# Long-time asymptotics of the Navier-Stokes equation in $\mathbb{R}^2$ and $\mathbb{R}^3$


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## 1 Introduction

Understanding the long-time evolution of fluid motions is facilitated by studying the coherent structures of the flow. From an experimental perspective this has long been recognized since visualizations of complicated flows often exhibit very obvious structures. Some of the most commonly observed structures are vortices. These vortices may appear initially through some instability in an underlying steady flow and then persist even after the flow becomes turbulent.

From a mathematical point of view, identifying simple, persistent features of the solutions of the equations describing fluid motions can also facilitate their analysis. These features might be the vortex solutions themselves, though we'll see that it is sometimes also possible and useful to focus on more abstract structures such as invariant manifolds in the phase space of the equations. Since fluid systems are governed by partial differential equations (which means that their phase space is infinite dimensional) it is particularly useful if one can identify *finite dimensional* invariant manifolds since this has the potential to greatly simplify the problem.

In this article, which reviews results previously obtained in [2], [3] [5], [4] and [6], we show that these two points of view can often be combined to yield fairly detailed insights into the solutions of the Navier-Stokes equation. In particular, in two dimensions this leads to a description of the long-time behavior of all solutions whose initial vorticity is integrable.

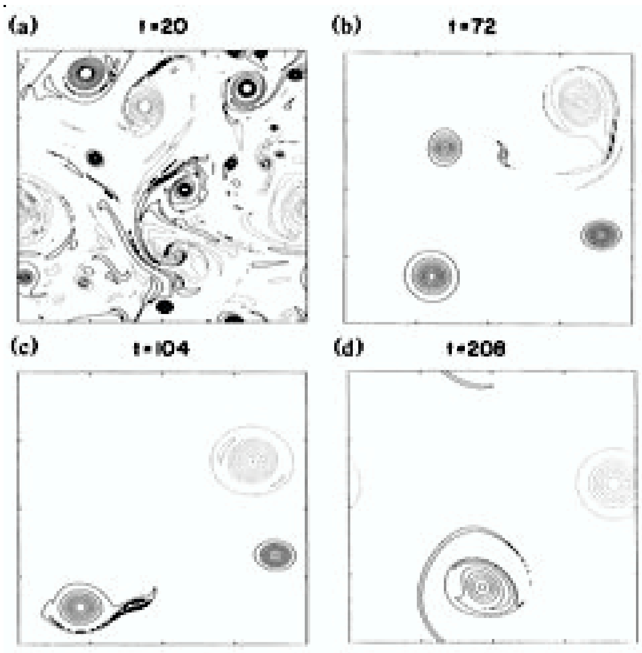


Figure 1: Plots of the iso-vorticity contours in a numerical simulation of a two-dimensional turbulent flow as a function of time. (Reprint of Figure 3 with permission from W. H. Matthaeus, et al. Phys. Rev. Lett., vol. 66, 2731 (1991), Copyright (1991) by the American Physical Society.)[9]

In three dimensions the resulting picture is necessarily much less complete but even there we will see that it gives a better understanding of both the existence and stability of the Burgers vortex and its variants.

The difference between two and three dimensions is not merely that the mathematical analysis is more difficult in three dimensions but reflects differences in the underlying physics. For instance, in two dimensions turbulent fluids undergo a so-called "inverse cascade" whereby energy flows from small scale motions to larger scales. This phenomenon is quite evident in numerical simulations of two-dimensional flows (Like that in Figure 1) where one observes that small scale structures originally present in the flow gradually coalesce into a smaller and smaller number of larger and larger features.

As an anonymous author poetically put it[1]:

> When little whirls meet little whirls,
> they show a strong affection;
> elope, or form a bigger whirl,
> and so on by advection.

[1]Quoted without attribution on `http://www.fluid.tue.nl/WDY/vort/2Dturb/2Dturb.html`

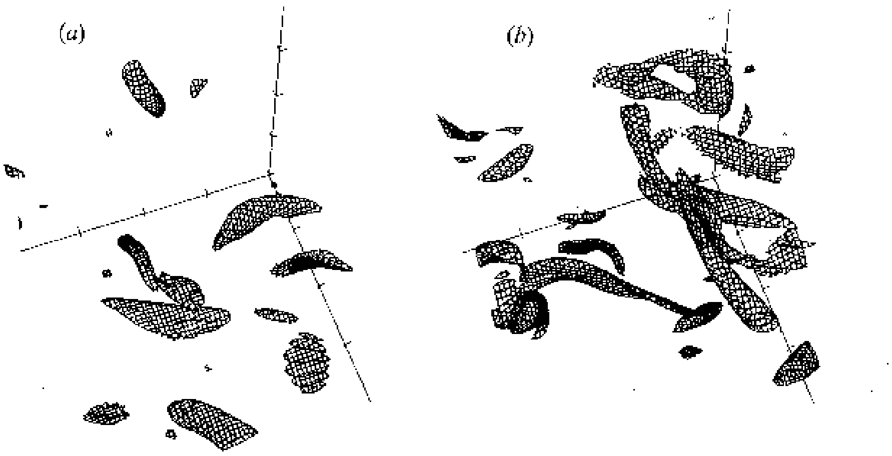


Figure 2: Plots of vorticity contours in a numerical simulation of a three-dimensional turbulent flow showing the presence of vortex tubes in the flow. (Reprint of Figure 6 with permission from Jimenez, et al. J. Fluid Mech. vol. 255, p. 65 (1993) Copyright (1993) by the Cambridge University Press.)[7]

In three dimensions on the other hand, energy typically flows from large scale structures to smaller and smaller scales until is is dissipated by viscosity. However, the vorticity of turbulent three-dimensional flows is not distributed randomly throughout the flow but tends to be organized in tubular structures. Figure 2 illustrates these tubes and also shows that they are roughly elliptical in cross-section.

