ON NONLINEAR STABILIZATION OF LINEARLY UNSTABLE MAPS

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Abstract. We examine the phenomenon of nonlinear stabilization, exhibiting a variety of related examples and counterexamples. For Gâteaux differentiable maps, we discuss a mechanism of nonlinear stabilization, in finite and infinite dimensions, which applies in particular to hyperbolic partial differential equations, and, for Fréchet differentiable maps with linearized operators that are normal, we give a sharp criterion for nonlinear exponential instability at the linear rate. These results highlight the fundamental open question whether Fréchet differentiability is sufficient for linear exponential instability to imply nonlinear exponential instability, at possibly slower rate.

1. Introduction

Since the pioneering work of Lyapunov, it has been classical to deduce stability properties of equilibria of dynamical systems from spectral information on the linearized operator. In his memoir [Ly], Lyapunov considers general systems of ordinary differential equations with analytic coefficients, in finite dimensions, and establishes several fundamental results on stability or instability of equilibria or time-periodic solutions, using spectral information on appropriate linearized systems. In particular, an equilibrium solution of an autonomous system is shown to be asymptotically stable if all eigenvalues of the associated linearized operator have strictly negative real parts, and to be unstable if at least one eigenvalue has a strictly positive real part. Such results were subsequently generalized to infinite-dimensional systems and less regular nonlinearities, under suitable spectral assumptions. The reader is referred to the monograph of Daleckii and Krein [DK] for a nice account of these early studies.

In the literature that follows Lyapunov's work, there is a striking asymmetry between the generalizations of the stability and the instability theorem, respectively. On the one hand, assuming the spectral mapping theorem and using a suitable norm, it is relatively straightforward to show that, for any autonomous dynamical system, an equilibrium is asymptotically stable if the vector field of the system is Fréchet differentiable at that point and if the spectrum of the derivative is entirely contained in the open left half-plane. In contrast, the Lyapunov instability theorem has no counterpart so far at this level of generality: many sufficient conditions for instability are known, but they all require either the existence of a spectral gap, or a somewhat restrictive assumption on the nonlinearity.

In fact, as of now, we are not aware of any example of nonlinear stabilization for a linearly unstable Fréchet differentiable dynamical system, nor of any result that would prevent such a phenomenon to occur under minimal and natural assumptions. The modest goal of this paper is to discuss the existing results in this direction, and to give a few generalizations. We also present examples that illustrate various aspects of this fundamental open question.

1.1. Formulation of the problem.

To avoid technicalities related to unbounded linear operators, we find it convenient to formulate the problem in terms of discrete time systems, namely difference equations, rather than differential equations as in Lyapunov's work. We thus consider a discrete

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evolutionary system of the form

$$u_{n+1} = \mathcal{F}(u_n), \qquad n \geq 0, \tag{1.1}$$

in a Banach space B, where $\mathcal{F} : B \to B$ is a nonlinear map. We always assume that $\mathcal{F}(0) = 0$, so that the origin $u = 0$ is an equilibrium point. This general framework includes in particular continuous time-evolutionary systems, such as partial differential equations with time-independent or time-periodic coefficients, in which case $\mathcal{F}$ is defined as the time-T solution map, see Section 1.5. As discussed in point (5) of Section 1.3 below, partial differential operators with smooth coefficients do not necessarily generate smooth solutions maps. We suppose nevertheless that there is associated with $\mathcal{F}$ a linearized map L, which is typically a Fréchet or Gâteaux derivative of $\mathcal{F}$ at the origin.

The main question is:

Problem 1.1. *Assuming that L has spectral radius $r(L) > 1$, corresponding to linear exponential instability, under what general conditions may we deduce also nonlinear instability?*

By nonlinear instability, we mean instability in the sense of Lyapunov of the equilibrium $u = 0$ for the evolutionary system (1.1). A related question is whether exponential instability occurs in the situation described by Problem 1.1, and, if so, whether it occurs at the linear rate $\rho = r(L)$. Here are the precise definitions:

Definition 1.2. The equilibrium $u = 0$ is

- *unstable (in the sense of Lyapunov)* if there exists $\varepsilon > 0$ such that, for any $\delta > 0$, one can find a sequence (u_n) solution of (1.1) such that $0 < |u_0|_B \leq \delta$ and $|u_n|_B \geq \varepsilon$ for some $n \in \mathbb{N}$.
- *exponentially unstable at rate $\rho > 1$* if there exists $\varepsilon > 0$ and $C > 0$ such that, for any $\delta > 0$, one can find a sequence (u_n) solution of (1.1) satisfying $0 < |u_0|_B \leq \delta$ and $|u_n|_B \geq C\rho^n |u_0|_B$ for all $n \in \mathbb{N}$ such that $\max(|u_0|_B, \ldots, |u_n|_B) \leq \varepsilon$.

It is obvious that exponential instability at any rate $\rho > 1$ implies instability in the sense of Lyapunov, with the same value of ε.

1.2. **Classical results.** Our understanding of Problem 1.1 is grounded in the following classical results:

(1) The *Lyapunov instability theorem* [Ly], which states that if B is finite-dimensional and if the linear operator $L : B \to B$ with $r(L) > 1$ approximates $\mathcal{F}$ near the origin in the sense that, for some $a > 0$ and $b > 0$ small enough,

$$|\mathcal{F}(u) - Lu|_B \leq b|u|_B \qquad \text{whenever} \quad |u|_B \leq a, \tag{1.2}$$

then the equilibrium $u = 0$ is exponentially unstable. Remark that condition (1.2) is always fulfilled if $\mathcal{F}$ is Fréchet differentiable at the origin and $L = d\mathcal{F}|_{u=0}$. Even in that particular case, instability may not occur at the linear rate, as is shown by Proposition 4.3.

(2) The extension of Lyapunov's instability theorem to the infinite-dimensional case under the assumption of a *spectral gap*, see e.g. [DK, Theorem VII.2.2]. Precisely, let L be a bounded linear operator on the Banach space B with spectral radius $r(L) > 1$, and assume that the spectrum $\sigma(L)$ does not intersect the unit circle $\{\lambda \in \mathbb{C} \,|\, |\lambda| = 1\}$. Then, if the map $\mathcal{F} : B \to B$ satisfies (1.2) for some $a > 0$ and $b > 0$ small enough, the equilibrium $u = 0$ is exponentially unstable. More generally, the same conclusion is obtained if one assumes that the linear operator L is ρ-pseudohyperbolic [Ru] instead of hyperbolic, namely if the spectrum $\sigma(L)$ does not intersect the circle of radius ρ, for some $\rho \in [1, r(L))$. It should be emphasized that the smallness assumption on the parameter b in (1.2) depends on the linear operator L, in such a way that $b \to 0$ when the spectral gap shrinks to zero.

(3) The *Rutman-Daleckii theorem* [DK, Theorem VII.2.3], and its more general version due to Henry [He, Theorem 5.1.5], which states that if $L : B \to B$ is a bounded linear operator which approximates the map $\mathcal{F}$ at a superlinear rate at the origin in the sense that

$$|\mathcal{F}(u) - Lu|_B \,\le\, b|u|_B^{1+p} \qquad \text{whenever} \quad |u|_B \le a\,, \tag{1.3}$$

for some $a > 0$, $b > 0$, and $p > 0$, then $r(L) > 1$ implies nonlinear exponential instability of the origin, at the linear rate $\rho = r(L)$. Condition (1.3) is of course stronger than (1.2), and implies in particular that $\mathcal{F}$ is Fréchet differentiable at the origin with $L = d\mathcal{F}|_{u=0}$. It holds typically if the map $\mathcal{F}$ has the regularity $C^{1,p}$ near the origin, for some $p \in (0, 1]$, namely if the Fréchet derivative $u \mapsto d\mathcal{F}(u)$ is Hölder continuous with exponent p. Note that, in the Rutman-Daleckii theorem, the spectral instability condition $r(L) > 1$ is the only assumption made on the linear operator L.

If the map $\mathcal{F}$ in (1.1) is Fréchet differentiable at the origin, the results above show that linear exponential instability implies nonlinear instability if the linearized operator $L = d\mathcal{F}|_{u=0}$ either has a spectral gap, or approximates the full map $\mathcal{F}$ at a superlinear rate. Both conditions are sufficient, but neither one is known to be necessary.

1.3. **Remarks and examples.** We do not give a definite answer to Problem 1.1, but only make a few remarks which, we hope, shed some light on what the solution may be.

(4) We first recall that even a very weak nonlinearity can stabilize a map that is linearly unstable, but not at exponential rate. This phenomenon already happens in finite dimensions, as is demonstrated by the following simple example

$$\mathcal{F}(v, w) \,=\, L\begin{pmatrix} v \\ w \end{pmatrix} - \begin{pmatrix} v^3 \\ w^3 \end{pmatrix}, \qquad L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{1.4}$$

for which the equilibrium $(v, w) = (0, 0) \in B = \mathbb{R}^2$ is linearly unstable, in the sense that L^n is unbounded as $n \to \infty$, and yet nonlinearly asymptotically stable, see Section 5. In that case we have of course $r(L) = 1$.

In Section 2 below, we construct in the same spirit an infinite-dimensional map $\mathcal{F}$ for which the origin $u = 0$ is stable, although the linearized operator $L = d\mathcal{F}|_{u=0}$ has the property that L^n grows nearly exponentially as $n \to \infty$. This shows that the linear exponential instability assumption $r(L) > 1$ is essential in Problem 1.1.

(5) Our next contribution is an intriguing example which indicates that, in the absence of a spectral gap, stabilization may be possible if the nonlinearity does not satisfy a superlinearity condition such as (1.3).

