta_q$  does not present singularities. If it did, as in the first observation above, all the discs  $\Delta_q$  would pass through this singular point, contradicting the lemma. We leave the details to the reader.

### The non rationally convex case

In this case  $a_n$  blows up. We want to prove that there exists a Riemann surface (holomorphic annulus or pair of holomorphic discs) attached to  $T$  and part of its rational hull. We look at the limit of  $\Delta_n$  in terms of currents. Consider  $\frac{[\Delta_n]}{a_n}$  the normalized current of integration on  $\Delta_n$ . We get a sequence of positive currents of mass 1 supported in the unit ball. Up to extracting it converges toward an Ahlfors current  $U$ . Recall that  $\partial\Delta_n = \partial V_n$  where  $V_n$  is the projection of the disc  $\tilde{V}_n$  bounded by  $\partial\tilde{\Delta}_n$  in  $S_n$ . Note that  $a_n$  is comparable to  $l_n$  (§2) and so to the maximal number of sheets of  $\tilde{V}_n$  over  $T_n$ . Hence  $\frac{[V_n]}{a_n}$  converges toward a current  $V$  supported on  $T$  such that  $dV = dU$ . Therefore  $\text{supp}(U) \subset r(T)$  (§1 b)). We have  $dU = \lim \frac{[\gamma_n]}{a_n}$  where  $\gamma_n = \phi_n(\partial\Delta_n \cap T_n)$  (§2). As  $a_n$  blows up we may even neglect parts of  $\gamma_n$  of bounded length in this limit. To exhibit Riemann surfaces in  $r(T)$  we further investigate the current  $U$ . We focus first on its boundary.

**a) Describing  $dU$ .** We will prove an integral formula of the form  $dU = \int_{\mathcal{G}} [\gamma] d\mu(\gamma)$ . Here  $\mathcal{G}$  is a compact space of Lipschitz curves in  $T$  and  $\mu$  a positive measure on it, supported on closed curves.

This requires an extra discussion of the characteristic foliation  $\mathcal{C}$ . By Denjoy theorem [9] any smooth foliation on  $T$  can be perturbed in order to get only a finite number of attracting or repulsive cycles (closed leaves). We may suppose that this holds true for  $\mathcal{C}$  as we already perturbed it (§2).

Call  $c$  such a characteristic cycle. Observe that the lifts of  $\phi_n^{-1}(c)$  cannot be closed in  $\tilde{T}_n$ . If it were the case such a lift would divide  $S_n$  out into two half spheres, one of which partially fillable. We would thus get a contact between this lift and the boundary of a disc of the filling  $\Sigma_n$ , contradicting the uniform transversality. Hence  $p^{-1}(\phi_n^{-1}(c))$  consists in finitely many periodic curves invariant by a power  $\tau^q$  of  $\tau$ .

It follows that the number of intersection points between  $\gamma_n$  and  $c$  is bounded. Indeed each lift of  $\phi_n^{-1}(c)$  cuts at most once  $\partial\tilde{\Delta}_n$  by transversality and because  $\partial\tilde{\Delta}_n$

separates  $S_n$ . Consider now thin tubes along the cycles in  $T$ . They divide  $T$  in a finite number of annuli. By uniform transversality  $\gamma_n$  cuts the tubes in a bounded number of short arcs. We may neglect them for the computation of  $dU$ . Hence the relevant part of  $\gamma_n$  consists in a bounded number of long arcs contained in the annuli.