What is not obvious from this "snapshot" of the fluid at one instant in time is that these tubes persist in the fluid. They are advected about by the background flow but they display a remarkable stability in their form for relatively long periods of time. This fact is quite striking if one looks at animations of three-dimensional turbulent flows. The prominence of these vortex tubes led to them being referred to as the "sinews of turbulence" in the memorable phase of [11] and since shortly after the discovery by Burgers [1] of the explicit vortex solutions of the three-dimensional Navier-Stokes equation which now bear his name, these solutions have been used to quantitatively model various aspects of turbulent flows [15].

# 2 The Navier-Stokes equations

The motion of an incompressible, viscous fluid is described by the Navier-Stokes equations, a system of nonlinear partial differential equations for the fluid velocity $\mathbf{u}(x,t)$ and pressure $p(x,t)$. Depending on the context $x$ and $\mathbf{u}$ are vectors in either $\mathbb{R}^2$ or $\mathbb{R}^3$. These equations take the form

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \nu \Delta \mathbf{u} - \frac{1}{\rho} \nabla p \ , \quad \nabla \cdot \mathbf{u} = 0 \ .$$

Here, $\nu$ is the kinematic viscosity of the fluid and $\rho$ its (constant) density.

The first of these equations is basically Newton's Law – the terms $\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}$ represent the acceleration of the fluid, while the terms on the right are the forces acting on it – the first term the internal frictional forces in the fluid and the second the pressure forces. The second equation just enforces the fact that the fluid is incompressible. We will study flows in unbounded domains – either $\mathbb{R}^2$ or $\mathbb{R}^3$ and thus by rescaling the spatial variable as $x \to x/\sqrt{\nu}$ we can assume without loss of generality that the viscosity is equal to one which we do from now on. Our goal in what follows is to understand the long-time behavior of solutions of these equations. More precisely, given the initial state of the fluid, we will attempt to characterize the behavior of the resulting solution as $t$ becomes large. In two dimensions we will find that we can characterize this asymptotic behavior for essentially any initial conditions, while in three dimensions we will be restricted to consider initial conditions which are either small in norm or close to a vortex solution.

# 3 Two-dimensional flows

We'll begin by focusing on the motion of two-dimensional fluids, including possibly turbulent motions. Although, we live in a three-dimensional world, many phenomena occur in regions in which one dimension is much smaller than the other two and my therefore be treated as essentially two-dimensional – for instance, the behavior of the atmosphere on large scales may for many purposes be treated as two-dimensional.

To study the long-time behavior of solutions of the Navier-Stokes equation in $\mathbb{R}^2$ or $\mathbb{R}^3$ it is convenient to work with the vorticity of the fluid rather than directly with the velocity. This is particularly true in two dimensions where the vorticity can be treated as a scalar. Roughly speaking, the vorticity describes how much "swirl" there is in the fluid. Mathematically, it is defined as the curl of the velocity:

$$\omega = \nabla \times \mathbf{u} = (0, 0, \partial_1 u_2 - \partial_2 u_1) \ .$$

If we let $\omega$ denote the one non-zero component of the vorticity, then taking the curl of the Navier-Stokes equation we find:

$$\partial_t \omega + (\mathbf{u} \cdot \nabla)\omega = \Delta \omega \ .$$

One problem with the vorticity formulation of the Navier-Stokes equation is that the fluid velocity still appears in the nonlinear term. However, we can recover the velocity given the vorticity via the Biot-Savart law:

$$\mathbf{u}(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(\mathbf{x} - \mathbf{y})^{\perp}}{|x - y|^2} \omega(y) dy \ , \quad x \in \mathbb{R}^2 \ .$$

Here and in the sequel, if $x = (x_1, x_2) \in \mathbb{R}^2$, we denote $\mathbf{x}^\perp = (-x_2, x_1)^{\mathrm{T}}$.

Note that this means that the non-linear term is still quadratic (in the vorticity) but now nonlocal. The velocity field constructed via the Biot-Savart law is automatically divergence free, so if we can solve the vorticity equation the corresponding solution of the Navier-Stokes equation can be reconstructed using the Biot-Savart law.

The mathematical study of the vorticity equation requires fairly detailed estimates which relate the decay and smoothness properties of the velocity field to those of the vorticity. We will not go into those details in this review but a number of such estimates are collected in Appendix B of [2].

The numerical experiments on two-dimensional flows illustrated in Figure 1 indicate that vortex solutions play an important role in the long-time asymptotics of the solutions. There exists a family of explicit vortex solutions of the 2D Navier-Stokes equations known as the Oseen vortices,

$$\Omega^\alpha(x,t) = \frac{\alpha}{4\pi(t+1)} e^{-\frac{x^2}{4(1+t)}} \ , \tag{1}$$

with the associated velocity field

$$\mathbf{v}^\alpha(x,t) = \frac{\alpha}{2\pi} \frac{e^{-\frac{x^2}{4(1+t)}} - 1}{|x|^2} \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} \ . \tag{2}$$

Because of the normalization of the Gaussian, we see that $\int_{\mathbb{R}^2} \Omega^\alpha(x,t) dx = \alpha$, so $\alpha$ measures the total circulation of the vortex. Note too that the formula for the Oseen vortices shows that the spatial size of the vortex increases with time (like $\sqrt{t}$ ). This is consistent with the simulations mentioned above and suggests that the analysis of these vortices may be more natural in rescaled coordinates. With this in mind we introduce "scaling variables" or "similarity variables":

$$\xi = \frac{x}{\sqrt{1+t}} \ , \qquad \tau = \log(1+t) \ .$$

We also rescale the dependent variables. If $\omega(x,t)$ is a solution of the vorticity equation and if $\mathbf{u}(x,t)$ is the corresponding velocity field, we introduce new functions $w(\xi,\tau)$, $\mathbf{v}(\xi,\tau)$ by

$$\omega(x,t) \quad = \quad \frac{1}{1+t} w(\frac{x}{\sqrt{1+t}}, \log(1+t)) \ ,$$

and analogously for $\mathbf{u}$:

$$\mathbf{u}(x,t) \quad = \quad \frac{1}{\sqrt{1+t}} \mathbf{v}(\frac{x}{\sqrt{1+t}}, \log(1+t)) \ .$$

In terms of these new variables the vorticity equation becomes

$$\partial_\tau w = \mathcal{L}w - (\mathbf{v}\cdot\nabla_\xi)w \ , \tag{3}$$

where

$$\mathcal{L}w = \Delta_\xi w + \frac{1}{2}\xi\cdot\nabla_\xi w + w \ .$$

We note that the new variables were chosen in such a way that $\mathbf{v}$ is still related to $w$ via the Biot-Savart law. Also, in terms of these variables the Oseen vortices take the form

$$W^\alpha(\xi,\tau) \equiv \alpha G(\xi) \equiv \frac{\alpha}{4\pi}e^{-\frac{\xi^2}{4}} \ ,$$

Thus, they are *fixed points* of the vorticity equation in this formulation.

