Lectures on Fuchsian groups and their Moduli

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Chapter 1

Fuchsian groups

1.1 The geometry of the hyperbolic plane

1.1.1

The most common model for the hyperbolic plane is the upper half-space $\mathcal{H} = \{z \in \mathbb{C} \ ; \ \mathrm{Im}(z) > 0\}$, endowed with the metric $ds^2 = \frac{dx^2+dy^2}{y^2}$. If $\mathrm{SL}(2,\mathbb{R})$ denote the Lie group of 2×2 matrices of determinant 1 then one sets $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm 1\}$. The natural action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{C}$ by mean Mbius transformations factors through an action of $\mathrm{PSL}(2,\mathbb{R})$ on the upper half-plane, as follows

$$w = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d};$$

As $\mathrm{Im}\, w = \mathrm{Im}\, z \cdot \frac{1}{(cz+d)^2}$ we find that $\mathrm{PSL}(2,\mathbb{R})$ actually acts by homeomorphisms on $\mathcal{H}$.

Proposition 1.1.1

i) *$\mathcal{H}$ is a complete Riemannian manifold of constant curvature -1.*

ii) $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{Isom}^+(\mathcal{H})$.

iii) *The geodesics of $\mathcal{H}$ are semi-circles (half-lines) orthogonal to the real axis.*

iv) $\mathrm{PSL}(2,\mathbb{R})$ *maps geodesics into geodesics.*

v) *The hyperbolic distance between two points $z, w \in \mathcal{H}$ is given by*

$$d(z,w) = \log \frac{|z-\overline{w}| + |z-w|}{|z-\overline{w}| - |z-w|} = \log[w, z^*, z, w^*]$$

where $[z_1, z_2, z_3, z_4] = \frac{(z_1-z_2)(z_3-z_4)}{z_2-z_3)(z_1-z_4)}$ and the geodesic joining z to w intersects the real axis into z^* and w^*.

Remarks.

1. $\mathrm{PSL}(2,\mathbb{R}) \subset \mathrm{Isom}(\mathcal{H})$ by direct computation.

2. The geodesics are semi-circles because one computes easily a lower bound for the length $l(\gamma)$ of a curve γ which joins $\gamma(0) = ia$ to $\gamma(1) = ib$, for $i = \sqrt{-1}$, $a, b \in \mathbb{R}_+$, as follows:

$$\ell_h(\gamma; \gamma(0) = ia, \gamma(1) = ib) \geq \int_0^1 \frac{\dot{y}}{y}\, dt = \log \frac{b}{a}$$

   The equality is attained only when γ is the vertical segment of half-line joining the two points. Further $\mathrm{PSL}(2,\mathbb{R})$ sends semi-circles into half-lines orthogonal to the real axis and in meantime it acts transitively on pairs of points in the half-plane, thus a geodesic joining two arbitrary points is the image of a vertical segment by such a Möbius transformation.

3. The full group of isometries of the upper half-plane is $\mathrm{Isom}(\mathcal{H}) = \mathrm{SL}^*(2,\mathbb{R})/\{\pm 1\}$ where $\mathrm{SL}^*(2,\mathbb{R}) = \{A \in \mathrm{GL}(2,\mathbb{R}) \ ; \ \det A \in \{\pm 1\}$.

1.1.2

Another well-known model for the hyperbolic geometry is the unit disk model $D = \{z \in \mathbb{C} \ ; \ |z| < 1\}$, with the metric $dr^2 = \frac{2(dx^2+dy^2)}{1-(x^2+y^2)}$, where we write $z = x + iy$. The map $f : \mathcal{H} \to \mathcal{D}$ given by $f(z) = \frac{z \cdot i+1}{z+i}$ is therefore an isometry. The geodesics are semi-circles orthogonal to the boundary circle (which is also called the principal circle).

Remarks.

1. The action of $\mathrm{PSL}(2,\mathbb{R})$ on $\mathcal{D}$ can be deduced from above:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} w = \frac{w(a+d+(b-c)i)+(b+c+(a-d)i)}{w(b+c-(a-d)i)+((a+d)-(b-c)i)} = \frac{w\alpha+\bar{\beta}}{w\beta+\bar{\alpha}}$$

   where $\alpha, \beta \in \mathbb{C}$, $\alpha\bar{\alpha} - \beta\bar{\beta} = 1$.

2. The $\mathrm{PSL}(2,\mathbb{R})$ action on $\mathcal{D}$ extends continuously to an action on the boundary circle, hence to the closed 2-disk $\overline{\mathcal{D}}$.

1.2 $\mathrm{PSL}(2,\mathbb{R})$ and Fuchsian groups

1.2.1

Set $\mathrm{Tr} : \mathrm{PSL}(2,\mathbb{R}) \to \mathbb{R}_+$ for the function $\mathrm{Tr}(A) = |\operatorname{tr}(\widetilde{A})|$ where $\widetilde{A}$ is an arbitrary lift of A in $\mathrm{SL}(2,\mathbb{R})$. Then Tr is well-defined. The elements of $\mathrm{PSL}(2,\mathbb{R})$ are classified as follows:

A) elliptic if $\mathrm{Tr}\, A < 2$, thus conjugate to a unique matrix of the form $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, with $\theta \in (0, 2\pi)$.

B) hyperbolic if $\mathrm{Tr}\, A > 2$, thus conjugate to a unique matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, where $\lambda \neq 1$, $\lambda \in \mathbb{R}$, hence diagonalizable over $\mathbb{R}$.

C) parabolic if $\mathrm{Tr}\, A = 2$, thus conjugate to either the positive or the negative translation along the real axis.

Remarks.

1. A transformation $A \in \mathrm{PSL}(2,\mathbb{R})$ is
   - elliptic iff it has a unique fixed point in $\mathcal{D}$ (or $\mathcal{H}$).
   - hyperbolic iff it has exactly two fixed points in $\partial\overline{\mathcal{D}}$ (or equivalently $\partial\overline{\mathcal{H}}$).
   - parabolic iff it has a unique fixed point in $\partial\overline{\mathcal{D}}$ (or equivalently $\partial\overline{\mathcal{H}}$).

2. Conjugacy classes in $\mathrm{SL}(2,\mathbb{R})$ (respectively $\mathrm{PSL}(2,\mathbb{R})$ are essentially determined by the trace (respectively Tr); this is really true for hyperbolic or elliptic transformations and up to sign ambiguity for the parabolic ones. Notice that parabolics are always conjugate within the larger group $\mathrm{PSL}^*(2,\mathbb{R}) = \mathrm{Isom}(\mathcal{H})$.

Remark.

1. $\mathrm{PSL}(2,\mathbb{R})$ acts on the unit tangent bundle $S\mathcal{H}$, by $A(z, v) = (Az, dA(v))$. This action is simply transitive, identifying $\mathrm{PSL}(2,\mathbb{R})$ and $S\mathcal{H}$. In particular $\mathrm{PSL}(2,\mathbb{R})$ is an open solid torus.

2. If γ_1, γ_2 are two geodesics in $\mathcal{H}$ and $z_i \in \gamma_i$ are two points on them then there exists $A \in \mathrm{PSL}(2,\mathbb{R})$ such that $A(\gamma_1) = \gamma_2$ and $Az_1 = z_2$.

1.2.2

$\mathrm{SL}(2,\mathbb{R})$ inherits a topology when seen as a subset of the Euclidean space $\mathrm{SL}(2,\mathbb{R}) \subset \mathbb{R}^4$ by means of the obvious inclusion sending the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ into the point (a,b,c,d). Furthermore, the $\mathbb{Z}/2$-action $(a,b,c,d) \longrightarrow (-a,-b,-c,-d)$ identifies $\mathrm{PSL}(2,\mathbb{R})$ as a quotient of the topological space $\mathrm{SL}(2,\mathbb{R})$, which is then endowed with the quotient topology. Notice that $\mathrm{PSL}(2,\mathbb{R})$ is a topological group since the topology defined above is compatible with the group structure.

Definition 1.2.1 *The subgroup* $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ *is* discrete *if* Γ *is a discrete set in the topological space* $\mathrm{PSL}(2,\mathbb{R})$*. Discrete *subgroups of* $\mathrm{PSL}(2,\mathbb{R})$ *are usually called* Fuchsian *groups.*

Definition 1.2.2 *Let* G *be a group of homeomorphisms acting on the topological space* X*. Then* G acts properly discontinuously *an* X *if the* G*-orbit of any* $x \in X$*, i.e.* $Gx = \{gx; g \in G\}$*, is a* locally finite **family**.

Remarks.

1. One required that the family of orbits be a locally finite family and not only a locally finite set. Recall that the family $\{M_\alpha\}_{\alpha\in\mathcal{J}}$ is called *locally finite* if for any compact subset $K \subset X$ we have $M_\alpha \cap K \neq \emptyset$ only for finitely many values of $\alpha \in A$.
2. In Gx each point is contained with a *multiplicity* equal to the order of the stabilizer $G_x = \{g \in G; gx = x\}$.
3. In particular G acts properly discontinuously iff each orbit *is discrete* (as a set this time) and the *order of the stabilizer is finite.*
4. G acts properly discontinuously iff for all $x \in X$, there exists a neighborhood $V \ni x$, $V \subset X$ such that $\mathcal{J}(V) \cap V \neq \emptyset$ for only finitely many $g \in G$.

Proposition 1.2.1 *The subgroup* $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ *is Fuchsian iff it acts properly discontinuously on* $\mathcal{H}$*.*

Corollaire 1.2.1

1. *The* fixed points of elliptic *elements form a* discrete set *in* $\mathcal{H}$*.*
2. *If moreover* Γ does not contain elliptic elements *then* $\mathcal{H}/\Gamma$ *is a complete connected orientable Riemannian 2-manifold of curvature* -1*.*

Remark. In general, a discrete group might well act non-discontinuously on a topological space. For example, the action of the discrete subgroup $\mathrm{PSL}(2,\mathbb{Z})$ of $\mathrm{PSL}(2,\mathbb{R})$ on the boundary circle $\partial\overline{\mathcal{H}}$, which is induced by the action of $\mathrm{PSL}(2,\mathbb{R})$ on the boundary, has an orbit equal to $\mathbb{Q} \cup \{\infty\}$ which is dense in $\mathbb{R} \cup \{\infty\}$. □

Examples.

1. $\mathrm{PSL}(2,\mathbb{Z})$ is a Fuchsian group.
2. $\mathrm{PSL}(2,\mathbb{Z}[\sqrt{2}])$ is *not* a Fuchsian group.

1.3 Discreteness of subgroups $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$

Definition 1.3.1 *The limit set* $\Lambda(\Gamma) \subset \overline{\mathcal{H}}$ *is the set of accumulation points of* Γ*-orbits* Γz*, for* $z \in \mathcal{H}$*.*

Examples.

1. If Γ is the cyclic group generated by the homothety $z \to \lambda z$, with $\lambda > 1$ then $\Lambda(\Gamma) = \{0, \infty\}$.
2. $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$ then $\Lambda(\Gamma) = \mathbb{R} \cup \{\infty\}$.
3. If Γ is Fuchsian then $\Lambda(\Gamma) \subset \mathbb{R} \cup \{\infty\}$.

Definition 1.3.2 *The subgroup* Γ *is* elementary *if there exists a finite* Γ*-orbit in* $\overline{\mathcal{H}}$*. Equivalently, we have the equality* $\mathrm{Tr}[A,B] = 2$*, whenever* $A, B \in \Gamma$ *have infinite order.*

Elementary Fuchsian groups can be characterized completely. It is known that:

Proposition 1.3.1 *An* elementary Fuchsian *group is either cyclic or it is conjugate within* PSL$(2,\mathbb{R})$ *to the group generated by the two transformations* g *and* j *below:*

$$g(z) = \lambda z, \quad \lambda > 1 \quad and \quad j(z) = -\frac{1}{z}$$

Proposition 1.3.2 *If* $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ *contains only elliptic elements (or the identity* $\mathbb{1}$*) then* Γ *is cyclic elementary.*

Proposition 1.3.3 *Let* Γ *be a* non-elementary *subgroup of* PSL$(2,\mathbb{R})$. *Then* Γ *is* discrete *iff the following equivalent conditions are fulfilled:*

i) The fixed points of elliptic elements do not accumulate on $\mathcal{H}$.

ii) The elliptic elements do not accumulate on $\mathbb{1}$.

iii) Each elliptic element has finite order.

Proposition 1.3.4 *Let* $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ *be non-elementary. Then:*

1. Γ *contains hyperbolic elements.*
2. *If* Γ *does not contain elliptic elements then it is discrete.*
3. Γ *is discrete iff every cyclic subgroup of* Γ *is discrete.*

Remarks.

1. The proof of the third assertion above is due to T.Jorgensen. He used the following inequality valid actually more generally in PSL$(2,\mathbb{C})$:

$$|\operatorname{tr}^2(A) - 2| + |\operatorname{tr}([A,B]) - 2| \geq 1$$

   whenever the subgroup $\langle A, B\rangle \subset \mathrm{PSL}(2,\mathbb{C})$ generated by A and B is discrete. Observe that both $\operatorname{tr}^2(A)$ and $\operatorname{tr}([A,B])$ are well-defined when $A, B \in \mathrm{PSL}(2,\mathbb{C})$.

2. Notice that a subgroup $\Gamma \subset \mathrm{PSL}(2,\mathbb{C})$ is discrete if and only if every two generators subgroup of Γ is discrete. Moreover, this time the discreteness of its cyclic subgroups is not sufficient.