The crucial observation is that these arcs are *embedded* (up to splitting them into two pieces). To prove this we further analyse the situation upstairs. Call  $B_n$  the ball bounded by the sphere  $S_n$  in  $\tilde{\omega}$  and  $\Omega_n$  the pseudoconvex domain bounded by  $B_n \cup \Sigma_n$  in  $\tilde{\Omega}$ . Then  $\tau^q(S_n^-) \subset \Omega_n$  for large  $n$ . This is where the choice of a half sphere enters. Hence its partial filling, as part of its envelope of holomorphy (§1 a)), must also be contained in  $\Omega_n$ . To be fully correct this argument requires to push slightly  $S_n$  off  $B_n$  in a new sphere  $S'_n \subset \tilde{\omega}$ , verify that the partial filling of  $\tau^q(S_n^-)$  is contained in the corresponding pseudoconvex domain  $\Omega'_n$  and deform back  $S'_n$  to  $S_n$ . In particular we get that  $\tau^q(\tilde{\Delta}_n) \subset \Omega_n$ . Hence  $\tau^q(\tilde{\Delta}_n)$  remains always on the same side of  $\Sigma_n$ , meaning that  $\tau^q(\partial\tilde{\Delta}_n)$  crosses  $\partial\tilde{\Delta}_n$  always in the same direction (say entering  $\tilde{V}_n$ ). Look now at a given lift of  $\phi_n^{-1}(A)$  in  $\tilde{T}_n$  where  $A$  is one of the aforementioned annuli. This is a strip invariant by  $\tau^q$ . It can be parametrized by  $\mathbb{R} \times [0, 1]$  via a diffeomorphism sending the vertical foliation to the characteristic one,  $\tau^q$  corresponding to the translation by 1. By transversality any component of  $\partial\tilde{\Delta}_n$  in the strip is a graph (via the diffeomorphism) with, say,  $\tilde{V}_n$  above it. Thus the component and its image by  $\tau^q$  intersect at most once as the latter crosses the former always bottom up. This allows us to cut the component into two pieces, each of them disjoint from its image by  $\tau^q$ . Therefore these pieces project down to embedded arcs.

According to this discussion  $dU$  is a finite sum of currents of the form  $\lim \frac{[\alpha_n]}{a_n}$ , where  $\alpha_n$  is an embedded arc sitting in an annulus  $A$ . We are now in position to prove the integral formula for each such limit. Via the parametrization of the corresponding strip and thanks to the uniform transversality,  $\alpha_n$  splits up into a union of graphs of functions from  $[0, 1]$  to  $[0, 1]$  which are uniformly Lipschitz. Denote by  $\mathcal{G}$  the compact space of graphs  $\gamma$  of functions  $g: [0, 1] \rightarrow [0, 1]$  such that  $\text{Lip}(g) \leq C$  (for some large  $C$ ). We have  $\frac{[\alpha_n]}{a_n} = \int_{\mathcal{G}} [\gamma] d\mu_n(\gamma)$  where  $\mu_n$  is a positive measure with finite support and bounded mass on  $\mathcal{G}$ . Up to extracting  $\mu_n$  converges toward a positive measure  $\mu$  on  $\mathcal{G}$ . We infer that  $\lim \frac{[\alpha_n]}{a_n} = \int_{\mathcal{G}} [\gamma] d\mu(\gamma)$ . Moreover the support of  $\mu$  consists in closed curves (graphs of functions  $g$  such that  $g(0) = g(1)$ ). Indeed if the graph of  $g$  is in  $\text{supp}(\mu)$  then it is certainly the limit of at least two successive graphs (of say  $g_n$  and  $h_n$ ) of  $\alpha_n$  ( $a_n$  blows up). As  $g_n(1) = h_n(0)$  we get  $g(0) = g(1)$  in the limit.

**b) Describing  $U$ .** We will prove now an integral formula of the form  $U = \int_{\mathcal{P}} W d\nu(W)$ . Here  $\mathcal{P}$  is the compact space of positive currents of mass 1 supported in the unit ball and  $\nu$  is a probability measure on it. The point is that  $\text{supp}(\nu)$  consists only in normalized currents of integration on holomorphic discs or

annuli attached to  $T$  (or finite sums of them). This formula comes from a division process.

We show first that  $U$  can be split up into a sum of four positive currents  $W$  of mass at most  $\frac{1}{2}$ . These currents will be proportional to Ahlfors currents limit of pieces of  $\Delta_n$ . Precisely  $W = \lim \frac{[\delta_n]}{a_n}$  where  $\delta_n \subset \Delta_n$ ,  $\text{area}(\delta_n) \leq \frac{a_n}{2}$  and  $\text{length}(\partial\delta_n \setminus \partial\Delta_n) = o(a_n)$ . In addition we want  $\partial\delta_n \cap \partial\Delta_n$  connected.