With this observation it is very natural to ask if these fixed points are stable. In fact, they are actually globally stable. That is, *any solution of the two-dimensional vorticity equation whose initial vorticity is integrable will approach one of these Oseen vortices.*

The proof of this result is based on the construction of a pair of Lyapunov functionals for the two-dimensional vorticity equation. However, we also were able to give a more detailed analysis of the behavior of solutions near one of the Oseen vortices by studying the linearization of the vorticity equation about a vortex and we begin by describing that analysis.

## 3.1 Local Stability

We begin with the linearization about the vortex solution. Linearizing about the vortex $\alpha G$ the equation takes the form:

$$\partial_\tau w = \mathcal{L}w - \alpha\Lambda w \ , \tag{4}$$

where

$$\mathcal{L}w = \Delta w + \frac{1}{2}\xi\cdot\nabla w + w \ , \tag{5}$$

and

$$\Lambda w = \mathbf{V}^G\cdot\nabla w + \mathbf{v}\cdot\nabla G \ . \tag{6}$$

In this last expression $\mathbf{V}^G$ is the velocity field associated via the Biot-Savart law with the Gaussian vorticity, $G$, and $\mathbf{v}$ is the velocity field associated with $w$ via the Biot–Savart law.

### 3.1.1 The operator $\mathcal{L}$

In order to determine the local stability of the Oseen vortices we need to compute the spectrum of the operator $\mathcal{L} - \alpha\Lambda$. The analysis of the operator $\mathcal{L}$ is facilitated by the observation that it is can be rewritten as the quantum mechanical harmonic oscillator.

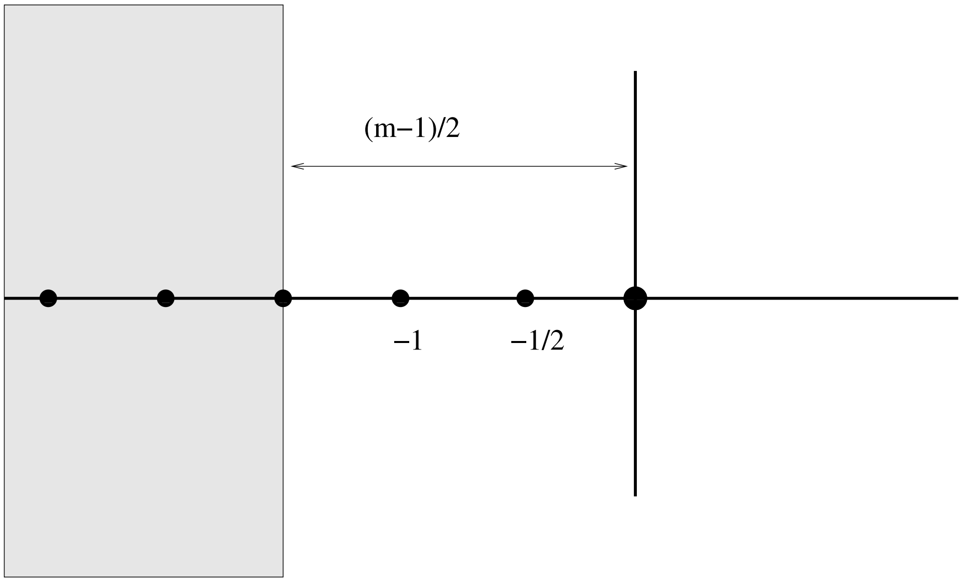


Figure 3: The spectrum of the operator $\mathcal{L}$ on the space $L^2(m)$ with eigenvalues at the non-positive half integers and a half plane of essential spectrum. (In the illustration, $m = 4$.)

In order to compute the spectrum we must specify precisely what function spaces we are working on. For our purposes, square integrable functions with some decay at large distances are appropriate and thus we define:

$$L^2(m) = \{f \in L^2(\mathbb{R}^2) \mid \|f\|_m < \infty\} ,$$

where

$$\|f\|_m = \left( \int_{\mathbb{R}^2} (1 + |\xi|^2)^m |f(\xi)|^2 d\xi \right)^{1/2} .$$

In these spaces, the spectrum of $\mathcal{L}$ consists of two pieces:

- Eigenvalues $\sigma_d = \{-\frac{k}{2} \mid k = 0, 1, 2, \ldots\}$,
- Essential spectrum $\sigma_c = \{\lambda \in \mathbb{C} \mid \Re\lambda \leq -(\frac{m-1}{2})\}$.

For the details of this calculation we refer the reader to Appendix A of [2], but remark that the basic idea is that if one rewrites the eigenvalue equation for $\mathcal{L}$ in Fourier transform variables one can solve the resulting partial differential equation explicitly and imposing the condition that the resulting solution lies in the spaces $L^2(m)$ results in the spectrum above. In particular, this calculation shows that all of the isolated eigenvalues are of finite multiplicity.