3. Another proof of the third part is due to Rosenberger [27], who showed that if we have a discrete subgroup $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ which is not isomorphic to $\mathbb{Z}/2 * \mathbb{Z}/2$ then

$$|\operatorname{Tr}([A,B]) - 2| \geq 2 - 2\cos\frac{\pi}{7}$$

   for any $A, B \in \Gamma$.

4. The *discreteness* of a two generator subgroup $\langle A, B\rangle$ can be algorithmically and effectively decided, as it was proved by J.Gilman (see [9]). By instance if A, B are hyperbolic elements with intersecting distinct axes and $\gamma^2 = [A,B]$, $A = E_1E_2$, $B = E_1E_3$, E_i having order 2, $\gamma = E_1E_2E_3$ and G denotes the subgroup $\langle A, B\rangle$, then we have:

   - if A, B, γ are primitive (i.e. hyperbolic, parabolic or elliptic of finite order) then G is discrete.
   - if $[A,B]$ is elliptic of infinite order then G is not discrete.
   - if $\operatorname{tr}[A,B] = -2\cos\frac{2k\pi}{n}$, $1 \leq k < \frac{n}{2}$,
     - if $k \notin \{2,3\}$ then G is not discrete.
     - if $k \in \{2,3\}$ then we have special triangular groups G which might be discrete, and the situation is completely understood (see [9]).

1.4 Freeness of subgroups of $\mathrm{PSL}(2,\mathbb{R})$ and $\mathrm{PSL}(2,\mathbb{C})$

Proposition 1.4.1

1. *The subgroup* $\Gamma_\lambda \subset \mathrm{PSL}(2,\mathbb{R})$, *which is generated by the two matrices* $B = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}$ *and* $A = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$ *is* free provide that $|\lambda| \geq 2$ *This was proved by I.N.Sanov in 1947 and by J.L.Brenner in 1955.*

2. *More generally the same holds true for* $\lambda \in \mathbb{C}$ *provided that one of the following conditions holds:*
   - λ *is outside the* unit disks centered at -1, 0 and 1.
   - λ *is* outside the disks radius $\frac{1}{2}$ centered at $\frac{-i}{2}$ and $\frac{i}{2}$ and outside *the unit open disks centered at* -1 *and* 1.
   - λ *lies outside the convex hull of the open disk radius* 1 *at origin and the points* ± 2.
   - $|\lambda| > 1$ *and* $|\operatorname{Im}(\lambda)| \geq \frac{1}{2}$.
   - $|\lambda - 1| > \frac{1}{2}$ *and* $1 \leq |\Re\lambda| < \frac{5}{4}$, *where* $\Re$ *denotes the real part and* Im *the imaginary part.*
   - λ *is* transcendental.

   Those $\lambda \in \mathbb{C}$ *for which the subgroup* Γ_λ *is free will be called free.*

3. *Moreover, it is known (see the papers [1, 2]) that*
   - *The set of algebraic complex numbers which are free points is dense in* $\mathbb{C}$.
   - *There exist sets* S_1, S_2 *consisting of algebraic irrational complex numbers so that* $\overline{S}_1 = (-2, 2)$, $\overline{S}_2 = (-i, i)$ *and* S_i *consist of non-free points.*
   - *The points* $\frac{1}{2}e^{2\pi i/k}$, $\frac{1}{k}$, $\frac{k}{k^2+1}$, *for* $k \in \mathbb{Z}$, $\frac{1}{\sqrt{2}}$, $\frac{9}{50}$, $\frac{8}{25}$, $\frac{25}{98}$ are non-free.
   - *If* $p^2 - Nq^2 = 1$, *where* p, q *are integers and* N *is a positive integer which is not a perfect square then* $\lambda = \frac{p}{q}$ *is non-free. In fact, we have*

$$W = A^{q^2} B^N A^{-1} = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$$

   and so

$$[W^{-1}BW, B] = 1$$

Remark. There exists a family of free subgroups, called Schottky groups, in $\mathrm{PSL}(2,\mathbb{C})$ generated by transformations $\gamma_1, \ldots, \gamma_g \in \mathrm{PSL}(2,\mathbb{C})$ with the property that $\gamma_i(\operatorname{int} D_{2i}) = \widehat{\mathbb{C}} \setminus D_{2i+1}$, where D_j are fixed disjoints disks in the plane. Then the ping-pong Lemma shows that this group is free. □

1.5 Arithmetic groups in $\mathrm{PSL}(2,\mathbb{R})$

Definition 1.5.1 *The subgroup* $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ *is called* arithmetic *if* Γ *is a subgroup of finite index in some group* $\varphi^{-1}(\mathrm{GL}(n,\mathbb{Z}))$ *arising as the inverse image of some finite dimensional representation*

$$\varphi : \Gamma \longrightarrow \mathrm{GL}(n,\mathbb{R})$$

Proposition 1.5.1 *An arithmetic Fuchsian group is commensurable with a Fuchsian group determined by a quaternion algebra (which are either* 2×2 *matrix algebras over some field or else division algebras). Moreover, all these Fuchsian arithmetic groups are either cocompact or commensurable with* $\mathrm{PSL}(2,\mathbb{Z})$.

Examples. The *Hecke groups*

$$\Gamma_\lambda = \left\langle \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \right\rangle \subset \mathrm{PSL}(2,\mathbb{R})$$

for the values of the parameter $\lambda = 2\cos\frac{\pi}{2}$ have different behaviour than those for which $|\lambda| \geq 2$ (which are free groups, by the results above). In fact, one has

$$\Gamma_{2\cos\frac{\pi}{q}} = \mathbb{Z}/q\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$$

and in particular, if $q = 3$ we obtain $\Gamma_3 = \mathrm{PSL}(2,\mathbb{Z})$.
Remark. If $q \notin \{3,4,6\}$ then the Hecke groups are *not* arithmetic. In fact the Hecke groups are of finite covolume, not cocompact and not commensurable with $\mathrm{PSL}(2,\mathbb{Z})$. For instance, if we consider the element $\gamma = \begin{pmatrix} -2\cos\frac{\pi}{5} & 1 \\ -1 & 0 \end{pmatrix} \in \Gamma_{\cos\frac{2\pi}{5}}$ then $\mathrm{Tr}\,\gamma^n \notin \mathbb{Q}$ for any $n \in \mathbb{Z} \setminus \{0\}$. This implies that no finite index subgroup of $\Gamma_{\cos\frac{2\pi}{5}}$ could be contained in $\mathrm{PSL}(2,\mathbb{Z})$. Thus the group is not arithmetic by the criterion provided above. □

1.6 Fundamental regions for Fuchsian groups

Definition 1.6.1 *We say that a closed region $X \subset \mathcal{H}$ is a fundamental region for Γ if*

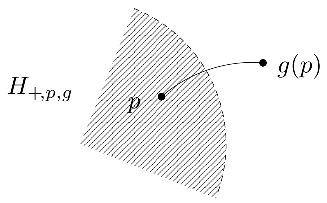
1. $\bigcup_{g\in\Gamma} gX = \mathcal{H}$
2. $\mathrm{int}(X) \cap g(\mathrm{int}(X)) = \emptyset$, *for all* $g \in \Gamma$, $g \neq 1$.

Remark. If X, Y are both fundamental regions for the group Γ and $\mu(X) < \infty$, then $\mu(X) = \mu(Y)$. □

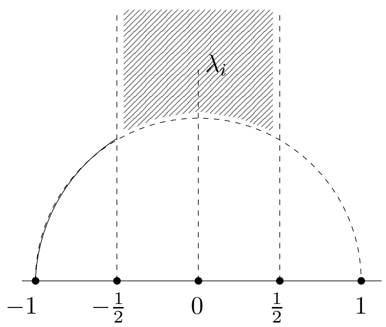
Let us provide examples of fundamental regions. The *Dirichlet fundamental region* associated to a Fuchsian group Γ is constructed as follows:

1. Pick up $p \in \mathcal{H}$ such that p is not fixed by any $g \in \Gamma \setminus \{0\}$.
2. Set then $D(\Gamma, p) = \{z \in \mathcal{H}; d(p,z) \leq d(z, g(p)),\ \text{for all } g \in \Gamma\}$.

Then $D(\Gamma, p)$ is a fundamental region for Γ, called the Dirichlet region.
Alternatively let $H_{+,p,g}$ be the half-space containing p and having as boundary the bisector of the segment of geodesic which joins p to $g(p)$. Then $D(\Gamma,p) = \bigcap_{g\in\Gamma\setminus\{0\}} H_{+,p,g}$, in particular it is geodesically convex. It is easy to prove that $D(\Gamma, p)$ is a connected fundamental region for Γ.



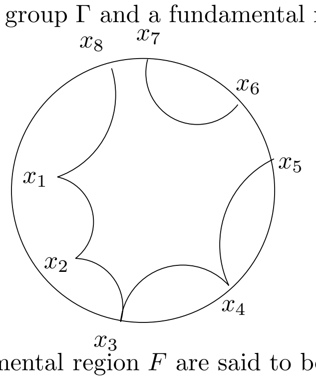
Example. Let $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$ and pick-up $p = \lambda i$, where $\lambda > 1$. Thus p is not fixed by any element of Γ. Then $\mathcal{D}(\mathrm{PSL}(2,\mathbb{Z})), \lambda_i) = \{z \in \mathcal{H}; |z| \geq 1, |\Re(z)| \leq \frac{1}{2}\}$



Remarks.

1. The tesselation $\{g(D(\Gamma,p)),\ g \in \Gamma\}$ is locally finite since Γ acts properly discontinuously.
2. It is important to notice that $\mathcal{H}/\Gamma$ *is homeomorphic then to* D/Γ for any fundamental region D of Γ (locally finite). Moreover D/Γ is obtained from D by using self-identifications of the boundary arcs, hence we can expect a complete topological description of $\mathcal{H}/\Gamma$.

Consider now a Fuchsian group Γ and a fundamental region F for Γ.



Two points of the fundamental region F are said to be congruent if they are equivalent under the action of Γ. They should therefore belong to the boundary ∂F of the fundamental region. We have then the following type of vertices of the curved polygon F:

- *elliptic* vertices of F corresponding to fixed points of elliptic elements;
- *parabolic* vertices of F corresponding to fixed points of parabolic elements.

Given a congruence class we consider the Γ-orbit of one vertex in that class and obtain a cycle. Two cycles are then different if they correspond to different classes of vertices modulo Γ. The cycles are said to be elliptic or parabolic according to their vertices.
Example. In the picture above x_1, x_2 are elliptic and x_3, x_4 are parabolic.
Remarks.

1. Elliptic cycles correspond to conjugacy classes of nontrivial maximal finite cyclic subgroups of Γ, while parabolic cycles correspond to maximal (cyclic) parabolic subgroups.
2. If $\theta_1, \ldots, \theta_t$ are the internal angles of the polygon F at the vertices $x_1, \ldots, x_t$ which are congruent to each other in F and form a cycle, then $\theta_1 + \cdots + \theta_t = \frac{2\pi}{m}$, where m is the order of the stabilizer in Γ of any of these vertices.

Definition 1.6.2 *The subgroup Γ is* geometrically finite *if there exists a convex fundamental region of Γ with finitely many sides.*

Let μ denote the Lebesgue measure on the quotient $\mathcal{H}/\Gamma$ by a Fuchsian group, induced from that of $\mathcal{H}$. We have the following characterization of geometric finiteness due to Siegel:

Theorem 1.6.1 (Siegel) *If $\mu(\mathcal{H}/\Gamma) < \infty$ then Γ is geometrically finite. Actually, for any fundamental region F we have the more precise estimation:*

$$\sum_{v \text{ vertex of } F} (\pi - \theta_v) \leq \mu(F) + 2\pi$$

where θ_v denotes the internal angle of the polygon F at the vertex v.

Remarks.

1. If Γ has a compact fundamental region then Γ has non parabolic elements.
2. Some (equivalently, any) fundamental region of Γ is noncompact iff the quotient $\mathcal{H}/\Gamma$ is noncompact.
3. Moreover, if $\mu(\mathcal{H}/\Gamma) < \infty$ but $\mathcal{H}/\Gamma$ is noncompact then there exist vertices at infinity in any fundamental region for Γ, which are parabolic vertices.
4. If some fundamental region F for Γ is compact then all fundamental regions of Γ are compact; further, Γ is cocompact iff $\mu(\mathcal{H}/\Gamma) < \infty$ and Γ has no parabolics.