Example 1.3. Let $\chi : \mathbb{R} \to [0, 2]$ be a smooth function such that $\mathrm{supp}(\chi) \subset [-1, 1]$ and $\chi(0) = 2$. Let also $h : [0, \infty) \to [0, \infty)$ be an increasing continuous function such that $h(0) = 0$. In the Hilbert space $B = L^2(\mathbb{R})$, we consider the map $\mathcal{F} : B \to B$ defined by

$$\big(\mathcal{F}(u)\big)(x) \,=\, \chi(x)\, u\big(x - h(|u|_B)\big)\,, \qquad x \in \mathbb{R}\,, \tag{1.5}$$

for all $u \in B$. Then $\mathcal{F}(0) = 0$, and $\mathcal{F}$ is differentiable at the origin in the sense of Gâteaux, but not in the sense of Fréchet, with derivative L given by

$$(Lu)(x) \,=\, \chi(x)\, u(x)\,, \qquad x \in \mathbb{R}\,.$$

The operator L is clearly self-adjoint in B with spectrum $\sigma(L) = [0, 2]$, so that the origin $u = 0$ is linearly exponentially unstable. However, as is shown in Section 3 below, the origin is nonlinearly stable if $h(s)$ converges sufficiently slowly to zero as $s \to 0$, for instance if $h(s) \ge C|\ln s|^{-1}$ for some sufficiently large $C > 0$.

Example 1.3 shows in particular that nonlinear stabilization is possible if the linearization of the map $\mathcal{F}$ at the origin is understood in a weaker sense than the usual Fréchet derivative. We elaborate on that question in Section 3, where we first prove all assertions above and then discuss a few related examples. One of them corresponds to a seemingly minor modification of the map (1.5), namely

$$\big(\mathcal{F}(u)\big)(x) \;=\; \chi(x)\, u\big(2x - h(|u|_B)\big)\,, \qquad x \in \mathbb{R}\,, \tag{1.6}$$

which has nevertheless a very different behavior. Unlike in Example 1.3, nonlinear stabilization is now possible even for a very smooth nonlinearity, such as $h(s) = s^2$. More remarkably, we also provide an example of a simple hyperbolic partial differential equation, for which the time-one solution map behaves qualitatively as in (1.6), so that nonlinear stabilization occurs. The choice of a hyperbolic equation here is not accidental: in such systems, the flow is typically not Fréchet differentiable at equilibria, so that weaker notions of linear tangent maps have to be introduced. In contrast, parabolic equations with smooth nonlinearities typically generate Fréchet differentiable flows, in which case the Rutman-Daleckii theorem applies and shows, in view of (1.3), that nonlinear stabilization is impossible if the flow is sufficiently smooth, for instance if the differential $u \mapsto d\mathcal{F}(u)$ is Hölder continuous with exponent $p > 0$. Finally, we also give in Section 3 a finite-dimensional example of a Gâteaux differentiable map for which nonlinear stabilization occurs.

These examples show that, even in finite dimensions, it is necessary in Problem 1.1 to understand the linearized operator L in the usual Fréchet sense.

(6) In Section 4, we study in some detail the important particular case where B is an infinite-dimensional Hilbert space, the map $\mathcal{F} : B \to B$ is Fréchet differentiable at the origin, and the linearized operator $L = d\mathcal{F}|_{u=0}$ is selfadjoint or normal. In that situation, even in the absence of a spectral gap, it is possible to separate the unstable part of the spectrum using spectral projections. We prove that linear exponential instability implies nonlinear instability provided that, for some $a > 0$,

$$|\mathcal{F}(u) - Lu|_B \;\le\; \alpha(|u|_B)|u|_B\,, \qquad \text{whenever} \quad |u|_B \le a\,, \tag{1.7}$$

where $\alpha : [0, a] \to [0, +\infty)$ is a nondecreasing function satisfying the integrability condition

$$\int_0^a \frac{\alpha(s)}{s}\,\mathrm{d}s \;<\; \infty\,. \tag{1.8}$$

This condition is obviously satisfied if $\alpha(s) = bs^p$ for some $p > 0$, hence our result sharpens the Rutman-Daleckii theorem in the normal case by replacing (1.3) with the weaker assumption (1.7)-(1.8).

It is interesting to observe that the function $\alpha(s) = |\ln s|^{-\gamma}$ satisfies (1.8) for any $\gamma > 1$. As for the limiting case $\gamma = 1$, Example 1.3 shows on the contrary that nonlinear stabilization can occur for a map $\mathcal{F}$ such that

$$|\mathcal{F}(u) - Lu|_B \;\le\; C|\ln|u|_B|^{-1}|u|_X\,, \qquad \text{for all } u \in X \subset B\,, \tag{1.9}$$

where $X = H^1(\mathbb{R})$, $B = L^2(\mathbb{R})$, and $L : B \to B$ is a selfadjoint operator. This is an indication that condition (1.8) may be close to optimal in the normal case – an indication but not a proof, since the nonlinear map in Example 1.3 is not Fréchet differentiable, as is reflected in the fact that $X \neq B$ in (1.9).

1.4. **Conclusion.** We are now in position to give a more precise formulation of Problem 1.1:

Problem 1.4. *Assuming Fréchet differentiability of the map $\mathcal{F}$ at the origin and exponential instability of the linearized operator $L = d\mathcal{F}|_{u=0}$, corresponding to $r(L) > 1$, what are sharp conditions on the remainder $|\mathcal{F}(u) - Lu|_B$ for* (a) *nonlinear instability,* (b) *nonlinear exponential instability,* (c) *nonlinear exponential instability at linear rate $\rho = r(L)$?*

In the normal case, the results of Section 4 give a satisfactory answer to (c), and based on Example 1.3 we conjecture that conditions (1.7), (1.8) are also sharp for (b). In the general case, these questions remain essentially open, although one may conjecture that nonlinear stabilization, if it occurs at all, requires both nonseparability of the spectrum and a very slow vanishing rate of the nonlinearity.

1.5. **Further comments and applications.** We try here to make a connexion between the abstract questions discussed in this paper and some concrete problems studied in the literature, especially in fluid mechanics where stability issues are of great theoretical and practical importance. Consider equations in the general form

$$\frac{\mathrm{d}u}{\mathrm{d}t} = Au + f(u)\,, \qquad t \geq 0\,, \tag{1.10}$$

where the linear operator $A : D(A) \to B$ is the generator of a strongly continuous semigroup in a Banach space B, and the nonlinearity $f : B \to B$ is locally Lipschitz and satisfies

$$|f(u)|_B = o(|u|_B), \qquad \text{as } |u|_B \to 0\,. \tag{1.11}$$

In this case, any solution $u_n = u(n)$ of (1.10) evaluated at integer times satisfies the recursion relation (1.1), where $\mathcal{F}$ is the time-1 solution map, and at the linear level one has the relation $L = d\mathcal{F}|_{u=0} = \exp(A)$. Most of the results presented above for the difference equation (1.1) have their counterpart for the differential equation (1.10)-(1.11), in particular the Lyapunov instability theorem under the assumption of a spectral gap and the Rutman-Daleckii theorem [SS].

Many time-evolutionary partial differential equations, while admitting the general form (1.10) (with the possible addition of constraint equations and boundary conditions), have nonlinearities f which are not Lipschitz. Often $f(u)$ involves spatial derivatives of the unknown u, and satisfies an estimate of the form

$$|f(u)|_B \lesssim |u|_X |u|_B\,, \qquad \text{for small enough } |u|_X\,, \text{ with } X \hookrightarrow B\,. \tag{1.12}$$

The loss of regularity in estimate (1.12) can sometimes be compensated for by regularizing estimates for the semigroup. For the Navier-Stokes equations, these are provided by analytic semigroup theory; for Schrödinger and related dispersive operators, by Strichartz-type estimates. In the absence of regularizing estimates, particular features of the system can be exploited, such as the cancellation $(f(u), u)_B = 0$, with $B = L^2$, which holds for perfect incompressible fluids governed by the Euler equations and delimited by impermeable boundaries. In any case, it is necessary to deal with several function spaces. In particular, the assumptions of stability and instability theorems become complicated and case-dependent when formulated at the level of the differential equation (1.10)-(1.12). For examples of such statements, we refer to [SS].

A great advantage of working with the "integrated" formulation (1.1) is that all results can be stated in a single Banach space, which typically consists of sufficiently regular functions so that the Cauchy problem for (1.10) is locally well-posed in this space. This point of view is satisfying in terms of simplicity and generality, and avoids any confusion between the assumptions of the stability/instability theorems and what is needed to solve the Cauchy problem. However, in most applications, spectral information on the linearized problem is easier to obtain at the level of the generator A, which is typically an explicit differential operator, whereas the semigroup $\exp(A)$ has no simple representation. In addition, it often happens that the spectrum of A is more conveniently studied in a low regularity space such as L^2, while the Cauchy problem is only known to be well-posed in a smaller space, such as the Sobolev space H^s for sufficiently large $s > 0$. Finally, the notions themselves of stability or instability may be sensitive to the choice of the function space (see e.g. [Li1]), and it is not clear that using a framework where the Cauchy problem is well-posed gives the most appropriate definition of nonlinear instability. In fact, a very conclusive notion of instability is obtained if small initial perturbations in a strong norm (e.g. in H^s for large

$s > 0$) are shown to evolve into large discrepancies measured in a weak norm (e.g. in L^2). Strong instabilities in this sense have been established by Grenier [Gr] for the 2D Euler equation, and the same approach was subsequently used to establish transverse instabilities of travelling waves in dispersive Hamiltonian models [RT1, RT2]. In sum, we point out that the instability theorems formulated for system (1.1) in a single Banach space, although potentially applicable to a variety of situations, do not subsume the numerous results obtained for particular PDEs, especially in fluid mechanics [FSV, Yu, BGS, VF, Li2, FPS] and for related models [GS, LS, FPV].

