# Global stability of vortex solutions of the two-dimensional Navier-Stokes equation

Thierry Gallay
Institut Fourier
Université de Grenoble I
BP 74
38402 Saint-Martin d'Hères
France

C. Eugene Wayne
Department of Mathematics
and Center for BioDynamics
Boston University
111 Cummington St.
Boston, MA 02215, USA

June 15, 2004

**Abstract**

Both experimental and numerical studies of fluid motion indicate that initially localized regions of vorticity tend to evolve into isolated vortices and that these vortices then serve as organizing centers for the flow. In this paper we prove that in two dimensions localized regions of vorticity do evolve toward a vortex. More precisely we prove that any solution of the two-dimensional Navier-Stokes equation whose initial vorticity distribution is integrable converges to an explicit self-similar solution called "Oseen's vortex". This implies that the Oseen vortices are dynamically stable for all values of the circulation Reynolds number, and our approach also shows that these vortices are the only solutions of the two-dimensional Navier-Stokes equation with a Dirac mass as initial vorticity. Finally, under slightly stronger assumptions on the vorticity distribution, we give precise estimates on the rate of convergence toward the vortex.

## 1 Introduction

In this paper we consider the motion of an incompressible, viscous fluid in two-dimensional Euclidean space. The velocity of such a fluid is described by the Navier-Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \Delta\mathbf{u} - \nabla p \ , \quad \nabla\cdot\mathbf{u} = 0 \ , \tag{1}$$

where $\mathbf{u} = \mathbf{u}(x,t) \in \mathbf{R}^2$ is the velocity field, $p = p(x,t) \in \mathbf{R}$ is the pressure field, and $x \in \mathbf{R}^2$, $t \geq 0$. For simplicity, the kinematic viscosity has been rescaled to 1.

We prove two basic results about the solutions of (1). First we show that for any initial velocity field whose vorticity is integrable, the solution of (1) with this initial velocity approaches an Oseen vortex, an explicit solution of (1) exhibited below. As we also show, the Oseen vortex is in fact the unique solution of (1) with a Dirac mass as initial vorticity. We then examine in more detail the approach toward the vortex by studying the spectrum of the linearized equation around the vortex solution. If we assume that the initial vorticity distribution lies in a weighted $L^2$ space, we can derive estimates on the spectrum of this linearized operator which allow us to prove optimal bounds on the rate of convergence toward the vortex.

We now describe our results in more detail. As we have argued in [14] and [15] it is often easier to understand the asymptotics of solutions of (1) by studying the evolution of the vorticity, rather than the velocity. This is especially true in two-dimensions where the vorticity is a scalar. Taking the curl of (1) we find that the vorticity $\omega = \partial_1 u_2 - \partial_2 u_1$ satisfies:

$$\frac{\partial \omega}{\partial t} + (\mathbf{u} \cdot \nabla)\omega = \Delta\omega \ , \quad x \in \mathbf{R}^2 \ , \quad t \geq 0 \ . \tag{2}$$

The velocity field $\mathbf{u}$ is defined in terms of the vorticity via the Biot-Savart law

$$\mathbf{u}(x) = \frac{1}{2\pi}\int_{\mathbf{R}^2} \frac{(\mathbf{x}-\mathbf{y})^\perp}{|x-y|^2}\omega(y)\,\mathrm{d}y \ , \quad x \in \mathbf{R}^2 \ . \tag{3}$$

Here and in the sequel, if $x = (x_1, x_2) \in \mathbf{R}^2$, we denote $\mathbf{x} = (x_1, x_2)^{\mathrm{T}}$ and $\mathbf{x}^\perp = (-x_2, x_1)^{\mathrm{T}}$.

The vorticity equation is globally well-posed in the space $L^1(\mathbf{R}^2)$. In particular, the results of Ben-Artzi [2], Brezis [7] and Kato [21] imply that:

**Theorem 1.1** *For all initial data $\omega_0 \in L^1(\mathbf{R}^2)$, equation (2) has a unique global solution $\omega \in C^0([0,\infty), L^1(\mathbf{R}^2)) \cap C^0((0,\infty), L^\infty(\mathbf{R}^2))$ such that $\omega(0) = \omega_0$. Moreover, for all $p \in [1, +\infty]$, there exists $C_p > 0$ such that*

$$|\omega(\cdot, t)|_p \leq \frac{C_p|\omega_0|_1}{t^{1-\frac{1}{p}}} \ , \quad t > 0 \ . \tag{4}$$

Here and in the remainder of the paper $|\cdot|_p$ denotes the norm on $L^p(\mathbf{R}^2)$. If $\mathbf{u} \in L^q(\mathbf{R}^2)^2$, we set $|u|_q = |\,|\mathbf{u}|\,|_q$, where $|\mathbf{u}| = (u_1^2 + u_2^2)^{1/2}$.

Among its other properties the semi-flow defined by (2) in $L^1(\mathbf{R}^2)$ preserves mass, i.e.

$$\int_{\mathbf{R}^2} \omega(x,t)\,\mathrm{d}x = \int_{\mathbf{R}^2} \omega_0(x)\,\mathrm{d}x \ , \quad t \geq 0 \ . \tag{5}$$

Furthermore, if the solution is sufficiently spatially localized so that the first moments of the vorticity distribution are finite then these are also preserved:

$$\int_{\mathbf{R}^2} x_j\omega(x,t)\,\mathrm{d}x = \int_{\mathbf{R}^2} x_j\omega_0(x)\,\mathrm{d}x \ , \quad t \geq 0 \ , \quad j = 1, 2 \ . \tag{6}$$

It is important to realize that the solutions of (2) given by Theorem 1.1 correspond to *infinite energy* solutions of the Navier-Stokes equations (1). More precisely, if $\omega(x,t)$ is a solution of (2) such that $\int \omega(x,t)\,\mathrm{d}x \neq 0$, then the velocity field $\mathbf{u}(x,t)$ given by (3) satisfies $|\mathbf{u}(\cdot,t)|_2 = \infty$ for all $t$. Explicit examples of such infinite energy solutions are the so-called *Oseen vortices*:

$$\omega(x,t) = \frac{\alpha}{4\pi t}\, e^{-|x|^2/(4t)} \ , \quad \mathbf{u}(x,t) = \frac{\alpha}{2\pi}\frac{\mathbf{x}^\perp}{|x|^2}\Big(1 - e^{-|x|^2/(4t)}\Big) \ , \tag{7}$$

where $|x|^2 = x_1^2 + x_2^2$ and $\alpha \in \mathbf{R}$ is a parameter which is often referred to as the "circulation Reynolds number". These solutions are "trivial" in the sense that $\mathbf{u}(x,t) \cdot \nabla\omega(x,t) \equiv 0$, so that (2) reduces to the linear heat equation. However, they play a prominent role in the long-time asymptotics of (2). Indeed, let

$$G(\xi) = \frac{1}{4\pi}\, e^{-|\xi|^2/4} \;, \quad \mathbf{v}^G(\xi) = \frac{1}{2\pi}\frac{\boldsymbol{\xi}^\perp}{|\xi|^2}\Big(1 - e^{-|\xi|^2/4}\Big) \;, \quad \xi \in \mathbf{R}^2 \;. \tag{8}$$

The following is the main result of this paper:

**Theorem 1.2** *If $\omega_0 \in L^1(\mathbf{R}^2)$, the solution $\omega(x,t)$ of (2) satisfies*

$$\lim_{t\to\infty} t^{1-\frac{1}{p}} \Big|\omega(\cdot,t) - \frac{\alpha}{t} G(\frac{\cdot}{\sqrt{t}})\Big|_p = 0 \;, \quad \text{for} \quad 1 \le p \le \infty \;, \tag{9}$$

*where $\alpha = \int_{\mathbf{R}^2} \omega_0(x)\,\mathrm{d}x$. If $\mathbf{u}(x,t)$ is the solution of (1) obtained from $\omega(x,t)$ via the Biot-Savart law (3), then*

$$\lim_{t\to\infty} t^{\frac{1}{2}-\frac{1}{q}} \Big|\mathbf{u}(\cdot,t) - \frac{\alpha}{\sqrt{t}}\mathbf{v}^G(\frac{\cdot}{\sqrt{t}})\Big|_q = 0 \;, \quad \text{for} \quad 2 < q \le \infty \;. \tag{10}$$

In other words, the solutions of (2) in $L^1(\mathbf{R}^2)$ behave asymptotically as the solutions of the linear heat equation $\partial_t\omega = \Delta\omega$ with the same initial data. For small solutions, this result has been obtained by Giga and Kambe in [17], see also [14]. As was observed by Carpio [10] (see also [16]), there is a deep connection between the asymptotics (9) and the uniqueness of the fundamental solution of the vorticity equation. More precisely, Carpio proved that (9) holds provided the Oseen vortex (7) is the unique solution of (2) with initial data $\alpha\delta$, where $\delta$ is Dirac's measure. As is shown in [12], [18], [21], this is true at least if $|\alpha|$ is sufficiently small. In this paper, we use a different method which allows us to obtain (9) without any restriction on $\alpha$. As a by-product of our analysis, we prove the uniqueness of the solution of (2) with a (large) Dirac mass as initial condition:

**Proposition 1.3** *Let $\omega \in C^0((0,T), L^1(\mathbf{R}^2) \cap L^\infty(\mathbf{R}^2))$ be a solution of (2) which is bounded in $L^1(\mathbf{R}^2)$, and assume that $\omega(\cdot,t)$ (considered as a finite Radon measure on $\mathbf{R}^2$) converges weakly to $\alpha\delta$ as $t \to 0+$, for some $\alpha \in \mathbf{R}$. Then*

$$\omega(x,t) = \frac{\alpha}{t}\, G\Big(\frac{x}{\sqrt{t}}\Big) \;, \quad x \in \mathbf{R}^2 \;, \quad 0 < t < T \;.$$

Theorem 1.2 has a number of important consequences. Recalling that $\alpha$ can be thought of as the Reynolds number of the flow, Theorem 1.2 says in more physical terms that the Oseen vortices are globally stable for any value of this number. In contrast to many situations in hydrodynamics, such as the Poiseuille or the Taylor-Couette flows, increasing the Reynolds number does not produce any instability. From another point of view, our result is compatible with the "inverse cascade" of energy in two-dimensional turbulence theory. In contrast to the situation in three dimensions where energy injected into the system at large scales flows to smaller and smaller scales until it is dissipated by viscosity, in two dimensions both experimental and numerical results indicate that even for very turbulent, high Reynolds number flows, there is a tendency for smaller vortices to coalesce and form larger and larger vortices. In this context, Theorem 1.2 says that in the whole space $\mathbf{R}^2$ this process continues until only a single vortex remains.

Another consequence of this result is that the Oseen vortices are the only self-similar solutions of the Navier-Stokes equations in $\mathbf{R}^2$ such that the vorticity field is integrable, see [28] for a related result. For completeness, we mention that these equations have many other self-similar solutions with nonintegrable vorticities. Indeed, adapting the results of [8] to the two-dimensional case, it is easy to verify that the Cauchy problem for (1) is globally well-posed for small data in a Besov space which contains homogeneous functions of degree $-1$. For such initial data, the velocity $\mathbf{u}(x,t)$ and the vorticity $\omega(x,t)$ are automatically self-similar, due to scaling invariance. For instance, given any continuous function $\varphi : S^1 \to \mathbf{R}$ with zero mean, there exists $\varepsilon > 0$ such that (1) has a self-similar solution $\mathbf{u}(x,t)$ with initial data $\mathbf{u}_0(x) = \varepsilon \mathbf{x}|x|^{-2}\varphi(x/|x|)$. If $\varphi$ is nonzero, the associated vorticity $\omega(x,t)$ decays like $1/|x|^2$ as $|x| \to \infty$ (at least in some directions), so that $\omega(\cdot,t) \notin L^1(\mathbf{R}^2)$.

To prove Theorem 1.2, our strategy is to study a rescaled version of (2) which is suggested by the form of the vortex solution (7). Thus, we introduce the "scaling variables" or "similarity variables":

$$\xi = \frac{x}{\sqrt{t}}\ , \qquad \tau = \log t\ .$$

If $\omega(x,t)$ is a solution of (2) and if $\mathbf{u}(x,t)$ is the corresponding velocity field, we define new functions $w(\xi,\tau)$, $\mathbf{v}(\xi,\tau)$ by

$$\omega(x,t) = \frac{1}{t}\, w\Big(\frac{x}{\sqrt{t}}, \log t\Big)\ , \quad \mathbf{u}(x,t) = \frac{1}{\sqrt{t}}\, \mathbf{v}\Big(\frac{x}{\sqrt{t}}, \log t\Big)\ . \tag{11}$$

Then $w(\xi,\tau)$ satisfies the equation

$$\partial_\tau w + (\mathbf{v}\cdot\nabla_\xi) w = \mathcal{L} w\ , \tag{12}$$

where

$$\mathcal{L} w = \Delta_\xi w + \frac{1}{2}(\boldsymbol{\xi}\cdot\nabla_\xi) w + w\ . \tag{13}$$

The rescaled velocity $\mathbf{v}$ is reconstructed from the rescaled vorticity $w$ by the Biot-Savart law:

$$\mathbf{v}(\xi) = \frac{1}{2\pi}\int_{\mathbf{R}^2} \frac{(\boldsymbol{\xi}-\boldsymbol{\eta})^\perp}{|\xi-\eta|^2}\, w(\eta)\,\mathrm{d}\eta\ , \quad \xi\in\mathbf{R}^2\ . \tag{14}$$

From (11) and Theorem 1.1, it is clear that the Cauchy problem for (12) is globally well-posed in $L^1(\mathbf{R}^2)$. Remark that $w(\cdot,0) = \omega(\cdot,1)$, hence imposing initial conditions to (12) at time $\tau = 0$ corresponds to imposing initial conditions to (2) at time $t = 1$. This is of course harmless since (2) is autonomous. Observe also that the Oseen vortices $\{\alpha G\}_{\alpha\in\mathbf{R}}$ are by construction a family of equilibria of (12).

To prove Theorem 1.2, we first study the long-time asymptotics of solutions whose vorticity distribution is more strongly localized than is necessary just to be in $L^1$. For any $m \ge 0$, we introduce the weighted Lebesgue space $L^2(m)$ defined by

$$\begin{aligned} L^2(m) &= \{f \in L^2(\mathbf{R}^2)\,|\, \|f\|_m < \infty\}\ , \quad \text{where} \\ \|f\|_m &= \Big(\int_{\mathbf{R}^2} (1+|\xi|^2)^m |f(\xi)|^2\,\mathrm{d}\xi\Big)^{1/2} = |b^m f|_2\ . \end{aligned}$$

Here and in the sequel, we denote $b(\xi) = (1+|\xi|^2)^{1/2}$. If $m > 1$, then $L^2(m) \hookrightarrow L^1(\mathbf{R}^2)$. In this case, we denote by $L^2_0(m)$ the closed subspace of $L^2(m)$ given by

$$L^2_0(m) = \left\{ w \in L^2(m) \,\Big|\, \int_{\mathbf{R}^2} w(\xi)\,\mathrm{d}\xi = 0 \right\} . \tag{15}$$

For $\mathbf{v} \in (L^2(m))^2$, we set $\|\mathbf{v}\|_m = \|\,|\mathbf{v}|\,\|_m$, where $|\mathbf{v}| = (v_1^2 + v_2^2)^{1/2}$.

As we observed in [14], a crucial advantage of using the vorticity formulation of the Navier-Stokes equations is that the spatial decay of the vorticity field is preserved under the evolution of (2). This remark is especially useful if one is interested in the long-time asymptotics since for parabolic equations there is a close relationship between the spatial and temporal decay of the solutions. The following result shows that the Cauchy problem for (12) is globally well-posed in the weighted space $L^2(m)$ if $m > 1$.

**Theorem 1.4 ([14], Theorem 3.2)** *Suppose that $w_0 \in L^2(m)$ for some $m > 1$. Then (12) has a unique global solution $w \in C^0([0,\infty), L^2(m))$ with $w(0) = w_0$, and there exists $C_1 = C_1(\|w_0\|_m) > 0$ such that*

$$\|w(\tau)\|_m \le C_1 \ , \quad \tau \ge 0 \ . \tag{16}$$

*Moreover, $C_1(\|w_0\|_m) \to 0$ as $\|w_0\|_m \to 0$. Finally, if $w_0 \in L^2_0(m)$, then $\int_{\mathbf{R}^2} w(\xi,\tau)\,\mathrm{d}\xi = 0$ for all $\tau \ge 0$, and $\lim_{\tau\to\infty} \|w(\tau)\|_m = 0$.*

Note that in contrast to this result the semi-flow defined by the Navier-Stokes equation does not preserve the spatial localization of the velocity field [14]. For instance, if the initial velocity $\mathbf{u}_0$ lies in $L^2(m)$ for some $m > 2$, then in general the solution $\mathbf{u}(\cdot,t)$ of (1) with initial data $\mathbf{u}_0$ will not be in $L^2(m)$ for $t > 0$. For a detailed study of the localization properties of solutions of the Navier-Stokes and vorticity equations, we refer to the recent work of Brandolese [5], [4], [6].

Our proof of Theorem 1.2 begins with a proof that the Oseen vortices attract all solutions of (12) with initial data in $L^2(m)$.

**Proposition 1.5** *Let $m > 1$, $w_0 \in L^2(m)$, and let $w \in C^0([0,+\infty), L^2(m))$ be the solution of (12) with initial data $w_0$. Then*

$$\|w(\tau) - \alpha G\|_m \to 0 \quad \textit{as } \tau \to +\infty \ ,$$

*where $\alpha = \int_{\mathbf{R}^2} w_0(\xi)\,\mathrm{d}\xi$.*

Thus, any solution of the vorticity equation which is sufficiently localized to be in $L^2(m)$ for $m > 1$ will converge, as time tends to infinity, toward one of the Oseen vortices, regardless of how large the Reynolds number is. As is shown in Section 3, this global convergence result can then be extended to all solutions in $L^1(\mathbf{R}^2)$. Returning to the original variables, we thus obtain Theorem 1.2 as a corollary.