Definition 1.6.3 *The orders of elliptic elements fixing the elliptic vertices (in each congruence class) are called the periods of the Fuchsian group. One can add ∞ as period of each parabolic vertex.*

Example. $\mathrm{PSL}(2,\mathbb{Z})$ has periods $2, 3$ and ∞.
Remark. Let T_i be elements of Γ which are pairing the sides of the fundamental region F. Then the set of elements $\{T_i\}$ generate the group Γ. In particular if F has finitely many sides (e.g. when Γ is geometrically finite) then Γ is finitely generated. □

Proposition 1.6.1

1. *If the Fuchsian group Γ is finitely generated then it is geometrically finite.*

2. *Moreover, if the Fuchsian group Γ is cocompact and $\mathcal{H}/\Gamma$ is a surface of genus g with r singular points corresponding to the periods $m_1, \dots, m_r$ then we have the following Siegel formula:*

$$\mu(\mathcal{H}/\Gamma) = 2\pi \left(2g - 2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i} \right) \right)$$

3. *For any $(g, m_1, \dots, m_r)$ with $2q - 2t + \sum_{i=1}^{r} 1 - \frac{1}{m_i} > 0$, $g \geq 0$, $r \geq 0$, $m_i \geq 2$, there exists a Fuchsian group of signature $(g, m_1, \dots, m_r)$, i.e. such that $\mathcal{H}/\Gamma$ is a surface of genus g with r singular points corresponding to the periods $m_1, \dots, m_r$.*

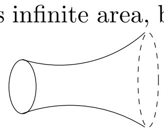
Remark. Notice that there exist subgroups of $\mathrm{GL}(2,\mathbb{R})$ which are not finitely generated, e.g.

$$\Gamma = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \; ; \; a = 2^{\alpha}, \; b = \frac{p}{2^q}, \; \alpha, p, q \in \mathbb{Z} \right\}$$

□

Remark. If Γ is not cocompact then one should consider the Nielsen core $K(\Gamma)$ which is the convex hull of $\Lambda(\Gamma)$. Then Γ is finitely generated iff $\mu(K(\Gamma)/\Gamma) < \infty$, which is equivalent to have a Dirichlet region with finitely many sides. Furthermore, Siegel's formula above extends to the finitely generated situation as follows:

$$\mu(K(\Gamma)/\Gamma) = 2\pi \left(2g - 2 + t + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i} \right) \right)$$

where $t = p + b$, where p denotes the number of parabolic cycles and b the number of boundary hyperbolics; one remarks that $K(\Gamma)/\Gamma$ is a surface with p punctures and b boundary curves, which is a core for the non-compact surface $\mathcal{H}/\Gamma$. The surface $\mathcal{H}/\Gamma$ has infinite area, but if we cut open the components corresponding to boundary hyperbolics, which have the shape  we obtain a cusped surface with boundary of finite area. □

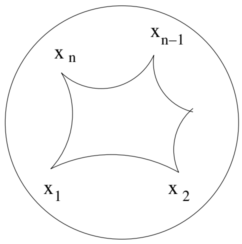
1.7 The Poincaré theorem: construction of Fuchsian groups

1.7.1 The finite case

Consider P a compact convex polygon with $n \geq 3$ vertices $x_1, x_2, \dots, x_n$ such that the interior angle at x_j is $\frac{\pi}{p_j}$, $p_j \in \mathbb{Z}_+$. There exists such a polygon with prescribed angles in the hyperbolic plane $\mathcal{H}$ a polygon exists if and only if (by the Gauss-Bonnet theorem) the following condition is fulfilled:

$$\sum_{j=1}^{n} \frac{1}{p_j} < n - 2$$

Remark that there exists a polygon P with these prescribed angles in the Euclidean plane $\mathbb{E}^2$ iff $\sum_{j=1}^{n} \frac{1}{p_j} = n - 2$, while the existence of a spherical polygon P on the sphere S^2 is equivalent to $\sum_{j=1}^{n-2} \frac{1}{p_j} > n - 2$. We will stick to the hyperbolic case.



Set σ_j for the reflection of $\mathcal{H}$ with respect to the side $x_j x_{j+1}$ and $\Gamma \subset \mathrm{PSL}^*(2, \mathbb{R})$ for the group of isometries generated by $\{\sigma_1, \ldots, \sigma_n\}$.

Proposition 1.7.1 *The group Γ admits the following presentation by means of the generators $\sigma_1, \ldots, \sigma_n$ and relations:*

$$\sigma_j^2 = 1, \ j = 1, n$$

$$(\sigma_{j-1}\sigma_j)^{p_j} = 1, \ j = 1, n$$

Moreover Γ acts properly on $\mathcal{H}$ and P is a fundamental region for Γ.

Remark that Γ is a *hyperbolic Coxeter group*.
Example. If $2 \leq a \leq b \leq c \in \mathbb{Z}$ so that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$, then the group

$$T^*_{a,b,c} = \left\langle x, y, z \mid x^2 = y^2 = z^2 = (yz)^a = (zx)^b = (xy)^c = 1 \right\rangle$$

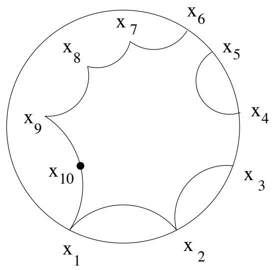
acts on $\mathcal{H}$ and has a fundamental domain given by a triangle whose angles are $\frac{\pi}{a}$, $\frac{\pi}{b}$, $\frac{\pi}{c}$. In particular, the area of the fundamental domain is $\pi(1 - \frac{1}{a} - \frac{1}{b} - \frac{1}{c})$. The index 2 subgroup $T_{a,b,c} \subset T^*_{a,b,c}$ consisting of orientation preserving isometries has the presentation

$$T_{a,b,c} = \left\langle u, v \mid u^c = v^a = (uv)^b = 1 \right\rangle, \ u = xy, \ v = yz$$

and a fundamental domain constructed by gluing two adjacent triangles with a common edge. The area of this quadrilateral is then $2\pi(1 - \frac{1}{a} - \frac{1}{b} - \frac{1}{c})$. In particular, its area is bounded from below by $\frac{2\pi}{21}$, with equality for $(a, b, c) = (2, 3, 7)$. It can be shown that $\inf \mu(\mathcal{H}/\Gamma) = \frac{2\pi}{21}$, where the infimum is taken over *all Fuchsian groups* Γ, not only over the triangular groups.

1.7.2 The infinite case

One can also consider noncompact polygons P



having a number of proper vertices at infinity (e.g. x_1, x_2 on the picture) but having improper vertices at infinity as well (like x_3, x_4, x_5, x_6 on the figure). The sides of P are the geodesic segments joining x_k and x_{k+1} where at least one of the vertices x_k, x_{k+1} should be proper. In the picture above we have therefore 8 sides. Notice that we might have vertices whose associated angle is π, like the vertex x_{10} from above.
Consider now that we are given the following data. We have first an involution ι from the set of the sides of P onto itself, and for each side e we are given an isometry $\sigma_e \in \mathrm{PSL}^*(2, \mathbb{R})$ of the hyperbolic plane $\mathcal{H}$, supposed to satisfy the conditions $\sigma_e(e) = \iota(e)$ and $\sigma_{\iota(e)} = \sigma_e^{-1}$. In general, the group Γ generated by the isometries σ_e, with e running over the set of sides of P is not discrete. However, the discreteness can be decided effectively for Γ, as follows. We assume that the following conditions are satisfied:

1. **The cycle condition** For *any cycle* $(x_1, x_2, \ldots, x_p)$ *of vertices at finite distance there exist* $m \in \mathbb{Z}$, $m \geq 1$ such that
$$\sum_{j=1}^{p} \theta_j = \frac{2\pi}{m}$$

2. **The cusp condition :** For *any cycle* $(x_1, x_2, \ldots, x_p)$ *of proper vertices at infinite* the transformation $\sigma_{e_p}\sigma_{e_{p-1}} \cdots e_{e_1}$ is *parabolic.*

Theorem 1.7.1 (Poincaré) *If the data* (P, ι, σ_e) *satisfies* the cycle and cusp conditions *above then the group* Γ *generated by the* σ_e *(with e running over the set of sides of* P*) is finitely presented by means of the generators* σ_e *and relations*

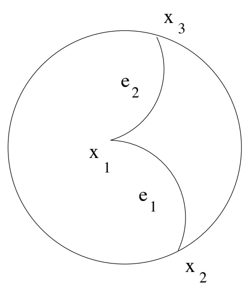
$\sigma_e^2 = 1$ *whenever* $\iota(e) = e$

$(\sigma_{e_p}\sigma_{e_{p-1}} \cdots \sigma_{e_1})^m = 1$ *for any cycle of vertices at finite distance* $(x_1, \ldots, x_p)$ *where* m *is the positive integer defined by the cycle condition.*

Moreover Γ *is acting properly discontinuously on* $\mathcal{H}$ *and* P *is a fundamental domain for* Γ*.*

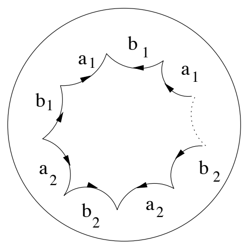
Complement. The vertices at finite distance contain the set of *elliptic fixed points* of Γ and possibly some regular points.
The cycles above are constructed so that they give elliptic or parabolic cycles. Specifically, we start with x_1 vertex of P, e_1 edge of P incident to x_1. Define next $x_2 = \sigma_{e_1}(x_1)$; let e_2 be the other side, different from $\sigma_{e_1}(e_1)$ which is pending at x_2. Then put $x_3 = \sigma_{e_2}(x_2)$ and e_3 be the new edge pending at x_3, and so on, until we get $(x_{p+1}, e_{p+1}) = (x_1, e_1)$. The sequence $x_1, \ldots, x_p$ is called a cycle.
Examples.

1. Take for P the polygon with 2 sides:



   where the angle between the sides e_1 and e_2 is $\frac{2\pi}{p}$. Let ι be the identity $\iota : \{e_1, e_2\} \to \{e_1, e_2\}$. Then Γ is given by
$$\Gamma = \left\langle \sigma_{e_1}, \sigma_{e_2} \mid \sigma_{e_1}^2 = \sigma_{e_2}^2 = (\sigma_{e_1}\sigma_{e_2})^p = 1 \right\rangle$$
   and thus it is isomorphic to the dihedral group of order $2p$.

2. Consider now the polygon P with $4g$ sides.



The $4g$ edges $e_1, e_2, e_3, \dots, e_{4g}$ of P are labeled counterclockwise as $a_1, b_1, a_1^{-1}, b_1^{-1}, a_2, b_2, a_2^{-1}, b_2^{-1}, \dots$ by giving them a label and an orientation. Let ι be the involution which interchanges the edges where the same letter (with opposite exponent) appear. Let σ_{a_i} be the hyperbolic isometry which sends the edge labeled a_i into the edge labeled a_i^{-1} with the reversed orientation. Similarly define the isometries σ_{b_i} associated to the edges labeled b_i.

Assume that the internal angles α_j of the polygon P verify the identity:

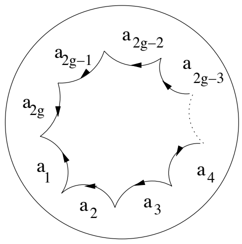
$$(*) \qquad \sum_{j=1}^{4g} \alpha_j = 2\pi$$

Then the cycle conditions are satisfied: there is only one cycle and the corresponding $m = 1$. In this case the vertices are regular points of the quotient (degenerate elliptic fixed points, since $m = 1$). Thus the group Γ is Fuchsian, given by the presentation:

$$\Gamma = \left\langle \sigma_{a_i}, \sigma_{b_j} \mid \sigma_{a_1}\sigma_{b_1}\sigma_{a_1}^{-1}\sigma_{b_1}^{-1}\sigma_{a_2}\sigma_{b_2}\sigma_{a_2}^{-1}\sigma_{b_2}^{-1} \cdots a_{a_g}\sigma_{b_g}\sigma_{a_g}^{-1}\sigma_{b_g}^{-1} = 1 \right\rangle$$

and having P as fundamental domain. In particular, $\mathcal{H}/\Gamma$ is the orientable surface of genus g obtained from the polygon P by identifying the edges according to the labeling.

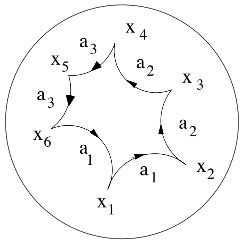
3. One remarks that different identifications of the sides might yield isomorphic Fuchsian groups with different presentations. Take for instance the pairing of the $4g$-gon induced by the counterclockwise labeling $a_1, a_2, \dots, a_{2g}, a_1^{-1}, a_2^{-1}, \dots, a_{2g}^{-1}$:



In this case we have
$$\Gamma = \left\langle \sigma_{a_{1,\dots,\sigma_{a_{2q}}}} \mid \sigma_{a_1}\sigma_{a_2}\cdots\sigma_{a_{2q}}\sigma_{a_1}^{-1}\sigma_{a_2}^{-1}\cdots\sigma_{a_{2q}}^{-1} = 1 \right\rangle.$$

In both cases from above $\mathcal{H}/\Gamma$ is a hyperbolic surface of genus g.

4. Let now consider the polygon P with the labeling of edges from the figure, inducing a natural involution, as above.



In this case we want that the isometry σ_{a_1} identifies the side x_1x_2 with the side x_2x_3 and so on. Thus σ_{a_1} does not preserve the orientation of the plane. The group Γ that we obtain is presented by

$$\Gamma = \left\langle \sigma_{a_1}, \sigma_{a_2}, \sigma_{a_3} \mid \sigma_{a_1}^2\sigma_{a_2}^2\sigma_{a_3}^2 = 1 \right\rangle \subset \mathrm{PSL}^*(2, \mathbb{R})$$

In fact, the quotient $\mathcal{H}/\Gamma$ is the non-orientable surface of genus 3.

5. A general finitely generated Fuchsian group has the following presentation:

   (a) Generators $a_1, b_1, a_2, b_2, \dots, a_g, b_g, e_1, \dots, e_r, p_1, \dots, p_s, h_1, \dots, h_i$

   (b) Relations:

   - $e_i^{m_i} = 1$
   - $\left(\prod_{i=1}^{g} [a_i, b_i]\right) e_1 \cdots e_r p_1 \cdots p_s h_1 \cdots h_i = 1$, where e_i are corresponding to the elliptic elements (which are of finite orders m_i), p_j to the parabolics and thus in one-to-one correspondence with the cusps, h_j to the boundary hyperbolic, and a_i, b_i are hyperbolic.

1.8 Applications of the Fuchsian groups

Proposition 1.8.1 (Hurwitz) *If Σ is a closed Riemann surface of genus $g \geq 2$ then* $\mathrm{Aut}(\Sigma)$ *(the group of holomorphic automorphisms) is finite and its order is uniformly bounded*

$$\mathrm{card}(\mathrm{Aut}(\Sigma)) \leq 84(g-1)$$

Equality holds for infinitely many g, but there exist infinitely many g for which the inequality is strict.