We add a few comments concerning the scope of the results presented in this paper. As for the linear part of the system, emphasis is put on the situation where no spectral gap exists, so that no version of the Lyapunov instability theorem can be invoked. This point of view is quite reasonable, because in applications the linearized operator often has continuous spectrum without any gap, especially in nonparabolic equations or for systems on unbounded spatial domains. In such situations, the Rutman-Daleckii theorem is applicable if the solution map is sufficiently smooth, and this is why our results concentrate on systems where the solution map is merely C^1, or is even less regular so that the linearization at the origin has to be understood in a weak sense, for instance as a Gâteaux derivative. Such a general point of view is not of pure academic interest: there are many examples of partial differential equations for which the solution map is not even of class C^1, no matter how smooth the nonlinearities. Nonlinear hyperbolic systems typically belong to this category [Ka2], whereas parabolic equations usually generate smooth solution maps if the nonlinearities are smooth.

We conclude this introduction with two remarks on related questions. First, we recall that a possible approach to prove nonlinear instability in a system such as (1.1) or (1.10) is to construct an unstable invariant set, which contains negative trajectories of the system that converge to the equilibrium as $n \to -\infty$ or $t \to -\infty$. As is well known, if the equilibrium is hyperbolic, the unstable set is a manifold that is as smooth as allowed by the nonlinearity [Ru]. Interestingly enough, if B is a Hilbert space and if the linearized operator is normal, it is possible to construct an unstable invariant set even in the absence of spectral gap, see [EZ, Theorem 7.4], provided the nonlinearity is sufficiently smooth. Such a result strengthens the conclusion of the Rutman-Daleckii theorem in the normal case. From another perspective, we would like to mention that Problem 1.1 is strongly related to a different question that can be formulated for the differential equation (1.10): *assuming that the linearized equation $u' = Au$ is ill-posed, under what conditions can we deduce that the nonlinear system* (1.10) *is also ill-posed?* This question does not make sense for the difference equation (1.1), and will therefore not be discussed here. We refer the interested reader to [BS, GT] for the analysis of a few examples.

2. Stabilization of near-exponentially unstable linear maps

The classical example (1.4) of stabilization of a neutrally unstable equilibrium can be extended in the infinite-dimensional case to a much more dramatic example of stabilization of a near-exponentially unstable linear map. This shows that, for nonnormal operators in infinite dimensions, any linear instability that is less than exponential, in the sense that $r(L) = 1$, is susceptible of nonlinear stabilization by a map $\mathcal{F}$ satisfying estimate (1.3) for some arbitrarily large $p > 0$. In our example, we have the lower bound

$$|L^n|_{B\to B} \;\ge\; \prod_{1\le k\le n} m_k\,, \qquad \text{for all } n \ge 1\,,$$

where (m_k) is a nonincreasing sequence of real numbers converging to 1 arbitrarily slowly as $k \to \infty$. An appropriate choice of the sequence (m_k) thus leads to linear instability at any sub-exponential rate.

Consider the space $B = \ell_2(\mathbb{N})$ of square-integrable real sequences $u = (u^0, u^1, u^2, \dots)$, and let $S : B \to B$ be the right shift defined by $Su = (0, u^0, u^1, \dots)$ for all $u \in B$. Given a nonincreasing sequence (m_n) of real numbers such that $m_1 \le 2$ and $m_n \to 1$ as $n \to \infty$, we denote by $M : B \to B$ the associated multiplication operator: $Mu = (m_0 u^0, m_1 u^1, \dots)$. We consider the map $\mathcal{F} : B \to B$ defined by

$$\mathcal{F}(u) \;=\; (1 - |u|_B^p) MSu\,, \qquad u \in B\,, \tag{2.1}$$

where $p > 0$. It is clear that $\mathcal{F}$ is Fréchet differentiable at the origin, with linear tangent map $L = MS$. An easy calculation shows that, for any $n \ge 1$,

$$|(MS)^n|_{B\to B} \;=\; |(MS)^n\big(1,0,0,\dots\big)|_B \;=\; \prod_{1\le k\le n} m_k\,, \tag{2.2}$$

and this implies that the spectral radius of MS is equal to one, because $m_n \to 1$ as $n \to \infty$.

Proposition 2.1. *For any $p > 0$ and any sequence (m_n) as above, the origin $u = 0$ is a stable fixed point of system* (1.1) *with $\mathcal{F}$ given by* (2.1)*. In addition, for the particular choice*

$$m_n \;=\; 1 + \frac{1}{\ln(n+2)}\,, \tag{2.3}$$

there exists $C > 0$ such that

$$|(MS)^n|_{B\to B} \;=\; |(MS)^n\big(1,0,0,\dots\big)|_B \;\ge\; Ce^{n/(2\ln n)}, \qquad \textit{for all } n \ge 2\,, \tag{2.4}$$

indicating near-exponential linear instability.

Proof. We first remark that, if $|u|_B \le 1$, then

$$|\mathcal{F}(u)|_B \;\le\; 2|u|_B\,, \tag{2.5}$$

because $m_n \le m_1 \le 2$ for all $n \ge 1$. Let (u_n) be the solution of (1.1) with initial data $u_0 \in B$. Then u_n has the form

$$u_n \;=\; \Big(\underbrace{0,\dots,0}_{n\text{ terms}},\star,\dots\Big)\,,$$

due to the repeated action of the right shift. In particular, as long as $|u_n|_B \le 1$, there holds

$$|u_{n+1}|_B \;\le\; m_{n+1}(1 - |u_n|_B^p)|u_n|_B\,, \qquad \text{hence} \quad |u_n|_B \;\le\; |u_0|_B \prod_{1\le k\le n} m_k\,. \tag{2.6}$$

Now, given any $\varepsilon \in (0,1)$, there exists $N = N(\varepsilon) \in \mathbb{N}^*$ such that

$$m_N\Big(1 - \frac{\varepsilon^p}{2^p}\Big) \;<\; 1\,. \tag{2.7}$$

Assume that $|u_0|_B < \delta$, where $\delta > 0$ is small enough so that

$$\delta \prod_{1\le n\le N} m_n \;<\; \varepsilon/2\,.$$

Then, by the second inequality in (2.6), for all $n \in \{1,\dots,N\}$ there holds $|u_n|_B < \varepsilon/2$. In fact, we shall show that $|u_n|_B < \varepsilon$ for all $n \in \mathbb{N}$, which proves that the origin is a stable equilibrium.

Indeed, if this is not the case, there exists a smallest integer $N_0 > N$ such that $|u_{N_0}|_B \ge \varepsilon$. Let $n = N_0 - 1 \ge N$, so that $|u_n|_B < \varepsilon$. If $|u_n|_B < \varepsilon/2$, estimate (2.5) implies that

$$|u_{N_0}|_B \;=\; |u_{n+1}|_B \;=\; |\mathcal{F}(u_n)|_B \;\le\; 2|u_n|_B \;<\; \varepsilon\,,$$

which is a contradiction. On the other hand, if $\varepsilon/2 \le |u_n|_B < \varepsilon$, it follows from (2.7) that

$$m_{n+1}(1 - |u_n|_B^p) \;\le\; m_N(1 - \varepsilon^p/2^p) \;<\; 1\,,$$

because $m_{n+1} = m_{N_0} \le m_N$ by construction. Using the first inequality in (2.6), we deduce that $|u_{N_0}|_B = |u_{n+1}|_B < |u_n|_B < \varepsilon$, which is again a contradiction. Thus there exists no such N_0, and nonlinear stability is established.

We next derive a lower bound on the linear growth rate, in the particular case of the sequence (m_n) given by (2.3). In view of (2.2) we have

$$\ln|(MS)^n(1,0,0,\dots)|_B \;=\; \sum_{1\le k\le n} \ln(1+\alpha_k)\,, \qquad \text{where} \quad \alpha_k \;=\; \frac{1}{\ln(k+2)}\,.$$

But $\ln(1+\alpha_k) \ge \alpha_k/2$ because $\alpha_k \in [0,2]$, hence

$$\ln|(MS)^n(1,0,0,\dots)|_B \;\ge\; \frac{1}{2}\sum_{1\le k\le n} \alpha_k\,.$$

Since $x \mapsto \ln(x+2)^{-1}$ is a decreasing function of $x \ge 0$, we deduce

$$\ln|(MS)^n(1,0,0,\dots)|_B \;\ge\; \frac{1}{2}\int_1^{n+1} \frac{\mathrm{d}x}{\ln(x+2)} \;\ge\; \frac{1}{2}\frac{x+2}{\ln(x+2)}\Big|_{x=1}^{x=n+1} \;=\; \frac{1}{2}\frac{n+3}{\ln(n+3)} - \frac{3}{2\ln 3}\,,$$

and estimate (2.4) easily follows. $\square$

In our example the linear operator $L = MS$ is not normal, because the adjoint operator $L^* = S^*M$ (where S^* is the left shift) does not commute with L. In fact L is a compact perturbation of the right shift S, which in turn corresponds to an infinite-dimensional Jordan block. Remark that the phenomenon described in Proposition 2.1 cannot occur in the normal case, because spectral stability of a normal operator L implies that $|L|_{B\to B} = r(L) \le 1$. From this point of view, the example above is rather related to the stabilization of unstable pseudospectra [TE]. Note finally that Proposition 2.1 holds for arbitrarily large values of $p > 0$, and this is in sharp contrast with the situation described in the Rutman-Daleckii theorem, where the superlinearity condition (1.3) with any $p > 0$ is sufficient to prevent nonlinear stabilization.

3. Examples of nonlinear stabilization

We discuss here Example 1.3 and several variants illustrating the possibility of nonlinear stabilization for linearly exponentially unstable equilibria. In all these examples, however, the map $\mathcal{F}$ is not Fréchet differentiable and the linearization has to be understood in a weaker sense, typically as a Gâteaux derivative.