The proof of Proposition 1.5 is based on the existence of a pair of Lyapunov functions for the rescaled vorticity equation (12). The first Lyapunov function, which is just the $L^1$ norm, is nonincreasing due to the maximum principle. It implies that the $\omega$-limit set of any solution must lie in the subset of solutions which are either everywhere positive or everywhere negative.

On the subset of positive solutions, we use a second Lyapunov function which is motivated by a formal analogy between (12) and some kinetic models such as the Vlasov-Fokker-Planck equation. Given $w : \mathbf{R}^2 \to \mathbf{R}_+$, we set

$$H(w) \;=\; \int_{\mathbf{R}^2} w(\xi) \log\Bigl(\frac{w(\xi)}{G(\xi)}\Bigr) \,\mathrm{d}\xi \;.$$

This quantity is often called the *relative entropy* (or relative information) of the vorticity distribution $w$ with respect to the Gaussian $G$, see e.g. [30]. A direct calculation shows that $H$ is non-increasing along the trajectories of (12):

$$\frac{\mathrm{d}}{\mathrm{d}\tau} H(w) \;=\; -\int_{\mathbf{R}^2} w \Bigl|\nabla \log\Bigl(\frac{w}{G}\Bigr)\Bigr|^2 \,\mathrm{d}\xi \le 0 \;.$$

More precisely, this formula shows that $H$ is strictly decreasing except along the family of Oseen vortices, which play the role of the Maxwellian distributions in kinetic theory. By LaSalle's invariance principle, the $\omega$-limit set of any (nonnegative) solution of (12) with total vorticity $\alpha$ is thus reduced to a single point $\{\alpha G\}$, which proves Proposition 1.5.

We next investigate the rate at which the solution $w(\tau)$ of (12) approaches the vortex $\alpha G$ as $\tau \to \infty$. This can be done by studying the linearization of (12) at the vortex. Under the assumptions of Proposition 1.5, there exists $\mu > 0$ such that $\|w(\tau) - \alpha G\|_m = \mathcal{O}(e^{-\mu\tau})$ as $\tau \to \infty$. Convergence is thus exponential in the rescaled time $\tau = \log t$, hence algebraic in the original time $t$. As in the case of the linear heat equation, the convergence rate satisfies $\mu < (m-1)/2$. This limitation originates from the essential spectrum of the linearized operator at the vortex, and is related to the spatial decay of the vorticity. In addition, discrete eigenvalues prevent convergence to be arbitrarily fast even for exponentially localized solutions. The following statement generalizes to arbitrary data in $L^2(m)$ the local results of [14].

**Proposition 1.6** *Fix $m > 2$. For any $w_0 \in L^2(m)$, the solution $w \in C^0([0,+\infty), L^2(m))$ of (12) with initial data $w_0$ satisfies*

$$\|w(\tau) - \alpha G\|_m = \mathcal{O}(e^{-\tau/2}) \;, \quad \textit{as } \tau \to \infty \;, \tag{17}$$

*where $\alpha = \int_{\mathbf{R}^2} w_0(\xi)\,\mathrm{d}\xi$. Moreover, if $m > 3$ and $\beta_1 = \beta_2 = 0$ where $\beta_i = \int_{\mathbf{R}^2} \xi_i w_0(\xi)\,\mathrm{d}\xi$, then*

$$\|w(\tau) - \alpha G\|_m = \mathcal{O}(e^{-\tau}) \;, \quad \textit{as } \tau \to \infty \;. \tag{18}$$

As we shall see in Section 4, if $m > 2$ and $(\beta_1, \beta_2) \neq (0,0)$, then $\|w(\tau) - \alpha G\|_m$ decays exactly like $e^{-\tau/2}$ as $\tau \to \infty$, so that (17) is sharp. Similarly, if $m > 3$, (18) is sharp in the sense that there is in general a correction to the Gaussian asymptotics decaying exactly like $e^{-\tau}$ as $\tau \to \infty$. If $\alpha \neq 0$, there is no loss of generality in assuming that $\beta_1 = \beta_2 = 0$, since this can always been achieved by an appropriate choice of the origin in the original variable $x \in \mathbf{R}^2$ (see Section 4). The situation is different if $\alpha = 0$, see [14]. In this case, if $(\beta_1, \beta_2) \neq (0,0)$, the solution converges to zero at the rate $e^{-\tau/2}$, and the next correction is $\mathcal{O}(\tau e^{-\tau})$ due to secular terms in the asymptotics.

If we reexpress estimate (17) in terms of the original dependent and independent variables, we find that

$$\Big|\omega(\cdot,t) - \frac{\alpha}{t}G(\frac{\cdot}{\sqrt{t}})\Big|_p = \mathcal{O}(t^{-(\frac{3}{2}-\frac{1}{p})}) \ , \quad \Big|\mathbf{u}(\cdot,t) - \frac{\alpha}{\sqrt{t}}\mathbf{v}^G(\frac{\cdot}{\sqrt{t}})\Big|_q = \mathcal{O}(t^{-(1-\frac{1}{q})}) \ ,$$

as $t \to \infty$, which represents a considerable sharpening of the decay rates in (9). Similarly, (18) implies that the quantities above are $\mathcal{O}(t^{-(2-\frac{1}{p})})$ and $\mathcal{O}(t^{-(\frac{3}{2}-\frac{1}{q})})$, respectively.

The rest of this paper is organized as follows. In Section 2, we study the compactness properties of the solutions of (12) in both $L^2(m)$ and $L^1(\mathbf{R}^2)$. In Section 3, we show that (12) has two Lyapunov functions, and we prove Proposition 1.5, Theorem 1.2 and Proposition 1.3. Finally, in Section 4, we study the spectrum of the linearization of (12) at the Oseen vortex and we obtain precise estimates of the rate at which solutions of (12) approach the vortex. In particular, we prove Proposition 1.6.

**Acknowledgements.** The first author is indebted to J. Dolbeault and, especially, to C. Villani for suggesting the beautiful idea of using the Boltzmann entropy functional in the context of the two-dimensional Navier-Stokes equation. The research of C.E.W. is supported in part by the NSF under grant DMS-0103915.

# 2 Smoothing and compactness properties

In this section we discuss some general properties of solutions of the vorticity equation (12) which we will need to establish the convergence results of the Section 3. To control the nonlinear terms in (2) or (12), we will need estimates on the velocity in terms of the vorticity. Let $\omega$ and $\mathbf{u}$ be related via the Biot-Savart law (3). If $\omega \in L^p(\mathbf{R}^2)$ for some $p \in (1,2)$, it follows from the classical Hardy-Littlewood-Sobolev inequality that $\mathbf{u} \in L^q(\mathbf{R}^2)^2$ where $\frac{1}{q} = \frac{1}{p} - \frac{1}{2}$, and there exists $C > 0$ such that

$$|\mathbf{u}|_q \le C|\omega|_p \ . \tag{19}$$

Of course, a similar result holds if $w$ and $\mathbf{v}$ are related via (14). Further estimates are collected in ([14], Lemma 2.1 and Appendix B).

## 2.1 Compactness in $L^2(m)$

If $w_0 \in L^2(m)$ for some $m > 1$, we know from Theorem 1.4 that equation (12) has a unique global solution $w \in C^0([0,\infty), L^2(m))$ with initial data $w_0$. As is explained in [14], $w$ is in fact a solution of the associated integral equation

$$\begin{aligned} w(\tau) &= \mathcal{S}(\tau)w_0 - \int_0^\tau \mathcal{S}(\tau-s)\mathbf{v}(s)\cdot\nabla w(s)\,\mathrm{d}s \\ &= \mathcal{S}(\tau)w_0 - \int_0^\tau e^{-\frac{1}{2}(\tau-s)}\nabla\cdot\mathcal{S}(\tau-s)\mathbf{v}(s)w(s)\,\mathrm{d}s \ , \end{aligned} \tag{20}$$

where $\mathcal{S}(\tau) = \exp(\tau\mathcal{L})$ is the $C_0$-semigroup generated by the operator $\mathcal{L}$. Remark that, since $\mathcal{S}(\tau)$ is not an analytic semigroup, the solution $w$ of (20) is not (in general) a classical solution of (12). In particular, $\tau \mapsto w(\tau)$ is not differentiable in $L^2(m)$. For later use, we recall the following results ([14], Appendix A):

**1.** If $m > 1$, there exists $C > 0$ such that, for all $f \in L^2(m)$,

$$\|\mathcal{S}(\tau)f\|_m \le C\|f\|_m \ , \quad \|\nabla\mathcal{S}(\tau)f\|_m \le \frac{C}{a(\tau)^{1/2}}\|f\|_m \ , \quad \tau > 0 \ , \tag{21}$$

where $a(\tau) = 1 - e^{-\tau}$.

**2.** If $0 < \mu \le 1/2$ and $m > 1 + 2\mu$, there exists $C > 0$ such that, for all $f \in L^2_0(m)$,

$$\|\mathcal{S}(\tau)f\|_m \le Ce^{-\mu\tau}\|f\|_m \ , \quad \|\nabla\mathcal{S}(\tau)f\|_m \le \frac{Ce^{-\mu\tau}}{a(\tau)^{1/2}}\|f\|_m \ , \quad \tau > 0 \ . \tag{22}$$

**3.** Let $1 \le q \le p \le \infty$ and $T > 0$. For all $\alpha \in \mathbf{N}^2$ there exists $C > 0$ such that

$$|b^m\partial^\alpha S(\tau)f|_p \le \frac{C}{a(\tau)^{(\frac{1}{q}-\frac{1}{p})+\frac{|\alpha|}{2}}}\,|b^m f|_q \ , \quad 0 < \tau \le T \ , \tag{23}$$

where $b(\xi) = (1 + |\xi|^2)^{1/2}$.

From Theorem 1.4 we also know that there exists $K_1 > 0$ such that $\|w(\tau)\|_m \le K_1$ for all $\tau \ge 0$. The aim of this section is to prove that the trajectory $\{w(\tau)\}_{\tau\ge0}$ is in fact *relatively compact* in $L^2(m)$. This is because equation (12) is both regularizing and "asymptotically confining", in the sense that solutions of (12) in $L^2(m)$ are asymptotically (as $\tau \to +\infty$) well localized in space. The localization effect originates in the dilation term $\frac{1}{2}\xi\cdot\nabla$ in the linear operator $\mathcal{L}$, and hence does not occur in the original vorticity equation (2).

We first prove that, for positive times, the solution $w(\tau)$ belongs to the weighted Sobolev space

$$H^1(m) = \{w \in L^2(m) \,|\, \partial_i w \in L^2(m) \text{ for } i = 1, 2\} \ , \tag{24}$$

which we equip with the norm $\|w\|_{H^1(m)} = (\|w\|_m^2 + \|\nabla w\|_m^2)^{1/2}$.

**Lemma 2.1** *Let $w_0 \in L^2(m)$ with $m > 1$, and let $w \in C^0([0,\infty), L^2(m))$ be the solution of (12) with initial data $w_0$. Then there exists $K_2 > 0$ such that*

$$\|\nabla w(\tau)\|_m \le \frac{K_2}{a(\tau)^{1/2}} \ , \quad \textit{for all } \tau > 0 \ , \tag{25}$$

*where $a(\tau) = 1 - e^{-\tau}$.*

**Proof:** Consider the Banach space $X = C^0([0,T], L^2(m)) \cap C^0((0,T], H^1(m))$ equipped with the norm

$$\|w\|_X = \sup_{\tau\in[0,T]} \|w(\tau)\|_m + \sup_{\tau\in(0,T]} a(\tau)^{1/2}\|\nabla w(\tau)\|_m \ .$$

We shall prove that there exist $T > 0$ and $K > 0$ such that, for all initial data $\tilde{w}_0 \in L^2(m)$ with $\|\tilde{w}_0\|_m \le K_1$, equation (20) has a unique solution $\tilde{w} \in X$, which satisfies $\|\tilde{w}\|_X \le K$. We then apply this result to $\tilde{w}_0 = w(nT/2)$ with $n \in \mathbf{N}$. By uniqueness, we have $\tilde{w}(\tau) = w(\tau + nT/2)$ for $\tau \in [0,T]$, hence

$$\sup_{0<\tau\le T} a(\tau)^{1/2}\|\nabla w(\tau + nT/2)\|_m \le K \ , \quad \text{for all } n \in \mathbf{N} \ ,$$

which implies (25).

To prove that (20) has a unique solution in $X$, we proceed as in ([14], Lemma 3.1). Given $w_1, w_2 \in X$, we define

$$R(\tau) = \int_0^\tau \mathcal{S}(\tau - s)\mathbf{v}_1(s) \cdot \nabla w_2(s)\,\mathrm{d}s\ , \quad 0 \le \tau \le T\ ,$$

where $\mathbf{v}_1(s)$ is the velocity field obtained from $w_1(s)$ via the Biot-Savart law (14). If $q \in (1,2)$, we know from [14] that

$$\|R(\tau)\|_m \le C\Big(\int_0^\tau \frac{1}{a(s)^{1/q}}\,\mathrm{d}s\Big)\|w_1\|_X\|w_2\|_X\ .$$

It remains to estimate

$$\nabla R(\tau) = \int_0^\tau \nabla\mathcal{S}(\tau - s)\mathbf{v}_1(s) \cdot \nabla w_2(s)\,\mathrm{d}s\ .$$

Applying (23) with $p = 2$ and $q$ as above, we obtain

$$\|\nabla\mathcal{S}(\tau - s)\mathbf{v}_1(s) \cdot \nabla w_2(s)\|_m \le \frac{C}{a(\tau - s)^{1/q}}|b^m\mathbf{v}_1(s)\nabla w_2(s)|_q\ .$$

Next, using Hölder's inequality, estimate (19) and the fact that $L^2(m) \hookrightarrow L^q(\mathbf{R}^2)$ when $2/(m+1) < q \le 2$, we find

$$\begin{aligned} |b^m\mathbf{v}_1(s) \cdot \nabla w_2(s)|_q \le |\mathbf{v}_1(s)|_{\frac{2q}{2-q}}|b^m\nabla w_2(s)|_2 &\le C|w_1(s)|_q\|\nabla w_2(s)\|_m \\ &\le C\|w_1(s)\|_m\|\nabla w_2(s)\|_m\ . \end{aligned} \tag{26}$$

Thus

$$\begin{aligned} a(\tau)^{1/2}\|\nabla R(\tau)\|_m &\le \int_0^\tau \frac{Ca(\tau)^{1/2}}{a(\tau - s)^{1/q}}\|w_1(s)\|_m\|\nabla w_2(s)\|_m\,\mathrm{d}s \\ &\le C\Big(\int_0^\tau \frac{a(\tau)^{1/2}}{a(\tau - s)^{1/q}a(s)^{1/2}}\,\mathrm{d}s\Big)\|w_1\|_X\|w_2\|_X\ . \end{aligned} \tag{27}$$

Summarizing, we have shown that $\|R\|_X \le C(T)\|w_1\|_X\|w_2\|_X$, where $C(T) \to 0$ as $T \to 0$. Using this bilinear estimate together with the bound (21) on the linear semigroup, it is straightforward to verify by a fixed point argument that, if $T > 0$ is sufficiently small, then (20) has a unique solution in $X$ with the desired properties. $\square$

Next, we decompose $w(\xi,\tau) = \alpha G(\xi) + R(\xi,\tau)$, where $G(\xi)$ is given by (8) and $\alpha = \int_{\mathbf{R}^2} w(\xi,\tau)\,\mathrm{d}\xi$ is time-independent due to (5). Then $R(\cdot,\tau) \in L^2_0(m)$ for all $\tau \ge 0$, where $L^2_0(m)$ is defined in (15). Moreover, $R$ satisfies the evolution equation

$$\partial_\tau R \,=\, \mathcal{L}R - \alpha\Lambda R - N(R)\ , \tag{28}$$

where

$$\Lambda R = \mathbf{v}^G \cdot \nabla R + \mathbf{v}^R \cdot \nabla G\ , \quad N(R) = \mathbf{v}^R \cdot \nabla R\ . \tag{29}$$

Here $\mathbf{v}^G$ is given by (8) and $\mathbf{v}^R$ denotes the velocity field associated to the vorticity $R$ by the Biot-Savart law (14). The corresponding integral equation is

$$R(\tau) \;=\; \mathcal{S}(\tau)R_0 - \alpha\int_0^\tau \mathcal{S}(\tau-s)\Lambda R(s)\,\mathrm{d}s - \int_0^\tau \mathcal{S}(\tau-s)N(R(s))\,\mathrm{d}s\;. \tag{30}$$

Equations (12) and (28) are clearly equivalent. In particular, Theorem 1.4 implies that, given any $\alpha \in \mathbf{R}$, the Cauchy problem for (28) is globally well-posed in $L^2_0(m)$ if $m > 1$.

We now prove that positive trajectories of (12) in $L^2(m)$ are relatively compact.

**Lemma 2.2** *Let $w_0 \in L^2(m)$ with $m > 1$, and let $w \in C^0([0,\infty), L^2(m))$ be the solution of (12) with initial data $w_0$. Then the trajectory $\{w(\tau)\}_{\tau\ge 0}$ is relatively compact in $L^2(m)$.*

**Proof:** Since $w \in C^0([0,\infty), L^2(m))$, it is sufficient to show that $\{w(\tau)\}_{\tau\ge 1}$ is relatively compact in $L^2(m)$. We decompose $w(\xi,\tau) = \alpha G(\xi) + R(\xi,\tau)$ as above, and consider the integral equation (30) satisfied by $R$. Let $R_1(\tau) = R(\tau) - \mathcal{S}(\tau)R_0$, so that $w(\tau) = \alpha G + \mathcal{S}(\tau)R_0 + R_1(\tau)$. Since $R_0 \in L^2_0(m)$, it follows from (22) that $\mathcal{S}(\tau)R_0$ converges to zero in $H^1(m)$ as $\tau \to \infty$. In particular, since $\{w(\tau)\}_{\tau\ge 1}$ is bounded in $H^1(m)$ by Lemma 2.1, we see that $\{R_1(\tau)\}_{\tau\ge 1}$ is also bounded in $H^1(m)$. Now, we shall prove that $\{R_1(\tau)\}_{\tau\ge 0}$ is bounded in $L^2(m+1)$. By Rellich's criterion (see [27], Theorem XIII.65), this implies that $\{R_1(\tau)\}_{\tau\ge 1}$ is relatively compact in $L^2(m)$. Since $\mathcal{S}(\tau)R_0$ converges to zero as $\tau \to \infty$, it follows that $\{w(\tau)\}_{\tau\ge 1}$ is also relatively compact in $L^2(m)$, which is the desired result.