Idea of proof. If $\Sigma = \mathcal{H}/\Gamma$, Γ Fuchsian and $N(\Gamma)$ is the normalizer of Γ in $\mathrm{PSL}(2,\mathbb{R})$ then the first observation is that $N(\Gamma)$ is also Fuchsian. In fact, if $n_k \in N(\Gamma)$ is a sequence of elements such that $n_k \to 1$, then one knows that $\lim_{k\to\infty} n_k \gamma n_k^{-1} = \gamma$, for all $\gamma \in \Gamma$. Now Γ being discrete implies that $n_k \gamma n_k^{-1} = \gamma$ for large k. Thus n_k and γ commute and thus they have the same fixed points, but there exist two γ_1, $\gamma_2 \in \Gamma$ which have not the same fixed points, contradiction.
Further if we choose $n \in N(\Gamma)$ then we have $n(\Gamma z) = \Gamma \cdot nz$ and thus there is an induced map $n : \mathcal{H}/\Gamma \to \mathcal{H}/\Gamma$, because n sends Γ-orbits into Γ-orbits. Thus we obtained an automorphism $n_* \in \mathrm{Aut}\,\Sigma$. The map $\eta : N(\Gamma) \to \mathrm{Aut}\,\Sigma$ sending n into n_* is a group homomorphism, which is surjective with $\ker \eta = \Gamma$ so that we obtain an isomorphism $\mathrm{Aut}(\Sigma) = N(\Gamma)/\Gamma$.
Moreover $N(\Gamma)$ is Fuchsian so that

$$\mathrm{card}(\mathrm{Aut}\,\Sigma) = \mathrm{card}(N(\Gamma)/\Gamma) = \frac{\mu(\mathcal{H}/\Gamma)}{\mu(\mathcal{H}/N(\Gamma))}$$

By Siegel's formula we have $\mu(\mathcal{H}/\Gamma) = 4\pi(g-1)$ since Γ has neither elliptics nor parabolics. Further

$$\mu(\mathcal{H}/N(\Gamma)) = 2\pi\left(2g' - 2 + t + \sum_{i=1}^{r}\left(1 - \frac{1}{\mathfrak{m}_i}\right)\right) = c_{q',\mathfrak{m}_i,t}$$

where $N(\Gamma)$ has signature $(g', \mathfrak{m}_i, t)$. An elementary computation shows that

$$\mu(\mathcal{H}/N(\Gamma)) \geq c_{g'=0,\mathfrak{m}_i=(2,3,7)} = \pi/21$$

with equality iff $N(\Gamma)$ is the triangle group $(2, 3.7) = \langle 2, 3, 7\rangle$.
Moreover, $\langle 2, 3, 7\rangle$ is a Coxeter group of matrices and hence it is residually finite. Thus there exists finite quotients $\frac{\langle 2,3,7\rangle}{T_i}$ of arbitrary large index. This proves that there exist infinitely many g for which we have equality.
If $g = p + 1$, with prime $p > 84$ then there is no such quotient of order $84(g-1)$ because first there is no such group of order 84 and second, any Sylow p-subgroup should be normal.
Remark. The first part of the theorem is classical and due to Hurwitz and the second part is due to Macbeath. L.Greenberg proved also that for any finite group G there exists a closed Riemann surface Σ such that $\mathrm{Aut}\,\Sigma \simeq G$.
□

Chapter 2

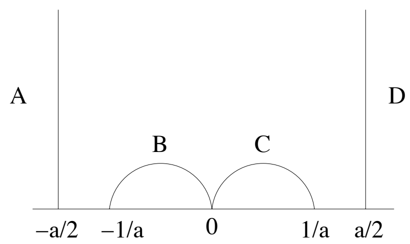
Spaces of discrete groups

2.1 Non rigidity phenomena for subgroups of $\mathrm{PSL}(2,\mathbb{R})$

1. We consider first the family of Hecke groups which we already encountered before,

$$\Gamma_a = \left\langle \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \right\rangle \subset \mathrm{PSL}(2,\mathbb{R}).$$

One knows that all Γ_a are free if $|a| \geq 2$ hence isomorphic to each other. If $a \in \mathbb{Z}$ then $\Gamma_a \subset \mathrm{PSL}(2,\mathbb{Z})$ and thus they are also discrete. However, one remarks that Γ_2 *and* Γ_a*,* $a > 2$ *are not conjugate inside* $\mathrm{PSL}(2,\mathbb{R})$. Using the Poincaré theorem we can construct Γ_a by making use of fundamental regions; it is easy to verify that the following domain P_a:



is a fundamental region for Γ_a having the sides A, B, C, D. Let $\iota : \{A, B, C, D\} \to \{A, B, C, D\}$ be the involution given by $\iota(A) = D$, $\iota(B) = C$, and let us define $\sigma_A = \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix}$ and $\sigma_B = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$. Then Γ_a is naturally identified with the group generated by σ_A and σ_B.
Now, if $a > 2$ then $\mu(P_a) = +\infty$, while for $a = 2$ we have $\frac{a}{2} = \frac{1}{a}$ and thus the quotient surface is a non-compact cusped surface of finite volume. Thus, $\mathrm{vol}(\mathcal{H}/\Gamma_2) \neq \mathrm{vol}(\mathcal{H}/\Gamma_a)$ and thus Γ_2 and Γ_a cannot be conjugate.
2. Let now consider the group $\Gamma_2' = \left\langle \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \right\rangle \subset \mathrm{PSL}(2,\mathbb{R})$. Then Γ_2' has the same fundamental region P_2 as Γ_2.
Moreover the involution $\iota : \{A, B, C, D\} \to \{A, B, C, D\}$ which yields Γ_2' is a different one, namely $\iota(A) = C$, $\iota(B) = D$. If we consider the matrices ${\sigma'}_A^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$, $\sigma_B' = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ then one verifies easily that Γ_2' is the group generated by σ_A' and σ_B'. The Poincaré theorem implies that Γ_2' is also free.
However, despite the fact that Γ_1 *and* Γ_2 *share the same fundamental region and thus* $\mu(\mathcal{H}/\Gamma_2) = \mu(\mathcal{H}/\Gamma_2')$, these groups are not conjugate within $\mathrm{PSL}(2,\mathbb{R})$. The reason is that $\mathcal{H}/\Gamma_2$ is homeomorphic to a 2-sphere with 3 cusps (*i.e.* $S^2 - \{0, 1, \infty\}$) while $\mathcal{H}/\Gamma_2'$ is a torus with 1-cusp. This follows immediately by looking at the identifications of sides of the respective fundamental domains induced by the involution.
3. *Conclusion*: If one seeks for families of Fuchsian groups then one needs to fix both the isomorphism type of the abstract group Γ as well as the homeomorphism type of the quotient surface $\mathcal{H}/\Gamma$. If Γ has no elliptic points then $\mathcal{H}/\Gamma$ is an orientable surface, with the orientation inherited by taking the quotient. It makes sense to set:

Definition 2.1.1 *The* Teichmüller space $\mathcal{T}(\Sigma)$ *of the oriented surface* Σ *is the space of marked Fuchsian groups* $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ *such that* $\pi_1\Sigma \xrightarrow{\sim} \Gamma$ *and* Σ *is orientation-preserving homeomorphic to* $\mathcal{H}/\Gamma$*. Notice that a marking of* Γ *is provided by a system of generators.*

An equivalent definition is to set:

Definition 2.1.2 *The Teichmüller space* $\mathcal{T}(\Sigma)$ *is the set of marked complex structures on* Σ *up to the equivalence relation below. A marked complex structure is a homotopy equivalence* $f : \Sigma \to M$ *where* M *is an arbitrary Riemann surface and two such* f *and* $f' : \Sigma \to M'$ *are equivalent* $f \sim f'$ *if there exist a conformal equivalence* $h : M \to M'$ *such that* $f' \simeq f \circ h$*,* $\simeq$ *denoting homotopy equivalence.*

Remark. By Riemann's uniformization theorem we can always write $M = \mathcal{H}/\Gamma$ where $\Phi \subset \mathrm{PSL}(2,\mathbb{R})$ acts by isometries. In particular, we have an identification:

$$\mathcal{T}(\Sigma) = \mathrm{Hom}^{+}_{f,d}(\pi_1, \Sigma, \mathrm{PSL}(2,\mathbb{R}))/\text{conjugacy within } \mathrm{PSL}(2,\mathbb{R})$$

where $\varphi = \pi_1\Sigma \to \mathrm{PSL}(2,\mathbb{R})$ belongs to $\mathrm{Hom}^{+}_{f,d}$ if φ preserves the orientation *i.e.* there exists $\phi : \Sigma \to \mathcal{H}/\Gamma$, $\Gamma = \varphi(\pi_1\Sigma)$ preserving the orientation, homeomorphism and such that

$$\phi_* : \pi_1\Sigma \to \pi_1(\mathcal{H}/\Gamma) = \Gamma \subset \mathrm{PSL}(2,\mathbb{R})$$

is identified with φ (up to a $\mathrm{PSL}(2,\mathbb{R})$-conjugacy). □

2.2 Thurston-Bonahon-Penner-Fock coordinates on the Teichmüller spaces

2.2.1 Preliminaries on fatgraphs

Let Γ be a finite graph. We denote by V_Γ and E_Γ the set of its vertices and edges respectively.

Definition 2.2.1 *An* orientation *at a vertex* v *is a cyclic ordering of the (half-) edges incident at* v*. A* fatgraph *(sometimes called* ribbon graph*) is a graph endowed with an orientation at each vertex of* Γ*. A* left-hand-turn path *in* Γ *is a directed closed path in* Γ *such that if* e_1, e_2 *are successive edges in the path meeting at* v*, then* e_2, e_1 *are successive edges with respect to the orientation at* v*. The ordered pair* e_1, e_2 *is called a left-turn. We sometimes call* faces *of* Γ *the left-hand-turn paths and denote them by* F_Γ*.*

A fatgraph is usually represented in the plane, by assuming that the orientation at each vertex is the counter-clockwise orientation induced by the plane, while the intersections of the edges at points other than the vertices are ignored. There is a natural surface, which we denote by Γ^t obtained by thickening the fatgraph. We usually call Γ^t the ribbon graph associated to Γ. We replace the half-edges around a vertex by thin strips joined at the vertex, whose boundary arcs have natural orientations. For each edge of the graph we connect the thin strips corresponding to the vertices by a ribbon which follows the orientation of their boundaries. We obtain an oriented surface with boundary. The boundary circles are in one-to-one correspondence with the left-hand-turn paths. If one caps each left-hand-turn path by a 2-disk we find a closed surface Γ^c, and this explains why we called these paths faces. The centers of the 2-disks will be called punctures of Γ^c and $\Gamma^o = int(\Gamma^t)$ is homeomorphic to the punctured surface.

There is a canonical embedding $\Gamma \subset \Gamma^t$, and one can associate to each edge e of Γ a properly embedded orthogonal arc $e^\perp$ which joins the two boundary components of the thin strip lying over e. The dual arcs $e^\perp$ divide the ribbon Γ^t into hexagons. When we consider the completion Γ^c, we join the boundary points of these dual arcs to the punctures within each 2-disk face and obtain a set of arcs connecting the punctures, denoted by the same symbols. Then the dual arcs divide Γ^c into triangles. We set $\Delta(\Gamma)$ for the triangulation obtained this way. The vertices of $\Delta(\Gamma)$ are the punctures of Γ^c. Remark that $\Delta(\Gamma)$ is well-defined up to isotopy. Now the fatgraph $\Gamma \subset \Gamma^t$ can be recovered from $\Delta(\Gamma)$ as follows. Mark a point in the interior of each triangle, and connect points corresponding to adjacent triangles. This procedure works for any given triangulation Δ of an oriented surface and produces a fatgraph $\Gamma = \Gamma(\Delta)$ with the property that $\Delta(\Gamma) = \Delta$. The orientation of Γ comes from the surface.

If Γ^o is the surface Σ_g^s of genus g with s punctures then by Euler characteristic reasons we have: $\sharp V_\Gamma = 4g-4+2s$, $\sharp E_\Gamma = 6g-6+3s$, $\sharp F_\Gamma = s$.

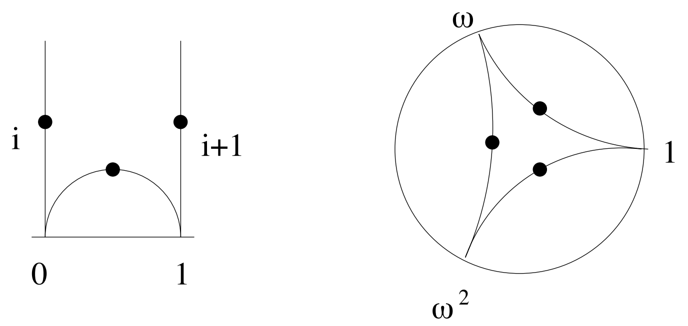


Figure 2.1: The standard marked ideal triangle

2.2.2 Coordinates on Teichmüller spaces

Marked ideal triangles Let us denote by $\mathbb{D}$ the unit disk, equipped with the hyperbolic metric. Recall that any two ideal triangles are isometric, since we may find a Möbius transformation, which takes one onto the other. Choose a point on each edge of the ideal triangle. The chosen points will be called *tick-marks*.