3.1. The main example.

In $B = L^2(\mathbb{R})$, we consider the map $\mathcal{F} : B \to B$ defined by (1.5). We recall that $\chi : \mathbb{R} \to [0,2]$ is a smooth function such that $\mathrm{supp}(\chi) \subset [-1,1]$ and $\chi(0) = 2$, and that $h : [0,\infty) \to [0,\infty)$ is an increasing continuous function satisfying $h(0) = 0$.

Proposition 3.1. *Given any $u_0 \in B$, the solution of (1.1) satisfies $|u_n|_B \to 0$ as $n \to \infty$. In addition, for any $\delta > 0$ and any $u_0 \in B$ with $|u_0|_B \le \delta$, the sequence (u_n) issued from u_0 satisfies*

$$\max_{n\in\mathbb{N}} |u_n|_B \;\le\; \delta\, 2^{\frac{2}{h(\delta)}+1}\,, \tag{3.1}$$

which proves nonlinear stability if $h(s)$ converges sufficiently slowly to zero as $s \to 0$.

Proof. Using the specific form of the map $\mathcal{F}$ defined in (1.5), we establish by induction on k the following representation formula for the solution of (1.1):

$$u_n(x) \;=\; \Bigl(\prod_{j=n-k+1}^{n} \chi(x - S_j^n)\Bigr)\, u_{n-k}(x - S_{n-k}^n)\,, \qquad x \in \mathbb{R}\,, \quad n \ge k \ge 1\,, \tag{3.2}$$

where the spatial shifts S_j^n are defined by

$$S_j^n \;=\; \sum_{\ell=j}^{n-1} h(|u_\ell|_B) \;\ge\; 0\,, \qquad 0 \le j \le n\,,$$

with the convention that $S_j^n = 0$ if $j = n$.

Given $\varepsilon > 0$, let $I_\varepsilon = \{n \ge 1 \,|\, |u_n|_B \ge \varepsilon\} \subset \mathbb{N}$. We claim that I_ε is a finite set, with cardinality

$$N_\varepsilon \;=\; \mathrm{card}(I_\varepsilon) \;\le\; \frac{2}{h(\varepsilon)} + 1\,. \tag{3.3}$$

This implies in particular that $|u_n|_B \to 0$ as $n \to \infty$. To prove (3.3), we first observe that, if $n \in I_\varepsilon$, then $S_1^n \le 2$. Indeed, if $S_1^n > 2$, then $\chi(x)\chi(x - S_1^n) = 0$ for all $x \in \mathbb{R}$ by the support property of χ, and the relation (3.2) with $k = n$ shows that $u_n \equiv 0$, so that $n \notin I_\varepsilon$. Using the monotonicity of the function h, we deduce that, for any $n \in I_\varepsilon$,

$$2 \;\ge\; S_1^n \;=\; \sum_{j=1}^{n-1} h(|u_j|_B) \;\ge\; \mathrm{card}\big(I_\varepsilon \cap \{1,\dots,n-1\}\big) h(\varepsilon)\,,$$

and (3.3) easily follows.

We next turn to a proof of (3.1). Given any $\delta > 0$ and any $u_0 \in B$ with $|u_0|_B \le \delta$, we take $N \in \mathbb{N}$ such that

$$|u_N|_B \;=\; \max_{n\in\mathbb{N}} |u_n|_B\,.$$

Such an N exists since the sequence $(|u_n|_B)_n$ converges to zero. If $|u_N|_B \le \delta$, then (3.1) is proved. Otherwise, in the backward sequence of real numbers $|u_N|_B$, $|u_{N-1}|_B, \dots, |u_0|_B$ we find the first term or terms to be greater than δ, but not all, since $|u_0|_B \le \delta$. That is, there exists an integer $k \le N$ such that $|u_{N-k}|_B \le \delta$ and $|u_j|_B > \delta$ for $j \in [N{-}k{+}1, N]$. By definition we have $k \le N_\delta$, where N_δ is defined by (3.3) with $\varepsilon = \delta$. Using the representation (3.2) and the fact that $\chi \le 2$, we conclude that

$$|u_N|_B \;\le\; 2^k |u_{N-k}|_B \;\le\; 2^{N_\delta}\delta \;\le\; \delta\, 2^{\frac{2}{h(\delta)}+1}\,,$$

which proves (3.1). Note that the right-hand side of (3.1) converges to zero as $\delta \to 0$ provided

$$\frac{2\ln(2)}{h(\delta)} + \ln(\delta) \;\to\; -\infty\,, \quad \text{as} \quad \delta \to 0\,,$$

which is the case, for instance, if $h(s) \ge C|\ln s|^{-1}$ for some $C > 2\ln(2)$. $\square$

Remark 3.2. More generally, if $\mathrm{supp}(\chi) \subset [-a, a]$ for some $a > 0$ and $0 \le \chi \le b$ for some $b > 1$, the bound (3.1) becomes

$$\max_{n\in\mathbb{N}} |u_n|_B \;\le\; \delta\, b^{\frac{2a}{h(\delta)}+1}\,, \qquad \text{provided} \quad |u_0|_B \le \delta\,.$$

In Example 1.3, the map $\mathcal{F}$ defined by (1.5) is not Fréchet differentiable at the origin. Indeed, there holds

$$\lim_{\lambda\to 0+} \lambda^{-1}\mathcal{F}(\lambda u) \;=\; Lu \;=\; \chi u\,, \qquad \text{for any } u \in B\,,$$

but the convergence is not uniform in u over the unit sphere $S = \{u \in B \,|\, |u|_B = 1\}$. Uniform convergence holds on any subset of S that is bounded in the Sobolev space $H^s(\mathbb{R})$ for some $s > 0$, but such subsets are not invariant under the evolution defined by $\mathcal{F}$. Note however that, in view of (3.2), the large Fourier modes of the initial data u_0 are only moderately amplified under evolution, because the function χ is assumed to be smooth. If on the contrary we take $\chi = 2\mathbf{1}_{[-1,1]}$, then Proposition 3.1 remains valid, but the spectrum of L now consists of two eigenvalues 0 and 2, so that L has a spectral gap, and the generalized Lyapunov instability theorem mentioned in the introduction shows that nonlinear stabilization is impossible if the map is Fréchet differentiable. In that case, large Fourier modes are immediately created because the Fourier transform of χ decays slowly at infinity.

3.2. **A variant with additional contraction of the support.** The following variant of Example 1.3 exhibits an even more efficient mechanism of nonlinear stabilization. Consider the Banach space $B = C_0^0([-1,0];\mathbb{R})$ of all continuous functions $u : [-1,0] \to \mathbb{R}$ satisfying $u(-1) = 0$, equipped with the norm $|u|_\infty = \max_{-1\le x\le 0} |u(x)|$. Let E be the extension operator defined for any $u \in B$ by $(Eu)(x) = u(x)$ if $x \in [-1,0]$, and $(Eu)(x) = 0$ if $x \in \mathbb{R} \setminus [-1,0]$. We define a map $\mathcal{F} : B \to B$ by

$$(\mathcal{F}(u))(x) \,=\, 2(Eu)(2x - |u|_\infty^2)\,, \qquad x \in [-1,0]\,, \tag{3.4}$$

and we introduce the associated linearized operator

$$(Lu)(x) \,=\, 2(Eu)(2x)\,, \qquad x \in [-1,0]\,.$$

As before, it is easy to verify that L is the Gâteaux derivative of $\mathcal{F}$ at $u = 0$, but that $\mathcal{F}$ is not Fréchet differentiable at the origin. Moreover, we obviously have $|Lu|_\infty = 2|u|_\infty$ for any $u \in B$, hence $|L^n u|_\infty = 2^n |u|_\infty$ for all $n \ge 1$. Thus $r(L) = 2$, indicating linear exponential instability.

Proposition 3.3. *Given $u_0 \in B$ the solution of (1.1) satisfies, for any $\alpha \in [0,1]$,*

$$|u_n|_\infty \,\le\, 2^{3(1-\alpha)/2}\, 2^{n(3\alpha-1)/2}\, |u_0|_\infty^\alpha\,, \qquad n \,\ge\, 1\,. \tag{3.5}$$

Taking $0 < \alpha < 1/3$ proves nonlinear asymptotic stability.

Proof. We have, evidently,

$$|u_n|_\infty \,\le\, 2^n |u_0|_\infty\,, \qquad \text{and} \quad \operatorname{supp} u_n \,\subset\, [-2^{-n},0]\,. \tag{3.6}$$

Let $I = \{n \in \mathbb{N} \,|\, |u_n|_\infty \le 2^{-n/2}\}$. If $n \notin I$, then $|u_n|_\infty^2 > 2^{-n}$, and using the information on the support in (3.6) together with definition (3.4) we deduce that $u_{n+1} \equiv 0$, so that $n+1 \in I$. As a consequence, if $n \notin I$ and $n \ge 1$, we must have $n-1 \in I$, hence $|u_n|_\infty \le 2|u_{n-1}|_\infty \le 2 \cdot 2^{-(n-1)/2}$. Thus we have shown that

$$|u_n|_\infty \,\le\, 2^{-(n-3)/2}\,, \qquad \text{for all } n \ge 1\,,$$

and interpolating with the bound in (3.6) we easily obtain estimate (3.5). □

As in Example 1.3 above, nonlinear stabilization occurs here because the map $\mathcal{F}$ involves a translation of the argument u, whose support (under the repeated action of $\mathcal{F}$) eventually leaves the interval $[-1,0]$ where linear instability is at play. The novelty is that $\mathcal{F}$ also shrinks the support of u by a factor of 2, so that the support of u_n for large n is contained in a very small one-sided neighborhood of the origin, and can thus easily be pushed away from the interval $[-1,0]$. This explains why stabilization can be realized here with a very small spatial shift $|u|_\infty^2$, whereas a larger translate $h(|u|_B)$ was necessary in Example 1.3. However, the contraction of the support has also the effect of creating large Fourier modes in the solution u_n, so that the modified map (3.4) is certainly further away from being Fréchet differentiable than the original map (1.5). As was already mentioned, failure of Fréchet differentiability at the origin implies failure of the remainder bound (1.3), hence Proposition 3.3 does not contradict the Rutman-Daleckii theorem mentioned in the introduction.