To prove the claim, we decompose $R_1(\tau) = R_2(\tau) + R_3(\tau) + R_4(\tau)$ where

$$\begin{aligned}
R_2(\tau) &= -\alpha\int_0^\tau e^{-\frac{1}{2}(\tau-s)}\nabla\cdot\mathcal{S}(\tau-s)\mathbf{v}^G R(s)\,\mathrm{d}s\;,\\
R_3(\tau) &= -\alpha\int_0^\tau e^{-\frac{1}{2}(\tau-s)}\nabla\cdot\mathcal{S}(\tau-s)\mathbf{v}^R(s) G\,\mathrm{d}s\;,\\
R_4(\tau) &= -\int_0^\tau e^{-\frac{1}{2}(\tau-s)}\nabla\cdot\mathcal{S}(\tau-s)\mathbf{v}^R(s) R(s)\,\mathrm{d}s\;.
\end{aligned}$$

To bound $R_2(\tau)$, we observe that $b\mathbf{v}^G \in L^\infty(\mathbf{R}^2)$. Using (21), we thus find

$$\begin{aligned}
\|R_2(\tau)\|_{m+1} &= |\alpha|\,\Big|b^{m+1}\int_0^\tau e^{-\frac{1}{2}(\tau-s)}\nabla\cdot\mathcal{S}(\tau-s)\mathbf{v}^G R(s)\Big|_2\,\mathrm{d}s\\
&\le C|\alpha|\int_0^\tau e^{-\frac{1}{2}(\tau-s)}\frac{1}{a(\tau-s)^{\frac{1}{2}}}|b^{m+1}\mathbf{v}^G R(s)|_2\,\mathrm{d}s\\
&\le C|\alpha|\int_0^\tau e^{-\frac{1}{2}(\tau-s)}\frac{1}{a(\tau-s)^{\frac{1}{2}}}|b\mathbf{v}^G|_\infty\|R(s)\|_m\,\mathrm{d}s \le CK_1^2\;,
\end{aligned} \tag{31}$$

for all $\tau \ge 0$, where $K_1 = \sup_{\tau\ge 0}\|w(\tau)\|_m$. To bound $R_3(\tau)$, we proceed in the same way. Using Hölder's inequality, estimate (19) and the embedding $L^2(m) \hookrightarrow L^{4/3}$, we obtain

$$|b^{m+1}\mathbf{v}^R G|_2 \le |\mathbf{v}^R|_4|b^{m+1}G|_4 \le C|R|_{4/3} \le C\|R\|_m\;, \tag{32}$$

so that $\|R_3(\tau)\|_{m+1} \le CK_1^2$ for all $\tau \ge 0$. Finally, to bound $R_4(\tau)$ we fix $q \in (1,2)$ such that $q \ge 2/m$. Combining (21) and (23), we obtain

$$\|R_4(\tau)\|_{m+1} \le C \int_0^\tau e^{-\frac{1}{2}(\tau-s)} \frac{1}{a(\tau-s)^{1/q}} |b^{m+1}\mathbf{v}^R(s)R(s)|_q \,\mathrm{d}s \;. \tag{33}$$

By Hölder's inequality, $|b\mathbf{v}^R b^m R|_q \le |b\mathbf{v}^R|_{\frac{2q}{2-q}} \|R\|_m$. Applying Proposition B.1(2) in [14], we also obtain $|b\mathbf{v}^R|_{\frac{2q}{2-q}} \le C\|R\|_{\frac{2}{q}} \le C\|R\|_m$. Inserting these bounds into (33), we find that $\|R_4(\tau)\|_{m+1} \le CK_1^2$ for all $\tau \ge 0$. Summarizing, we have shown that $\|R_1(\tau)\|_{m+1} \le CK_1^2$ for all $\tau \ge 0$, which concludes the proof. □

Finally, we show that negative or complete trajectories of (12) in $L^2(m)$ that are bounded in $L^2(m)$ for some $m > 1$ are also relatively compact.

**Lemma 2.3** *Assume that $m > 1$ and that $w \in C^0(\mathbf{R}, L^2(m))$ is a solution of (12) which is bounded in $L^2(m)$. Then $\{w(\tau)\}_{\tau\in\mathbf{R}}$ is relatively compact in $L^2(m)$.*

**Proof:** By assumption, there exists $K_1 > 0$ such that $\|w(\tau)\|_m \le K_1$ for all $\tau \in \mathbf{R}$. As in the proof of Lemma 2.2, we decompose $w(\xi,\tau) = \alpha G(\xi) + R(\xi,\tau)$, where $\alpha = \int_{\mathbf{R}^2} w(\xi,\tau)\,\mathrm{d}\xi$. The remainder $R(\tau)$ satisfies the integral equation

$$R(\tau) \;=\; \mathcal{S}(\tau-\tau_0)R(\tau_0) - \alpha\int_{\tau_0}^\tau \mathcal{S}(\tau-s)\Lambda R(s)\,\mathrm{d}s - \int_{\tau_0}^\tau \mathcal{S}(\tau-s)N(R(s))\,\mathrm{d}s \;, \tag{34}$$

for all $\tau_0 < \tau$. Since $R(\tau_0) \in L^2_0(m)$ and $\|R(\tau_0)\|_m \le CK_1$ for all $\tau_0 \in \mathbf{R}$, it follows from (22) that $\|\mathcal{S}(\tau-\tau_0)R(\tau_0)\|_m \to 0$ as $\tau_0 \to -\infty$. Moreover, proceeding as in the proof of Lemma 2.2 and using the analogues of estimates (31), (33), it is easy to see that both integrals in the right-hand side of (34) have a limit in $L^2(m)$ (and even in $L^2(m+1)$) as $\tau_0 \to -\infty$. Thus, we have the representation

$$R(\tau) \;=\; -\alpha\int_{-\infty}^\tau \mathcal{S}(\tau-s)\Lambda R(s)\,\mathrm{d}s - \int_{-\infty}^\tau \mathcal{S}(\tau-s)N(R(s))\,\mathrm{d}s \;, \quad \tau\in\mathbf{R}\;, \tag{35}$$

which implies that $\|R(\tau)\|_{m+1} \le CK_1^2$ for all $\tau \in \mathbf{R}$. This shows that $\{w(\tau)\}_{\tau\in\mathbf{R}}$ is bounded in $L^2(m+1)$. On the other hand, it follows from Lemma 2.1 that $\{w(\tau)\}_{\tau\in\mathbf{R}}$ is bounded in $H^1(m)$, hence $\{w(\tau)\}_{\tau\in\mathbf{R}}$ is relatively compact in $L^2(m)$ by Rellich's criterion. □

**Remark 2.4** *By a bootstrap argument, it is clear from the proof of Lemma 2.3 that $\{w(\tau)\}_{\tau\in\mathbf{R}}$ is bounded in $H^k(m')$ for all $k \in \mathbf{N}$ and all $m' \in \mathbf{N}$. In other words, the trajectory $\{w(\tau)\}_{\tau\in\mathbf{R}}$ is bounded in the Schwartz space $\mathcal{S}(\mathbf{R}^2)$.*

## 2.2 Compactness in $L^1(\mathbf{R}^2)$

We now study the compactness properties of the solutions of (12) in $L^1(\mathbf{R}^2)$. We first recall two important estimates for the solutions of the original vorticity equation (2), and use the change of variables (11) to obtain the corresponding bounds on the solutions of (12).

The first is a smoothing estimate, see for instance [21]. Under the assumptions of Theorem 1.1, the vorticity $\omega(x,t)$ satisfies for all $p \in (1,\infty]$

$$|\nabla\omega(\cdot,t)|_p \le \frac{C'_p}{t^{\frac{3}{2}-\frac{1}{p}}}\ , \quad t > 0\ , \tag{36}$$

where $C'_p$ depends only on $|\omega_0|_1$. The second is a nice pointwise estimate due to Carlen and Loss, see Theorem 3 in [9]. For any $\beta \in (0,1)$, there exists $C_\beta > 0$ (depending on $|\omega_0|_1$) such that

$$|\omega(x,t)| \le C_\beta \int_{\mathbf{R}^2} \frac{1}{t}\exp\Bigl(-\beta\frac{|x-y|^2}{4t}\Bigr)|\omega_0(y)|\,\mathrm{d}y\ , \quad x \in \mathbf{R}^2\ , \quad t > 0\ . \tag{37}$$

Assume now that $w_0 \in L^1(\mathbf{R}^2)$, and let $w \in C^0([0,\infty), L^1(\mathbf{R}^2))$ be the solution of (12) with initial data $w_0$. Then the function $\omega(x,t)$ defined by (11) is a solution of (2) on the time interval $[1,\infty)$ with initial data $\omega(x,1) = w_0(x)$. Applying (36) and returning to the rescaled variables, we obtain

$$|\nabla w(\cdot,\tau)|_p \le \frac{C'_p}{a(\tau)^{\frac{3}{2}-\frac{1}{p}}}\ , \quad \tau > 0\ , \tag{38}$$

where $a(\tau) = 1 - e^{-\tau}$. Similarly, we deduce from (37) that

$$|w(\xi,\tau)| \le C_\beta \int_{\mathbf{R}^2} \frac{1}{a(\tau)}\ \exp\Bigl(-\beta\frac{|\xi - ye^{-\tau/2}|^2}{4a(\tau)}\Bigr)|w_0(y)|\,\mathrm{d}y\ , \quad \xi \in \mathbf{R}^2\ , \quad \tau > 0\ . \tag{39}$$

With these estimates at hand, we are now ready to prove the main result of this section.

**Lemma 2.5** *Let $w_0 \in L^1(\mathbf{R}^2)$, and let $w \in C^0([0,\infty), L^1(\mathbf{R}^2))$ be the solution of (12) with initial data $w_0$. Then $\{w(\tau)\}_{\tau\ge 0}$ is relatively compact in $L^1(\mathbf{R}^2)$.*

**Proof:** Again, it is sufficient to show that $\{w(\tau)\}_{\tau\ge 1}$ is relatively compact. Fix $R > 0$. From (39) one has $|w(\xi,\tau)| \le w_1(\xi,\tau) + w_2(\xi,\tau)$ where

$$\begin{aligned} w_1(\xi,\tau) &= C_\beta \int_{|y|\le R} \frac{1}{a(\tau)}\ \exp\Bigl(-\beta\frac{|\xi - ye^{-\tau/2}|^2}{4a(\tau)}\Bigr)|w_0(y)|\,\mathrm{d}y\ , \\ w_2(\xi,\tau) &= C_\beta \int_{|y|\ge R} \frac{1}{a(\tau)}\ \exp\Bigl(-\beta\frac{|\xi - ye^{-\tau/2}|^2}{4a(\tau)}\Bigr)|w_0(y)|\,\mathrm{d}y\ . \end{aligned}$$

If $|\xi| \ge 2R$, then $|\xi - ye^{-\tau/2}| \ge |\xi| - |y| \ge |\xi|/2$ whenever $|y| \le R$ and $\tau \ge 0$. It follows that

$$w_1(\xi,\tau) \le C_\beta |w_0|_1 \frac{1}{a(\tau)}\ \exp\Bigl(-\beta\frac{|\xi|^2}{16a(\tau)}\Bigr)\ , \quad |\xi| \ge 2R\ ,$$

hence

$$\int_{|\xi|\ge 2R} w_1(\xi,\tau)\,\mathrm{d}\xi\ \le\ C_\beta |w_0|_1 \int_{|z|\ge 2R} \exp\Bigl(-\beta\frac{|z|^2}{16}\Bigr)\,\mathrm{d}z\ =:\ \varepsilon_1(R)\ .$$

Moreover, using Fubini's theorem, we find

$$\int_{\mathbf{R}^2} w_2(\xi,\tau)\,\mathrm{d}\xi \;\le\; C_\beta \frac{4\pi}{\beta} \int_{|y|\ge R} |w_0(\xi)|\,\mathrm{d}\xi \;=:\; \varepsilon_2(R)\;.$$

Thus we have shown

$$\sup_{\tau>0} \int_{|\xi|\ge 2R} |w(\xi,\tau)|\,\mathrm{d}\xi \;\le\; \varepsilon_1(R) + \varepsilon_2(R) \to 0 \quad \text{as } R\to\infty\;. \tag{40}$$

On the other hand, using (38) with $p=\infty$, we see that

$$K := \sup_{\tau\ge 1}\sup_{\xi\in\mathbf{R}^2} |\nabla w(\xi,\tau)| < \infty\;.$$

Fix $\varepsilon>0$. According to (40), there exists $R>1$ such that

$$\sup_{\tau\ge 1}\int_{|\xi|\ge R-1} |w(\xi,\tau)|\,\mathrm{d}\xi \le \varepsilon/3\;.$$

Let $\delta = \min(1, \varepsilon(3K\pi R^2)^{-1})$. If $y\in\mathbf{R}^2$ satisfies $|y|\le\delta$, then for all $\tau\ge 1$

$$\int_{|\xi|\ge R} |w(\xi-y,\tau)-w(\xi,\tau)|\,\mathrm{d}\xi \;\le\; 2\int_{|\xi|\ge R-1} |w(\xi,\tau)|\,\mathrm{d}\xi \le \frac{2\varepsilon}{3}\;,$$

and

$$\int_{|\xi|\le R} |w(\xi-y,\tau)-w(\xi,\tau)|\,\mathrm{d}\xi \;\le\; \pi R^2|y| \sup_{|\xi|\le R+1} |\nabla w(\xi,\tau)| \le \pi R^2\delta K \le \frac{\varepsilon}{3}\;,$$

so that $\int_{\mathbf{R}^2} |w(\xi-y,\tau)-w(\xi,\tau)|\,\mathrm{d}\xi \le \varepsilon$. Thus we have shown

$$\sup_{\tau\ge 1}\sup_{|y|\le\delta}\int_{\mathbf{R}^2} |w(\xi-y,\tau)-w(\xi,\tau)|\,\mathrm{d}\xi \to 0 \quad \text{as } \delta\to 0\;. \tag{41}$$

By the Riesz criterion ([27], Theorem XIII.66), it follows from (40), (41) that $\{w(\tau)\}_{\tau\ge 1}$ is relatively compact in $L^1(\mathbf{R}^2)$, which is the desired result. □

To conclude this subsection, we remark that Lemma 2.3 has no analogue in $L^1(\mathbf{R}^2)$, namely negative trajectories of (12) that are bounded in $L^1(\mathbf{R}^2)$ need not be relatively compact in that space. To see this, let $\omega_0 \in L^1(\mathbf{R}^2)$ and let $\omega(x,t)$ be the solution of (2) given by Theorem 1.1. If $w(\xi,\tau) = e^\tau\omega(\xi e^{\tau/2}, e^\tau)$, then $\{w(\cdot,\tau)\}_{\tau\le 0}$ is a bounded negative trajectory in $L^1(\mathbf{R}^2)$. However, $w(\cdot,\tau)$ is "evanescent" as $\tau\to-\infty$, hence $\{w(\cdot,\tau)\}_{\tau\le 0}$ is not relatively compact in $L^1(\mathbf{R}^2)$ unless $\omega_0\equiv 0$.

## 2.3 Preservation of positivity

A more qualitative property that will be essential for our analysis is the fact that solutions of (2), and hence of (12), satisfy a maximum principle. We state this property in the original variables, and for a generalized version of (2) where the velocity and vorticity field are not necessarily connected to each other. This generalization will be useful in the next section.

**Proposition 2.6** *Assume that* $\tilde{\mathbf{u}} \in C_b^0(\mathbf{R}^2 \times [0,\infty), \mathbf{R}^2)$ *and that* $\omega \in C_b^2(\mathbf{R}^2 \times [0,\infty), \mathbf{R})$ *is a solution of*

$$\partial_t \omega(x,t) + \tilde{\mathbf{u}}(x,t) \cdot \nabla \omega(x,t) = \Delta \omega(x,t) \ , \quad x \in \mathbf{R}^2 \ , \ t \geq 0 \ . \tag{42}$$

*If* $\omega(x,0) \geq 0$ *for all* $x \in \mathbf{R}^2$, *then either* $\omega(x,t) \equiv 0$ *or* $\omega(x,t) > 0$ *for all* $x \in \mathbf{R}^2$ *and all* $t > 0$.

**Proof:** This classical result is obtained for instance by combining Theorems 3.5 and 3.10 in the book of Protter and Weinberger [26]. □

As a corollary, we obtain that (2) preserves positivity. The same property holds for (12) and is proved using the change of variables (11).

**Corollary 2.7** *Assume that* $\omega_0 \in L^1(\mathbf{R}^2)$ *satisfies* $\omega_0(x) \geq 0$ *for almost all* $x \in \mathbf{R}^2$, *and that* $\omega_0(x)$ *does not vanish almost everywhere. Then the solution of (2) given by Theorem 1.1 satisfies* $\omega(x,t) > 0$ *for all* $x \in \mathbf{R}^2$ *and all* $t > 0$.

**Proof:** If $\omega_0 \in \mathcal{S}(\mathbf{R}^2)$ and $\omega_0(x) \geq 0$ for all $x \in \mathbf{R}^2$, the solution $\omega(x,t)$ of (2) and the corresponding velocity field $\mathbf{u}(x,t)$ satisfy the assumptions of Proposition 2.6, hence $\omega(x,t) \geq 0$ for all $x \in \mathbf{R}^2$ and all $t > 0$. Since solutions of (2) depend continuously on the initial data in $L^1(\mathbf{R}^2)$, the same result holds for any solution of (2) with initial data $\omega_0 \in L^1(\mathbf{R}^2)$ such that $\omega_0(x) \geq 0$ almost everywhere. Moreover, given any $t_0 > 0$, this solution satisfies $\omega \in C_b^2(\mathbf{R}^2 \times [t_0,\infty), \mathbf{R})$ and the corresponding velocity field satisfies $\mathbf{u} \in C_b^0(\mathbf{R}^2 \times [t_0,\infty), \mathbf{R}^2)$. Applying Proposition 2.6 again, we deduce that $\omega(x,t) > 0$ for all $x \in \mathbf{R}^2$ and all $t > t_0$, unless $\omega(x,t_0) \equiv 0$. Since this is true for any $t_0 > 0$, we conclude that $\omega(x,t) > 0$ for all $x \in \mathbf{R}^2$ and all $t > 0$, unless $\omega_0(x)$ vanishes almost everywhere. □

# 3 Global convergence results

This section is devoted to the proofs of Proposition 1.5, Theorem 1.2 and Proposition 1.3. The argument relies on the compactness properties of the previous section, and uses the crucial fact that system (12) has two Lyapunov functions which we introduce now.