For this, parametrize  $\Delta_n$  by the unit disc via a holomorphic map  $f_n: D \rightarrow B$  such that the images by  $f_n$  of the four half discs cut out in  $D$  by  $\mathbb{R}$  or  $i\mathbb{R}$  have the same area  $\frac{a_n}{2}$ . Denote by  $X$  the cross  $(\mathbb{R} \cap D) \cup (i\mathbb{R} \cap D)$ . According to the next lemma, we may pick a generic angle  $\theta$  close to  $\frac{\pi}{4}$  such that  $\text{length}(f_n(e^{i\theta}X)) = o(a_n)$ . The rotated cross  $e^{i\theta}X$  divides  $D$  out in four quarter discs  $d$ . Put  $\delta_n = f_n(d)$  and  $W = \lim \frac{[\delta_n]}{a_n}$ . The currents  $W$  have all the desired properties. Here is the precise statement we used.

**Lemma.** *Let  $f_n: D \rightarrow B$  be a sequence of holomorphic discs (piecewise) smooth up to  $\partial D$ . Put  $a_n = \text{area}(f_n(D))$ ,  $l_n(\theta) = \text{length}(f_n([0, e^{i\theta}]))$  and suppose that  $a_n$  blows up. Then  $l_n(\theta) = o(a_n)$  for almost all  $\theta$  (up to extracting a subsequence).*

*Proof.* We have  $l_n(\theta) = \int_0^1 \|f'_n(re^{i\theta})\| dr \leq l + \int_{1/2}^1 \|f'_n(re^{i\theta})\| dr$  for some constant  $l$  as  $\|f'_n\|$  is uniformly bounded in the disc of radius  $\frac{1}{2}$ . On the other hand, let  $a_n(\theta)$  be the area of the image by  $f_n$  of the sector between  $[0, 1]$  and  $[0, e^{i\theta}]$ . Then  $\frac{da_n}{d\theta}(\theta) = \int_0^1 \|f'_n(re^{i\theta})\|^2 r dr$ . By Cauchy–Schwarz inequality  $(l_n(\theta))^2 \leq 2l^2 + 2 \ln(2) \frac{da_n}{d\theta}(\theta)$ . Integrating, we get  $\int_0^{2\pi} (l_n(\theta))^2 d\theta \leq 4\pi l^2 + 2 \ln(2) a_n$ , so  $\lim \int_0^{2\pi} \left(\frac{l_n(\theta)}{a_n}\right)^2 d\theta = 0$ . By Fatou's lemma  $\int_0^{2\pi} \liminf \left(\frac{l_n(\theta)}{a_n}\right)^2 d\theta = 0$ , which concludes.

Iterating this process we may write  $U$  as a sum of  $4^k$  positive currents of mass at most  $2^{-k}$  proportional to Ahlfors currents coming from  $(\Delta_n)$ . Hence  $U = \int_{\mathcal{P}} W d\nu_k(W)$  where  $\nu_k$  is a probability measure supported on these Ahlfors currents. By compactness of  $\mathcal{P}$  we may suppose that  $(\nu_k)$  converges toward a probability measure  $\nu$  on  $\mathcal{P}$  and we get our integral formula  $U = \int_{\mathcal{P}} W d\nu(W)$ . Take now a current  $W$  in the support of  $\nu$ . By construction  $W$  is an Ahlfors current as a limit of Ahlfors currents. We will see below that  $dW$  is supported on a curve  $\gamma \subset T$  of finite length (with finitely many components). Hence  $\text{supp}(W) \subset \hat{\gamma}$  which by Alexander theorem (§1 b) is a Riemann surface. By §1 c) we conclude that  $W$  is actually supported in a finite union of holomorphic discs or annuli attached to  $T$ .