3.3. **A hyperbolic partial differential equation.** Interestingly enough, the phenomenon illustrated in the previous example can occur if $\mathcal{F}$ is the time-one map associated with an autonomous partial differential equation. To see this, consider the quasilinear hyperbolic equation

$$u_t + \big((-x+u^2)u\big)_x = 0\,, \qquad x \in [-1,0]\,, \qquad u(-1,t) \,=\, 0\,. \tag{3.7}$$

We assume that the initial data u_0 belong to the convex cone

$$B_+ \,=\, \{u \in B \,|\, u \text{ is nondecreasing}\}\,,$$

where $B = C_0^0([-1,0];\mathbb{R})$ is the same function space as in Section 3.2. For such data Eq. (3.7) has a unique global solution for positive times, which can be constructed by the method of characteristics and satisfies $u(\cdot,t) \in B_+$ for all $t \ge 0$.

Indeed, the characteristic curve $X(t)$ originating from point $x_0 \in [-1,0]$ satisfies the differential equation $X'(t) = -X(t) + 3u(X(t),t)^2$ with initial condition $X(0) = x_0$. Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\,u(X(t),t) \,=\, u_t(X(t),t) + [-X(t) + 3u(X(t),t)^2]u_x(X(t),t) \,=\, u(X(t),t)\,,$$

we have $u(X(t),t) = e^t u_0(x_0)$, which in turn implies that

$$X(t) \,=\, e^{-t}\Big(x_0 + u_0(x_0)^2(e^{3t}-1)\Big)\,, \qquad t \,\ge\, 0\,. \tag{3.8}$$

As $u_0 \in B_+$, for each $t \ge 0$ the right-hand side of equation (3.8) is a strictly increasing function of $x_0 \in [-1,0]$, which maps $[-1,0]$ onto $[-e^{-t}, a_0]$, with $a_0 = e^{-t}u_0(0)^2(e^{3t}-1) \ge 0$. In particular, for each $t \ge 0$ and each $x \in [-e^{-t},0]$ there exists a unique $x_0 \in [-1,0]$ such that $X(t) = x$. Denoting $x_0 = X_0(x,t)$, we obtain the representation formula

$$u(x,t) \,=\, \begin{cases} e^t u_0(X_0(x,t)) & \text{if } x \in [-e^{-t},0]\,, \\ 0 & \text{if } x \in [-1,-e^{-t}]\,, \end{cases} \tag{3.9}$$

which gives a global solution to (3.7) for positive times, such that $u(\cdot,t) \in B_+$ for all $t \ge 0$.

Moreover, the time-t map defined by Eq. (3.9) is Gâteaux differentiable at the origin in the cone B_+, with derivative given by the time-t map of the linearized equation $\tilde{u}_t - (x\tilde{u})_x = 0$. Indeed, given u_0 in B_+ and $\lambda > 0$, consider the function u_λ defined in such a way that $\lambda u_\lambda(x,t)$ is the (unique) solution of (3.7) with initial data $\lambda u_0 \in B_+$. For $t > 0$ and $x \in [-e^{-t},0]$, we know from (3.8), (3.9) that

$$u_\lambda(x,t) \,=\, e^t u_0(X_{0,\lambda}(x,t))\,, \qquad \text{with} \quad e^t x \,=\, X_{0,\lambda}(t,x) + \lambda^2 u_0(X_{0,\lambda}(t,x))^2(e^{3t}-1)\,. \tag{3.10}$$

Since $e^t x - \lambda^2|u_0|_\infty(e^{3t}-1) \le X_{0,\lambda}(x,t) \le e^t x$, we deduce from (3.9) and (3.10) that, for any fixed $t > 0$, the function $u_\lambda(\cdot,t)$ converges uniformly on $[-1,0]$ to $\tilde{u}(\cdot,t)$ as $\lambda \to 0$, where

$$\tilde{u}(x,t) \,=\, \begin{cases} e^t u_0(e^t x) & \text{if } x \in [-e^{-t},0] \\ 0 & \text{if } x \in [-1,-e^{-t}] \end{cases}$$

is the unique solution in B_+ to the linearized equation $\tilde{u}_t - (x\tilde{u})_x = 0$ with initial data u_0.

We observe that $|\tilde{u}(\cdot,t)|_\infty = e^t|u_0|_\infty$ for any $t \ge 0$, so that the linearized evolution is exponentially unstable, but the following result shows that the nonlinear evolution is asymptotically stable.

Proposition 3.4. *For all initial data $u_0 \in B_+$ the solution of* (3.7) *given by* (3.9) *satisfies, for any $\alpha \in [0,1]$,*

$$|u(\cdot,t)|_\infty \,\le\, C^{1-\alpha}\, e^{(3\alpha-1)t/2}\, |u_0|_\infty^\alpha\,, \qquad t \ge 1\,,$$

where $C = (1-e^{-3})^{-1/2}$. Taking $0 < \alpha < 1/3$ proves nonlinear asymptotic stability.

Proof. Given $t > 0$ and $x \in [-e^{-t}, 0]$, we denote $x_0 = X_0(x,t) \in [-1,0]$. We know from (3.8) that $x_0 + u_0(x_0)^2(e^{3t}-1) = e^t x \le 0$, hence we deduce from (3.9) that

$$0 \,\le\, u(x,t) \,=\, e^t u_0(x_0) \,\le\, \frac{e^t|x_0|^{1/2}}{(e^{3t}-1)^{1/2}}\,.$$

Assuming $t \ge 1$ this gives the estimate $|u(\cdot,t)|_\infty \le Ce^{-t/2}$, and interpolating with the trivial bound $|u(\cdot,t)|_\infty \le e^t|u_0|_\infty$ we obtain the desired result. $\square$

Remark 3.5. The restriction of the analysis to the convex cone B_+ represents a considerable simplification, because no shocks can develop and the solution can be constructed using the method of characteristics. For general initial data $u_0 \in B$ with no monotonicity assumption, global existence of a unique entropic weak solution to the scalar conservation law (3.7) can be established following the approach of Kruzkhov [Kr]. However this solution may now have discontinuities, hence it is necessary to work in a larger function space, such as $L^\infty([-1,0])$. In this more general setting, the time-t map associated with (3.7) is not differentiable at the origin, even in the sense of Gâteaux, so that the definition of the linearized system becomes a more delicate question. These technicalities are not related in any way to the nonlinear stabilization effect that we discuss here, hence we prefer avoiding them by working in the cone B_+, which however is not a Banach space.

3.4. **A finite-dimensional example.** To conclude this section, we exhibit a two-dimensional Gâteaux differentiable map for which nonlinear stabilization occurs via a similar mechanism as in the previous examples. In $B = \mathbb{R}^2$ we consider the map $\mathcal{F}$ defined by

$$\mathcal{F}(v,w) \,=\, (2v\mathbf{1}_{D^c}(v,w)\,,\, w/2 + v^2/4)\,, \qquad (v,w) \in B\,, \tag{3.11}$$

where $D = \{(v,w)\,|\,0 < |w| < v^2\} \subset \mathbb{R}^2$ and $\mathbf{1}_{D^c}$ denotes the indicator function of $D^c = \mathbb{R}^2 \backslash D$. Note that D^c contains the axis $w = 0$, and that $\mathcal{F}(v,0) = (2v, v^2/4)$. The map $\mathcal{F}$ is not continuous, but it is continuous at the origin and Gâteaux differentiable there with derivative $L(v,w) = (2v, w/2)$, which is both linear and exponentially unstable. However, it is easy to show that the origin is asymptotically stable for the dynamics associated with the full map (3.11).

Proposition 3.6. *For all initial data* $(v_0, w_0) \in \mathbb{R}^2$ *the solution of* (1.1) *with* $\mathcal{F}$ *given by* (3.11) *satisfies, for all* $n \in \mathbb{N}$,

$$|v_n|^2 + |w_n| \,\le\, 4\Bigl(\frac{3}{4}\Bigr)^{n-1}\bigl(v_0^2 + |w_0|\bigr)\,. \tag{3.12}$$

Proof. If $v_0 = 0$, then $v_n = 0$ and $w_n = 2^{-n}w_0$ for all $n \in \mathbb{N}$, hence (3.12) obviously holds. Moreover, if $(v_0,w_0) \in D$, then $v_1 = 0$ and $w_1 = w_0/2 + v_0^2/4$, so that (3.12) holds for $n \le 1$, and the subsequent values of n follow as in the previous case. Similarly, if $w_0 = 0$ and $v_0 \ne 0$, then $(v_1, w_1) = (2v_0, v_0^2/4) \in D$, and proceeding as above we conclude that (3.12) holds for all $n \in \mathbb{N}$. So it remains to consider the case where $|w_0| \ge v_0^2 > 0$. If $|w_n| \ge v_n^2$ for all $n \in \mathbb{N}$, we have $|w_{n+1}| \le |w_n|/2 + v_n^2/4 \le 3|w_n|/4$ for all n, hence

$$|w_n| \,\le\, \Bigl(\frac{3}{4}\Bigr)^n|w_0|\,, \qquad v_n^2 \,\le\, |w_n| \,\le\, \Bigl(\frac{3}{4}\Bigr)^n|w_0|\,, \tag{3.13}$$

and (3.12) follows. In the converse case, let $N \ge 1$ be the smallest integer for which $|w_N| < v_N^2$. Then the first inequality in (3.13) holds for all $n \le N$, and the second one for $n \le N-1$. Since $v_N^2 \le 4v_{N-1}^2 \le 4\cdot(3/4)^{N-1}|w_0|$, we deduce that (3.12) holds for all $n \le N$. We also know that $|w_N| \ge 4^{-N}|w_0| > 0$, hence $(v_N, w_N) \in D$, and this implies that $v_n = 0$ for all $n \ge N+1$. As

$$|w_{N+1}| \,\le\, \frac{|w_N|}{2} + \frac{v_N^2}{4} \,<\, \frac{3v_N^2}{4} \,\le\, 4\Bigl(\frac{3}{4}\Bigr)^N|w_0|\,,$$

we conclude that estimate (3.12) holds for $n = N+1$, hence also for all subsequent n. The proof is thus complete. $\square$