## 3.1 A pair of Lyapunov functions

Let $\Phi : L^1(\mathbf{R}^2) \to \mathbf{R}_+$ be the continuous function defined by

$$\Phi(w) \ = \ \int_{\mathbf{R}^2} |w(\xi)| \, \mathrm{d}\xi \ , \tag{43}$$

and let

$$\Sigma \ = \ \left\{ w \in L^1(\mathbf{R}^2) \,\middle|\, \int_{\mathbf{R}^2} |w(\xi)| \, \mathrm{d}\xi = \left| \int_{\mathbf{R}^2} w(\xi) \, \mathrm{d}\xi \right| \right\} \ .$$

In words, a function $w \in L^1(\mathbf{R}^2)$ belongs to $\Sigma$ if and only if $w(\xi)$ has (almost everywhere) a constant sign. Remark that, by Corollary 2.7, the set $\Sigma$ is positively invariant under the evolution of (12).

We first show that $\Phi$ is a Lyapunov function for the semiflow of (12). More precisely, $\Phi$ is strictly decreasing along the trajectories of (12) except on the invariant set $\Sigma$ where $\Phi$ is constant.

**Lemma 3.1** *Let $w_0 \in L^1(\mathbf{R}^2)$, and let $w \in C^0([0,\infty), L^1(\mathbf{R}^2))$ be the solution of (12) with initial data $w_0$. Then $\Phi(w(\tau)) \le \Phi(w_0)$ for all $\tau \ge 0$. Moreover $\Phi(w(\tau)) = \Phi(w_0)$ for all $\tau \ge 0$ if and only if $w_0 \in \Sigma$.*

**Proof:** If $w_0 \in \Sigma$, then $w(\tau) \in \Sigma$ for all $\tau \ge 0$. Using (5), we thus find

$$\Phi(w(\tau)) \;=\; \Big|\int_{\mathbf{R}^2} w(\xi,\tau)\,\mathrm{d}\xi\Big| \;=\; \Big|\int_{\mathbf{R}^2} w_0(\xi)\,\mathrm{d}\xi\Big| = \Phi(w_0)\;, \quad \text{for all } \tau \ge 0\;.$$

Assume now that $w_0 \notin \Sigma$. Then $w_0 = w_0^+ - w_0^-$, where

$$w_0^+(\xi) = \max(w_0(\xi),0) \ge 0\;, \quad w_0^-(\xi) = -\min(w_0(\xi),0) \ge 0\;.$$

By assumption, both $w_0^+$ and $w_0^-$ are nonzero on a set of positive Lebesgue measure. Let $w_1$ and $w_2$ be solutions of

$$\begin{aligned} \partial_\tau w_1 + \mathbf{v}\cdot\nabla w_1 &= \mathcal{L}w_1\;, \quad \tau \ge 0\;, \\ \partial_\tau w_2 + \mathbf{v}\cdot\nabla w_2 &= \mathcal{L}w_2\;, \quad \tau \ge 0\;, \end{aligned} \tag{44}$$

with initial data $w_1(0) = w_0^+$, $w_2(0) = w_0^-$, where $\mathbf{v}(\xi,\tau)$ is the velocity field associated with the solution $w(\xi,\tau)$ of (12). Following the proof of Theorem 1.1 (see [21]), one verifies that $w_1, w_2 \in C^0([0,\infty), L^1(\mathbf{R}^2))$ and that

$$\int_{\mathbf{R}^2} w_1(\xi,\tau)\,\mathrm{d}\xi \;=\; \int_{\mathbf{R}^2} w_0^+(\xi)\,\mathrm{d}\xi\;, \quad \int_{\mathbf{R}^2} w_2(\xi,\tau)\,\mathrm{d}\xi \;=\; \int_{\mathbf{R}^2} w_0^-(\xi)\,\mathrm{d}\xi\;, \tag{45}$$

for all $\tau \ge 0$. Moreover, using Proposition 2.6 and a density argument as in the proof of Corollary 2.7, it is straightforward to verify that $w_1(\xi,\tau) > 0$ and $w_2(\xi,\tau) > 0$ for all $\xi \in \mathbf{R}^2$ and all $\tau > 0$. Now, by construction we have $w(\xi,\tau) = w_1(\xi,\tau) - w_2(\xi,\tau)$, hence

$$|w(\xi,\tau)| \;=\; |w_1(\xi,\tau) - w_2(\xi,\tau)| \;<\; w_1(\xi,\tau) + w_2(\xi,\tau)\;,$$

for all $\xi \in \mathbf{R}^2$ and all $\tau > 0$. Integrating over $\mathbf{R}^2$ and using (45), we obtain

$$\begin{aligned} \int_{\mathbf{R}^2} |w(\xi,\tau)|\,\mathrm{d}\xi \;&<\; \int_{\mathbf{R}^2} (w_1(\xi,\tau) + w_2(\xi,\tau))\,\mathrm{d}\xi \\ &=\; \int_{\mathbf{R}^2} (w_0^+(\xi) + w_0^-(\xi))\,\mathrm{d}\xi \;=\; \int_{\mathbf{R}^2} |w_0(\xi)|\,\mathrm{d}\xi\;, \quad \tau > 0\;. \end{aligned}$$

This shows that $\Phi(w(\tau)) < \Phi(w_0)$ for all $\tau > 0$. □

Next we fix $m > 3$ and we consider solutions of (12) in the invariant cone $L^2(m) \cap \Sigma_+$, where

$$\Sigma_+ \;=\; \{w \in L^1(\mathbf{R}^2)\,|\, w(\xi) \ge 0 \text{ almost everywhere}\}\;.$$

We define $H : L^2(m) \cap \Sigma_+ \to \mathbf{R}$ by

$$H(w) \;=\; \int_{\mathbf{R}^2} w(\xi)\log\Big(\frac{w(\xi)}{G(\xi)}\Big)\,\mathrm{d}\xi\;. \tag{46}$$

Since

$$
\begin{aligned}
w\log\Big(\frac{w}{G}\Big) &= \Big(\frac{w}{G}\log\Big(\frac{w}{G}\Big)\Big)\, G \,\ge\, -\frac{1}{e}\, G \ , \\
w\log\Big(\frac{w}{G}\Big) &= w\log(4\pi w) + \frac{|\xi|^2}{4}\, w \,\le\, Cw^2 + \frac{|\xi|^2}{4}\, w \ ,
\end{aligned}
$$

it is clear that $H$ is well-defined and bounded from below by $-1/e$. Moreover, using for instance the inequality

$$
|w_1\log w_1 - w_2\log w_2| \,\le\, C\Big(|w_1-w_2|^{1/2} + |w_1-w_2|(w_1^{1/2}+w_2^{1/2})\Big) \ ,
$$

one verifies that $H$ is continuous on $L^2(m)\cap\Sigma_+$ (equipped with the topology of $L^2(m)$). We now show that $H$ is indeed a Lyapunov function for the semiflow defined by (12) on $L^2(m)\cap\Sigma_+$.

**Lemma 3.2** *Assume that $w_0\in L^2(m)\cap\Sigma_+$ with $m>3$, and let $w\in C^0([0,\infty),L^2(m))$ be the solution of (12) with initial data $w_0$. Then $H(w(\tau))\le H(w_0)$ for all $\tau\ge 0$. Moreover $H(w(\tau))=H(w_0)$ for all $\tau\ge 0$ if and only if $w_0=\alpha G$ for some $\alpha\ge 0$.*

**Proof:** If $w_0=\alpha G$ for some $\alpha\ge 0$, then $w(\tau)=\alpha G$ for all $\tau\ge 0$, hence obviously $H(w(\tau))=H(w_0)$ for all $\tau\ge 0$. Assume now that $w_0$ is not a multiple of $G$ (in particular, $w_0\neq 0$). Then the solution $w(\xi,\tau)$ of (12) is smooth and strictly positive for all $\tau>0$. We claim that $\tau\mapsto H(w(\tau))$ is differentiable for $\tau>0$, and that

$$
\frac{\mathrm{d}}{\mathrm{d}\tau}H(w(\tau)) \,=\, -I(w(\tau)) \ , \quad \tau>0 \ , \tag{47}
$$

where

$$
I(w) = \int_{\mathbf{R}^2} w(\xi)\Big|\nabla\log\Big(\frac{w(\xi)}{G(\xi)}\Big)\Big|^2\,\mathrm{d}\xi \ge 0 \ . \tag{48}
$$

Remark that $I(w)$ vanishes if and only if $w$ is proportional to $G$. Thus, under the assumptions above, it is clear that $I(w(\tau))>0$ at least for $\tau>0$ sufficiently small, hence $H(w(\tau))<H(w_0)$ for all $\tau>0$.

Thus, all that remains is to prove (47). Assume first that $w_0$ belongs to the Schwartz space $\mathcal{S}(\mathbf{R}^2)$. Then $w\in C^1([0,\infty),\mathcal{S}(\mathbf{R}^2))$ is a classical solution of (12) in $\mathcal{S}(\mathbf{R}^2)$. Moreover, $w(\xi,\tau)$ satisfies a *Gaussian lower bound* for any $\tau>0$, see [24] or ([18], Theorem 3.1). More precisely, there exist positive constants $\gamma$ and $C_\gamma$ (depending only on $|w_0|_1$) such that, for all $\xi\in\mathbf{R}^2$ and all $\tau>0$,

$$
\begin{aligned}
w(\xi,\tau) &\ge \frac{C_\gamma}{a(\tau)}\int_{\mathbf{R}^2}\exp\Big(-\gamma\frac{|\xi-ye^{-\tau/2}|^2}{2a(\tau)}\Big)w_0(y)\,\mathrm{d}y \ , \\
&\ge \frac{C_\gamma}{a(\tau)}\exp\Big(-\gamma\frac{|\xi|^2}{a(\tau)}\Big)\int_{\mathbf{R}^2}\exp\Big(-\gamma\frac{|y|^2}{a(\tau)}\Big)w_0(y)\,\mathrm{d}y \ ,
\end{aligned} \tag{49}
$$

see also (39). Using these properties, it is straightforward to verify that $\tau\mapsto H(w(\tau))$ is differentiable for $\tau>0$, and that

$$
\frac{\mathrm{d}}{\mathrm{d}\tau}H(w(\tau)) \,=\, \int_{\mathbf{R}^2}\Big(1+\log\frac{w}{G}\Big)\partial_\tau w\,\mathrm{d}\xi = \int_{\mathbf{R}^2}\Big(1+\log\frac{w}{G}\Big)(\mathcal{L}w - \mathbf{v}\cdot\nabla w)\,\mathrm{d}\xi \ .
$$

Next, using the identity $\mathcal{L}w = \mathrm{div}(G\nabla(\frac{w}{G}))$ and integrating by parts, we obtain

$$\begin{aligned}\int_{\mathbf{R}^2}\Big(1+\log\frac{w}{G}\Big)(\mathcal{L}w)\,\mathrm{d}\xi &= -\int_{\mathbf{R}^2}\nabla\Big(\log\frac{w}{G}\Big)\cdot\frac{G}{w}\nabla\Big(\frac{w}{G}\Big)w\,\mathrm{d}\xi \\ &= -\int_{\mathbf{R}^2} w\Big|\nabla\Big(\log\frac{w}{G}\Big)\Big|^2\,\mathrm{d}\xi \;=\; -I(w)\ .\end{aligned}$$

On the other hand, using $\mathbf{v}\cdot\nabla w = \mathrm{div}(\mathbf{v}w)$ and integrating by parts, we find

$$\begin{aligned}\int_{\mathbf{R}^2}\Big(1+\log\frac{w}{G}\Big)(\mathbf{v}\cdot\nabla w)\,\mathrm{d}\xi &= \int_{\mathbf{R}^2}(1+\log(4\pi w))(\mathbf{v}\cdot\nabla w)\,\mathrm{d}\xi + \int_{\mathbf{R}^2}\frac{|\xi|^2}{4}(\mathbf{v}\cdot\nabla w)\,\mathrm{d}\xi \\ &= -\int_{\mathbf{R}^2}\mathbf{v}\cdot\nabla w\,\mathrm{d}\xi - \frac{1}{2}\int_{\mathbf{R}^2}(\boldsymbol{\xi}\cdot\mathbf{v})w\,\mathrm{d}\xi.\end{aligned}$$

We claim that both integrals in the last expression vanish. This is obvious for the first one, since $\mathbf{v}\cdot\nabla w = \mathrm{div}(\mathbf{v}w)$. As for the second one, using (14) and Fubini's theorem we obtain

$$\begin{aligned}\int_{\mathbf{R}^2}(\boldsymbol{\xi}\cdot\mathbf{v}(\xi))w(\xi)\,\mathrm{d}\xi &= \frac{1}{2\pi}\int_{\mathbf{R}^2\times\mathbf{R}^2}\boldsymbol{\xi}\cdot\frac{(\boldsymbol{\xi}-\boldsymbol{\eta})^\perp}{|\xi-\eta|^2}w(\eta)w(\xi)\,\mathrm{d}\eta\,\mathrm{d}\xi \\ &= \frac{1}{4\pi}\int_{\mathbf{R}^2\times\mathbf{R}^2}(\boldsymbol{\xi}-\boldsymbol{\eta})\cdot\frac{(\boldsymbol{\xi}-\boldsymbol{\eta})^\perp}{|\xi-\eta|^2}w(\eta)w(\xi)\,\mathrm{d}\eta\,\mathrm{d}\xi = 0\ .\end{aligned}$$

Summarizing, we have shown that

$$H(w(\tau_1)) - H(w(\tau_0)) = -\int_{\tau_0}^{\tau_1} I(w(\tau))\,\mathrm{d}\tau\ , \tag{50}$$

for all $\tau_1 > \tau_0 > 0$.

We now return to the general case where $w_0 \in L^2(m)\cap\Sigma_+$. Given $\tau_1 > \tau_0 > 0$, the solution $w$ of (12) satisfies $w \in C^0([\tau_0,\tau_1], H^k(m))$ for any $k \in \mathbf{N}$, where $H^k(m)$ is the weighted Sobolev space defined in analogy with (24). Moreover, the map

$$w_0 \in L^2(m) \quad \mapsto \quad w \in C^0([\tau_0,\tau_1], H^k(m))$$

is continuous. On the other hand, using for instance [22], it is not difficult to verify that the quantity $I(w)$ is finite for any positive $w \in H^k(m)$ if $k \geq 2$, and that $I(w)$ depends continuously on $w$ in that topology. Thus we see that both sides of (50) depend continuously on the initial data $w_0$ in the topology of $L^2(m)$. Since (50) holds for all $w_0$ in the dense subset $\mathcal{S}(\mathbf{R}^2)\cap\Sigma_+$, it follows that (50) is valid for all $w_0 \in L^2(m)\cap\Sigma_+$ and all $\tau_1 > \tau_0 > 0$. This concludes the proof. $\square$

## 3.2 Convergence in $L^2(m)$

Using the compactness properties of Section 2 and the two Lyapunov functions of the previous subsection, we are now able to prove Proposition 1.5.

**Lemma 3.3** *Assume that $m > 1$ and that $\{w(\tau)\}_{\tau\in\mathbf{R}}$ is a complete trajectory of (12) which is bounded in $L^2(m)$. Then $w(\tau) = \alpha G$ for all $\tau \in \mathbf{R}$, where $\alpha = \int_{\mathbf{R}^2} w(\xi,0)\,\mathrm{d}\xi$.*

**Proof:** We know from Lemma 2.3 that $\{w(\tau)\}_{\tau\in\mathbf{R}}$ is relatively compact in $L^2(m)$. In view of Remark 2.4, we can assume without loss of generality that $m > 3$. Let $\Omega$ be the $\omega$-limit set of the trajectory $\{w(\tau)\}_{\tau\in\mathbf{R}}$. Since by Lemma 3.1 $\Phi$ is a Lyapunov function which is strictly decreasing except on $\Sigma$, it follows from LaSalle's invariance principle that $\Omega \subset \Sigma$. In particular, since the total mass is conserved, any $\bar{w} \in \Omega$ satisfies $\Phi(\bar{w}) = |\int_{\mathbf{R}^2} \bar{w}(\xi)\,\mathrm{d}\xi| = |\alpha|$. The same is true for any function $\underline{w}$ in the $\alpha$-limit set $\mathcal{A}$. As $\tau \mapsto \Phi(w(\tau))$ is non-increasing, it follows that $\Phi(w(\tau)) = |\alpha|$ for all $\tau \in \mathbf{R}$. By Lemma 3.1 again, we conclude that $w(\tau) \in \Sigma$ for all $\tau \in \mathbf{R}$. Thus, upon replacing $w(\xi_1, \xi_2, \tau)$ by $-w(\xi_2, \xi_1, \tau)$ if necessary, we can assume that $\{w(\tau)\}_{\tau\in\mathbf{R}} \subset L^2(m) \cap \Sigma_+$.