If we think of this spectral picture in terms of dynamical systems theory we expect that we should be able to construct finite dimensional invariant manifolds tangent at the origin to the eigenspaces of the isolated eigenvalues, at least if we can ignore the effects of the additional term $\alpha\Lambda$ in (4). For solutions whose total initial vorticity is small one can show that this term is a small perturbation of $\mathcal{L}$ and in [2] we proved that the intuition provided by dynamical systems theory is correct in the sense that one has:

**Theorem 1** *Fix $k \in \mathbf{N}$ and $m > k+2$. Then in a sufficiently small neighborhood of the origin in $L^2(m)$ there exists a submanifold that is invariant with respect to the semiflow defined by (3). This manifold is tangent at the origin to the spectral subspaces corresponding to the eigenvalues $\lambda = -j/2$ , $j = 0, 1, 2 \ldots, k$. Furthermore, any solution of (3) in this neighborhood of the origin either lies on this invariant manifold or approaches the manifold at a rate $\mathcal{O}(t^{-\mu})$ for some $\mu > k/2$.*

Thus, if we fix in advance some decay rate $\mathcal{O}(t^{-\mu})$ up to which we wish to compute the asymtotics of the solutions this theorem shows that at least for small solutions (or, as we show in [2], for any solution of finite energy) the asymptotic behavior up to this order is governed by the behavior of the solutions on a finite dimensional invariant manifold. Since the evolution of solutions on this manifold can be computed as solutions of a finite dimensional system of *ordinary* differential equations this means we have reduced the task of computing the asymptotic behavior of (small) solutions of the Navier-Stokes equations to the task of analyzing the behavior of solutions of ordinary differential equations – a much easier job. Using this approach we showed:

- One could systematically compute the asymptotics of solutions of the Navier-Stokes equation. This lead to phenomena which to the best of our knowledge had not been observed before such as the appearance of logarithmic terms in the asymptotic expansion.
- One could give a geometrical interpretation of analytical conditions previously derived by Miyakawa and Schonbek, [10] on the optimal decay rate of solutions of the Navier-Stokes equation. From this dynamical systems perspective, the conditions on moments of the solution that Miyakawa and Schonbek derived are an analytical representation of the fact that the initial conditions of the solution lie on particular invariant manifolds in the phase space of the equation.
- One can also extend these methods to small solutions of the three-dimensional Navier-Stokes equation [3].

### 3.1.2 The operator $\Lambda$

While the results described above give very precise information about solutions near the origin if we want a more global view of the behavior of solutions of these equations we must include the effects of the term $\alpha\Lambda$ in (4). As mentioned above $\alpha$ describes the total circulation of the vortex. Thus, if we are looking at perturbations of "strong" vortices this term could at least formally be larger than $\mathcal{L}$. However, the effects of $\Lambda$ are "localized" in the sense that the first term in (6) contains a factor of $\mathbf{V}^G$ while the second contains the factor $\nabla G$, both of which go to zero as $|\xi| \to \infty$. This allows one to show that $\alpha\Lambda$ is a relatively compact perturbation of $\mathcal{L}$, regardless of how large $\alpha$ is and thus, the essential spectrum of $\mathcal{L} - \alpha\Lambda$ is the same as that of $\mathcal{L}$, namely, $\sigma_c = \{\lambda \in \mathbb{C} \mid \Re\lambda \leq -(\frac{m-1}{2})\}$, regardless of the size of $\alpha$. Thus, the only way that $\alpha\Lambda$ can cause a vortex to become unstable is if one of the eigenvalues is "pushed" into the right half plane.

To analyze the behavior of the eigenvalues of $\mathcal{L} - \alpha\Lambda$ recall the earlier remark that $\mathcal{L}$ is equivalent to the Hamiltonian operator of the quantum mechanical harmonic oscillator. This fact is more obvious if we consider the action of $\mathcal{L}$ not on the Hilbert space $L^2(m)$, but rather on the space

$$X = \left\{ w \in L^2(\mathbb{R}^2) \,\middle|\, G^{-1/2} w \in L^2(\mathbb{R}^2) \right\} ,$$

equipped with the scalar product

$$(w_1, w_2)_X \;=\; \int_{\mathbb{R}^2} \frac{1}{G(\xi)} \, \bar{w}_1(\xi) w_2(\xi) d\xi \; .$$

We also introduce the closed subspace $X_0$ defined by

$$X_0 = \left\{ w \in X \,\middle|\, \int_{\mathbb{R}^2} w(\xi) d\xi = 0 \right\} .$$

One can show that all the eigenfunctions outside of the essential spectrum of $\mathcal{L} - \alpha\Lambda$ lie in this Hilbert space. If we conjugate the operator $\mathcal{L}$ with the Gaussian weight function we find:

$$L \;=\; G^{-1/2} \mathcal{L} \, G^{1/2} \;=\; \Delta - \frac{|\xi|^2}{16} + \frac{1}{2} \; .$$

But this is just the usual representation of the quantum mechanical harmonic oscillator's Hamiltonian operator and using well known facts about this operator one can show without difficulty that:

- The linear operator $\mathcal{L}$ is self-adjoint in $X$, and $\mathcal{L} \leq 0$.
- $\mathcal{L} \leq -1/2$ on $X_0$.

One can show by explicit computation that 0 is an eigenvalue of the combined operator $\mathcal{L} - \alpha\Lambda$ for all $\alpha$, with eigenfunction the Gaussian, $G$ and that the projection onto this eigenspace just consists of multiplying by 1 and integrating over all of $\mathbb{R}^2$. (This eigenvalue corresponds to the fact that the vorticity equation conserves total circulation.) Thus, if we focus now on the non-zero eigenvalues of $\mathcal{L} - \alpha\Lambda$ it suffices to restrict our attention to the Hilbert space $X_0$ of functions with zero average.

One can now prove by repeated integration by parts (see [5] for details) that the linear operator $\Lambda$ is *skew-symmetric* on $X_0$.

Now we can demonstrate the local stability of the Oseen vortices for any value of $\alpha$. Suppose that $\lambda$ is a non-zero eigenvalue of $\mathcal{L} - \alpha\Lambda$ in $L^2(m)$, with $\Re(\lambda) > \frac{1-m}{2}$. Let $w$ be the corresponding eigenfunction – i.e assume $(\mathcal{L} - \alpha\Lambda)w = \lambda w$. Note that by the remark above, $w \in X_0$. Then

$$\lambda(w,w)_X \;=\; (w, \mathcal{L}w)_X - \alpha(w, \Lambda w)_X \ ,$$

and hence

$$\Re(\lambda)(w,w)_X \;=\; (w, \mathcal{L}w)_X \;\le\; -\frac{1}{2}(w,w)_X \ ,$$

since $\Lambda$ is skew-symmetric and $\mathcal{L} \le -1/2$ on $X_0$. Thus, $\Re(\lambda) \le -1/2$.