Remark 3.7. The map $\mathcal{F}$ defined in (3.11) is not continuous, but we believe that the same stabilization phenomenon can occur for a map that is Hölder continuous and smooth outside the origin. Lipschitz regularity cannot be achieved, because in a finite-dimensional space a Lipschitz map $\mathcal{F}$ that is Gâteaux differentiable at the origin with a linear derivative $L = d\mathcal{F}|_{u=0}$ is automatically Fréchet differentiable [AH, Appendix A], in which case nonlinear stabilization is precluded by the Lyapunov instability theorem.

4. The case of normal linear tangent maps

In this final section, we assume that B is a Hilbert space, and that L is a bounded linear operator in B which is *normal*, in the sense that $LL^* = L^*L$. We consider a map $\mathcal{F} : B \to B$ with $\mathcal{F}(0) = 0$ satisfying (1.7), namely

$$|\mathcal{F}(u) - Lu|_B \,\le\, \alpha(|u|_B)|u|_B\,, \qquad \text{whenever} \quad |u|_B \le a\,, \tag{4.1}$$

where $a > 0$ and $\alpha : [0, a] \to [0, +\infty)$ is a nondecreasing function such that $\alpha(s) \to 0$ as $s \to 0$. This of course implies that $\mathcal{F}$ is Fréchet differentiable at the origin with $L = d\mathcal{F}|_{u=0}$. In this situation, the Rutman-Daleckii theorem quoted in the introduction can be sharpened as follows.

Proposition 4.1. *If the function α satisfies the integrability condition* (1.8)*, then the linear instability assumption $r(L) > 1$ implies implies nonlinear exponential instability of system* (1.1)*, at the linear rate $\rho = r(L)$.*

Proof. We adapt the proof given in [He, Theorem 5.1.5]. Given $u_0 \in B$, let (u_n) be the sequence defined by (1.1). By straightforward induction, there holds

$$u_n \,=\, L^n u_0 \,+ \sum_{0\le k\le n-1} L^{n-k-1}\bigl(\mathcal{F}(u_k) - Lu_k\bigr)\,, \qquad n \,\ge\, 1\,, \tag{4.2}$$

where L^0 is the identity map. Since the operator L is normal, the norm $|L|_{B\to B}$ is equal to the spectral radius $r(L)$, see [Ka1, Section V.2.1], which gives the bound

$$|Lu|_B \,\le\, r(L)|u|_B\,, \qquad \text{for all } u \in B\,. \tag{4.3}$$

From (4.2), (4.3), and assumption (4.1), we deduce the upper bound

$$|u_n|_B \,\le\, r(L)^n|u_0|_B \,+ \sum_{0\le k\le n-1} r(L)^{n-k-1}\alpha(|u_k|_B)|u_k|_B\,, \tag{4.4}$$

which holds provided $|u_k|_B \le a$ for $0 \le k \le n-1$.

From now on, we assume that $r(L) > 1$ and that the function α satisfies the integrability assumption (1.8). We fix $\eta \in (0, a]$ small enough so that

$$\frac{2}{r(L)\ln r(L)} \int_0^\eta \frac{\alpha(s)}{s}\,\mathrm{d}s \,\le\, \frac{1}{4}\,. \tag{4.5}$$

Given any sufficiently small $\delta > 0$, we denote by $N = N(\delta)$ the unique positive integer such that

$$2r(L)^N\delta \,\le\, \eta \,<\, 2r(L)^{N+1}\delta\,. \tag{4.6}$$

In a first step, given any initial data $u_0 \in B$ with $|u_0|_B \le \delta$, we prove by induction that

$$|u_k|_B \,\le\, 2r(L)^k|u_0|_B\,, \quad 0 \le k \le N\,. \tag{4.7}$$

Indeed the inequality in (4.7) obviously holds for $k = 0$. Given $1 \le n \le N$, assume that the inequality holds for $0 \le k \le n-1$. Then $|u_k|_B \le 2r(L)^N\delta \le \eta$ for $k \le n-1$, and using in addition (4.4) we deduce that

$$|u_n|_B \,\le\, r(L)^n\Bigl(1 + 2r(L)^{-1} \sum_{0\le k\le n-1} \alpha\bigl(2r(L)^k|u_0|_B\bigr)\Bigr)|u_0|_B\,. \tag{4.8}$$

As the function α is nondecreasing by assumption, we can bound

$$\sum_{0\le k\le n-1} \alpha\big(2r(L)^k|u_0|_B\big) \;\le\; \int_0^n \alpha(2r(L)^x|u_0|_B)\,\mathrm{d}x \;=\; \frac{1}{\ln r(L)}\int_{2|u_0|_B}^{2r(L)^n|u_0|_B} \frac{\alpha(s)}{s}\,\mathrm{d}s$$

$$\le\; \frac{1}{\ln r(L)}\int_0^{\eta} \frac{\alpha(s)}{s}\,\mathrm{d}s \;\le\; \frac{r(L)}{8}\,, \qquad (4.9)$$

where we have used (4.5) and (4.6) in the last inequalities. Combining (4.8) and (4.9), we obtain $|u_n|_B \le (5/4)r(L)^n|u_0|_B \le 2r(L)^n|u_0|_B$, which completes the inductive proof of (4.7).

In a second step, we prove nonlinear exponential instability of system (1.1) by considering specific initial data. As the spectrum of L is a compact subset of the complex plane, there exists $\lambda \in \sigma(L)$ such that $|\lambda| = r(L)$. Moreover λ is an approximate eigenvalue of L in the sense that, for any $\nu > 0$, there exists $v_\nu \in B$ such that

$$|v_\nu|_B = 1\,, \qquad \text{and} \quad |(L-\lambda)v_\nu|_B \;\le\; \nu\,. \qquad (4.10)$$

Indeed, take any $\mu \in \mathbb{C}$ with $|\mu| > r(L)$ and $|\mu - \lambda| < \nu/2$. The norm of the resolvent operator $(L-\mu)^{-1}$ is equal to $\mathrm{dist}(\mu,\sigma(L))^{-1} \ge |\mu-\lambda|^{-1} > 2/\nu$, hence there exists $v_\nu \in B$ with $|v_\nu|_B = 1$ such that $|(L-\mu)v_\nu|_B \;\le\; \nu/2$, and (4.10) follows. Using the factorization

$$L^n - \lambda^n \;=\; (L^{n-1} + L^{n-2}\lambda + \cdots + L\lambda^{n-2} + \lambda^{n-1})(L-\lambda)\,,$$

together with the bound (4.3) and the fact that $|\lambda| = r(L)$, we easily obtain

$$|(L^n - \lambda^n)v_\nu|_B \;\le\; \nu n r(L)^{n-1}\,, \qquad \text{for all } n \;\ge\; 1\,. \qquad (4.11)$$

Now, given $\delta > 0$ arbitrarily small, we define $N \in \mathbb{N}$ as in (4.6) and take $v_\nu \in B$ such that (4.10) holds with $\nu = r(L)/(4N)$. We choose as initial data $u_0 = \delta v_\nu$, so that $|u_0|_B = \delta$. In view of (4.11), we have

$$|L^n u_0|_B \;\ge\; r(L)^n\Big(1 - \frac{\nu n}{r(L)}\Big)|u_0|_B \;\ge\; \frac{3}{4}r(L)^n|u_0|_B\,, \qquad 1 \le n \le N\,.$$

Thus, using (4.2) as well as the estimates (4.7) and (4.9), we obtain the lower bound

$$|u_n|_B \;\ge\; r(L)^n\Big(\frac{3}{4} - 2r(L)^{-1}\sum_{0\le k\le n-1}\alpha\big(2r(L)^k|u_0|_B\big)\Big)|u_0|_B \;\ge\; \frac{1}{2}r(L)^n|u_0|_B\,,$$

for $1 \le n \le N$. In particular, in view of (4.6), there holds $|u_N|_B \ge \varepsilon := \eta/(4r(L))$, and this shows that system (1.1) is exponentially unstable at the linear rate $\rho = r(L)$ in the sense of Definition 1.2. $\square$

Remark 4.2. We develop below a more geometric approach to nonlinear instability, based on the construction of invariant regions, which gives in particular an alternative proof of Proposition 4.1. The above proof is more straightforward, and it is interesting to note that the normality of the linearized operator L is only used to establish estimate (4.3), which is a first step in the derivation of the upper bound (4.4).

The following example shows that Proposition 4.1 is sharp in terms of preserving the linear rate.

Proposition 4.3. *For the scalar map $\mathcal{F} : \mathbb{R} \to \mathbb{R}$ defined by*

$$\mathcal{F}(u) \;=\; r(L)u - \alpha(|u|)u\,, \qquad (4.12)$$

where $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing on the interval $[0,a]$ with $\alpha(s) \to 0$ as $s \to 0$, nonlinear exponential instability of system (1.1) *at the linear rate $r(L) > 1$ implies the integrability condition* (1.8).