We now use the second Lyapunov function $H$. By Lemma 3.2 and LaSalle's principle, $\mathcal{A}$ and $\Omega$ are contained in the line of equilibria $\{\alpha' G\}_{\alpha'\geq 0}$. Since the total mass is conserved, we necessarily have $\mathcal{A} = \Omega = \{\alpha G\}$. As $H$ is non-increasing, it follows that $H(w(\tau)) = H(\alpha G) = \alpha \log(\alpha)$ for all $\tau \in \mathbf{R}$. By Lemma 3.2 again, we conclude that $w(\tau) = \alpha G$ for all $\tau \in \mathbf{R}$. ☐

**Proof of Proposition 1.5:** Let $w_0 \in L^2(m)$ with $m > 1$, and let $w \in C^0([0, +\infty), L^2(m))$ be the solution of (12) with initial data $w_0$. From Lemma 2.2, we know that $\{w(\tau)\}_{\tau\geq 0}$ is relatively compact in $L^2(m)$. Let $\Omega \subset L^2(m)$ denote the $\omega$-limit set of this trajectory. As is well-known, $\Omega$ is non-empty, compact, fully invariant under the evolution of (12), and $\Omega$ attracts $w(\tau)$ in $L^2(m)$ as $\tau \to +\infty$. If $\bar{w} \in \Omega$, there exists a complete trajectory $\{\bar{w}(\tau)\}_{\tau\in\mathbf{R}}$ of (12) such that $\bar{w}(\tau) \in \Omega$ for all $\tau \in \mathbf{R}$ and $\bar{w}(0) = \bar{w}$. By Lemma 3.3, $\bar{w}(\tau) = \alpha G$ for all $\tau \in \mathbf{R}$, where $\alpha = \int_{\mathbf{R}^2} \bar{w}(\xi)\,\mathrm{d}\xi = \int_{\mathbf{R}^2} w_0(\xi)\,\mathrm{d}\xi$. Thus $\Omega = \{\alpha G\}$, which is the desired result. ☐

## 3.3 Convergence in $L^1(\mathbf{R}^2)$

We now study the behavior of the solutions of (12) in $L^1(\mathbf{R}^2)$ and prove Theorem 1.2 and Proposition 1.3.

**Proposition 3.4** *Let $w_0 \in L^1(\mathbf{R}^2)$, and let $w \in C^0([0,\infty), L^1(\mathbf{R}^2))$ be the solution of (12) with initial data $w_0$. Then $|w(\tau) - \alpha G|_1 \to 0$ as $\tau \to \infty$, where $\alpha = \int_{\mathbf{R}^2} w_0(\xi)\,\mathrm{d}\xi$.*

**Proof:** We know from Lemma 2.5 that $\{w(\tau)\}_{\tau\geq 0}$ is relatively compact in $L^1(\mathbf{R}^2)$. Let $\Omega$ be the $\omega$-limit set of this trajectory. Then $\Omega$ is non-empty, compact, fully invariant under the evolution of (12), and $\Omega$ attracts $w(\tau)$ in $L^1(\mathbf{R}^2)$ as $\tau \to +\infty$. If $\bar{w} \in \Omega$, there exists a sequence $\tau_n \to \infty$ such that $|w(\tau_n) - \bar{w}|_1 \to 0$ and $w(\xi, \tau_n) \to \bar{w}(\xi)$ for almost all $\xi \in \mathbf{R}^2$. Using (39) and Lebesgue's dominated convergence theorem, we obtain

$$|\bar{w}(\xi)| = \lim_{n\to\infty} |w(\xi, \tau_n)| \leq C_\beta |w_0|_1\, e^{-\beta|\xi|^2/4}~, \quad \xi \in \mathbf{R}^2~, \tag{51}$$

since $a(\tau_n) \to 1$ as $n \to \infty$. In particular, this shows that $\Omega$ is bounded in $L^2(m)$ for all $m > 1$.

Now, since $\Omega$ is invariant under the semiflow of (12), there exists a complete trajectory $\{\bar{w}(\tau)\}_{\tau\in\mathbf{R}}$ such that $\bar{w}(\tau) \in \Omega$ for all $\tau \in \mathbf{R}$ and $\bar{w}(0) = \bar{w}$. As we just observed, $\{\bar{w}(\tau)\}_{\tau\in\mathbf{R}}$ is bounded in $L^2(m)$ for all $m > 1$. Applying Lemma 3.3, we conclude that $\bar{w}(\tau) = \alpha G$ for all $\tau \in \mathbf{R}$, where $\alpha = \int_{\mathbf{R}^2} \bar{w}(\xi)\,\mathrm{d}\xi = \int_{\mathbf{R}^2} w_0(\xi)\,\mathrm{d}\xi$. This proves that $\Omega = \{\alpha G\}$, which is the desired result. ☐

**Proof of Theorem 1.2:** Let $\omega \in C^0([0,\infty), L^1(\mathbf{R}^2))$ be the solution of (2) with initial data $\omega_0$. If we set $w(\xi,\tau) = e^\tau \omega(\xi\, e^{\tau/2}, e^\tau - 1)$, then $w \in C^0([0,\infty), L^1(\mathbf{R}^2))$ is the solution of (12) with initial data $w_0 = \omega_0$. Applying Proposition 3.4 and returning to the original function $\omega(x,t)$, we obtain

$$\lim_{t\to\infty} \Big|\omega(\cdot,t) - \frac{\alpha}{t+1} G\Big(\frac{\cdot}{\sqrt{t+1}}\Big)\Big|_1 = 0 \ ,$$

which is equivalent to (9) for $p = 1$. Next, interpolating between (9) for $p = 1$ and (4) for $p = \infty$, we obtain (9) for $p \in (1,\infty)$. Then, using (19) if $2 < q < \infty$ or Lemma 2.1(b) in [14] if $q = \infty$, we arrive at (10). Finally, using the previous results and the integral equation satisfied by $\omega(x,t)$, it is straightforward to show that (9) also holds for $p = \infty$. □

As we already observed, a bounded negative trajectory of (12) in $L^1(\mathbf{R}^2)$ need not be relatively compact. However, if we assume that the trajectory is relatively compact or at least that its $\alpha$-limit set is nonempty, then we have the following result which generalizes Lemma 3.3.

**Proposition 3.5** *Let $\{w(\tau)\}_{\tau\in\mathbf{R}}$ be a complete trajectory of (12) in $L^1(\mathbf{R}^2)$, and assume that $w(\tau)$ has a convergent subsequence in $L^1(\mathbf{R}^2)$ as $\tau \to -\infty$. Then $w(\tau) = \alpha G$ for all $\tau \in \mathbf{R}$, where $\alpha = \int_{\mathbf{R}^2} w(\xi,0)\,\mathrm{d}\xi$.*

**Proof:** By assumption, there exists $\bar w$ in $L^1(\mathbf{R}^2)$ and a sequence $\tau_n \to -\infty$ such that $|w(\tau_n) - \bar w|_1 \to 0$ and $w(\xi,\tau_n) \to \bar w(\xi)$ for almost all $\xi \in \mathbf{R}^2$. Fix $\tau \in \mathbf{R}$, and take $n \geq 0$ sufficiently large so that $\tau_n < \tau$. In view of (39), we have for all $\xi \in \mathbf{R}^2$

$$\begin{aligned}
|w(\xi,\tau)| \ &\leq\ C_\beta \int_{\mathbf{R}^2} \frac{1}{a(\tau-\tau_n)} \exp\Big(-\beta\frac{|\xi - ye^{-(\tau-\tau_n)/2}|^2}{4a(\tau-\tau_n)}\Big)|w(y,\tau_n)|\,\mathrm{d}y \\
&\leq\ C_\beta \int_{\mathbf{R}^2} \frac{1}{a(\tau-\tau_n)} \exp\Big(-\beta\frac{|\xi - ye^{-(\tau-\tau_n)/2}|^2}{4a(\tau-\tau_n)}\Big)|w(y,\tau_n) - \bar w(y)|\,\mathrm{d}y \\
&+\ C_\beta \int_{\mathbf{R}^2} \Big|\frac{1}{a(\tau-\tau_n)} \exp\Big(-\beta\frac{|\xi - ye^{-(\tau-\tau_n)/2}|^2}{4a(\tau-\tau_n)}\Big) - e^{-\beta|\xi|^2/4}\Big||\bar w(y)|\,\mathrm{d}y \\
&+\ C_\beta |\bar w|_1 e^{-\beta|\xi|^2/4}\ .
\end{aligned}$$

Taking the limit $n \to \infty$ and using Lebesgue's dominated convergence theorem, we obtain

$$|w(\xi,\tau)| \leq C_\beta |\bar w|_1 e^{-\beta|\xi|^2/4}\ , \quad \xi \in \mathbf{R}^2\ ,\ \tau \in \mathbf{R}\ .$$

This shows that the trajectory $\{w(\tau)\}_{\tau\in\mathbf{R}}$ is bounded in $L^2(m)$ for any $m > 1$, and the result follows from Lemma 3.3. □

As is clear from the change of variables (11), results about negative trajectories of (12) give information on the behavior of solutions of (2) as $t \to 0+$. In particular, Lemma 3.3 and Proposition 3.5 show that solutions of (2) with Dirac masses as initial data are unique in a certain class. A generalization of these results allows to prove Proposition 1.3.

**Proof of Proposition 1.3:** In view of Theorem 1.1, we can assume without loss of generality that $T = \infty$. The solution $\omega(x,t)$ of (2) can be represented as

$$\omega(x,t) = \int_{\mathbf{R}^2} \Gamma_u(x,t;y,s)\omega(y,s)\,\mathrm{d}y\ , \quad x \in \mathbf{R}^2\ , \quad t > s > 0\ .$$

Here $\Gamma_u$ is the fundamental solution of the time-dependent linear operator $\partial_t - \Delta + \mathbf{u} \cdot \nabla$, and $\mathbf{u}(x,t)$ is the velocity field obtained from $\omega(x,t)$ via the Biot-Savart law. By assumption, there exists $K > 0$ such that $|\omega(\cdot,t)|_1 \le K$ for all $t > 0$. From [9], we know that for any $\beta \in (0,1)$ there exists $C_\beta > 0$ (depending on $K$) such that

$$|\Gamma_u(x,t;y,s)| \le \frac{C_\beta}{t-s} \exp\Bigl(-\beta \frac{|x-y|^2}{4(t-s)}\Bigr) , \tag{52}$$

for all $x, y \in \mathbf{R}^2$ and all $t > s > 0$, see (37). Moreover, it is shown in [24] (see also Theorem 3.1 in [18]) that $\Gamma_u$ is a Hölder continuous function of its arguments. More precisely, there exists $\gamma \in (0,1)$ (depending only on $K$) and, for any $\tau > 0$, a constant $C > 0$ (depending only on $K$ and $\tau$) such that

$$|\Gamma_u(x,t;y,s) - \Gamma_u(x,t;y',s')| \le C\Bigl(|y-y'|^\gamma + |s-s'|^{\gamma/2}\Bigr) , \tag{53}$$

whenever $t - s \ge \tau$ and $t - s' \ge \tau$. In particular, if $x, y \in \mathbf{R}^2$ and $t > 0$, the function $s \mapsto \Gamma_u(x,t;y,s)$ can be continuously extended to $s = 0$, and this extension (still denoted by $\Gamma_u$) satisfies (52), (53) with $s = 0$.

Now, fix $x \in \mathbf{R}^2$ and $t > 0$. Then for any $s \in (0,t)$ we have

$$\begin{aligned} \omega(x,t) &= \int_{\mathbf{R}^2} \Gamma_u(x,t;y,0)\omega(y,s)\,\mathrm{d}y \\ &+ \int_{\mathbf{R}^2} (\Gamma_u(x,t;y,s) - \Gamma_u(x,t;y,0))\omega(y,s)\,\mathrm{d}y . \end{aligned}$$

In view of (53), the second integral in the right-hand side converges to zero as $s \to 0+$, since $|\omega(\cdot,s)|_1 \le K$ for all $s > 0$. On the other hand, since $y \mapsto \Gamma_u(x,t;y,0)$ is continuous and vanishes at infinity, and since $\omega(\cdot,s)$ converges weakly to $\alpha\delta$ as $s \to 0+$, the first integral converges to $\alpha\Gamma_u(x,t;0,0)$. Thus

$$|\omega(x,t)| = |\alpha||\Gamma_u(x,t;0,0)| \le \frac{C_\beta|\alpha|}{t}\, e^{-\beta|x|^2/(4t)} , \quad x \in \mathbf{R}^2 \quad t > 0 .$$

Finally, let $w(\xi,\tau) = e^\tau \omega(\xi e^{\tau/2}, e^\tau)$ for $\xi \in \mathbf{R}^2$, $\tau \in \mathbf{R}$. Then $w \in C^0(\mathbf{R}, L^1(\mathbf{R}^2))$ is a solution of (12) which satisfies $|w(\xi,\tau)| \le C_\beta|\alpha|\, e^{-\beta|\xi|^2/4}$ for all $\xi \in \mathbf{R}^2$, $\tau \in \mathbf{R}$. In particular, $\{w(\cdot,\tau)\}_{\tau\in\mathbf{R}}$ is bounded in $L^2(m)$ for any $m > 1$, hence by Lemma 3.3 $w(\xi,\tau) = \alpha' G(\xi)$ for some $\alpha' \in \mathbf{R}$. Clearly $\alpha' = \alpha$, and the proof is complete. □

**Remark 3.6** *As is clear from the proof, Proposition 1.3 remains true if one assumes only that $\omega(\cdot,t)$ stays bounded in $L^1(\mathbf{R}^2)$ and that $\omega(\cdot,t_n)$ converges weakly to $\alpha\delta$ for some sequence $t_n \to 0$.*

**Remark 3.7** *A slight extension of the techniques developed in this section allows to prove that the Oseen vortices are stable in the sense of Lyapunov: for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for all initial data $w_0 \in L^1(\mathbf{R}^2)$ with $|w_0 - \alpha G|_1 \le \delta$, the solution of (12) satisfies $|w(\tau) - \alpha G|_1 \le \varepsilon$ for all $\tau \ge 0$. Note that this does not follow from Proposition 3.4.*

### 3.4 Convergence rate for positive solutions

If we restrict ourselves to *nonnegative* solutions of (12), then combining the entropy dissipation law (47) with a few classical inequalities in information theory we can obtain an explicit estimate on the time needed for a solution to approach the Oseen vortex. This is the so-called "entropy dissipation method", which is by now a classical approach in kinetic theory, see for instance [29, 1, 30].

Let $w \in L^1(\mathbf{R}^2)$, $w \geq 0$, and assume that $\alpha = \int_{\mathbf{R}^2} w(\xi)\,\mathrm{d}\xi > 0$. In information theory, the quantity $H(w)$ defined in (46) is called the *relative Kullback entropy* of $w$ with respect to the Gaussian $G$. Similarly, $I(w)$ defined in (48) is called the *relative Fisher information* of $w$ with respect to $G$. The difference between the entropy $H(w)$ and its minimal value $H(\alpha G)$ is bounded from below by the Csiszár-Kullback inequality

$$\frac{1}{2\alpha}\|w - \alpha G\|_{L^1}^2 \leq H(w) - H(\alpha G)\ , \tag{54}$$

and from above by the Stam-Gross logarithmic Sobolev inequality

$$H(w) - H(\alpha G) \leq I(w)\ . \tag{55}$$

Assume now that $w_0 \in L^2(m) \cap \Sigma_+$ for some $m > 3$, and that $\alpha = \int_{\mathbf{R}^2} w_0(\xi)\,\mathrm{d}\xi > 0$. Let $w \in C^0([0,\infty), L^2(m))$ be the solution of (12) with initial data $w_0$. Combining (47) with (55), we immediately obtain

$$H(w(\tau)) - H(\alpha G) \leq (H(w_0) - H(\alpha G))\, e^{-\tau}\ , \quad \tau \geq 0\ .$$

Applying (54), we conclude that

$$\|w(\tau) - \alpha G\|_{L^1} \leq \sqrt{2\alpha}\,(H(w_0) - H(\alpha G))^{1/2}\, e^{-\tau/2}\ , \quad \tau \geq 0\ . \tag{56}$$

This shows that $w(\tau)$ converges to $\alpha G$ at the rate $e^{-\tau/2}$, which is optimal in general (see Section 4). Moreover, (56) gives an explicit estimate of the prefactor in terms of the initial data. In particular, this provides an explicit upper bound of the time needed for the solution to enter a given neighborhood of the vortex.

Unfortunately, we do not know how to extend the entropy dissipation method to the general case where the vorticity may change sign. In the next section, we obtain local convergence rates by studying (12) in a neighborhood of the family of Oseen vortices, but this approach does not provide any explicit estimate in the sense of (56).

## 4 Local convergence rates

From the results of the previous section we know that any solution of the Navier-Stokes equation whose initial vorticity distribution lies in $L^1(\mathbf{R}^2)$ will converge toward the Oseen vortex with the same total vorticity. In the present section we show that for solutions in the weighted space $L^2(m)$ with $m > 1$ we can derive precise estimates on the rate at which solutions approach the vortices.

Our analysis proceeds by first analyzing the linearization of (12) at a vortex solution. We prove estimates on the location of the spectrum of linearized operator which in particular imply that the vortex solutions are spectrally stable for all values of the circulation Reynolds number. We then show that these bounds also imply decay estimates for the full nonlinear evolution in a neighborhood of the vortex.

### 4.1 Eigenvalue estimates

Fixing $\alpha \in \mathbf{R}$ and linearizing (12) around $w = \alpha G$, we find

$$\partial_\tau w + \alpha(\mathbf{v}^G \cdot \nabla w + \mathbf{v} \cdot \nabla G) \;=\; \Delta w + \frac{1}{2}\xi \cdot \nabla w + w \;, \tag{57}$$

where as usual $\mathbf{v}$ is the velocity field associated to $w$ via (14) and $\mathbf{v}^G$ is the velocity field of the Oseen vortex. This equation can be rewritten as $\partial_\tau w = \mathcal{L}w - \alpha\Lambda w$, where

$$\mathcal{L}w \;=\; \Delta w + \frac{1}{2}\xi \cdot \nabla w + w \;, \quad \Lambda w = \mathbf{v}^G \cdot \nabla w + \mathbf{v} \cdot \nabla G \;, \tag{58}$$

see (28), (29).

The linear operator $\mathcal{L}$ in $L^2(m)$ is studied in detail in ([14], Appendix A). It is defined on the maximal domain

$$\mathcal{D}_m(\mathcal{L}) = \Big\{ w \in L^2(m) \,\Big|\, \Delta w + \frac{1}{2}\xi \cdot \nabla w \in L^2(m) \Big\} \;.$$

If $w \in \mathcal{D}_m(\mathcal{L})$, one can show that $\Delta w \in L^2(m)$, so that $\mathcal{D}_m(\mathcal{L}) \subset H^2(m)$. The essential spectrum of $\mathcal{L}$ is given by

$$\sigma_m^{\mathrm{ess}}(\mathcal{L}) = \Big\{ \lambda \in \mathbf{C} \,\Big|\, \mathrm{Re}\,(\lambda) \le \frac{1-m}{2} \Big\} \;.$$

In addition, $\mathcal{L}$ has a sequence of eigenvalues $0, -1/2, -1, \ldots$ whose eigenfunctions are rapidly decreasing at infinity.