Definition 2.2.2 *A* marked ideal triangle *is an ideal triangle with a tick-mark on each one of its three sides. An isomorphism between two marked ideal triangles is an isomorphism between the ideal triangles which preserves the tick-marks.* A standard marked ideal triangle *is one which is isometric to the marked ideal triangle whose vertices in the disk model are given by* $v_1 = 1,\ v_2 = \omega,\ v_3 = \omega^2$ *and whose tick-marks are* $t_1 = -(2-\sqrt{3}),\ t_2 = -(2-\sqrt{3})\omega,\ t_3 = -(2-\sqrt{3})\omega^2$*, where* $\omega = \mathrm{e}^{2\pi \mathrm{i}/3}$.

The ideal triangle and its tick-marks are pictured in figure 2.1 in both the half-plane model and the disk model; they correspond each other by the map $z \mapsto \frac{z-(\omega+1)}{z-(\bar{\omega}+1)}$.

Coordinates on the Teichmüller space of punctured surfaces Set $\mathcal{T}_g^s$ for the Teichmüller space of surfaces of genus g with s punctures. Let Γ be a fatgraph with the property that Γ^c is a surface of genus g with s punctures and let S denote the surface Γ^c endowed with a hyperbolic structure of finite volume, having the cusps at the punctures.

As already explained above we have a triangulation $\Delta(\Gamma)$ associated to Γ. One deforms the arcs of $\Delta(\Gamma)$ within their isotopy class in order to make them geodesic. We shall associate a real number $t_e \in \mathbb{R}$ to each edge of $\Delta(\Gamma)$ (equivalently, to each edge of Γ). Set Δ_v and Δ_w for the two triangles sharing the edge $e^\perp$. We consider next two adjacent lifts of these triangles (which we denote by the same symbols) to the hyperbolic space $\mathbb{H}^2$. Then both Δ_v and Δ_w are isometric to the standard ideal triangle of vertices v_1, v_2 and v_3. These two isometries define (by pull-back) canonical tick-marks t_v and respectively t_w on the geodesic edge shared by Δ_v and Δ_w. Set t_e for the (real) length of the translation along this geodesic needed to shift t_v to t_w. Notice that this geodesic inherits an orientation as the boundary of the ideal triangle Δ_v in $\mathbb{H}^2$ which gives t_e a sign. If we change the role of v and w the number t_e is preserved.

An equivalent way to encode the translation parameters is to use the cross-ratios of the four vertices of the glued quadrilateral $\Delta_v \cup \Delta_w$, which are considered as points of $\mathbb{R}P^1$. It is convenient for us to consider $\mathbb{R}P^1$ as the boundary of the upper half-plane model of $\mathbb{H}^2$, and hence the ideal points have real (or infinite) coordinates. Let assume that Δ_v is the ideal triangle determined by $[p_0 p_{-1} p_\infty]$ and Δ_w is $[p_0 p_\infty p]$. We consider then the following cross-ratios:

$$z_e = [p_{-1}, p_\infty, p, p_0] = [p, p_0, p_{-1}, p_\infty] = \log - \frac{(p_0 - p)(p_{-1} - p_\infty)}{(p_\infty - p)(p_{-1} - p_0)}.$$

This cross-ratio reflects both the quadrilateral geometry and the decomposition into two triangles. In fact the other possible decomposition into two triangle of the same quadrilateral leads to the value z_e.

The relation between the two translation parameters t_e and z_e is immediate. Consider the ideal quadrilateral of vertices $-1, 0, e^z$ and ∞, whose cross-ratio is $z_e = z$, where $e = [0\infty]$. The left triangle tick-mark is located at i, while the right one is located at ie^{-z}, after the homothety sending the triangle into the standard triangle.

Taking in account that the orientation of the edge e is up-side one derives that t_e is the signed hyperbolic distance between i and $e^{-z_e}i$, which is z_e.

Proposition 2.2.1 *The map $\mathbf{t}_\Gamma : \mathcal{T}_g^s \to \mathbb{R}^{E_\Gamma}$ given by $t_\Gamma(S) = (t_e)_{e\in E_\Gamma}$ is a homeomorphism onto the linear subspace $\mathbb{R}^{E_\Gamma/F_\Gamma} \subset \mathbb{R}^{E_\Gamma}$ given by equations:*

$$t_\gamma := \sum_{k=1}^{n} t_{e_k} = 0,$$

for all left-hand-turn closed paths $\gamma \in F_\Gamma$, which is expressed as a cyclic chain of edges $e_1, ..., e_n$.

Remark. Notice that there are exactly s left-hand-turn closed paths, which lead to s independent equations hence the subspace $\mathbb{R}^{E_\Gamma/F_\Gamma}$ from above is of dimension $6g - 6 + 2s$. □

Proof. The map $\mathbf{t}_\Gamma$ is continuous, and it suffices to define an explicit inverse for it. Let Γ be a trivalent fatgraph whose edges are labeled by real numbers $\mathbf{r} = (r_e)_{e\in E_\Gamma}$. We want to paste one copy Δ_v of the standard marked ideal triangle on each vertex v of Γ and glue together by isometries these triangles according to the edges connections. Since the edges of an ideal triangle are of infinite length we have the freedom to use arbitrary translations along these geodesics when gluing together adjacent sides. If $e = [vw]$ is an edge of Γ then one can associate a real number $t_e \in \mathbb{R}$ as follows. There are two tick-marks, namely t_v and t_w on the common side of Δ_v and Δ_w. We denote by t_e the amount needed for translating t_v into t_w according to the orientation inherited as a boundary of Δ_v. Given now the collection of real numbers $\mathbf{r}$ we can construct unambiguously our Riemann surface $S(\Gamma, \mathbf{r})$, which moreover has the property that $t_\Gamma(S(\Gamma, \mathbf{r})) = \mathbf{r}$. Furthermore it is sufficient now to check whenever this constructions yields a complete Riemann surfaces. The completeness at the puncture determined by the left-hand-turn path γ is equivalent to the condition $t_\gamma = 0$, and hence the claim. The cusps of $S(\Gamma)$ are in bijection with the left-hand-turn paths in Γ, and the triangulation of $S(\Gamma)$ obtained by our construction corresponds to Γ. □

Remark. W.Thurston associated to an ideal triangulation a system of shearing coordinates for the Teichmüller space in mid eighties (see [29]). However, the systematic study of such coordinates appeared only later in the papers of F.Bonahon [4] and from a slightly different perspective in Penner's treatment of the decorated Teichmüller spaces ([24]). V.Fock unraveled the elementary aspects of this theory which lead him further to the quantification of the Teichmüller space. □

The Fuchsian group associated to Γ and $\mathbf{r}$ The surface $S(\Gamma, \mathbf{r})$ is uniformized by a Fuchsian group $G = G(\Gamma, \mathbf{r}) \subset \mathrm{PSL}(2, \mathbb{R})$, i.e. $S(\Gamma, \mathbf{r}) = \mathbb{H}^2/G(\Gamma, \mathbf{r})$. We can explicitly determine the generators of the Fuchsian group, as follows.

We have natural isomorphisms between the fundamental group $\pi_1(S(\Gamma, \mathbf{r}) \cong \pi_1(\Gamma^t) \cong \pi_1(\Gamma)$. Any path γ in Γ is a cyclic sequence of adjacent directed edges $e_1, e_2, e_3, ..., e_n$, where e_i and e_{i+1} have the vertex v_i in common. We insert between e_i and e_{i+1} the symbol lt if e_i, e_{i+1} is a left-hand-turn, the symbol rt if it is a right-hand-turn and no symbol otherwise (i.e. when e_{i+1} is e_i with the opposite orientation). Assume now that we have a Riemann surface whose coordinates are $\mathbf{t}_\Gamma(S) = \mathbf{r}$. We define then a representation $\rho_\mathbf{r} : \Pi_1(\Gamma) \to \mathrm{PSL}(2, \mathbb{R})$ of the path groupoid $\Pi_1(\Gamma)$ by the formulas:

$$\rho_\mathbf{r}(e) = \begin{pmatrix} 0 & e^{\frac{r_e}{2}} \\ -e^{-\frac{r_e}{2}} & 0 \end{pmatrix}, \text{ and } \rho_\mathbf{r}(lt) = \rho_\mathbf{r}(rt)^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

This is indeed well-defined since $\rho_\mathbf{r}(e)^2 = -1 = 1 \in \mathrm{PSL}(2, \mathbb{R})$, and hence the orientation of the edge does not matter, and $\rho_\mathbf{r}(lt)^3 = \rho_\mathbf{r}(rt)^3 = 1$. Furthermore the fundamental group $\pi_1(\Gamma)$ is a subgroup of $\Pi_1(\Gamma)$.

Proposition 2.2.2 *The Fuchsian group $G(\Gamma, \mathbf{r})$ is $\rho_\mathbf{r}(\pi_1(\Gamma)) \subset \mathrm{PSL}(2, \mathbb{R})$.*

Proof. We can begin doing the pasting without leaving the hyperbolic plane, until we get a polygon P, together with a side pairing. We may think of each triangle as having a white face and a black face, and build the polygon P such that all the triangles have white face up. We attach to each side pairing (s_i, s_j) an orientation preserving isometry A_{ij}, such that $A_{ij}(s_i) = s_j$, A_{ij} sends tick-marks into the tick-marks shifted by r_e, and $P \cap A_{ij}(P) = \emptyset$. Denote by G the subgroup of $\mathrm{ISO}^+(\mathbb{D})$ generated by all the side-pairing transformations. In order to apply the Poincaré Theorem all the vertex-cycle transformations must be parabolic. This amounts to ask that for every left-hand-turn closed path γ we have $t_\gamma = 0$. Then by the Poincaré theorem G is a discrete group of isometries with P as its fundamental domain and $\mathbb{H}^2/G$ is the complete hyperbolic Riemann surface $S(\Gamma, \mathbf{r})$.

We need now the explicit form of the matrices A_{ij}. We obtain them by composing the isometries sending a marked triangle into the adjacent one, in a suitable chain of triangles, where consecutive ones have a common edge. If e is such an edge we remark that $\rho_{\mathbf{r}}(e)$ do the job we want, because it sends the triangle $[-1, 0, \infty]$ into $[e^{r_e}, \infty, 0]$. Moreover the quadrilateral $[-1, 0, e^{r_e}, \infty]$, with this decomposition into two triangles, has associated the cross-ratio r_e. We need next to use $\rho_{\mathbf{r}}(lt)$ which permutes counter-clockwise the tick-marks and the vertices $-1, 0$ and ∞ of the ideal triangle. Then one identifies the matrices A_{ij} with the images of the closed paths by $\rho_{\mathbf{r}}$. □

Remark. We observe that the left-hand-turn paths are preserved under an isomorphism of graphs which preserves the cyclic orientation at each vertex. Thus any automorphism of the fatgraph Γ induces an automorphism of $S(\Gamma)$. □

Coordinates on the Teichmüller space of surfaces with geodesic boundary Set $\mathcal{T}_{g,s;or}$ for the Teichmüller space of surfaces of genus g with s *oriented* boundary components. Here *or* denotes the choice of one orientation for each of the boundary components. Since the surface has a canonical orientation, we can set unambiguously $or : \{1, 2, ..., s\} \to \mathbb{Z}/2\mathbb{Z}$ by assigning $or(j) = +1$ if the orientation of the j-th component agrees with that of the surface and $or(j) = -1$, otherwise. We suppose that each boundary component is a geodesic in the hyperbolic metric, and possibly a cusp (hence in some sense this space is slightly completed). Let Γ be a fatgraph with the property that Γ^t is a surface of genus g with s boundary components and let S denote the surface Γ^t endowed with a hyperbolic structure, for which the boundary is geodesic. Assume that, in this metric, the boundary geodesics b_j have length l_j.

Consider the restriction of the hyperbolic metric to $int(\Gamma^t) = \Gamma^o$. Then Γ^o is canonically homeomorphic to the punctured surface $\Gamma^c - \{p_1, ..., p_s\}$. In particular there is a canonically induced hyperbolic metric on $\Gamma^c - \{p_1, ..., p_s\}$, which we denote by S^*. Moreover this metric is not complete at the punctures p_j. Suppose that the punctures p_j corresponds to the left-hand-turn closed paths γ_j, or equivalently the boundary components geodesics b_j, of length l_j. Assume that we have an ideal triangulation of S^* by geodesic simplices, whose ideal vertices are the punctures p_j. Then the holonomy of the hyperbolic structure around the vertex p_j is a non-trivial, and it can be calculated in the following way (see [28], Prop.3.4.18, p.148). Consider a geodesic edge α entering the puncture and a point $p \in \alpha$. Then the geodesic spinning around p_j in the positive direction (according to the orientation of the boundary circle) is intersecting again α a first time in the point $h_{p_j}(p)$. The hyperbolic distance between the points p and $h_{p_j}(p)$ is the length l_j of the boundary circle in the first metric. Moreover the point $h_{p_j}(p)$ lies in the ray determined by p and the puncture p_j. Notice that if we had chose the loop encircling the puncture to go in opposite direction then the iterations $h_{p_j}(p)$ would have gone faraway from the puncture, and the length would have been given the negative sign. Set therefore l_j^o* for the signed length.

We construct as above the geodesic ideal triangulation $\Delta(\Gamma)$ of the non-complete hyperbolic punctured surface S^*. We can therefore compute the holonomy map using the thick-marks on some edge abutting to the puncture p_j. It is immediately that the the holonomy displacement on this edge is given by t_{γ_j}, where γ_j is the left-hand-turn closed path corresponding to this puncture. In particular we derive that:

$$|t_{\gamma_j}| = l_j, \text{ for all } j \in \{1, 2, ..., s\}.$$

Using the method from the previous section we know how to associate to any edge e of Γ a real number $t_e = t_e(S^*)$ measuring the shift between two ideal triangles in the geodesic triangulation of the surface S^*.