Proof. Assume nonlinear exponential instability of system (1.1) at the linear rate: there exists $\varepsilon > 0$ and $C > 0$ such that, for arbitrarily small initial data $u_0 \in \mathbb{R}$, the solution of (1.1) satisfies $|u_n| \ge Cr(L)^n|u_0|$ as long as $|u_n| \le \varepsilon$. Without loss of generality, we suppose that that $\varepsilon \le a$ and that $\alpha(s) \le r(L)/2$ for all $s \in [0,\varepsilon]$. Let (u_n) be an unstable sequence as described above, and assume for definiteness that $u_0 > 0$. Let N be the largest nonnegative integer such that

$$Cr(L)^n u_0 \;\le\; u_n \;\le\; \varepsilon\,, \qquad \text{for} \quad 0 \le n \le N\,.$$

By definition of $\mathcal{F}$ in (4.12), there holds

$$u_n \;=\; r(L)^n u_0 \prod_{0\le k\le n-1}\Bigl(1 - \frac{\alpha(u_k)}{r(L)}\Bigr)\,, \qquad n \in \mathbb{N}\,, \tag{4.13}$$

so that $u_n \le r(L)^n u_0$ for $0 \le n \le N+1$. It follows that

$$\ln u_n \;=\; \ln u_0 + n\ln r(L) \;+ \sum_{0\le k\le n-1} \ln\Bigl(1 - \frac{\alpha(u_k)}{r(L)}\Bigr)\,, \qquad 1 \le n \le N\,,$$

and since $u_n \ge Cr(L)^n u_0$ this implies

$$\frac{1}{r(L)} \sum_{0\le k\le n-1} \alpha(u_k) \;\le\; -\sum_{0\le k\le n-1} \ln\Bigl(1 - \frac{\alpha(u_k)}{r(L)}\Bigr) \;\le\; -\ln C\,, \qquad 1 \le n \le N\,, \tag{4.14}$$

where the first inequality results from the fact that $x \le -\ln(1-x)$ for $x \in [0,1)$. Using the monotonicity assumption on α and the lower bound $u_k \ge Cr(L)^k u_0$, we deduce from (4.14) that

$$\int_{-1}^{N-1} \alpha\bigl(Cr(L)^x u_0\bigr)\,\mathrm{d}x \;\le \sum_{0\le k\le N-1} \alpha\bigl(Cr(L)^k u_0\bigr) \;\le\; -r(L)\ln C\,,$$

hence also

$$\int_{Cr(L)^{-1}u_0}^{Cr(L)^{N-1}u_0} \frac{\alpha(s)}{s}\,\mathrm{d}s \;\le\; -r(L)\ln r(L)\ln C\,.$$

Our choice of N implies that $r(L)^{N-1}u_0 \ge \varepsilon/r(L)^2$, whereas the lower bound $Cr(L)^{-1}u_0$ in the above integral can be taken arbitrarily small. We conclude that

$$\int_0^{C\varepsilon/r(L)^2} \frac{\alpha(s)}{s}\,\mathrm{d}s \;\le\; -r(L)\ln r(L)\ln C\,,$$

which of course implies (1.8). ☐

Remark 4.4. If $\alpha(s)$ converges arbitrarily slowly to zero as $s \to 0$, it is clear from the representation (4.13) that the origin $u = 0$ is exponentially unstable at any rate $\rho < r(L)$, but Proposition 4.3 shows that one can take $\rho = r(L)$ only if the integrability condition (1.8) is satisfied. In the particular case where $\alpha(s) = |\ln s|^{-\gamma}$ near $s = 0$, we thus have exponential instability at the linear rate $r(L)$ if $\gamma > 1$, and exponential instability at any rate $\rho < r(L)$ if $0 < \gamma \le 1$. In the limiting case $\gamma = 1$, a more detailed analysis shows that solutions of (1.1) with small initial data $u_0 > 0$ satisfy a lower bound of the form

$$\frac{u_n}{|\ln u_n|^\sigma} \;\ge\; r(L)^n \frac{u_0}{|\ln u_0|^\sigma}\,,$$

for some $\sigma > 0$, as long as u_n remains sufficiently small.

In the rest of this section, we establish a general nonlinear instability result which encompasses the particular situation considered in Proposition 4.1. Let B_1, B_2 be two Banach spaces, with $B_1 \neq \{0\}$. We consider the following discrete dynamical system in the product space $B_1 \times B_2$

$$\begin{cases} v_{n+1} \,=\, \mathcal{F}_1(v_n, w_n) \,:=\, L_1 v_n + \mathcal{N}_1(v_n, w_n)\,, \\ w_{n+1} \,=\, \mathcal{F}_2(v_n, w_n) \,:=\, L_2 w_n + \mathcal{N}_2(v_n, w_n)\,, \end{cases} \tag{4.15}$$

where L_1, L_2 are bounded linear operators in B_1, B_2, respectively, satisfying the following estimates

$$|L_1 v| \,\ge\, \rho|v|\,, \quad |L_2 w| \,\le\, \rho|w|\,, \quad \text{for all } (v,w) \in B_1 \times B_2\,, \quad \text{where } \rho > 1. \tag{4.16}$$

We assume that the nonlinear maps $\mathcal{N}_1 : B_1 \times B_2 \to B_1$ and $\mathcal{N}_2 : B_1 \times B_2 \to B_2$ vanish at the origin, and for any $r > 0$ we denote

$$\alpha(r) \,=\, \sup\left\{ \frac{|\mathcal{N}_1(v,w)| + |\mathcal{N}_2(v,w)|}{|v|} \;\Big|\; 0 < |v| \le r\,,\ |w| \le |v| \right\}. \tag{4.17}$$

The nondecreasing function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ measures the size of the nonlinearity $\mathcal{N} = (\mathcal{N}_1, \mathcal{N}_2)$ only inside the truncated cone $\{(v,w)\,|\,0 < |v| \le r,\ |w| \le |v|\}$. Therefore, assuming that $\alpha(r) \to 0$ as $r \to 0$ does not necessarily imply that the map $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$ is Fréchet differentiable at the origin, with derivative $L = (L_1, L_2)$. Nevertheless, the following result shows that, if $\alpha(s)/s$ is integrable at the origin, the linear exponential instability condition (4.16) implies nonlinear instability of the origin for the full system (4.15).

Proposition 4.5. *Assume that* (4.16) *holds for some* $\rho > 1$ *and that the function* α *defined in* (4.17) *satisfies the integrability condition* (1.8) *for some* $a > 0$. *Then the origin* $(0,0)$ *is an unstable equilibrium of* (4.15).

Proof. The idea is to construct an invariant region D in a careful way, depending on the function α which measures the size of the nonlinearity. Let $r_0 > 0$ and $\beta : [0, r_0] \to [0, \infty)$ be a nondecreasing continuous function satisfying $\beta(r) \le r$ for $0 \le r \le r_0$. We denote

$$D \,=\, \Big\{ (v,w) \in B_1 \times B_2 \,\Big|\, 0 < |v| \le r_0\,,\ |w| \le \beta(|v|) \Big\}. \tag{4.18}$$

If $(v_n, w_n) \in D$, then

$$|v_{n+1}| \,\ge\, |L_1 v_n| - |\mathcal{N}_1(v_n, w_n)| \,\ge\, (\rho - \alpha(|v_n|))|v_n|\,, \tag{4.19}$$

$$|w_{n+1}| \,\le\, |L_2 w_n| + |\mathcal{N}_2(v_n, w_n)| \,\le\, \rho\beta(|v_n|) + |v_n|\alpha(|v_n|)\,. \tag{4.20}$$

To ensure that $|w_{n+1}| \le \beta(|v_{n+1}|)$ if $|v_{n+1}| \le r_0$, we impose the functional inequality

$$\rho\beta(r) + r\alpha(r) \,\le\, \beta(\rho r - r\alpha(r))\,, \qquad 0 \le r \le r_0\,. \tag{4.21}$$

It is not clear a priori that (4.21) has any solution β that is continuous and nondecreasing, but we shall see that the integrability condition (1.8) is a necessary and sufficient condition for the solvability of (4.21). For the moment we just observe that, since α a nondecreasing function, the hypothesis (1.8) implies that $\alpha(r) \to 0$ as $r \to 0$. Thus, taking r_0 small enough, we can suppose that $\rho - \alpha(r_0) > 1$.

We now look for a solution of (4.21) in the form

$$\beta(r) \,=\, Cr \int_0^r \frac{\alpha(s)}{s}\,\mathrm{d}s\,, \qquad 0 \le r \le r_0\,,$$

where $C > (\rho \ln(\rho))^{-1}$. For $r \le r_0$ we then have

$$\begin{aligned}
\beta(\rho r - r\alpha(r)) - \rho\beta(r) &= C(\rho r - r\alpha(r)) \int_0^{\rho r - r\alpha(r)} \frac{\alpha(s)}{s}\, \mathrm{d}s - C\rho r \int_0^r \frac{\alpha(s)}{s}\, \mathrm{d}s \\
&= C\rho r \int_r^{\rho r - r\alpha(r)} \frac{\alpha(s)}{s}\, \mathrm{d}s - Cr\alpha(r) \int_0^{\rho r - r\alpha(r)} \frac{\alpha(s)}{s}\, \mathrm{d}s \\
&\ge Cr\alpha(r)\Big(\rho \ln(\rho - \alpha(r)) - \int_0^{\rho r - r\alpha(r)} \frac{\alpha(s)}{s}\, \mathrm{d}s\Big) ,
\end{aligned}$$

where in the last line the monotonicity of α was used. Since the last integral converges to zero as $r \to 0$, and since $C\rho \ln(\rho) > 1$, the right-hand side is larger than $r\alpha(r)$ if r is sufficiently small. Thus, if $r_0 > 0$ is small enough, then $\beta(r) \le r$ for $0 \le r \le r_0$ and (4.21) is satisfied.