Note that since we have rescaled all physical parameters like the viscosity to be equal to 1 in our equation, the parameter $\alpha$ can be thought of as the Reynolds number for this solution and we see that this result says that all the non-zero eigenvalues of the linearization of the vorticity equation around the Oseen vortex lie in the left half plane, regardless of how large the Reynolds number is. This should be contrasted with the behavior of other two-dimensional flows like the plane Poiseuille flow in which increasing the Reynolds number has a destabilizing effect on the flow.

## 3.2 Global Stability

While the previous results about the linearization about the Oseen vortex can be used to prove the local stability of the vortex solutions if we want more global stability results we must use other methods, namely Lyapunov functionals. In [5] we developed two Lyapunov functionals to prove that the Oseen vortices are not only locally but also globally stable.

The first of these functionals is motivated by the observation that the two-dimensional vorticity equation is a nonlinear version of the heat equation and recalling that the heat equation (and also the vorticity equation) satisfy a maximum principle. Decomposing the solution of (3) into its positive and negative parts and applying the maximum principle to each piece of the solution one sees that the function

$$\Phi_1[w](\tau) = \int_{\mathbb{R}^2} |w(\xi,\tau)| d\xi$$

is a Lyapunov function for (3), namely it is monotonic non-increasing along solutions of this equation and in fact is strictly decreasing except on solutions which are either everywhere non-negative or everywhere non-positive. This last fact implies (by way of the LaSalle invariance principle) that the $\omega$-limit set of any solution, that is the function or functions which determine its long-time behavior, must either be the zero function (which can occur only if the total vorticity of the initial condition is zero) or be strictly positive or strictly negative. For the remainder of this section we assume without loss of generality that the $\omega$-limit set is strictly positive.

To determine precisely what positive function, or functions we approach we draw inspiration for our second Lyapunov functional from the kinetic theory of gases. There one also is searching for convergence to a Gaussian distribution of velocities, much as we are searching for convergence to a Gaussian distribution of vorticity and in kinetic theory the *relative entropy function* has been an extremely powerful and useful tool [16]. Motivated by this example we define our second Lyapunov functional to be

$$\Phi_2[w](\tau) = \int_{\mathbb{R}^2} w(\xi,\tau) \log\left(\frac{w(\xi,\tau)}{G(\xi)}\right) d\xi \ .$$

If we now compute the time derivative of $\Phi_2$ we find that

$$\frac{d}{d\tau}\Phi_2[w](\tau) = -\int_{\mathbb{R}^2} w(\xi,\tau)|\nabla \log \frac{w(\xi,\tau)}{G(\xi)}|^2 d\xi \leq 0 \ .$$

Furthermore, $\frac{d}{d\tau}\Phi_2[w](\tau)$ is strictly less than zero unless the solution $w(\xi,\tau)$ is proportional to the Gaussian. Thus, appealing again to the LaSalle Principle, we see that the $\omega$-limit set of this solution must coincide with the a multiple of the Gaussian – i.e. as time goes to infinity, the vorticity of the solution will approach one of the Oseen vortices.

In fact, to make the above argument mathematically rigorous requires a certain amount of additional work, but the two essential ideas are as above, namely one first uses the Lyapunov functional based on the maximum principle to conclude that the limiting behavior of any solution lies in the set of solutions with all one sign, and then uses the relative entropy function to conclude that any solution that is everywhere positive (or everywhere negative) will approach one of the Oseen vortices. More precisely, one can conclude:

**Theorem 2** *If $\omega_0 \in L^1(\mathbf{R}^2)$, with $\alpha = \int_{\mathbf{R}^2} \omega_0(x)dx$, the two-dimensional vorticity equation has a unique solution with initial condition $\omega_0$. This solution satisfies*

$$\lim_{t\to\infty} t^{1-\frac{1}{p}} |\omega(\cdot,t) - \frac{\alpha}{t} G(\frac{x}{\sqrt{t}})|_{L^p} = 0 \ ,$$

*for $1 \leq p \leq \infty$.*

One can draw a number of corollaries from this theorem – for instance, it implies that the only self similar solutions of the the two-dimensional Navier-Stokes equation with integrable vorticity are the Oseen vortices. Furthermore, we had mentioned earlier that the Oseen vortices remain locally stable for any Reynolds number, in contrast to many other two-dimensional flows, and from this theorem we see that this remains true even if the perturbations we consider are no longer small perturbations of the vortices.

# 4 Three-dimensional flows

In three dimensions one cannot, of course, hope for as complete a picture of the long-time behavior of all flows as we obtained in two dimensions. For small initial data, one can again construct finite dimensional invariant manifolds which describe the long-time asymptotics of the solutions, just as in two dimensions [3]. However, for large initial data no such description exists – indeed, the question of whether or not solutions with arbitrary smooth initial velocity fields remain smooth for all time is one of the Clay Mathematics Institute's million dollar "Millennium Prize Problems". Nonetheless, it has been realized for some time on the basis of both numerical and experimental studies that vortices play an important role in understanding turbulent fluid motions. The difference between the behavior of two and three-dimensional flows is due to a difference in the form of the vorticity equation in the two cases. If one again computes the equation for the evolution of the vorticity by taking the curl of the Navier-Stokes equation one finds that the (vector) vorticity evolves according to the equation:

$$\partial_t \omega + (\mathbf{u} \cdot \nabla)\omega - (\omega \cdot \nabla)\mathbf{u} = \Delta\omega \ .$$

We have again rescaled lengths here so that the kinematic viscosity $\nu = 1$. The additional nonlinear term in this equation allows the velocity field to "amplify" the vorticity and is known as the "vorticity stretching" term.