Proposition 2.2.3 *The map $\mathbf{t}_\Gamma : \mathcal{T}_{g,s;or} \to \mathbb{R}^{E_\Gamma}$ given by $t_\Gamma(S) = (t_e)_{e \in E_\Gamma}$ is a homeomorphism.*

Proof. The construction of an inverse map proceeds as above. Given $\mathbf{r} \in \mathbb{R}^{E_\Gamma}$ we construct a non-complete hyperbolic surface S^* with s punctures with the given parameters, by means of gluing ideal triangles. As shown in ([28], Prop. 3.4.21, p.150) we can complete this hyperbolic structure to a surface with geodesic boundary S, such that $int(S) = S^*$. Further if $t_{\gamma_j} > 0$, then we assign the orientation of γ_j for the boundary component b_j, otherwise we assign the reverse orientation. When $t_{\gamma_j} = 0$ it means that we have a cusp at p_j. □

Remark. The two points of $\mathcal{T}_{g,s;or}$ given by the same hyperbolic structure on the surface $\Sigma_{g,s}$ but with distinct orientations of some boundary components lie in the same connected component. Nevertheless the previous formulas shows that a path connecting them must pass through the points of $\mathcal{T}_{g,s;or}$ corresponding to surfaces having a cusp at the respective puncture. □

Set $\mathcal{T}_{g,s}$ for the Teichmüller space of surfaces of genus g with s *non-oriented* boundary components, i.e. hyperbolic metrics for which the boundary components are geodesic. There is a simple way to recover coordinates on $\mathcal{T}_{g,s}$ from its oriented version. Let $\psi : \mathbb{R}^{E_\Gamma} \to \mathbb{R}^{F_\Gamma}$ be the map $\psi(\mathbf{t}) = (t_{\gamma_i})_{\gamma_i \in F_\Gamma}$. Choose a projector $\psi^* : \mathbb{R}^{E_\Gamma} \to \ker \psi = \mathbb{R}^{E_\Gamma / F_\Gamma}$, and set $\iota_{|.|} : \mathbb{R}^{F_\Gamma} \to \mathbb{R}^{F_\Gamma}$ for the map given on coordinates by $\iota_{|.|}(y_j)_{j=1,\sharp F_\Gamma} = (|y_j|)_{j=1,\sharp F_\Gamma}$. Then $\mathcal{T}_{g,s}$ is the quotient by the $(\mathbb{Z}/2\mathbb{Z})^{F_\Gamma}$-action on $\mathcal{T}_{g,s;or}$ which changes the orientation of the boundary components.

Proposition 2.2.4 *We have a homeomorphism $\mathbf{t}_\Gamma : \mathcal{T}_{g,s} \to \mathbb{R}^{6g-6+2s} \oplus \mathbb{R}^s$, which is induced from the second line of the following commutative diagram:*

$$\begin{array}{ccc}
\mathcal{T}_{g,s;or} & \xrightarrow{(\psi^*\oplus\psi)\circ\mathbf{t}_\Gamma} & \mathbb{R}^{E_\Gamma/F_\Gamma} \oplus \mathbb{R}^{F_\Gamma} \\
\downarrow & & \downarrow id \oplus \iota_{|.|} \\
\mathcal{T}_{g,s} & \longrightarrow & \mathbb{R}^{E_\Gamma/F_\Gamma} \oplus \mathbb{R}_+^{F_\Gamma} \\
\uparrow & & \uparrow id \oplus 0 \\
\mathcal{T}_g^s & \longrightarrow & \mathbb{R}^{E_\Gamma/F_\Gamma}
\end{array}$$

Remark. Observe that the embedding $\mathcal{T}_g^s \hookrightarrow \mathcal{T}_{g,s}$ given in terms of coordinates by adding on the right a string of zeroes lifts to an embedding $\mathcal{T}_g^s \hookrightarrow \mathcal{T}_{g,s;or}$. □

Putting together the results of the last two sections we derive that:

Proposition 2.2.5 *The map $\mathbf{t}_\Gamma : \mathcal{T}_{g,n;or}^s \to \mathbb{R}^{E_\Gamma}$ given by $t_\Gamma(S) = (t_e)_{e \in E_\Gamma}$ is a homeomorphism of the Teichmüller space of surfaces of genus g with n oriented boundary components and s punctures onto the linear subspace $\mathbb{R}^{E_\Gamma / F^*\Gamma}$ of dimension $6g - 6 + 3n + 2s$ given by the equations: $t_{\gamma_j} = 0$, for those left-hand-turn closed paths γ_j corresponding to the punctures, $\gamma_j \in F_\Gamma^* \subset F_\Gamma$.*

Chapter 3

The interplay between mapping class groups and Teichmüller spaces

3.1 Mapping class groups

3.1.1 General facts about mapping class groups

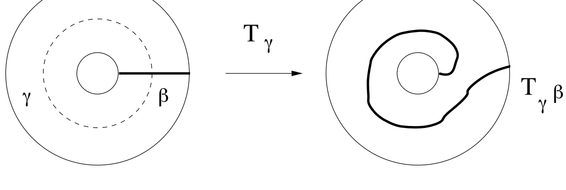
Consider Σ surface, possibly with boundary and punctures or marked points, compact and orientable. We denote by $\mathrm{Homeo}^+(\Sigma)$ the group of homeomorphisms of Σ preserving the orientation, endowed with the compact-open topology.

Definition 3.1.1 *The mapping class group of Σ is $\mathrm{Mod}(\Sigma) = \mathrm{Homeo}^+(\Sigma)/\simeq$, where $f, g : \Sigma \to \Sigma$ are equivalent if they are homotopic. This is equivalent to consider the quotient $\mathrm{Homeo}^+(\Sigma)/\mathrm{Homeo}_0(\Sigma)$, where $\mathrm{Homeo}_0(\Sigma)$ is the connected component of identity. If Σ has boundary or marked points then one requires that the homeomorphisms and the homotopies we are concerned of to fix this boundary/marking data (pointwise or setwise).*

Remarks.

1. An equivalent definition is $\mathrm{Mod}(\Sigma) = \mathrm{Diffeo}^+(\Sigma)/\simeq$, where $\mathrm{Diffeo}^+(\Sigma)$ denotes the group of diffeomorphisms of Σ and $\simeq$ is the smooth homotopy equivalence. The smoothness class might be chosen to be any $\mathcal{C}^r$, with $r \geq 1$.
2. Topological groups of homeomorphisms like $\mathrm{Homeo}(\Sigma)$, or Banach-Frechet groups of diffeomorphisms like $\mathrm{Diffeo}(\Sigma)$ are huge groups and little is known about. Here is a sample of results:
   i) If we replace in the definition above *homotopy* by *isotopy* (*i.e.* homotopy among homeomorphisms/diffeomorphisms) we still obtain the same quotient $\mathrm{Mod}(\Sigma)$; however, one requires that Σ be compact and $\Sigma \neq D^2, S' \times [0,1]$. This is a classical result due to Baer, Epstein.
   ii) If the genus of Σ is $g \geq 2$ then Eells and Sampson proved that the connected component $\mathrm{Diffeo}_0(\Sigma)$ is *contractible*. Moreover $\mathrm{Diffeo}_0(\Sigma)$ is a simple group, as well as the diffeomorphism groups of arbitrary compact manifolds, as was shown by Banyaga and Thurston. An old result of Anderson and Fisher established that the group of homeomorphisms $\mathrm{Homeo}_0(\Sigma)$ is also a simple group.
3. $\mathrm{Mod}(\Sigma)$ *is a discrete group.*

Elements of $\mathrm{Mod}(\Sigma)$ are usually constructed via the following procedure: pick-up a simple closed curve $\gamma \subset \Sigma$ and an annulus A having the core γ. We perform then the following transformation supported in A:



We drawn on the left side the curve β transversal to γ which joins the boundary circles and on the right side the image of β by T_γ. We extend further T_γ by identity to all of Σ. The transformation T_γ is called the *left Dehn twist around* Γ.
Remarks.

1. In order to define T_γ one needs to specify the orientation of Σ but γ is not oriented.

2. Actually the mapping class group $\mathrm{Mod}(A)$ of the annulus is the cyclic group $\langle T_\gamma\rangle \simeq \mathbb{Z}$ generated by the Dehn twist. In fact, if $\varphi \in \mathrm{Mod}\, A$ then look at the image $\varphi(\beta)$ which is well-defined (and independent on the homeomorphism representing φ in $\pi_1(A, \partial A)$. Then there exists n such that $T_\gamma^{-n}\varphi(\beta) = \beta \in \pi_1(A, \partial A)$. Moreover, in the case of the annulus we find that the curves $T_\gamma^{-n}\varphi(\beta)$ and β are not only homotopic but isotopic. Cut open A along β in order to obtain a disk. By the Alexander trick, which says that homeomorphisms of the 2-disk are homotopic/isotopic to identity we find that $T_\gamma^n \simeq \varphi$, as claimed.

Example.

1. Id Σ_g denotes the closed surface of genus g then we have a natural isomorphism $\mathrm{Mod}(\Sigma_1) \cong \mathrm{SL}(2, \mathbb{Z})$, which sends the class of the homeomorphism φ into the map $\varphi_* : H_1(\Sigma_1) \to H_1(\Sigma_1)$ induced on the homology.

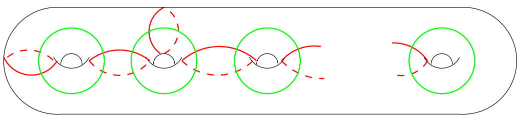
2. Notice that if the genus $g \geq 2$ then $\mathrm{Mod}(\Sigma)$ is *not* an arithmetic group, as it was proved by N.Ivanov.

Proposition 3.1.1 (M.Dehn, J.Nielsen, R.Baer, D.B.A.Epstein) *For any $g \geq 1$ there is a natural isomorphism*

$$\mathrm{Mod}(\Sigma_g) \to \mathrm{Out}^+(\pi_1(\Sigma_g)) = \mathrm{Aut}(\pi_1(\Sigma_g))/\,\mathrm{Inn}(\pi_1\Sigma_g)$$

which sends the class of the homeomorphism φ into the class of the map $\varphi_* : \pi_1\Sigma_g \to \pi_1\Sigma_g$ induced by φ in homotopy. Here $\mathrm{Inn}(\Gamma)$ is the set of inner automorphisms (acting by conjugacy), which corresponds to the freedom in choosing the base point for the fundamental group. The result can be stated in the case when Σ_g has boundary but one needs to add extra conditions on the automorphisms in the right hand side, by asking them to preserve the conjugacy classes of boundary loops.

Proposition 3.1.2 (M.Dehn, W.L.B.Lickorish, S.Humpreis) *$\mathrm{Mod}(\Sigma_g)$ is finitely presented. A convenient system of generators is given by the Dehn twists around the $2g + 1$ curves below:*



This is sharp, as Humpries proved that $\mathrm{Mod}(\Sigma_g)$ cannot be generated by $2g$ (or less) Dehn twists, if $g \geq 2$. However, one can do better if we accept generators which are not necessary Dehn twists. B.Wajnryb showed that two elements can generate $\mathrm{Mod}(\Sigma_g)$, and M. Korkmaz proved that one of the two generators can be taken to be a Dehn twist, and also that both generators can be torsion elements.

Open questions.

1. It is still unknown whether the *Torelli subgroup* $Tor(\Sigma_g) \subset \mathrm{Mod}(\Sigma_g)$ – which is the subgroup generated by the Dehn twists along separating simple curves, and the elements $T_\gamma T_\delta^{-1}$ where γ, δ are boundary circles of some subsurface of Σ – *is finitely presented*. It is known it is finitely generated if $g \geq 3$ and free on infinitely many generators $g = 2$ by the results of G.Mess. Notice that $Tor(\Sigma_g)$ is also the kernel of the natural morphism $\mathrm{Mod}(\Sigma_g) \to Aut(H_1(\Sigma_g))$.

2. There is an interesting subgroup $\mathcal{K}(\Sigma_g) \subset Tor(\Sigma_g)$, which is the subgroup generated by the Dehn twists along separating curves. Recently, D.Biss and B.Farb proved that $\mathcal{K}(\Sigma_g)$ is not finitely generated, leaving unsettled the problem on whether its abelianization is finitely generated or not when $g \geq 3$.

3.1.2 Mapping class groups acting on Teichmüller spaces

There is a close relation between mapping class groups and Teichmüller spaces. By using the identification between $\mathrm{Mod}(\Sigma)$ and $\mathrm{Out}^+(\pi, \Sigma)$ the mapping class group acts by left composition on the space $\mathcal{T}(\Sigma)$ which can be seen as a space of group representation:

$$\mathcal{T}(\Sigma) = \mathrm{Hom}^+_{f,d}\left(\pi_1\Sigma, \mathrm{PSL}(2,\mathbb{R})\right) / \mathrm{PSL}(2,\mathbb{R})$$

Specifically, we have

$$\mathrm{Out}^+(\pi,\Sigma) \times \mathrm{Hom}^+_{f,d}\left(\pi_1\Sigma, \mathrm{PSL}(2,\mathbb{R})\right) / \mathrm{PSL}(2,\mathbb{R}) \to \mathrm{Hom}^+_{f,d}\left(\pi_1\Sigma, \mathrm{PSL}(2,\mathbb{R})\right) / \mathrm{PSL}(2,\mathbb{R})$$

given by

$$(\varphi, [\rho]) \longrightarrow [\varphi \circ \rho]$$

Moreover, $\mathrm{Mod}(\Sigma)$ acts by real analytic homeomorphisms. This action is important in understanding both the algebraic structure of the mapping class group using the geometry of the Teichmüller space, because of the following basic result going back to F.Klein and R.Fricke.