With r_0 and β as above, consider the solution (v_n, w_n) of (4.15) originating from arbitrarily small initial data $(v_0, w_0) \in D$. As long as $|v_n| \le r_0$, the solution (v_n, w_n) remains in the region D defined by (4.18), as can be seen using the bounds (4.19), (4.20), the functional inequality (4.21), and a straightforward induction. In that region, the lower bound (4.19) implies that $|v_n| \ge (\rho - \alpha(r_0))^n |v_0|$, which proves that the origin is exponentially unstable at rate $\rho - \alpha(r_0) > 1$. □

Remark 4.6. Assume that we are given a solution β of (4.21). Since β is nondecreasing, we have in particular $\rho\beta(r) + r\alpha(r) \le \beta(\rho r)$ for $0 \le r \le r_0$. Thus, for $\varepsilon > 0$ small enough,

$$\int_\varepsilon^{r_0} \frac{\alpha(r)}{r}\, \mathrm{d}r \le \int_\varepsilon^{r_0} \frac{\beta(\rho r)}{r^2}\, \mathrm{d}r - \rho \int_\varepsilon^{r_0} \frac{\beta(r)}{r^2}\, \mathrm{d}r = \rho \int_{\rho\varepsilon}^{\rho r_0} \frac{\beta(r)}{r^2}\, \mathrm{d}r - \rho \int_\varepsilon^{r_0} \frac{\beta(r)}{r^2}\, \mathrm{d}r ,$$

hence

$$\int_\varepsilon^{r_0} \frac{\alpha(r)}{r}\, \mathrm{d}r + \rho \int_\varepsilon^{\rho\varepsilon} \frac{\beta(r)}{r^2}\, \mathrm{d}r \le \rho \int_{r_0}^{\rho r_0} \frac{\beta(r)}{r^2}\, \mathrm{d}r .$$

Taking $\varepsilon \to 0$, we obtain $\int_0^{r_0} \alpha(r)/r\, \mathrm{d}r < \infty$. The integrability condition (1.8) is thus necessary and sufficient for a solution of (4.21) to exist.

As a conclusion, we briefly indicate how Proposition 4.5 can be used to establish the nonlinear instability result in Proposition 4.1. In the Hilbert space B, consider the map $\mathcal{F}(u) = Lu + \mathcal{N}(u)$ whose Fréchet derivative $L = d\mathcal{F}|_{u=0}$ is a normal operator with spectral radius $r(L) > 1$. Given $1 < \rho < r(L)$, let $P_1 : B \to B$ be the spectral projection corresponding to the (nonempty) subset of $\sigma(L)$ contained in the annulus $\{\lambda \in \mathbb{C} \,|\, \rho \le |\lambda| \le r(L)\}$, and let $P_2 = 1 - P_1$. Even in the absence of spectral gap, the (orthogonal) projections P_1, P_2 can be constructed using the spectral theorem for bounded normal operators. Let $B_1 = P_1 B$, $B_2 = P_2 B$ so that $B_1 \ne \{0\}$ and $B = B_1 \oplus B_2$. If we denote by L_1, L_2 the restrictions of L to the invariant subspaces B_1, B_2, respectively, then estimates (4.16) hold by construction because L_1, L_2 are normal operators. Finally, denoting $\mathcal{N}_1 = P_1\mathcal{N}$, $\mathcal{N}_2 = P_2\mathcal{N}$, and writing $u_n = v_n + w_n$ with $v_n = P_1 u_n$ and $w_n = P_2 u_n$, we see that system (1.1) takes the form (4.15), and assumption (4.1) on the nonlinearity $\mathcal{N}$ shows that the right-hand side of (4.17) is bounded from above by a multiple of $\alpha(r)$, where α is as in (4.1). Thus, under the integrability condition (1.8), Proposition 4.5 implies that the origin $u = 0$ is nonlinearly exponentially unstable for system (1.1), at any rate $\rho < r(L)$.

Remark 4.7. Proposition 4.5 deals in principle with a more general situation than Proposition 4.1, but except in the normal case considered above it is not clear under which assumptions a linear operator L without spectral gap can be decomposed into a strongly unstable part L_1 and a weakly unstable part L_2 satisfying estimates of the form (4.16).

5. Appendix : Stabilization in a two-dimensional system

We briefly analyze the two-dimensional dynamical system associated with the map (1.4), namely

$$v_{n+1} \,=\, v_n + w_n - v_n^3\,, \qquad w_{n+1} \,=\, w_n - w_n^3\,. \tag{5.1}$$

Elementary calculations show that the origin $(u,v) = (0,0)$ in (5.1) is asymptotically stable.

Proposition 5.1. *If* $|v_0| \le 1/2$ *and* $|w_0| \le 1/8$*, the solution of* (5.1) *satisfies*

$$|v_n| \,\le\, \max(|v_0|, |w_0|^{1/3})\,, \qquad |w_n| \,\le\, |w_0|\,, \qquad \text{for all } n \in \mathbb{N}\,. \tag{5.2}$$

In addition $|v_n| + |w_n| \to 0$ *as* $n \to \infty$.

Proof. We first assume that $0 \le v_0 \le 1/2$ and $0 \le w_0 \le 1/8$. Since $w_{n+1} = w_n(1 - w_n^2)$, it is clear that the sequence (w_n) is nonincreasing and converges to zero as $n \to \infty$, see also Remark 5.2 below. In particular we have $0 \le w_n \le w_0$ for all $n \in \mathbb{N}$. As for the first component, we show by induction that

$$0 \,\le\, v_n \,\le\, \max(v_0, w_0^{1/3})\,, \qquad \text{for all } n \in \mathbb{N}\,. \tag{5.3}$$

Indeed, assume that (5.3) holds for some $n \in \mathbb{N}$. If $w_n \le v_n^3$, then $v_{n+1} \le v_n \le \max(v_0, w_0^{1/3})$ and $v_{n+1} = v_n(1 - v_n^2) + w_n \ge 0$. If $w_n \ge v_n^3$, then $v_{n+1} \ge v_n \ge 0$ and

$$v_{n+1} \,=\, v_n + w_n - v_n^3 \,\le\, \max_{0 \le u \le w_n^{1/3}} \bigl(u + w_n - u^3\bigr) \,=\, w_n^{1/3} \,\le\, w_0^{1/3}\,,$$

because the map $u \mapsto u - u^3$ is increasing on the interval $[0, w_n^{1/3}] \subset [0, 1/2]$. This shows that the bounds (5.3), which hold by assumption for $n = 0$, remain valid for all $n \in \mathbb{N}$. We also note that the inequality $w_n > v_n^3$ cannot hold for all $n \in \mathbb{N}$, because in that case the sequence (v_n) would be strictly increasing, and we know that the sequence (w_n) decreases to zero. So there exists $n \in \mathbb{N}$ such that $\delta_n := v_n^3 - w_n \ge 0$. Using (5.1) we deduce that

$$\begin{aligned}
\delta_{n+1} \,&=\, v_{n+1}^3 - w_{n+1} \,=\, (v_n - \delta_n)^3 - w_n + w_n^3 \\
&=\, \delta_n(1 - 3v_n^2 + 3v_n\delta_n - \delta_n^2) + w_n^3 \,\ge\, \delta_n(1 - 3v_n^2 - \delta_n^2) \,\ge\, 0\,,
\end{aligned}$$

because $3v_n^2 + \delta_n^2 \le 3v_n^2 + v_n^6 \le 3/4 + 1/64 < 1$ (this follows from (5.3) and from our assumptions on the initial data). Thus we necessarily have $\delta_n \ge 0$ for all sufficiently large $n \in \mathbb{N}$, which implies that the sequence (v_n) is eventually nonincreasing, hence converges to some limit $\bar{u} \ge 0$ as $n \to \infty$. As $w_n \to 0$, it follows from (5.1) that $\bar{u} = \bar{u} - \bar{u}^3$, hence $\bar{u} = 0$. This concludes the proof in the case where $v_0 \ge 0$ and $w_0 \ge 0$.

We next assume that $-1/2 \le v_0 \le 0$ and $0 \le w_0 \le 1/8$. In that case, as long as $v_n \le 0$, the sequence (v_n) is nondecreasing because $w_n - v_n^3 \ge 0$. So either $v_n \le 0$ for all $n \in \mathbb{N}$, in which case $v_n \to 0$ as $n \to \infty$, or there exists a first integer $n \in \mathbb{N}$ such that $v_n \le 0$ and $v_{n+1} > 0$. In that case we have $v_{n+1} = v_n(1 - v_n^2) + w_n \le w_n$, hence $v_{n+1} \in [0, 1/2]$, $w_{n+1} \in [0, 1/8]$, and we are back to the situation studied above where both components are nonnegative. We deduce that (5.2) holds in all cases and that $|v_n| + |w_n| \to 0$ as $n \to \infty$.

Finally, the same conclusions hold if $w_0 \le 0$, because system (5.1) is clearly invariant under the transformation $(v_n, w_n) \mapsto (-v_n, -w_n)$. This concludes the proof. □

Remark 5.2. In fact, a standard comparison argument with the continuous time ODE $w' = -w^3$ shows that, if $3w_0^2 \le 1$, the solution w_n of (5.1) satisfies

$$|w_n| \,\le\, \frac{|w_0|}{\sqrt{1 + 2|w_0|^2 n}}\,, \qquad \text{for all } n \in \mathbb{N}\,.$$

In particular $|w_n| = \mathcal{O}(n^{-1/2})$ as $n \to \infty$.

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