Because of the spatial decay of $\mathbf{v}^G$ and $G$, the operator $\Lambda$ is a relatively compact perturbation of $\mathcal{L}$ and hence $\sigma_m^{\mathrm{ess}}(\mathcal{L} - \alpha\Lambda)$ does not depend on $\alpha$. In particular, we can always push this essential spectrum far away from the imaginary axis by taking $m > 0$ sufficiently large. Thus the spectral stability of the vortex solutions will be determined by the isolated eigenvalues of $\mathcal{L} - \alpha\Lambda$ in $L^2(m)$. As we shall see, the corresponding eigenfunctions have a Gaussian decay at infinity so that, in contrast to the essential spectrum, these isolated eigenvalues do not depend on $m$.

We next observe that, due to symmetries of equation (2), some eigenvalues of $\mathcal{L} - \alpha\Lambda$ are in fact independent of $\alpha$. For instance, if $m > 1$, then $\lambda = 0$ is a simple eigenvalue of $\mathcal{L}$ in $L^2(m)$, with eigenfunction $G$. Since $\mathbf{v}^G \cdot \nabla G = 0$, it is clear from (58) that $\Lambda G = 0$, so that 0 is an eigenvalue of $\mathcal{L} - \alpha\Lambda$ for any $\alpha \in \mathbf{R}$. This zero eigenvalue is due to the fact that the Oseen vortices form a one-parameter family of equilibria of (12). The associated spectral projection $P_0$ reads (for any $\alpha$)

$$(P_0 w)(\xi) = G(\xi) \int_{\mathbf{R}^2} w(\xi')\,\mathrm{d}\xi' \;.$$

Thus, it will be sufficient to study the spectrum of $\mathcal{L} - \alpha\Lambda$ in the spectral subspace $L^2_0(m)$ defined in (15), which by (5) is also invariant under the nonlinear evolution (28).

Similarly, if $m > 2$, $\lambda = -1/2$ is a double eigenvalue of $\mathcal{L}$ with eigenfunctions $F_1, F_2$, where $F_j = -\partial_j G$ ($j = 1, 2$). Differentiating the identity $\mathbf{v}^G \cdot \nabla G = 0$ with respect to $\xi_j$, we see that $\Lambda F_j = 0$ ($j = 1, 2$). It follows that $-1/2$ is still an eigenvalue of $\mathcal{L} - \alpha\Lambda$ for any $\alpha \in \mathbf{R}$. This eigenvalue originates in the translation invariance of (2) with respect to $\xi \in \mathbf{R}^2$. The associated spectral projection $P_1$ reads (for any $\alpha$)

$$(P_1 w)(\xi) = F_1(\xi) \int_{\mathbf{R}^2} \xi_1' w(\xi')\, \mathrm{d}\xi' + F_2(\xi) \int_{\mathbf{R}^2} \xi_2' w(\xi')\, \mathrm{d}\xi' \;.$$

Thus, it is again sufficient to study the spectrum of $\mathcal{L} - \alpha\Lambda$ in the spectral subspace $L^2_1(m)$ defined by

$$L^2_1(m) \;=\; \Bigl\{ w \in L^2_0(m) \,\Big|\, \int_{\mathbf{R}^2} \xi_j w(\xi)\, \mathrm{d}\xi = 0 \text{ for } j = 1, 2 \Bigr\} \;, \tag{59}$$

which by (6) is also invariant under the nonlinear evolution (28).

Finally, if $m > 3$, $\lambda = -1$ is a triple eigenvalue of $\mathcal{L}$ with eigenfunctions $\Delta G$, $(\partial_1^2 - \partial_2^2)G$, and $\partial_1\partial_2 G$. Since $\Delta G = \frac{1}{4}(|\xi|^2 - 4)G$ is radially symmetric, it is clear that $\Lambda(\Delta G) = 0$, so that $-1$ is still an eigenvalue of $\mathcal{L} - \alpha\Lambda$ for any $\alpha \in \mathbf{R}$. This is due the fact that (2) is autonomous and invariant under the rescaling $\omega(x,t) \mapsto \lambda^2 \omega(\lambda x, \lambda^2 t)$. However, as we shall see, the eigenvalue $-1$ is *simple* if $\alpha \neq 0$. The associated spectral projection $P_2$ reads (for any $\alpha \neq 0$)

$$(P_2 w)(\xi) = \Delta G(\xi) \int_{\mathbf{R}^2} \frac{1}{4}(|\xi'|^2 - 4) w(\xi')\, \mathrm{d}\xi' \;.$$

Thus, if $\alpha \neq 0$, it is sufficient to study the spectrum of $\mathcal{L} - \alpha\Lambda$ in the spectral subspace $L^2_2(m)$ defined by

$$L^2_2(m) \;=\; \Bigl\{ w \in L^2_1(m) \,\Big|\, \int_{\mathbf{R}^2} |\xi|^2 w(\xi)\, \mathrm{d}\xi = 0 \Bigr\} \;, \tag{60}$$

which (as can be verified by a direct calculation) is also invariant under the nonlinear evolution (28).

The principal result of this subsection is:

**Proposition 4.1** *Fix $m > 1$ and $\alpha \in \mathbf{R}$. Then any eigenvalue $\lambda$ of $\mathcal{L} - \alpha\Lambda$ in $L^2_0(m)$ satisfies*

$$\mathrm{Re}\,(\lambda) \le \max\Bigl(-\frac{1}{2}\,,\; \frac{1-m}{2}\Bigr) \;. \tag{61}$$

*If moreover $m > 2$, then any eigenvalue $\lambda$ of $\mathcal{L} - \alpha\Lambda$ in $L^2_1(m)$ satisfies*

$$\mathrm{Re}\,(\lambda) \le \max\Bigl(-1\,,\; \frac{1-m}{2}\Bigr) \;. \tag{62}$$

**Remark 4.2** *In view of the preceding remarks estimates (61) and (62) are sharp. If $m > 3$ and $\alpha \neq 0$, the proof shows that any eigenvalue of $\mathcal{L} - \alpha\Lambda$ in $L^2_2(m)$ satisfies* $\mathrm{Re}\,(\lambda) < -1$, *but we are not able to give a sharp estimate in that case. Numerical calculations in [25] indicate that the eigenvalues that are not frozen by symmetries have a real part that converges to $-\infty$ as $|\alpha| \to \infty$, thereby suggesting that a fast rotation has a stabilizing effect on the vortex. Proposition 4.1 shows rigorously that at the spectral level perturbations of the vortex solutions decay at least as fast when $\alpha$ is large as when $\alpha = 0$.*

To prove Proposition 4.1, we proceed in three steps. First, we observe that the linear operators $\mathcal{L}$ and $\Lambda$ are invariant under the group of rotations $SO(2)$. Thus, using polar coordinates in $\mathbf{R}^2$ and expanding the angular variable in Fourier series, we reduce the eigenvalue equation for the operator $\mathcal{L}-\alpha\Lambda$ to a (nonlocal) ordinary differential equation in the radial variable. Next, a careful study of this differential equation reveals that, if $\lambda \in \mathbf{C}$ is an isolated eigenvalue of $\mathcal{L}-\alpha\Lambda$, the corresponding eigenfunction has a Gaussian decay at infinity. Finally, we prove that the operators $\mathcal{L}$ and $\Lambda$ are respectively self-adjoint and skew-symmetric in a weighted $L^2$ space with appropriate Gaussian weight, and Proposition 4.1 then follows from elementary considerations.

### 4.1.1 Polar coordinates

Fix $m > 0$. For any $n \in \mathbf{Z}$, let $P_n$ be the orthogonal projection in $L^2(m)$ defined by

$$\begin{aligned} (P_n w)(r\cos\theta, r\sin\theta) &= \omega_n(r)e^{in\theta} \ , \quad \text{where} \\ \omega_n(r) &= \frac{1}{2\pi}\int_0^{2\pi} w(r\cos\theta, r\sin\theta)e^{-in\theta} \,\mathrm{d}\theta \ . \end{aligned} \tag{63}$$

Clearly, $P_n P_{n'} = \delta_{nn'}P_n$ and $\sum_{n\in\mathbf{Z}} P_n = 1$. If $w \in L^2(m)$, the functions $\omega_n : \mathbf{R}_+ \to \mathbf{C}$ in (63) belong to the Hilbert space

$$Z(m) \;=\; \Big\{\omega : \mathbf{R}_+ \to \mathbf{C} \,\Big|\, \int_0^\infty r(1+r^2)^m |\omega(r)|^2 \,\mathrm{d}r < \infty\Big\} \ . \tag{64}$$

For any $n \in \mathbf{Z}$, let $\mathcal{L}_n$ be the linear operator on $Z(m)$ defined by

$$\mathcal{L}_n\omega \;=\; \partial_r^2\omega + \Big(\frac{r}{2}+\frac{1}{r}\Big)\partial_r\omega + \Big(1-\frac{n^2}{r^2}\Big)\omega \ . \tag{65}$$

Let also $\Lambda_n$ be the (bounded) linear operator on $Z(m)$ defined by $\Lambda_0 = 0$ and

$$\Lambda_n\omega \;=\; in(\varphi\omega - g\Omega) \ , \quad n \neq 0 \ , \tag{66}$$

where

$$\varphi(r) \;=\; \frac{1}{2\pi r^2}(1-e^{-r^2/4}) \ , \quad g(r) \;=\; \frac{1}{4\pi}\,e^{-r^2/4} \ ,$$

and

$$\Omega(r) \;=\; \frac{1}{4|n|}\left(\int_0^r \Big(\frac{z}{r}\Big)^{|n|} z\omega(z)\,\mathrm{d}z + \int_r^\infty \Big(\frac{r}{z}\Big)^{|n|} z\omega(z)\,\mathrm{d}z\right) \ . \tag{67}$$

It is easy to see that the operator $\Lambda_n$ is indeed well-defined:

**Lemma 4.3** *If $n \in \mathbf{Z}^*$ and $\omega \in Z(m)$ for some $m > 0$, then (67) defines a continuous function $\Omega : \mathbf{R}_+ \to \mathbf{C}$. Moreover, $\Omega(r)/r$ converges to zero as $r \to \infty$ and is at most logarithmically divergent as $r \to 0$.*

**Proof:** The proof is straightforward using (64), (67) and Hölder's inequality. □

We now show that $\mathcal{L}_n$ and $\Lambda_n$ are the expressions of $\mathcal{L}$ and $\Lambda$ in polar coordinates:

**Lemma 4.4** *The operators $\mathcal{L}$ and $\Lambda$ commute with the projections $P_n$. If $n \in \mathbf{Z}$ and $w \in \mathcal{D}_m(\mathcal{L})$ for some $m > 0$, then*

$$(\mathcal{L}P_n w)(r\cos\theta, r\sin\theta) \;=\; e^{in\theta}(\mathcal{L}_n\omega_n)(r)\ , \tag{68}$$
$$(\Lambda P_n w)(r\cos\theta, r\sin\theta) \;=\; e^{in\theta}(\Lambda_n\omega_n)(r)\ , \tag{69}$$

*where $\omega_n(r) = e^{-in\theta}(P_n w)(r\cos\theta, r\sin\theta)$.*

**Proof:** All we need is to prove (68) and (69). The first relation follows from (58) by an elementary calculation. To prove (69), assume that $w(r\cos\theta, r\sin\theta) = \omega_n(r)e^{in\theta}$ for some $n \in \mathbf{Z}$ and some $\omega_n \in Z(m)$. Then

$$\mathbf{v}^G \cdot \nabla w \;=\; \frac{1}{2\pi r}(1 - e^{-r^2/4})\frac{1}{r}\partial_\theta w \;=\; ine^{in\theta}\varphi\omega_n\ .$$

On the other hand, the velocity field $v$ corresponding to $w$ satisfies $\partial_1 v_2 - \partial_2 v_1 = w$, $\partial_1 v_1 + \partial_2 v_2 = 0$. In polar coordinates, these relations become

$$\frac{1}{r}\partial_r(rv_\theta) - \frac{1}{r}\partial_\theta v_r \;=\; w\ , \quad \frac{1}{r}\partial_r(rv_r) + \frac{1}{r}\partial_\theta v_\theta \;=\; 0\ .$$

We look for a solution of the form $v_r = \bar{v}_r(r)\, e^{in\theta}$, $v_\theta = \bar{v}_\theta(r)\, e^{in\theta}$. Then

$$(r\bar{v}_\theta)' - in\bar{v}_r \;=\; r\omega_n\ , \quad (r\bar{v}_r)' + in\bar{v}_\theta \;=\; 0\ .$$

Eliminating $\bar{v}_\theta$, we find the following ODE for $h = r\bar{v}_r$:

$$(rh')' - \frac{n^2}{r}h + inr\omega_n \;=\; 0\ .$$

The general solution is:

$$h(r) \;=\; \frac{in}{2|n|}\left(\int_0^r \left(\frac{z}{r}\right)^{|n|} z\omega_n(z)\,\mathrm{d}z + \int_r^\infty \left(\frac{r}{z}\right)^{|n|} z\omega_n(z)\,\mathrm{d}z\right) + A_1 r^n + A_2 r^{-n}\ ,$$

where $A_1, A_2 \in \mathbf{C}$. Since we want a velocity $\bar{v}_r = h/r$ that is locally integrable and converges to zero at infinity, we must choose $A_1 = A_2 = 0$. Setting $\Omega = \frac{1}{2in}h = \frac{1}{2in}r\bar{v}_r$, we finally obtain:

$$\mathbf{v} \cdot \nabla G \;=\; -\frac{1}{2}rv_r g \;=\; -ine^{in\theta}g\Omega\ .$$

This concludes the proof of (69). □

### 4.1.2 Gaussian decay of eigenfunctions

The aim of this paragraph is to prove:

**Lemma 4.5** *Fix $m > 0$, and assume that $w \in \mathcal{D}_m(\mathcal{L})$ satisfies $(\mathcal{L} - \alpha\Lambda)w = \mu w$, where $\alpha \in \mathbf{R}$ and $\mathrm{Re}\,(\mu) > \frac{1-m}{2}$. Then there exist $C > 0$ and $\gamma \geq 0$ such that*

$$|w(\xi)| \leq C(1 + |\xi|^2)^\gamma\, e^{-|\xi|^2/4}\ , \quad \xi \in \mathbf{R}^2\ .$$

**Proof:** We use the notations of the preceding paragraph. According to Lemma 4.4, we can assume that the eigenfunction $w$ satisfies $w = P_n w$ for some $n \in \mathbf{Z}$. Thus, there exists $\omega \in Z(m)$ such that $w(r\cos\theta, r\sin\theta) = \omega(r)e^{in\theta}$. In view of (65), (66) and Lemma 4.4, $\omega$ satisfies the (inhomogeneous) ordinary differential equation

$$\omega''(r) + \left(\frac{r}{2} + \frac{1}{r}\right)\omega'(r) + \left(1 - \mu - \frac{n^2}{r^2} - in\alpha\varphi\right)\omega(r) + in\alpha g\Omega(r) = 0 \;, \qquad (70)$$

where $\Omega$ is defined in (67).

The basic idea is now to use the classical results of Coddington and Levinson [11] to show that, for $r$ large, any solution of (70) either decays like $\omega(r) \sim r^{2\gamma}e^{-r^2/4}$ for some $\gamma \geq 0$, or like $\omega(r) \sim r^{2\mu-2}$ in which case it cannot belong to $L^2(m)$. However, a certain amount of preliminary work is needed in order to bring (70) into a form to which we can apply the results of [11].

We begin by introducing new variables $f$, $F$ and $t$ via the definitions

$$\omega(r) = f(r^2/4) \;,\; \Omega(r) = F(r^2/4) \;,\; t = r^2/4 \;.$$

In terms of these new variables, (70) takes the form

$$f''(t) + \left(1 + \frac{1}{t}\right)f'(t) + \left(\frac{1-\mu}{t} - a(t)\right)f(t) + b(t) = 0 \;,\quad t > 0 \;, \qquad (71)$$

where

$$a(t) = \frac{n^2}{4t^2} + \frac{in\alpha}{8\pi t^2}(1 - e^{-t}) \;,\quad b(t) = \frac{in\alpha}{4\pi t}e^{-t}F(t) \;.$$

Recall that we are interested in the behavior of solutions of this equation for $t$ large. We first consider the behavior of the homogeneous part of this equation and then construct the solution of the full equation via the method of variation of parameters.