If one examines numerical simulations of turbulent solutions of this equation one observes that the vorticity of the flow tends to be concentrated in long, roughly cylindrical, tubes[2]. Closer examination shows that these tubes:

- Have a relatively stable shape as they evolve.
- Are not quite circular in cross section.

In particular, note that in marked contrast to the Oseen vortices that occur in two-dimensional flows these vortices these vortices do not "spread out", but remain roughly constant in diameter. This is possible in three dimensions because of a balance between amplification due to the vortex stretching term and the diffusive spreading due to viscosity. An explicit example of such a stationary vortex solution is the Burgers vortex, an exact solution of the Navier-Stokes equation that is a superposition of a background strain field with a swirling motion in the plane perpendicular to the strain axis. The velocity field of the Burgers vortex has the form:

$$\mathbf{U}(x_1, x_2, x_3) = \begin{pmatrix} -\frac{1}{2}x_1 \\ -\frac{1}{2}x_2 \\ x_3 \end{pmatrix} + \alpha \begin{pmatrix} u_1(x_1, x_2) \\ u_2(x_1, x_2) \\ 0 \end{pmatrix} , \qquad (7)$$

where the components $u_1$ and $u_2$ of the velocity are exactly the same as those of the Oseen vortex *in rescaled coordinates*. Note that in this case they do *not* spread with time.

[2] A nice, animated example of such numerics due to Prof. L. Collins' research group can be seen at `http://gears.aset.psu.edu/viz/services/projectlist/lance_collins/`

The vorticity of the Burgers vortex has only a single non-zero component, and this component is a Gaussian, just as in the case of the Oseen vortex.

$$\mathbf{\Omega}(x_1, x_2, x_3; \alpha) \;=\; \alpha \begin{pmatrix} 0 \\ 0 \\ G(x_1, x_2) \end{pmatrix} \;, \quad \text{where} \quad G \;=\; \partial_1 u_2 - \partial_2 u_1 \;.$$

Note that background strain field in Burgers vortex is irrotational, so the components $u_1$ and $u_2$ can be recovered from the vorticity of the solution via the Biot-Savart law.

Note further that there is a family of vortices, parameterized by the total circulation $\alpha$. We will see below that in the non-symmetric case we will also find families of vortices parameterized by the total circulation but in that case, the families will no longer consist of multiples of a single function as is true of these classical Burgers vortices.

This connection between the Oseen vortex and the Burgers vortex is an example of a remarkable connection between two and three-dimensional flows discovered by Lundgren, [8]. Namely, if $\omega(x_1, x_2, t)$ is a solution of the two-dimensional vorticity equation and if $S(t) = \exp(\int_0^t \gamma(\tau) d\tau)$, then

$$\mathbf{\Omega}(x_1, x_2, x_3, t) \;=\; \begin{pmatrix} 0 \\ 0 \\ S(t)\omega(\sqrt{S(t)}x_1, \sqrt{S(t)}x_2, (\int_0^t S(t')dt')) \end{pmatrix}$$

is a solution of the three-dimensional vorticity equation in a time-dependent background strain field

$$\mathbf{u}^s(x_1, x_2, x_3, t) \;=\; \begin{pmatrix} -\frac{\gamma(t)}{2} x_1 \\ -\frac{\gamma(t)}{2} x_2 \\ \gamma(t) x_3 \end{pmatrix} \;.$$

Lundgren used this relationship to extend prior work of Townsend [15] which used random distributions of Burgers vortices to quantitatively, but non-rigorously, model turbulent flows. We will use it to transfer our understanding of the stability of Oseen vortices to questions about perturbations of Burgers vortices.

The fact that the vortices observed in numerical studies of turbulent three-dimensional fluids were not cylindrical as Burgers vortices are lead to a number of studies of perturbations of these explicit solutions. Among the many studies we note particularly those of:

- Robinson and Saffman, [13] who introduced asymmetric vortices as more realistic models for the vortices which appear in turbulent flows. These are vortices in which neither the strain field nor the vorticity of the vortex are axisymmetric. Robinson and Saffman:
  - Constructed perturbative approximations to these vortices for small Reynolds number and asymmetry parameter.

  - Conducted numerical investigations of their existence up to Reynolds number of about 100.

- Moffatt, Kida and Ohkitani, [11] developed formal asymptotic expansions for the vorticity field of these non-axisymmetric Burgers vortices for large Reynolds number.
- Prochazka and Pullin, [12] studied numerically the stability of these solutions with respect to two-dimensional perturbations in the plane transverse to the strain axis. They found that numerically the vortices were stable with respect to such perturbations for all values of the Reynolds number and for all values of the asymmetry parameter (which we define below) between zero and one.

With the aid of the results described in Section 3, and their extensions, along with the relationship between two and three-dimensional flows encapsulated in the Lundgren transformation we have recently shown ([4], [6]) how several aspects of this theory can be made rigorous.

We begin by discussing the existence of non-symmetric, Burgers vortices. Instead of considering a symmetric background strain field like that of the Burgers vortex, we consider a background field

$$\mathbf{u}^s(x_1,x_2,x_3) \;=\; \begin{pmatrix} -\frac{1}{2}(1+\lambda)x_1 \\ -\frac{1}{2}(1-\lambda)x_2 \\ x_3 \end{pmatrix} \;, \tag{8}$$

where the asymmetry parameter $\lambda \in [0,1)$. We assume that the swirling part of the flow, superimposed on this background strain, has only the first two components of its velocity non-zero and that these two components depend only on $x_1$ and $x_2$ as in the case of Burgers vortex, that is, we look for a stationary solution of the three-dimensional Navier-Stokes equation of the form

$$\mathbf{U}(x_1,x_2,x_3) \;=\; \begin{pmatrix} -\frac{1}{2}(1+\lambda)x_1 \\ -\frac{1}{2}(1-\lambda)x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} u_1(x_1,x_2) \\ u_2(x_1,x_2) \\ 0 \end{pmatrix} \;,$$

The vorticity $\mathbf{\Omega} = \nabla \times \mathbf{U}$ is aligned with the vertical axis and depends only on the horizontal variable, namely

$$\mathbf{\Omega}(x_1,x_2,x_3) \;=\; \begin{pmatrix} 0 \\ 0 \\ \omega(x_1,x_2) \end{pmatrix} \;, \quad \text{where} \quad \omega \;=\; \partial_1 u_2 - \partial_2 u_1 \;.$$

Inserting this expression into the three-dimensional vorticity equation we find that in order for such a solution to exist the vorticity must satisfy

$$(u_1 - \frac{1}{2}(1+\lambda)x_1)\partial_1\omega + (u_2 - \frac{1}{2}(1-\lambda)x_2)\partial_2\omega \;=\; \Delta\omega + \omega \;. \tag{9}$$

As usual, $u_1$ and $u_2$ can be recovered from the vorticity via the Biot-Savart law.