Let us describe  $dW$ . By construction  $W = \lim \frac{W_k}{\text{mass}(W_k)}$  where  $W_k = \lim \frac{[\delta_{n,k}]}{a_n}$  with  $\delta_{n,k} \subset \Delta_n$ ,  $\text{area}(\delta_{n,k}) \leq 2^{-k} a_n$ ,  $\text{length}(\partial\delta_{n,k} \setminus \partial\Delta_n) = o(a_n)$  and  $\partial\delta_{n,k} \cap \partial\Delta_n$  connected. We use the notations of the previous paragraph. Recall that we had singled out an annulus  $A$  outside thin tubes of the characteristic cycles in  $T$  and an arc  $\alpha_n$  of  $\gamma_n$  embedded in  $A$ . So  $\partial\delta_{n,k} \cap \partial\Delta_n$  gives rise to a subarc  $\alpha_{n,k}$  of  $\alpha_n$ . We check

now that  $\lim \frac{[\alpha_{n,k}]}{a_n}$  is supported in a set converging to a curve of finite length (with at most two components). This will conclude as  $dW$  is a finite sum of such limits.

Indeed  $\alpha_{n,k}$  is built out of graphs in  $\mathcal{G}$  and we have  $\lim \frac{[\alpha_{n,k}]}{a_n} = \int_{\mathcal{G}} [\gamma] d\mu_k(\gamma)$  for a positive measure  $\mu_k \leq \mu$  on  $\mathcal{G}$ . Note that we have a partial order on  $\mathcal{G}$  given by  $\alpha \leq \beta$  if the corresponding functions satisfy  $a \leq b$ . We may speak of intervals  $[\alpha, \beta]$  or  $] \alpha, \beta[ = [\alpha, \beta] \setminus \{\alpha, \beta\}$ . This order is total on the graphs appearing in  $\alpha_{n,k}$  ( $\alpha_n$  is embedded). Denote by  $\eta_{n,k}$  and  $\lambda_{n,k}$  the lowest and the highest of these graphs. By compactness of  $\mathcal{G}$  we may suppose that  $\eta_{n,k}, \lambda_{n,k}$  converge to  $\eta_k, \lambda_k$ , and that  $\eta_k, \lambda_k$  converge to  $\eta, \lambda$ . By construction  $\text{supp}(\mu_k) \subset ]\eta_k, \lambda_k[$  and, moreover,  $\mu_k = \mu$  on  $] \eta_k, \lambda_k[$ . As the mass of  $\mu_k$  goes to 0, it follows that  $\mu$  does not charge  $] \eta, \lambda[$ . Hence  $\text{supp}(\mu_k) \subset ]\eta_k, \lambda_k[ \setminus ] \eta, \lambda[$  which goes to  $\{\eta, \lambda\}$ . This concludes.

**c) End of the argument.** At this stage we do have compact Riemann surfaces (holomorphic discs or annuli) attached to  $T$  and contained in  $r(T)$ . We want more. We are looking for a compact Riemann surface  $C$  (holomorphic annulus or a pair of holomorphic discs) such that  $\partial C$  bounds in  $T$ . Here is how we proceed.

Choose a common orientation of the characteristic cycles. Note that the boundaries of our Riemann surfaces are parallel to these cycles. They also inherit a natural orientation from the Riemann surface. We speak of a *positive* boundary if the two orientations agree, or *negative* if not. Call positive (negative) an annulus or a disc with only positive (negative) boundaries, and *opposite* an annulus or a pair of discs with opposite boundaries. We are looking for an opposite annulus or a pair of opposite discs among our Riemann surfaces. Suppose we do not have any.

Recall that  $dU$  bounds in  $T$ . This implies that our Riemann surfaces cannot be all positive, or all negative. We have three possibilities left: either the presence among them of a positive annulus and a negative annulus, or of a positive annulus and a negative disc, or the converse. By symmetry we may suppose that we have a positive annulus  $A^+$  and a negative Riemann surface (annulus or disc)  $C^-$ . Observe now that two disjoint closed curves in  $T$  parallel to the characteristic cycles are necessarily linked in  $S^3$ . This can be checked for any pair of disjoint curves in the standard torus, as soon as they are not meridians (i.e. do not bound a disc in the complement of the standard torus).

Hence the boundaries of  $A^+$  and  $C^-$  are linked. This implies that  $A^+$  and  $C^-$  intersect inside the unit ball. But by construction  $A^+$  is contained in the support of an Ahlfors current coming from  $(\Delta_n)$ . As  $A^+$  intersects  $C^-$ , before the limit  $\Delta_n$  would have to intersect  $C^-$  (§1 c)). This is impossible as  $C^- \subset \hat{T}$  and  $\Delta_n \subset \Omega$ .