**Lemma 4.6** *The linear, homogeneous equation*

$$\tilde{f}''(t) + \left(1 + \frac{1}{t}\right)\tilde{f}'(t) + \left(\frac{1-\mu}{t} - a(t)\right)\tilde{f}(t) = 0 \;,$$

*has two linearly independent solutions $\varphi_1(t)$ and $\varphi_2(t)$ such that*

$$\lim_{t\to\infty} t^{1-\mu}\begin{pmatrix}\varphi_1(t)\\ \varphi_1{}'(t)\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \;,\quad \lim_{t\to\infty} t^{\mu}e^{t}\begin{pmatrix}\varphi_2(t)\\ \varphi_2{}'(t)\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix} \;.$$

**Proof:** If we define $x(t) = \begin{pmatrix}\tilde{f}(t)\\ \tilde{f}'(t)\end{pmatrix}$ we can rewrite the differential equation in the lemma as

$$x'(t) = (A + V(t) + R(t))x(t) \;,$$

where

$$A = \begin{pmatrix}0 & 1\\ 0 & -1\end{pmatrix} \;,\; V(t) = \begin{pmatrix}0 & 0\\ -\frac{1-\mu}{t} & -\frac{1}{t}\end{pmatrix} \;,\; R(t) = \begin{pmatrix}0 & 0\\ a(t) & 0\end{pmatrix} \;.$$

But in this form, the lemma follows immediately from ([11], Theorem 3.8.1, p. 92). □

We now derive the asymptotic form of the solution $f(t)$ of (71) by applying the method of variation of parameters. Set

$$f(t) = A(t)\varphi_1(t) + B(t)\varphi_2(t) \ , \tag{72}$$

where $A'(t)\varphi_1(t) + B'(t)\varphi_2(t) = 0$. Then differentiating $f$ and using (71) we find

$$A'(t)\varphi_1'(t) + B'(t)\varphi_2'(t) + b(t) = 0 \ .$$

Solving for $A'$ and $B'$ we obtain

$$\begin{pmatrix} A'(t) \\ B'(t) \end{pmatrix} = \frac{1}{W(t)} \begin{pmatrix} b(t)\varphi_2(t) \\ -b(t)\varphi_1(t) \end{pmatrix} \ ,$$

where $W(t) = -\frac{1}{t}e^{-t}$ is the Wronskian determinant of $\varphi_1$ and $\varphi_2$. Integrating both sides of this equation we find

$$A(t) = A_1 - \int_1^t s e^s b(s)\varphi_2(s)\,\mathrm{d}s \ , \quad B(t) = B_1 + \int_1^t s e^s b(s)\varphi_1(s)\,\mathrm{d}s \ .$$

Recall from the definition of $b(t)$ that $e^s b(s) = \frac{in\alpha}{4\pi}\frac{F(s)}{s}$. From Lemma 4.3 we know that $F(s)/s = \Omega(2\sqrt{s})/s$ converges to zero as $s \to \infty$, so that $e^s b(s)$ is bounded for $s \ge 1$. Thus, the asymptotic behavior of $\varphi_2$ implies that $A_1(t) \to A_\infty$ as $t \to \infty$. If $A_\infty \neq 0$, then $f(t) \sim A_\infty t^{\mu-1}$ as $t \to \infty$ and hence (reverting to the original polar coordinates) $\omega(r) \sim \omega_\infty r^{2\mu-2}$. But since $\mathrm{Re}\,(\mu) > \frac{1-m}{2}$, this would imply that $\int_0^\infty r(1+r^2)^m|\omega(r)|^2\,\mathrm{d}r = \infty$ and this in turn would violate the hypothesis that the eigenfunction $w$ is in $L^2(m)$. Thus, $A_\infty = 0$, and $A(t) = \int_t^\infty s e^s b(s)\varphi_2(s)\,\mathrm{d}s$ from which we conclude that $|A(t)| \le Ce^{-t}t^\gamma$, for some $\gamma \ge 0$. In analogous fashion one proves that $|B(t)| \le Ct^\gamma$. Inserting these bounds on $A$ and $B$ into (72) and using the asymptotic estimates on $\varphi_1$ and $\varphi_2$ we conclude that there exists $\gamma \ge 0$ such that $|f(t)| \le Ct^\gamma e^{-t}$, for $t \ge 1$, or

$$|\omega(r)| \le Cr^{2\gamma}\, e^{-r^2/4} \ , \quad r \ge 1 \ .$$

This is the desired estimate, since $|w(\xi)| = |\omega(|\xi|)|$. The proof of Lemma 4.5 is now complete. $\square$

### 4.1.3 Localization of eigenvalues

Let $X$ denote the (complex) Hilbert space

$$X = \Big\{ w \in L^2(\mathbf{R}^2) \,\Big|\, G^{-1/2}w \in L^2(\mathbf{R}^2) \Big\} \ ,$$

equipped with the scalar product

$$(w_1, w_2)_X = \int_{\mathbf{R}^2} \frac{1}{G(\xi)}\, \bar{w}_1(\xi) w_2(\xi)\,\mathrm{d}\xi \ .$$

We also introduce the closed subspaces $X_0, X_1$ defined by

$$\begin{aligned} X_0 &= \Big\{ w \in X \,\Big|\, \int_{\mathbf{R}^2} w(\xi)\,\mathrm{d}\xi = 0 \Big\} = X \cap L^2_0(m) \ , \\ X_1 &= \Big\{ w \in X_0 \,\Big|\, \int_{\mathbf{R}^2} \xi_j w(\xi)\,\mathrm{d}\xi = 0 \text{ for } j = 1,2 \Big\} = X \cap L^2_1(m) \ . \end{aligned}$$

According to Lemma 4.5, if $\mu$ is an eigenvalue of $\mathcal{L} - \alpha\Lambda$ in $L^2(m)$ with $\mathrm{Re}\,(\mu) > \frac{1-m}{2}$, the corresponding eigenfunction belongs to $X$. This result is very useful because both operators $\mathcal{L}$ and $\Lambda$ have nice properties in this space.

**Lemma 4.7** *The linear operator $\mathcal{L}$ is self-adjoint in $X$, and $\mathcal{L} \le 0$. Morevoer, $\mathcal{L} \le -1/2$ on $X_0$ and $\mathcal{L} \le -1$ on $X_1$.*

**Proof:** Define $L : \mathcal{D}(L) \to L^2(\mathbf{R}^2)$ by $\mathcal{D}(L) = \{\psi \in H^2(\mathbf{R}^2) \,|\, |\xi|^2\psi \in L^2(\mathbf{R}^2)\}$ and

$$L \,=\, G^{-1/2}\mathcal{L}\,G^{1/2} \,=\, \Delta - \frac{|\xi|^2}{16} + \frac{1}{2}\ .$$

In quantum mechanics, the operator $-L$ is (up to numerical constants) the Hamiltonian of the harmonic oscillator in $\mathbf{R}^2$. As is well-known (see for example [19]), $L$ is self-adjoint in $L^2(\mathbf{R}^2)$ and $\sigma(L) = \{-n/2 \,|\, n \in \mathbf{N}\}$. By construction, the operator $\mathcal{L} : \mathcal{D}(\mathcal{L}) \to X$ with domain

$$\mathcal{D}(\mathcal{L}) = \Big\{ w \in X \,\Big|\, |\xi|^2 w \in X\ ,\ \Delta w + \frac{1}{2}\xi\cdot\nabla w \in X \Big\}\ ,$$

is thus self-adjoint in $X$ with the same spectrum. In particular, $\mathcal{L} \le 0$. Now, observe that 0 is a simple eigenvalue of $\mathcal{L}$ with eigenfunction $G$, and that $X_0$ is just the orthogonal complement of the eigenspace $\mathbf{R}G$ in $X$. Thus $X_0$ is stable under $\mathcal{L}$ and the restriction of $\mathcal{L}$ to $X_0$ is a self-adjoint operator satisfying $\mathcal{L} \le -1/2$. Similarly, one can show that $\mathcal{L} \le -1$ on $X_1$. $\square$

**Lemma 4.8** *The linear operator $\Lambda$ is skew-symmetric in $X$.*

**Proof:** Since $\Lambda G = 0$ and since the subspace $X_0$ is stable under $\Lambda$, it is sufficient to show that $\Lambda$ is skew-symmetric on $X_0$. Let $w, \tilde{w} \in X_0 \cap \mathcal{D}(\mathcal{L})$, and denote by $\mathbf{v}, \tilde{\mathbf{v}}$ the corresponding velocity fields. Without loss of generality, we assume that $w, \tilde{w}$ (hence also $\mathbf{v}, \tilde{\mathbf{v}}$) are real functions. Then

$$(\tilde{w}, \Lambda w)_X = \int_{\mathbf{R}^2} \left( \frac{1}{G}\,\tilde{w}\mathbf{v}^G\cdot\nabla w - \frac{1}{2}\tilde{w}(\mathbf{v}\cdot\xi)\right) \mathrm{d}\xi\ ,$$

because $\nabla G = -\frac{\xi}{2}G$. Observe that $G^{-1}\mathbf{v}^G$ is a divergence free vector field, so that

$$\int_{\mathbf{R}^2} \frac{1}{G}\,\tilde{w}\mathbf{v}^G\cdot\nabla w\,\mathrm{d}\xi = -\int_{\mathbf{R}^2}\frac{1}{G}\,w\mathbf{v}^G\cdot\nabla\tilde{w}\,\mathrm{d}\xi\ . \tag{73}$$

On the other hand, the following identity is easy to check:

$$\tilde{w}(\mathbf{v}\cdot\xi) + w(\tilde{\mathbf{v}}\cdot\xi) \,=\, (\xi_1\partial_1 - \xi_2\partial_2)(v_1\tilde{v}_2 + v_2\tilde{v}_1) + (\xi_1\partial_2 + \xi_2\partial_1)(v_2\tilde{v}_2 - v_1\tilde{v}_1)\ .$$

Since $w, \tilde{w}$ have zero mean, it follows from ([14], Appendix B) that $\mathbf{v}, \tilde{\mathbf{v}}$ decay at least like $1/|\xi|^2$ as $\xi \to \infty$. Thus, integrating both sides, we obtain

$$\int_{\mathbf{R}^2} (\tilde{w}(\mathbf{v}\cdot\xi) + w(\tilde{\mathbf{v}}\cdot\xi))\,\mathrm{d}\xi \,=\, 0\ . \tag{74}$$

Combining (73) and (74), we see that $(\tilde{w}, \Lambda w)_X + (\Lambda\tilde{w}, w)_X = 0$. $\square$

**Proof of Proposition 4.1.** Fix $m > 1$ and assume that $\lambda$ is an eigenvalue of $\mathcal{L} - \alpha\Lambda$ in $L^2_0(m)$ with $\mathrm{Re}\,(\lambda) > \frac{1-m}{2}$. By Lemma 4.5, there exists a nonzero $w \in X_0 \cap \mathcal{D}(\mathcal{L})$ such that $(\mathcal{L} - \alpha\Lambda)w = \lambda w$. In particular,

$$\lambda(w,w)_X \;=\; (w,\mathcal{L}w)_X - \alpha(w,\Lambda w)_X \;,$$

hence

$$\mathrm{Re}\,(\lambda)(w,w)_X \;=\; (w,\mathcal{L}w)_X \;\le\; -\frac{1}{2}(w,w)_X \;,$$

since $\Lambda$ is skew-symmetric and $\mathcal{L} \le -1/2$ on $X_0$. Thus, $\mathrm{Re}\,(\lambda) \le -1/2$.

Similarly, if $m > 2$ and $\lambda$ is an eigenvalue of $\mathcal{L} - \alpha\Lambda$ in $L^2_1(m)$ with $\mathrm{Re}\,(\lambda) > \frac{1-m}{2}$, there exists a nonzero $w \in X_1 \cap \mathcal{D}(\mathcal{L})$ such that $(\mathcal{L} - \alpha\Lambda)w = \lambda w$. Proceeding as above and using the fact that $\mathcal{L} \le -1$ on $X_1$, we obtain $\mathrm{Re}\,(\lambda) \le -1$. □

**Remark 4.9** *If $m > 3$ and $\lambda$ is an eigenvalue of $\mathcal{L} - \alpha\Lambda$ in $L^2_2(m)$ with $\mathrm{Re}\,(\lambda) > \frac{1-m}{2}$, there exists a nonzero $w \in \mathcal{D}(\mathcal{L}) \cap L^2_2(m)$ such that $(\mathcal{L} - \alpha\Lambda)w = \lambda w$. The argument above shows that $\mathrm{Re}\,(\lambda) \le -1$, and that $\mathrm{Re}\,(\lambda) = -1$ if and only if $\mathcal{L}w = -w$. But this implies that $w$ is a linear combination of $(\partial_1^2 - \partial_2^2)G$ and $\partial_1\partial_2 G$, and a direct calculation shows that no such $w$ can be an eigenfunction of $\mathcal{L} - \alpha\Lambda$ if $\alpha \ne 0$. Thus any eigenfunction $\lambda$ of $\mathcal{L} - \alpha\Lambda$ in $L^2_2(m)$ satisfies $\mathrm{Re}\,(\lambda) < -1$ if $\alpha \ne 0$.*

## 4.2 Bounds on the linear evolution

In this subsection we prove that the eigenvalue estimates of the previous subsection imply analogous bounds on the linear evolution. Fix $\alpha \in \mathbf{R}$ and consider the linear equation $\partial_\tau R = \mathcal{L}R - \alpha\Lambda R$ which is the linearization of (12) a the vortex $\alpha G$, see (28), (57). The corresponding integral equation reads

$$R(\tau) = \mathcal{S}(\tau)R_0 - \alpha\int_0^\tau \mathcal{S}(\tau - s)\Lambda R(s)\,\mathrm{d}s \;, \tag{75}$$

where $\mathcal{S}(\tau) = \exp(\tau\mathcal{L})$. Proceeding as in Section 2.1, it is straightforward to show by a contraction mapping argument that this equation defines a strongly continuous semigroup $\mathcal{T}_\alpha(\tau)$ in $L^2(m)$ for any $m > 1$, namely $R(\tau) = \mathcal{T}_\alpha(\tau)R_0$. We first prove that $\mathcal{T}_\alpha(\tau)$ is a compact perturbation of $\mathcal{S}(\tau)$.

**Lemma 4.10** *Let $m > 1$. The linear operator $\mathcal{K}_\alpha(\tau) = \mathcal{T}_\alpha(\tau) - \mathcal{S}(\tau)$ is compact in $L^2(m)$ for each $\tau > 0$.*

**Proof:** All the necessary estimates are already contained in Section 2.1. Observe that the term $\mathcal{K}_\alpha(\tau)R_0$ in (75) is precisely what we called $R_2(\tau) + R_3(\tau)$ in the proof of Lemma 2.2. Repeating the estimates proved there, we obtain

$$\|\mathcal{K}_\alpha(\tau)R_0\|_{m+1} \le C(\tau)\|R_0\|_m \;, \quad \tau > 0 \;,$$

for some $C(\tau) > 0$. Similarly, the result of Lemma 2.1 applies to $\mathcal{T}_\alpha(\tau)R_0$, hence to $\mathcal{K}_\alpha(\tau)R_0$, and yields

$$\|\nabla\mathcal{K}_\alpha(\tau)R_0\|_m \le \frac{C(\tau)}{a(\tau)^{1/2}}\,\|R_0\|_m \;, \quad \tau > 0 \;,$$

where $a(\tau) = 1 - e^{-\tau}$. The conclusion then follows from Rellich's criterion. $\square$

We now use this result to bound the essential spectrum of $\mathcal{T}_\alpha(\tau)$. We recall that $\lambda \in \mathbf{C}$ is in the essential spectrum of a linear operator $L$ if $\lambda$ is not a *normal point* for $L$, i.e. if $\lambda$ is not in the resolvent set of $L$ and is not an isolated eigenvalue of $L$ of finite multiplicity, see e.g. [20]. For any $r > 0$, we denote by $\mathcal{B}(r)$ the closed disk of radius $r$ centered at the origin in $\mathbf{C}$:

$$\mathcal{B}(r) \;=\; \{z \in \mathbf{C} \,|\, |z| \le r\}\;.$$

**Lemma 4.11** *Let $m > 1$. For any $\tau > 0$, the essential spectrum of $\mathcal{T}_\alpha(\tau)$ in $L^2(m)$ satisfies*

$$\sigma_m^{\mathrm{ess}}(\mathcal{T}_\alpha(\tau)) \;=\; \mathcal{B}(e^{\tau(1-m)/2})\;. \tag{76}$$

**Proof:** Fix $\tau > 0$. The results of ([14], Appendix A) imply that the essential spectrum of $\mathcal{S}(\tau)$ in $L^2(m)$ is exactly $\mathcal{B}(e^{\tau(1-m)/2})$. Since $\mathcal{T}_\alpha(\tau)$ is a compact perturbation of $\mathcal{S}(\tau)$, it follows from ([20], Theorem A.1) that the complement of the ball $\mathcal{B}(e^{\tau(1-m)/2})$ in $\mathbf{C}$ either consists entirely of eigenvalues of $\mathcal{T}_\alpha(\tau)$, or entirely of normal points for $\mathcal{T}_\alpha(\tau)$. In our case, the first possibility is excluded. Indeed, assume that $\lambda \in \mathbf{C}$ is an eigenvalue of $\mathcal{T}_\alpha(\tau)$ with $|\lambda| > 1$. By the spectral mapping theorem, $\lambda = e^\nu$ where $\nu$ is an eigenvalue of $\mathcal{L} - \alpha\Lambda$ in $L^2(m)$ with $\mathrm{Re}\,(\nu) > 0$, which contradicts Proposition 4.1. Thus, $\sigma_m^{\mathrm{ess}}(\mathcal{T}_\alpha(\tau)) \subset \mathcal{B}(e^{\tau(1-m)/2})$, and since $\mathcal{S}(\tau)$ is also a compact perturbation of $\mathcal{T}_\alpha(\tau)$ the same argument shows that $\sigma_m^{\mathrm{ess}}(\mathcal{T}_\alpha(\tau)) = \mathcal{B}(e^{\tau(1-m)/2})$. $\square$

By construction, the spectral subspaces $L_0^2(m)$ and $L_1^2(m)$ are left invariant by the semigroup $\mathcal{T}_\alpha(\tau)$. Combining Lemma 4.11 with the eigenvalue estimates of Section 4.1, we obtain precise bounds on the growth of $\mathcal{T}_\alpha(\tau)$ in these subspaces:

**Proposition 4.12** *Fix $\alpha \in \mathbf{R}$. Assume that either*
**a)** *$0 < \mu \le 1/2$ and $R_0 \in L_0^2(m)$ for some $m > 1 + 2\mu$, or*
**b)** *$1/2 < \mu \le 1$ and $R_0 \in L_1^2(m)$ for some $m > 1 + 2\mu$.*
*Then there exists $C > 0$ (independent of $R_0$) such that*

$$\|\mathcal{T}_\alpha(\tau)R_0\|_m \le C\, e^{-\mu\tau}\|R_0\|_m\;, \quad \tau \ge 0\;. \tag{77}$$

**Proof:** Assume first that $0 < \mu < 1/2$. By Lemma 4.11, the essential spectrum of $\mathcal{T}_\alpha(1)$ in $L_0^2(m)$ satisfies $\sigma_m^{\mathrm{ess}}(\mathcal{T}_\alpha(1)) = \mathcal{B}(e^{(1-m)/2})$, and $e^{(1-m)/2} < e^{-\mu}$ since $m > 1 + 2\mu$. If $\lambda \in \sigma(\mathcal{T}_\alpha(1))$ satisfies $|\lambda| > e^{(1-m)/2}$, then $\lambda$ is an isolated eigenvalue of $\mathcal{T}_\alpha(1)$ and (by the spectral mapping theorem) there exists an eigenvalue $\nu$ of $\mathcal{L} - \alpha\Lambda$ in $L_0^2(m)$ such that $e^\nu = \lambda$. Applying Proposition 4.1, we obtain $\mathrm{Re}\,(\nu) \le 1/2$, hence $|\lambda| \le e^{-1/2} < e^{-\mu}$. Thus the spectral radius of $\mathcal{T}_\alpha(1)$ in $L_0^2(m)$ is strictly less than $e^{-\mu}$, and (77) follows (see e.g. [13], Proposition IV.2.2). A similar argument proves (77) if $R_0 \in L_1^2(m)$ and $1/2 < \mu < 1$.