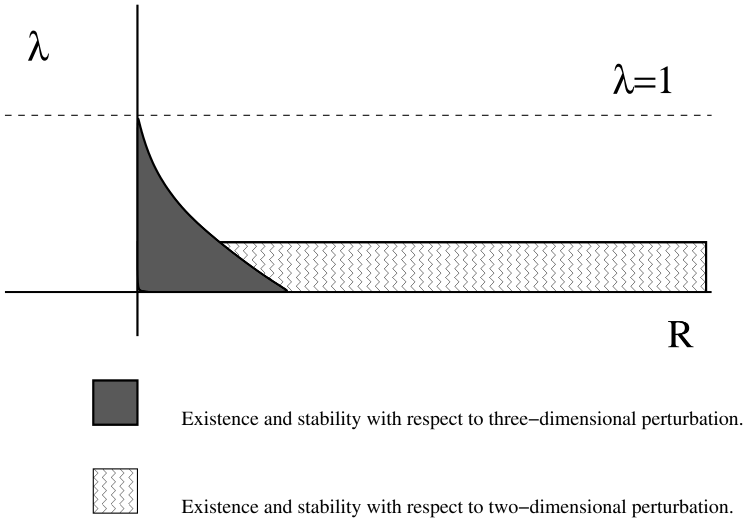


Figure 4: An illustration of the regions in which we can prove the existence of non-axisymmetric Burgers vortices

Our first result in this context proves rigorously that (9) has a solution if the asymmetry parameter $\lambda$ is not too large, for all Reynolds numbers. In Figure 4 this corresponds to the horizontal rectangle extending off to infinity. One important thing to note is that the allowed size of the asymmetry parameter for which we have existence of the vortex solution is uniform in the Reynolds number $R$, for $R$ large.

The existence proof is a rigorous perturbation argument, taking as our starting point the known, symmetric Burgers vortex. The main steps of the proof are:

- We write the vorticity of the asymmetric vortex as $\omega = \alpha G + w$ (i.e. we regard it as a perturbation of the Burgers vortex. (Here $\alpha$ is proportional to the Reynolds number and is chosen so that $w$ has zero average with respect to $x_1$ and $x_2$.)
- $w$ then satisfies the equation

$$(\mathcal{L} - \alpha\Lambda)w = -\lambda\mathcal{M}(\alpha G + w) + \mathbf{v}\cdot\nabla w \ ,$$

  where $\mathcal{M}w = (x_1\partial_{x_1}w - x_2\partial_{x_2}w)/2$ and $\mathcal{L}$ and $\Lambda$ are the same operators considered earlier in Section 3.
- Given the information derived earlier about the spectrum of $(\mathcal{L} - \alpha\Lambda)$ we see immediately that this operator is invertible on the space of functions of zero average and we can rewrite this equation as a fixed point problem

$$w = (\mathcal{L} - \alpha\Lambda)^{-1}\big( -\lambda\mathcal{M}(\alpha G + w) + \mathbf{v}\cdot\nabla w\big) \ .$$

- One now proves that the fixed point equation for the vorticity has a solution by the contraction mapping theorem. Just as in the case of the classical Burgers vortices we obtain (for each fixed, sufficiently small, value of $\lambda$) a family of vortex solutions, $\Omega^B(x_1, x_2; \alpha)$, parameterized by the total circulation, $\alpha$.

- The uniformity with respect to the Reynolds number comes from analyzing (more or less explicitly) the limit
$$\lim_{\alpha\to\infty} (\mathcal{L} - \alpha\Lambda)^{-1}\mathcal{M}(\alpha G) \ .$$

We refer the reader to [4] for more details but remark that as a by-product of the proof one obtains the asymptotic expressions derived by Moffatt, Kida and Ohkitani as the leading order terms in a rigorous expansion of these asymmetric vortices in the large Reynolds number limit.

Given the existence of these non-symmetric vortices, or of the classical Burgers vortices for that matter, it is natural to ask whether or not they are stable. As with any stability question it is important to specify exactly what the allowed class or perturbations is. If we consider only two-dimensional perturbations in the plane transverse to the vortex axis (i.e. only the first two components of the velocity are non-zero and these depend only on the variables $x_1$ and $x_2$) then Lundgren's transformation plus the results of Section 3 on the stability of two-dimensional vortices almost immediately implies stability in this three-dimensional context.

The stability issue is much more complicated if we allow full three-dimensional perturbations, however. The vorticity is a vector in this case and one can no longer reduce the problem to a two-dimensional one – one must consider the full three-dimensional vorticity equation. Nonetheless, also in this case we are able to show that the the Burgers vortices are "stable with shift" if the Reynolds number is not too large. By this we mean that for small Reynolds number, if one takes initial conditions which are a small perturbation of a vortex, the solution with this initial condition will tend, as $t \to \infty$, to one of these asymmetric vortices. In general one will not converge back to the vortex which one initially perturbed but our results include an explicit formula for the limiting vortex. The region in which we can prove stability in this sense is the triangular shaded region in Figure 4. Note in particular, that as a part of this investigation we also prove that one can find stable, asymmetric Burgers vortices for any asymmetry parameter $\lambda \in [0, 1)$, provided the Reynolds number is sufficiently small.