## References

- [1] H. Alexander, Gromov's method and Bennequin's problem. *Invent. Math.* **125** (1996), 135–148. [Zbl 0853.32003](#) [MR 1389963](#)

- [2] H. Alexander, Disks with boundaries in totally real and Lagrangian manifolds. *Duke Math. J.* **100** (1999), 131–138. [Zbl 0953.32026](#) [MR 1714757](#)
- [3] M. Audin and J. Lafontaine (ed.), *Holomorphic curves in symplectic geometry* Prog. Math. 117, Birkhäuser, Basel 1994. [Zbl 0802.53001](#) [MR 1274924](#)
- [4] E. Bedford, Stability of the polynomial hull of  $T^2$ . *Ann. Scuola Norm. Sup. Pisa* **8** (1981), 311–315 [Zbl 0472.32012](#) [MR 0623939](#)
- [5] E. Bedford and W. Klingenberg, On the envelope of holomorphy of a 2-sphere in  $\mathbb{C}^2$ . *J. Amer. Math. Soc.* **4** (1991), 623–646. [Zbl 0736.32009](#) [MR 1094437](#)
- [6] J. Duval, Singularités des courants d’Ahlfors. *Ann. Sci. Ecole Norm. Sup.* **39** (2006), 527–533. [Zbl 1243.32012](#) [MR 2265678](#)
- [7] J. Duval and N. Sibony, Polynomial convexity, rational convexity, and currents. *Duke Math. J.* **79** (1995), 487–513. [Zbl 0838.32006](#) [MR 1344768](#)
- [8] Y. Eliashberg, Filling by holomorphic discs and its applications. In *Geometry of low dimensional manifolds*, London Math. Soc. Lecture Note Ser. 151, Cambridge University Press, Cambridge 1990, 45–67. [Zbl 0731.53036](#) [MR 1171908](#)
- [9] C. Godbillon, *Dynamical systems on surfaces*. Universitext, Springer, Berlin 1983. [Zbl 0502.58002](#) [MR 0681119](#)
- [10] M. Gromov, Pseudoholomorphic curves in symplectic manifolds. *Invent. Math.* **82** (1985), 307–347. [Zbl 0592.53025](#) [MR 0809718](#)
- [11] S. Ivashkovich and V. Shevchishin, Reflection principle and  $J$ -complex curves with boundary on totally real immersions. *Commun. Contemp. Math.* **14** (2002), 65–106. [Zbl 1025.32024](#) [MR 1890078](#)
- [12] N. G. Kruzhilin, Two-dimensional spheres on the boundaries of pseudoconvex domains in  $\mathbb{C}^2$ . *Math. USSR-Izv.* **39** (1992), 1151–1187. [Zbl 0778.32003](#) [MR 1152210](#)
- [13] N. G. Kruzhilin, Holomorphic disks with boundaries in totally real tori in  $\mathbb{C}^2$ . *Math. Notes* **56** (1994), 1244–1248. [Zbl 0842.32011](#) [MR 1330599](#)
- [14] R. Nevanlinna, *Analytic functions*. Grundlehren Math. Wiss. 162, Springer, Berlin 1970. [Zbl 0199.12501](#) [MR 0279280](#)
- [15] E. L. Stout, *Polynomial convexity*. Prog. Math. 261, Birkhäuser, Boston 2007. [Zbl 1124.32007](#) [MR 2305474](#)
- [16] R. Ye, Filling by holomorphic curves in symplectic 4-manifolds. *Trans. Amer. Math. Soc.* **350** (1998), 213–250. [Zbl 0936.53047](#) [MR 1422913](#)

Received November 18, 2011

Julien Duval, Laboratoire de Mathématiques, Université Paris-Sud, 91405 Orsay cedex, France

E-mail: [julien.duval@math.u-psud.fr](mailto:julien.duval@math.u-psud.fr)

Damien Gayet, Institut Camille Jordan, Université Claude Bernard, 69622 Villeurbanne cedex, France

E-mail: [gayet@math.univ-lyon1.fr](mailto:gayet@math.univ-lyon1.fr)