Proposition 3.1.3 (R.Fricke, S.Kravetz) $\mathrm{Mod}(\Sigma)$ *acts properly discontinuously on* $\mathcal{T}(\Sigma)$.

Idea of proof. Let assume that there exist a sequence $\varphi_n \in \mathrm{Mod}\,\Sigma$ so that there exist two compacts C_1, C_2 in the Teichmüller space with the property $\varphi_n(C_1) \cap C_2 \neq \emptyset$ for all n. Then there exists a convergent sequence of points $z_n \to z \in \mathcal{T}(\Sigma)$ so that $\varphi_n(z_n)$ also converges to some point $w \in \mathcal{T}(\Sigma)$. Thus $\varphi_n^{-1}\varphi_{n-1}(z_n) \to z$. We will show that if $\xi_n \in \mathrm{Mod}\,\Sigma$ has the property that $\xi_n z_n \to z$ then $\xi_n = \mathbb{1}$ for large enough n.
This is a consequence of the following facts:

1. If Γ is a Fuchsian group then the set

   $$A(\Gamma) = \{\mathrm{Tr}(\gamma); \gamma \in \Gamma \subset \mathrm{PSL}(2,\mathbb{R})\} \subset \mathbb{R}_+$$

   is mapped by the function $\cosh \frac{1}{2}(x)$ bijectively into the marked set of lengths of geodesics of the surface $\mathcal{H}/\Gamma$ (indexed by elements of Γ. Moreover, these sets are discrete.

2. If $\xi \in \mathrm{Mod}(\Sigma)$ and $\Gamma = \rho(\pi_1\Sigma)$ is a Fuchsian group then the marked set $A(\xi\Gamma)$ is obtained from the marked set $A(\Gamma)$ by a permutation of its elements.

3. The regular functions $\mathrm{tr}(\rho(\gamma))$, $\gamma \in \pi_1\Sigma$, viewed as functions $\mathcal{T}(\Sigma) \to \mathbb{R}$ are generating a polynomial algebra which is finitely generated. The proof is based on the identity:

   $$\mathrm{tr}(x)\,\mathrm{tr}(y) = \mathrm{tr}(xy) \times \mathrm{tr}(xy^{-1}).$$

4. If $\xi_n z_n \to z$ then for large n

   $$A(\xi_n\Gamma_{zn}) \text{ and } A(\Gamma_s) \text{ agree on ther first } N \text{ items}$$

   It N is large enough in order that all generators of the algebra above are contained among the first N items then we find that $A(\xi_n\Gamma_{zn}) = A(\Gamma_z)$. Since ξ_n acts as a permutation on the marked sets of geodesics we derive that the permutation is the identity.

5. Two hyperbolic structures on a surface having the same marked lengths of geodesics are isometric. In fact, if the traces of two discrete faithful representations coincide i.e. $\mathrm{tr}(\rho(\gamma) = \mathrm{tr}(\rho'(\gamma))$ for any $\gamma \in \pi_1(\Sigma)$ then the representations are conjugate.

Remarks.

1. The $\mathrm{Mod}(\Sigma_g)$-action on the Teichmüller space is *effective* if $g \geq 3$. When $g = 1, 2$ the hyperelliptic involution acts trivially on $\mathcal{T}(\Sigma)$.

2. The quotient $\mathcal{T}(\Sigma)/\mathrm{Mod}(\Sigma)$ is naturally a complex space with orbifold singularities (at points where the $\mathrm{Mod}(\Sigma)$ action is not free). However, one knows that all stabilizers should be finite. In this respect the moduli space $\mathcal{M}(\Sigma) = \mathcal{T}(\Sigma)/\mathrm{Mod}(\Sigma)$ plays the role of a *classifying space for the mapping class group.* For instance, we have an isomorphism

   $$H^*(\mathcal{M}(\Sigma); \mathbb{Q}) \simeq H^*(\mathrm{Mod}(\Sigma); \mathbb{Q})$$

3. Since $\mathcal{T}(\Sigma)$ is a topological cell each torsion element of $\mathrm{Mod}(\Sigma)$ should fix a non-empty set. In particular, any periodic mapping class contains a periodic homeomorphism which is a conformal homeomorphism for some complex structure on Σ.

3.1.3 Stabilizers of the mapping class group action

The action of $\mathrm{Mod}(\Sigma)$ on $\mathcal{T}(\Sigma)$ is properly discontinuous and hence it has finite stabilizers. A point p in $\mathcal{T}(\Sigma)$ corresponds to a class of marked Riemann surface $p = [S]$, and we can identify the stabilizer $\mathrm{Mod}(\Sigma)_p$ of the point p, as follows:

$$\mathrm{Mod}(\Sigma)_p = \{\varphi \in \mathrm{Mod}(\Sigma) \text{ such that } [\varphi S] = [S]\}$$

Moreover, S is defined by the holonomy map $\rho_S : \pi_1\Sigma \to \mathrm{PSL}(2,\mathbb{R})$ and so we have $\rho_{\varphi S} = \varphi \circ \rho_S$, where φ is interpreted now as an element of $\mathrm{Out}^+(\pi_1\Sigma)$. Since the marked surfaces determined by $\rho_{\varphi S}$ and ρ_S are the same they should be obtained by means of a conjugation within $\mathrm{PSL}(2,\mathbb{R})$ *i.e.* there exists $\lambda = \lambda_\varphi \in \mathrm{PSL}(2,\mathbb{R})$ so that

$$\rho_{\varphi S} = \lambda_\varphi \rho_S \lambda_\varphi^{-1}$$

In particular, λ_φ belongs to the normalizer of the Fuchsian group $\rho_S(\pi_1 S)$ and it is immediate that the map

$$\lambda : \mathrm{Mod}(\Sigma)_p \longrightarrow N\left(\rho_S(\pi_1\Sigma)\right)/\rho_S(\pi_1\Sigma)$$

is a group homomorphism. Actually, we have a more precise result:

Proposition 3.1.4 *The stabilizer of the class of the marked Riemann surface* $[S]$ *is given by*

$$\mathrm{Mod}(\Sigma)_{p=[S]} = \mathrm{Aut}(S)$$

where Aut(S) *are the conformal (i.e. holomorphic) automorphism group of* S*.*

In fact, any element of $\mathcal{N}(\Gamma)/\Gamma$, Γ Fuchsian group corresponds to an automorphism of the Riemann surface (see the section 1.8).

Corollary 3.1.1 *For a generic Riemann surface* S *we have* $\mathrm{Aut}(S) = \{\mathbb{1}\}$.

Remark. It is known that, if the genus of Σ is $g =\geq 4$, then the local structure of $\mathcal{T}(\Sigma)/\mathrm{Mod}(\Sigma)$ around $p \in \mathcal{T}(\Sigma)/\mathrm{Mod}(\Sigma)$ is described by the quotient $\mathbb{R}^{6g-6}/F_p$, where $F_p \subset \mathrm{GL}(6g-6)$ is a finite group, which is the image of a faithful linear representation $\mathrm{Aut}(S) \to \mathrm{GL}(6g-6)$ (where $[S] = p$).
In particular, the point p is smooth in the quotient iff S has no automorphisms.
A more elaborate analysis shows that the space $\mathcal{T}(\Sigma)/\mathrm{Mod}(\Sigma)$ is singular at the points when S has automorphisms (as shown by E.Rauch in 1962) for $g \geq 4$. For $g = 2$ there is only one singular point, corresponding to the Riemann surface given by the equation:

$$y^2 = x^5 - 1$$

which has additional symmetries with respect to the rest of Riemann surfaces having only the hyperelliptic involution automorphism. For $g = 3$ the hyperelliptic locus consists of smooth points. □
Furthermore it is known that there exists a finite index subgroup of $\mathrm{Mod}(\Sigma)$ which acts freely on $\mathcal{J}(\Sigma)$. A quantitative estimate of the index follows from the following result due to J. P. Serre (1958):

Proposition 3.1.5 *If* $\varphi \in \mathrm{Mod}(\Sigma)_{[S]}$ *is an automorphism of the Riemann surface* S *and*

$$\varphi_* : H_1(\Sigma, \mathbb{Z}/\ell\mathbb{Z}) \longrightarrow H_1(\Sigma, \mathbb{Z}/\ell\mathbb{Z})$$

is the identity for some $\ell \geq 3$ *then* $\varphi = \mathbb{1}$*. In particular* $\ker(\mathrm{Mod}(\Sigma) \to Aut(H_1(\Sigma; \mathbb{Z}/\ell\mathbb{Z})) \cong Sp(2g, \ell))$ *acts freely on* $\mathcal{T}(\Sigma)$ *for any* $\ell \geq 3$.

Chapter 4

Computation of the mapping class group action on the Teichmüller spaces

4.1 The mapping class group action

4.1.1 The Ptolemy modular groupoid

The modular groupoid was considered by Mosher in his thesis and further as a key ingredient in [20, 21], it is implicit in Harer's paper on the arc complex (see [14]) and then studied by Penner (see [24, 25]; notice that the correct definition is that from [25]) who introduced also the terminology.

Recall that a groupoid is a category whose morphisms are invertible, such that between any two objects there is at least one morphism. The morphisms from an object to itself form a group (the group associated to the groupoid).

Remark. Suppose that we have an action of a group G on a set M. We associate a groupoid $\mathcal{G}(G, M)$ as follows: its objects are the G-orbits on M, and the morphisms are the G-orbits of the diagonal action on $M \times M$. If the initial action was free then G embeds in $\mathcal{G}(G, M)$ as the automorphisms group of any object. □

Assume that we have an ideal triangulation $\Delta(\Gamma)$ of a surface Σ_g^s. If e is an edge shared by the triangles Δ_v and Δ_w of the triangulation then we change the triangulation by excising the edge e and replacing it by the other diagonal of the quadrilateral $\Delta_v \cup \Delta_w$, as in figure 4.1. This operation $F[e]$ was called flip in [7] or elementary by Mosher and Penner.

Let $\mathcal{IT}(\Sigma_g^s))$ denote the set of isotopy classes of ideal triangulations of Σ_g^s. The reduced *Ptolemy groupoid* $\overline{P_g^s}$ is the groupoid generated by the flips action on $\mathcal{IT}(\Sigma_g^s))$. Specifically its elements are classes of sequences $\Delta_0, \Delta_1, ..., \Delta_m$, where Δ_{j+1} is obtained from Δ_j by using a flip. Two sequences $\Delta_0, ..., \Delta_m$ and $\Delta'_0, ..., \Delta'_n$ are equivalent if their initial and final terms coincide i.e. there exists a homeomorphism φ preserving the punctures such that $\varphi(\Delta_0) \cong \Delta'_0$ and $\varphi(\Delta_m) \cong \Delta'_n$, where $\cong$ denotes the isotopy equivalence. Notice that any two (isotopy classes of) ideal triangulations are connected by a chain of flips (see [15] for an elementary proof), and hence $\overline{P_g^s}$ is indeed an groupoid. Moreover $\overline{P_g^s}$ is the groupoid $\mathcal{G}(\mathcal{M}_g^s, \mathcal{IT}(\Sigma_g^s))$ associated to the obvious action of the mapping class group $\mathcal{M}_g^s$ on the set of isotopy classes of ideal triangulations $\mathcal{IT}(\Sigma_g^s))$. One problem in considering $\overline{P_g^s}$ is that the action of $\mathcal{M}_g^s$ on $\mathcal{IT}(\Sigma_g^s))$ is *not free* but there is a simple way to remedy it. For instance in [20, 21] one adds the extra structure coming from fixing an *oriented arc* of the ideal triangulation. A second problem is that we want that the mapping class group action on the Teichmüller space extends to a

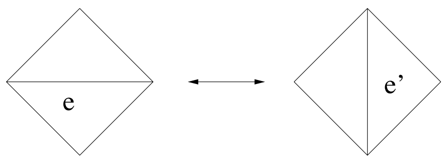


Figure 4.1: The flip

groupoid action.

Consider now an ideal triangulation $\Delta = \Delta(\Gamma)$, where Γ is its dual fatgraph. A labelling of Δ is a numerotation of its edges $\sigma_\Gamma : E_\Gamma \to \{1, 2, ..., \sharp E_\Gamma\}$. Set now $\mathcal{LIT}(\Sigma_g^s))$ for the set of labeled ideal triangulations. The Ptolemy groupoid P_g^s of the punctured surface Σ_g^s is the groupoid generated by flips on $\mathcal{LIT}(\Sigma_g^s))$. The flip $F[e]$ associated to the edge $e \in E_\Gamma$ acts on the labellings in the obvious way:

$$\sigma_{F[e](\Gamma)}(f) = \begin{cases} \sigma_\Gamma(f), & \text{if } f \neq e' = F[e](e) \\ \sigma_\Gamma(e), & \text{if } f = e', \end{cases}$$

According to ([25] Lemma 1.2.b), if $2g - 2 + s \geq 2$ then any two labeled ideal triangulations are connected by a chain of flips, and thus P_g^s is indeed a groupoid. Moreover, this allows us to identify P_g^s with $\mathcal{G}(\mathcal{M}_g^s, \mathcal{LIT}(\Sigma_g^s))$.
Remark. In the remaining cases, namely Σ_0^3 and Σ_1^1, the flips are not acting transitively on the set of labeled ideal triangulations. In this situation an appropriate labelling consist in an oriented arc, as in [20]. The Ptolemy groupoid associate to this labeling has the right properties, and it acts on the Teichmüller space. □

Proposition 4.1.1 *We have an exact sequence*

$$1 \to \mathcal{S}_{6g-6+3s} \to P_g^s \to \overline{P_g^s} \to 1,$$

where $\mathcal{S}_n$ denotes the symmetric group on n letters. Notice that $P_1^1 = \overline{P_1}^1$. If $(g, s) \neq (1, 1)$ then $\mathcal{M}_g^s$ naturally embeds in P_g^s as the group associated to the groupoid.