To state our stability result more precisely we limit ourselves to the classical symmetric Burgers vortices since it is slightly simpler to state the result in that context. Let $L^2(m)$ be the weighted Hilbert spaces introduced in Section 3. Note that if we look at the evolution of a three-dimensional vorticity field $\mathbf{\Omega}$ in a background strain field $\mathbf{u}^s$ as in (8) we find it satisfies the equation

$$\partial_t\mathbf{\Omega} + (\mathbf{U}\cdot\nabla)\mathbf{\Omega} - (\mathbf{\Omega}\cdot\nabla)\mathbf{U} + (\mathbf{u}^s\cdot\nabla)\mathbf{\Omega} - (\mathbf{\Omega}\cdot\nabla)\mathbf{u}^s \,=\, \Delta\mathbf{\Omega}\ , \quad \nabla\cdot\mathbf{\Omega} \,=\, 0\ . \qquad (10)$$

**Theorem 3** *Fix $m \ge 2$. There exists $R_0 > 0$ and $\delta_0 > 0$ such that if $|\alpha| \le R_0$ and*

$$\sup_{x_3\in\mathbb{R}} \|\omega^0(\cdot, x_3)\|_{L^2(m)} < \delta_0 \ ,$$

*then the solution of the three-dimensional vorticity equation, (10), with initial vorticity*

$$\mathbf{\Omega}^0(x_1,x_2,x_3) \;=\; \begin{pmatrix} 0 \\ 0 \\ \alpha G(x_1,x_2) \end{pmatrix} + \omega^0(x_1,x_2,x_3) \ ,$$

*converges as* $t \to \infty$ *to the Burgers vortex*

$$\tilde{\mathbf{\Omega}}(x_1,x_2,x_3) \;=\; \begin{pmatrix} 0 \\ 0 \\ (\alpha+\delta\alpha) G(x_1,x_2) \end{pmatrix} \ .$$

*The convergence toward the limiting vortex is with respect to the* $L^2(m)$ *norm in* $(x_1,x_2)$ *and uniformly on compact sets in* $x_3$*. The parameter* $\delta\alpha$ *gives the difference between the circulation of the initial vortex which we perturb and the limiting vortex toward which we converge and can be computed in terms of the initial perturbation (see (13) below.)*

The proof of this theorem is somewhat complicated and we refer the reader to [6] for details, but just mention one important and somewhat surprising aspect of the proof. If we linearize equation (10) about a Burgers vortex, then for small Reynolds number (or equivalently, small $\alpha$), one obtains a small perturbation of a linear operator for which we can compute an explicit formula for the kernel of the corresponding semi-group. This semigroup is decaying if it acts on perturbations whose third component has zero average with respect to the transverse variables $(x_1,x_2)$ in each $x_3$ cross-section, i.e. if

$$\int_{\mathbb{R}^2} \omega_3(x_1,x_2,x_3,t)\,dx_1 dx_2 = 0 \ , \tag{11}$$

for each $x_3 \in \mathbb{R}$. In general, the perturbations we consider will not satisfy (11), but we force it to hold by writing the third component of the vorticity as

$$(\alpha + \phi(x_3,t))G(x_1,x_2) + \omega_3(x_1,x_2,x_3,t) \ , \tag{12}$$

where $\phi(x_3,t)$ is chosen so that $\omega_3$ satisfies (11). One can think of this step as adjusting the total circulation of the vortex which we are perturbing (in each $x_3$-cross-section and at each time, $t$) in such a way that the third component of the perturbation has zero average. From the remark about the decay properties of the linearized semi-group this will insure that $\omega_j(\cdot,t)$, $j=1,2,3$, will decay with time, but it has the disadvantage that the background vortex is no longer a stationary solution of the Navier-Stokes equation (due to the time and space dependence of $\phi$.) Thus, we must next compute the time evolution of $\phi$.

Remarkably, its evolution decouples completely from that of the other components of the vorticity (see the calculation in Section 3 of [6]) and we find that $\phi$ satisfies the linear partial differential equation

$$\partial_t \phi + x_3 \partial_3 \phi = \partial_3^2 \phi \ .$$

This equation can be solved explicitly and we find that if the initial value of $\phi$ is $\phi^0$ (which can be computed from the initial value of the vorticity $\mathbf{\Omega}^0$) then the solution $\phi(x_3, t)$ converges as $t$ tends toward infinity to the constant value

$$\delta\alpha = \frac{1}{\sqrt{2\pi}} \int e^{-z^2/2} \phi^0(z) dz \ , \tag{13}$$

which from (12) implies that the circulation of the limiting vortex is $\alpha + \delta\alpha$.

# 5 Conclusions

In both the two and three-dimensional Navier-Stokes equations ideas from dynamical systems theory like invariant manifold theorems and Lyapunov functions give us insight into the long-time asymptotics of solutions and the existence and stability of vortices.

In two dimensions these methods describe the long-time asymptotics of any solution with integrable initial vorticity. However, it would be nice if one could also understand what might be termed the "intermediate" asymptotics. As a start, for instance, one could look in more detail at the interaction between a pair of vortices. The latter stages of such an interaction for "small" vortices (i.e. those with small total vorticity) can be determined from an analysis of the equations on the invariant manifolds near the origin described in Section 3, but for the moment we cannot extend this analysis to vortices of arbitrary size.

In three dimensions there are many more open questions. For instance, numerical investigations suggest that the non-axisymmetric vortices constructed in Section 4 should exist for all values of the asymmetry parameter $\lambda \in [0, 1)$, but so far we can only prove their existence for small values of $\lambda$, except in the small Reynolds number regime. It would also be very interesting to understand the stability properties of these vortices for large Reynolds number. Here, it is not so clear whether even the symmetric Burgers vortices are stable with respect to *three-dimensional* perturbations at large Reynolds number, though the persistence of these vortex tubes in numerical simulations of very turbulent flows as well as the direct numerical study of their linear stability by Schmid and Rossi [14] indicate that they have some sort of stability properties.

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