Now, assume that $\mu = 1/2$ and $m > 2$. Any $R_0 \in L_0^2(m)$ can be decomposed as $R_0 = \beta_1 F_1 + \beta_2 F_2 + \tilde{R}_0$, where $\beta_j = \int_{\mathbf{R}^2} \xi_j R_0 \,\mathrm{d}\xi$, $F_j = -\partial_j G$, and $\tilde{R}_0 \in L_1^2(m)$. It follows that

$$\mathcal{T}_\alpha(\tau)R_0 = e^{-\tau/2}(\beta_1 F_1 + \beta_2 F_2) + \mathcal{T}_\alpha(\tau)\tilde{R}_0\;,$$

and we already know that $\|\mathcal{T}_\alpha(\tau)\tilde{R}_0\|_m \le Ce^{-\nu\tau}\|\tilde{R}_0\|_m$ for some $\nu > 1/2$. Thus (77) holds and is sharp in this case. A similar argument shows that (77) holds with $\mu = 1$ if $R_0 \in L_1^2(m)$ for some $m > 3$. $\square$

Finally, we will need in the following subsection $L^p$-$L^q$ estimates of $\mathcal{T}_\alpha(\tau)$ and its derivatives, in the spirit of (23).

**Proposition 4.13** *Under the assumptions of Proposition 4.12, if $R_0$ satisfies in addition $b^m R_0 \in L^q(\mathbf{R}^2)$ for some $q \in (1,2)$, then*

$$\|\mathcal{T}_\alpha(\tau)R_0\|_m \le \frac{C\,e^{-\mu\tau}}{a(\tau)^{\frac{1}{q}-\frac{1}{2}}}\,|b^m R_0|_q \ , \quad \|\nabla\mathcal{T}_\alpha(\tau)R_0\|_m \le \frac{C\,e^{-\mu\tau}}{a(\tau)^{\frac{1}{q}}}\,|b^m R_0|_q \ , \quad \tau > 0 \ .$$

**Proof:** Fix $1 < q < 2$ and assume that $b^m R_0 \in L^q(\mathbf{R}^2)$. Given $T > 0$, we consider the function space

$$\begin{aligned} Y &= \{R \in C^0((0,T], H^1(m)) \,|\, \|R\|_Y < \infty\} \ , \quad \text{where} \\ \|R\|_Y &= \sup_{0<\tau\le T}\Big\{a(\tau)^{\frac{1}{q}-\frac{1}{2}}\|R(\tau)\|_m + a(\tau)^{\frac{1}{q}}\|\nabla R(\tau)\|_m\Big\} \ . \end{aligned}$$

Using (23) and proceeding as in the proof of Lemma 2.1, it is straightforward to show that, if $T > 0$ is sufficiently small, there exists $C > 0$ such that (75) has a unique solution in the ball of radius $C|b^m R_0|_q$ centered at the origin in $Y$. This proves the desired estimates for $0 < \tau \le T$, and the general case follows from Proposition 4.12 if one uses the semigroup property and the fact that the subspaces $L^2_0(m)$ and $L^2_1(m)$ are left invariant by $\mathcal{T}_\alpha(\tau)$. $\square$

## 4.3 Bounds on the nonlinear evolution

In this subsection, we show that our estimates on the linear semigroup $\mathcal{T}_\alpha(\tau)$ generated by $\mathcal{L} - \alpha\Lambda$ imply similar bounds on the full nonlinear evolution (28). As is easy to verify, the integral equation (30) satisfied by $R(\tau)$ is equivalent to

$$R(\tau) = \mathcal{T}_\alpha(\tau)R_0 - \int_0^\tau \mathcal{T}_\alpha(\tau-s)(\mathbf{v}^R(s)\cdot\nabla R(s))\,\mathrm{d}s \ . \tag{78}$$

The following result implies Proposition 1.6 as a particular case:

**Proposition 4.14** *Under the assumptions of Proposition 4.12, the solution $R(\tau)$ of (28) with initial data $R_0$ satisfies $\|R(\tau)\|_m = \mathcal{O}(e^{-\mu\tau})$ as $\tau \to \infty$.*

**Proof:** Let $R \in C^0([0,+\infty), L^2_0(m))$ be the solution of (28) with initial data $R_0$. We know from Proposition 1.5 that $\|R(\tau)\|_m$ converges to zero as $\tau \to \infty$, hence we can assume without loss of generality that $\|R_0\|_m$ is small. Given $T > 0$, we define

$$M(T) = \max\Big\{\sup_{0\le\tau\le T} e^{\mu\tau}\|R(\tau)\|_m \, , \, \sup_{0<\tau\le T} a(\tau)^{\frac{1}{2}}\,e^{\mu\tau}\|\nabla R(\tau)\|_m\Big\} \ .$$

By Lemma 2.1, $M(T) < \infty$ and there exists $C_0 > 0$ such that $M(T) \le C_0\|R_0\|_m$ if $T$ is sufficiently small. Applying Propositions 4.12 and 4.13, and using the fact that the subspaces $L^2_0(m)$ and $L^2_1(m)$ are stable under the nonlinearity $\mathbf{v}^R\cdot\nabla R$, we obtain from (78)

$$\|R(\tau)\|_m \le C\,e^{-\mu\tau}\|R_0\|_m \ + C\int_0^\tau e^{-\mu(\tau-s)}\frac{1}{a(\tau-s)^{\frac{1}{q}-\frac{1}{2}}}|b^m\mathbf{v}^R(s)\cdot\nabla R(s)|_q\,\mathrm{d}s \ ,$$

where $1 < q < 2$. As in the proof of Lemma 2.1, we can bound

$$|b^m \mathbf{v}^R(s) \cdot \nabla R(s)|_q \le C \|R(s)\|_m \|\nabla R(s)\|_m \le \frac{C\, e^{-2\mu s}}{a(s)^{\frac{1}{2}}} M(T)^2 ,$$

hence

$$\begin{aligned} \|R(\tau)\|_m &\le e^{-\mu\tau}\Big( C\|R_0\|_m + C M(T)^2 \int_0^\tau \frac{e^{-\mu s}}{a(\tau - s)^{\frac{1}{q}-\frac{1}{2}} a(s)^{\frac{1}{2}}}\, \mathrm{d}s \Big) \\ &\le e^{-\mu\tau}\Big( C_1 \|R_0\|_m + C_2 M(T)^2 \Big) , \quad 0 \le \tau \le T , \end{aligned}$$

where $C_1, C_2 > 0$ are independent of $T$. Without loss of generality, we assume in what follows that $C_1 \ge C_0$. Differentiating (78) and using similar estimates, we also obtain

$$\|\nabla R(\tau)\|_m \le \frac{e^{-\mu\tau}}{a(\tau)^{\frac{1}{2}}} \Big( C_1 \|R_0\|_m + C_2 M(T)^2 \Big) , \quad 0 < \tau \le T .$$

Summarizing, we have shown

$$M(T) \le C_1 \|R_0\|_m + C_2 M(T)^2 . \tag{79}$$

Now, assume that $R_0$ is small enough so that $4C_1C_2\|R_0\|_m < 1$. As $C_1 \ge C_0$, we also have $C_0\|R_0\|_m \le \bar{M}$, where

$$\bar{M} = \frac{1}{2C_2}\Big( 1 - \sqrt{1 - 4C_1C_2\|R_0\|_m} \Big) > 0 .$$

Thus $M(T) \le \bar{M}$ for $T > 0$ sufficiently small, and since $M(T)$ depends continuously on $T$ it follows from (79) that $M(T) \le \bar{M}$ for all $T > 0$. In particular, $\|R(\tau)\|_m \le \bar{M}\, e^{-\mu\tau}$ for all $\tau \ge 0$. □

**Remark 4.15** *Similarly, if $R_0 \in L^2_2(m)$ for some $m > 3$, the solution $R(\tau)$ of (28) with initial data $R_0$ satisfies $\|R(\tau)\|_m = \mathcal{O}(e^{-\mu\tau})$ as $\tau \to \infty$, for some $\mu \ge 1$ depending on $\alpha$. We know from Remark 4.9 that $\mu > 1$ if $\alpha \ne 0$, but we have no sharp result in that case.*

**Remark 4.16** *It follows directly from the proof that the Oseen vortices are stable equilibria in $L^2(m)$, see Remark 3.7.*

We conclude this section by showing that, in the case where the total vorticity is nonzero, there is no loss of generality in assuming that the perturbations of the vortex have vanishing first order moments. Indeed, assume that $1/2 < \mu \le 1$ and that $w_0 \in L^2(m)$ for some $m > 1 + 2\mu$. Let $w \in C^0([0,\infty), L^2(m)$ be the solution of (12) with initial data $w_0$. For any $b \in \mathbf{R}^2$, the function $\bar{w}(\xi,\tau) = w(\xi + b\,e^{-\tau/2}, \tau)$ is again a solution of (12) (because the original equation (2) is translation invariant in $x \in \mathbf{R}^2$). If $\alpha = \int_{\mathbf{R}^2} w_0(\xi)\,\mathrm{d}\xi \ne 0$, we can choose $(b_1, b_2) = \alpha^{-1}(\beta_1, \beta_2)$, where $\beta_i = \int_{\mathbf{R}^2} \xi_i w_0(\xi)\,\mathrm{d}\xi$. Then $\int_{\mathbf{R}^2} \xi_i \bar{w}(\xi, 0)\,\mathrm{d}\xi = 0$ for $i = 1, 2$, so that $\bar{w}(\cdot, 0) - \alpha G \in L^2_1(m)$. Applying Proposition 4.14, we obtain $\|\bar{w}(\tau) - \alpha G\|_m = \mathcal{O}(e^{-\mu\tau})$ as $\tau \to \infty$. Returning to the original function $w(\xi,\tau)$ and using a straightforward Taylor expansion, we obtain the second order asymptotics

$$\|w(\tau) - \alpha G - (\beta_1 F_1 + \beta_2 F_2)\, e^{-\tau/2}\|_m = \mathcal{O}(e^{-\mu\tau}), \quad \tau \to \infty , \tag{80}$$

where $F_j = -\partial_j G$, $j = 1, 2$. As was already mentioned, this result still holds when $\alpha = 0$ except that, if $\mu = 1$ and $(\beta_1, \beta_2) \ne (0, 0)$, the right-hand side of (80) should be replaced by $\mathcal{O}(\tau e^{-\tau})$.

**Remark 4.17** *Assume that $m > 3$, $\alpha \neq 0$, and $\int_{\mathbf{R}^2} |\xi|^2 w_0(\xi)\,\mathrm{d}\xi \neq 0$. If we translate and rescale $w_0$ appropriately, we can produce a new initial condition $\bar{w}_0$ with $\bar{w}_0 - \alpha G \in L^2_2(m)$. By Remark 4.15, the corresponding solution then satisfies $\|\bar{w}(\tau) - \alpha G\|_m = \mathcal{O}(e^{-\mu\tau})$ as $\tau \to \infty$ for some $\mu > 1$. Moreover $w(\xi,\tau)$ and $\bar{w}(\xi,\tau)$ are linked by a simple relation, due to the fact that the original equation (2) is translation and dilation invariant. Using this relation, we obtain that the the next correction to the asymptotics (80) is of the form $\gamma(\Delta G)\,e^{-\tau}$, for some $\gamma \in \mathbf{R}$.*

**Remark 4.18** *The connection of the translation invariance of the Navier-Stokes equation with the decay associated to the first moment of the vorticity seems first to have been remarked upon by Bernoff and Lingevitch, [3]. The connection between symmetries of the linear and nonlinear heat equation and Burgers' equation and the decay rates of the long-time asymptotics of solutions of these equations was systematically explored in [31] and [23].*

# References


[1] A. Arnold, P. Markowich, G. Toscani, and A. Unterreiter. On convex Sobolev inequalities and the rate of convergence to equilibrium for Fokker-Planck type equations. *Comm. Partial Differential Equations*, 26(1-2):43–100, 2001.

[2] M. Ben-Artzi. Global solutions of two-dimensional Navier-Stokes and Euler equations. *Arch. Rational Mech. Anal.*, 128(4):329–358, 1994.

[3] A. J. Bernoff and J. F. Lingevitch. Rapid relaxation of an axisymmetric vortex. *Phys. Fluids*, 6(11):3717–3721, November 1994.

[4] L. Brandolese. *Localisation, Oscillations et Comportement Asymptotique pour les Équations de Navier-Stokes.* PhD thesis, École Normale Supérieure de Cachan, 2001.

[5] L. Brandolese. On the localization of symmetric and asymmetric solutions of the Navier-Stokes equations in $\mathbf{R}^n$. *C. R. de l'Acad. de Science de Paris*, t. 332(Série 1):125–130, 2001.

[6] L. Brandolese. Space-time decay of Navier-Stokes flows invariant under rotations. Preprint, April 2003. arXiv:math.AP/0304436.

[7] H. Brezis. Remarks on the preceding paper by M. Ben-Artzi: "Global solutions of two-dimensional Navier-Stokes and Euler equations". *Arch. Rational Mech. Anal.*, 128(4):359–360, 1994.

[8] M. Cannone and F. Planchon. Self-similar solutions for Navier-Stokes equations in $\mathbf{R}^3$. *Commun. Partial Differ. Equations*, 21(1-2):179–193, 1996.

[9] E. A. Carlen and M. Loss. Optimal smoothing and decay estimates for viscously damped conservation laws, with applications to the 2-D Navier-Stokes equation. *Duke Math. J.*, 81(1):135–157 (1996), 1995.

[10] A. Carpio. Asymptotic behavior for the vorticity equations in dimensions two and three. *Comm. Partial Differential Equations*, 19(5-6):827–872, 1994.

[11] E. A. Coddington and N. Levinson. *Theory of ordinary differential equations.* McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955.

[12] G.-H. Cottet. Équations de Navier-Stokes dans le plan avec tourbillon initial mesure. *C. R. Acad. Sci. Paris Sér. I Math.*, 303(4):105–108, 1986.

[13] K.-J. Engel and R. Nagel. *One-Parameter Semigroups for Linear Evolution Equations.* Graduate Texts in Mathematics. Springer-Verlag, New York, 2000.

[14] Th. Gallay and C. E. Wayne. Invariant manifolds and the long-time asymptotics of the Navier-Stokes and vorticity equations on $\mathbf{R}^2$. *Arch. Ration. Mech. Anal.*, 163(3):209–258, 2002.

[15] Th. Gallay and C. E. Wayne. Long-time asymptotics of the Navier-Stokes and vorticity equations on $\mathbb{R}^3$. *R. Soc. Lond. Philos. Trans. Ser. A Math. Phys. Eng. Sci.*, 360(1799):2155–2188, 2002. Recent developments in the mathematical theory of water waves (Oberwolfach, 2001).

[16] M.-H. Giga and Y. Giga. *Nonlinear partial differential equations : asymptotic behaviour of solutions and self-similar solutions.* Book in preparation.

[17] Y. Giga and T. Kambe. Large time behavior of the vorticity of two-dimensional viscous flow and its application to vortex formation. *Comm. Math. Phys.*, 117(4):549–568, 1988.

[18] Y. Giga, T. Miyakawa, and H. Osada. Two-dimensional Navier-Stokes flow with measures as initial vorticity. *Arch. Rational Mech. Anal.*, 104(3):223–250, 1988.

[19] J. Glimm and A. Jaffe. *Quantum physics.* Springer-Verlag, New York, second edition, 1987. A functional integral point of view.

[20] D. Henry. *Geometric theory of semilinear parabolic equations.* Springer-Verlag, Berlin, 1981.

[21] T. Kato. The Navier-Stokes equation for an incompressible fluid in $\mathbf{R}^2$ with a measure as the initial vorticity. *Differential Integral Equations*, 7(3-4):949–966, 1994.

[22] P.-L. Lions and C. Villani. Régularité optimale de racines carrées. *C. R. Acad. Sci. Paris Sér. I Math.*, 321(12):1537–1541, 1995.

[23] J. Miller and A. Bernoff. Rates on convergence to self-similar solutions of Burgers' equation. *Stud. Appl. Math.*, 111(1):29–40, 2003.

[24] H. Osada. Diffusion processes with generators of generalized divergence form. *J. Math. Kyoto Univ.*, 27(4):597–619, 1987.

[25] A. Prochazka and D. I. Pullin. On the two-dimensional stability of the axisymmetric Burgers vortex. *Phys. Fluids*, 7(7):1788–1790, 1995.

[26] M. H. Protter and H. F. Weinberger. *Maximum principles in differential equations.* Prentice-Hall Inc., Englewood Cliffs, N.J., 1967.

[27] M. Reed and B. Simon. *Methods of modern mathematical physics. IV. Analysis of operators.* Academic Press, New York, 1978.

[28] L. Rossi and J. Graham-Eagle. On the existence of two-dimensional, localized, rotating, self-similar vortical structures. *SIAM J. Appl. Math.*, 62(6):2114–2128, 2002.

[29] G. Toscani. Entropy production and the rate of convergence to equilibrium for the Fokker-Planck equation. *Quart. Appl. Math.*, 57(3):521–541, 1999.

[30] C. Villani. A review of mathematical topics in collisional kinetic theory. In *Handbook of mathematical fluid dynamics, Vol. I*, pages 71–305. North-Holland, Amsterdam, 2002.

[31] Th. P. Witelski and A. J. Bernoff. Self-similar asymptotics for linear and nonlinear diffusion equations. *Studies in Applied Mathematics*, 100:153–193, 1998.