Proof. The first part is obvious. The following result is due to Penner ([25],Thm.1.3):

Lemma 4.1.1 *If $(g, s) \neq (1, 1)$ then $\mathcal{M}_g^s$ acts freely on $\mathcal{LIT}(\Sigma_g^s)$.*

Proof. A homeomorphism keeping invariant a labeled ideal triangulation either preserves the orientation of each arc or else it reverses the orientation of all arcs. In fact once the orientation of an arc lying in some triangle is preserved, the orientation of the other boundary arcs of the triangle must also be preserved. Further in the first situation either the surface is Σ_0^3 (when $\mathcal{M}_0^3 = 1$) or else each triangle is determined by its 1-skeleton, and the Alexander trick shows that the homeomorphism is isotopic to identity. In the second case we have to prove that $(g, s) = (1, 1)$. Since the arcs cannot have distinct endpoints we have $s = 1$. Let Δ_1 be an oriented triangle and $D \subset \Delta_1$ be a 2-disk which is a slight retraction of Δ_1 into its interior. The image D' of D cannot lie within Δ_1 because the homeomorphism is globally orientation preserving while the orientation of the boundary of D' is opposite to that of $\partial\Delta$. Thus D' lies outside Δ_1 and the region between $\partial D'$ and $\partial\Delta_1$ is an annulus, so the complementary of Δ_1 consists of one triangle. Therefore $g = 1$. □
Remark. The punctured torus Σ_1^1 has an automorphism which reverse the orientation of each of the three ideal arcs. □
□

The case of the punctured torus is settled by the following:

Proposition 4.1.2 *Let $\Delta_{st} = \{\alpha_1, \alpha_2, \alpha_3\}$, where $\alpha_1 = (1, 0), \alpha_2 = (1, 1), \alpha_3 = (0, 1)$ be the standard labeled ideal triangulation of the punctured torus $\Sigma_1^1 = \mathbb{R}^2/\mathbb{Z}^2 - \{0\}$.*

1. *If $\Delta = \{\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \alpha_{\sigma(3)}\}$ is flip equivalent to Δ_{st} then σ is the identity.*
2. *A mapping class which leaves invariant Δ_{st} is either identity or $-id \in SL(2, \mathbb{Z}) = \mathcal{M}_1^1$.*
3. *Let $\Delta = \{\gamma_1, \gamma_2, \gamma_3\}$ be an arbitrary ideal triangulation. Then there exists an unique $\sigma(\Delta) \in \mathcal{S}_3$ such that Δ is flip equivalent with the labeled diagram $\{\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \alpha_{\sigma(3)}\}$.*
4. *In particular if $\Delta = \varphi(\Delta_{st})$ then we obtain a group homomorphism $\sigma : SL(2, \mathbb{Z}) \to \mathcal{S}_3$, given by $\sigma(\varphi) = \sigma(\varphi(\Delta_{st})$, whose values can be computed from:*

$$\sigma\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = (23), \quad \sigma\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = (12), \quad \sigma\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = (13).$$

We need therefore another labeling for Σ_1^1, which amounts to fix a *distinguished oriented edge (d.o.e.)* of the triangulation. The objects acted upon flips are therefore pairs (Δ, e), where e is the d.o.e. of Δ. A flip acts on the set of labeled ideal triangulations with d.o.e. as follows. If the flip leaves e invariant then the new d.o.e. is the old one. Otherwise the flip under consideration is $F[e]$, and the new d.o.e. will be the image e' of e, oriented so that the frame (e, e') at their intersection point is positive with respect to the surface orientation. The groupoid Pt_g^s generated by flips on (labeled) ideal triangulations with d.o.e. of is called the *extended Ptolemy groupoid*. Since any edge permutation is a product of flips (when $(g, s) \neq 1$) it follows that any two labeled triangulations with d.o.e. can be connected by a chain of flips.

The case of the punctured torus is subjected to caution again: it is more convenient to define the groupoid Pt_1^1 as that generated by iterated compositions of flips on the standard (labeled or not) ideal triangulation Δ_{st} of Σ_1^1 with a fixed d.o.e., for instance α_1. In fact proposition 4.1.2 implies that there are three distinct orbits of the flips on triangulations with d.o.e., according to the the position of the d.o.e. within Δ_{st}.

Remark. For all (g, s) we have an exact sequence:

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \mathsf{Pt}_g^s \to P_g^s \to 1.$$

Moreover $\mathcal{M}_g^s \to P_g^s$ lifts to an embedding $\mathcal{M}_g^s \hookrightarrow \mathsf{Pt}_g^s$. □

Remark. We can define the groupoid $\overline{\mathsf{Pt}_g^s}$ by considering flips on ideal triangulations with d.o.e. without labellings. □

Remark. The kernel of the map $\mathcal{M}_1^1 \to P_1^1$ is the group of order two generated by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Therefore any (faithful) representation of P_1^1 induces a (faithful) representation of $\mathrm{PSL}(2, \mathbb{Z})$. □

Remark. One reason to consider P_g^s instead of $\overline{P_g^s}$ is that P_g^s acts on the Teichmüller space while $\overline{P_g^s}$ does not. The other reason is that $\mathcal{M}_g^s$ injects into P_g^s (if $(g, s) \neq (1, 1)$). The kernel of $\mathcal{M}_g^s \to \overline{P_g^s}$ is the image of the automorphism group $Aut(\Gamma)$ in $\mathcal{M}_g^s$. □

Proof. An automorphism of Γ is a combinatorial automorphism which preserves the cyclic orientation at each vertex. Notice that an element of $Aut(\Gamma)$ induces a homeomorphism of Γ^t and hence an element of $\mathcal{M}_g^s$. Now, if φ is in the kernel then φ is described by a permutation of the edges i.e. an element of $\varphi_* \in \mathcal{S}_{\sharp E_\Gamma}$. One can assume that the orientations of all arcs are preserved by φ when $(g, s) \neq (1, 1)$. Then φ_* completely determines φ, by the Alexander trick. Further φ induces an element of $Aut(\Gamma)$ whose image in $\mathcal{S}_{\sharp E_\Gamma}$ is precisely φ_*. This establishes the claim. Notice that the map $Aut(\Gamma) \to \mathcal{S}_{\sharp E_\Gamma}$ is injective for most but not for all fatgraphs Γ. The fatgraphs Γ for which the map $Aut(\Gamma) \to \mathcal{S}_{\sharp E_\Gamma}$ fails to be injective are described in [22]. □

We can state now a presentation for Pt_g^s which is basically due to Penner ([25]):

Proposition 4.1.3 Pt_g^s *is generated by the flips* $F[e]$ *on the edges. The relations are:*

1. *Set* J *for the change of orientation of the d.o.e. Then*

$$F[F[e]e]F[e] = \begin{cases} 1, & \text{if } e \text{ is not the d.o.e.} \\ J, & \text{if } e \text{ is the d.o.e.} \end{cases}$$

2. $J^2 = 1$.

3. *Consider the pentagon from picture 4.2, and* $F[e_j]$ *be the flips on the dotted edges. Let* $\tau_{(12)}$ *denote the transposition interchanging the labels of the two edges* e_1 *and* f_1 *from the initial triangulation. Then we have:*

$$F[e_1]F[e_2]F[e_3]F[e_4]F[e_5] = \begin{cases} J\tau_{(12)}, & \text{if } e_1 \text{ is not the d.o.e.} \\ \tau_{(12)}, & \text{if } e_1 \text{ is the d.o.e.} \end{cases}$$

   The action of $\tau_{(12)}$ *on triangulations with d.o.e. is at follows: if none of the permuted edges* e, f *is the d.o.e. then* $\tau_{(12)}$ *leaves the d.o.e. unchanged. If the d.o.e. is one of the permuted edges, say* e*, then the new d.o.e. is* f *oriented such that* e *(with the former d.o.e. orientation) and* f *with the given d.o.e. orientation form a positive frame on the surface. Notice that* $[F[e_1]F[e_2]F[e_3]F[e_4]F[e_5] = \tau_{(12)}$ *even if* f_1 *is the d.o.e.*

4. *If* e *and* f *are disjoint edges then* $F[e]F[f] = F[f]F[e]$.

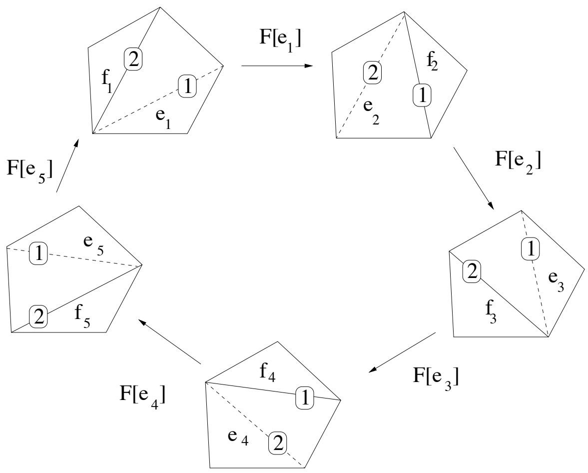


Figure 4.2: The pentagon relation

5. *The relations in a* $\mathbb{Z}/2\mathbb{Z}$ *extension of the symmetric group, expressed in terms of flips. To be more specific, les us assume that the edges are labeled and the d.o.e. is labelled* 0*. Then we have:*

$$\tau_{(0i)}^2 = J,\ \tau_{(ij)}^2 = 1,\ \textit{if } i,j \neq 0,\ \tau_{(st)}\tau_{(mn)} = \tau_{(mn)}\tau_{(st)}\ \textit{if } \{m,n\} \cap \{s,t\} = \emptyset,$$

$$\tau_{(st)}\tau_{(tv)}\tau_{(st)} = \tau_{(tv)}\tau_{(st)}\tau_{(tv)},\ \textit{if } s,t,v \textit{ are distinct.}$$

6. $F[\tau(e)]\tau F[e] = \tau$*, for any label transposition* τ *(expressed as a product of flips as above), which says that the symmetric group is a normal subgroupoid of* P_g^s*.*

Proof. We analyze first the case where labellings are absent:

Lemma 4.1.2 $\overline{P_g^s}$ *is generated by the flips on edges* $F[e]$*. The relations are:*

1. $F[e]^2 = 1$*, which is a fancy way to write that the composition of the flip on* $F[e](e)$ *with the flip on e is trivial.*

2. $F[e_1]F[e_2]F[e_3]F[e_4]F[e_5] = 1$*, where* $F[e_i]$ *are the flips considered in the picture 4.2.*

3. *Flips on two disjoint edges commute each other.*

Proof. This result is due to Harer (see [14]). It was further exploited by Penner ([24, 25]). □

The complete presentation is now a consequence of the two exact sequences from proposition 4.1.1 and remark 4.1.1, relating $\overline{P_g^s}$, P_g^s and P_g^s. □

Remark. By setting $J = 1$ above we find the presentation of P_g^s, with which we will be mostly concerned in the sequel. □

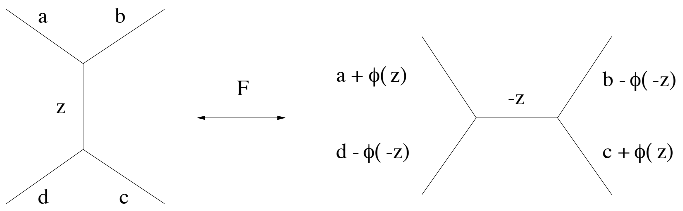
4.1.2 The mapping class group action on the Teichmüller spaces

In order to understand the action on $\mathcal{T}_g^s$ we to consider also $\mathcal{T}_{g,s;or}$.

The action of $\mathcal{M}_g^s$ on the Teichmüller space extends to an action of P_g^s to $\mathcal{T}_g^s$. Geometrically we can see it as follows. An element of $\mathcal{T}_g^s$ is a marked hyperbolic surface S. The marking comes from an ideal triangulation. If we change the triangulation by a flip, and keep the hyperbolic metric we obtain another element of $\mathcal{T}_g^s$.

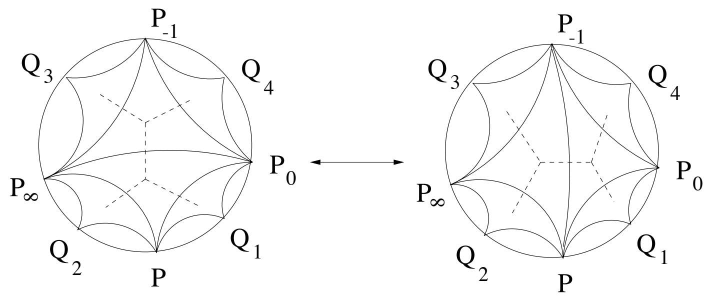
In the same way the $\mathcal{M}_{g,s}$ action on the Teichmüller space $\mathcal{T}_{g,s;or}$ extends to an action of the Ptolemy groupoid $P_{g,s}$. This action is very easy to understand in terms of coordinates. In more specific terms a flip between the graphs Γ and Γ' induces an analytic isomorphism $\mathbb{R}^{E_\Gamma} \to \mathbb{R}^{E_{\Gamma'}}$ by intertwining the coordinate systems t_Γ and $t_{\Gamma'}$. It is more convenient to identify $\mathbb{R}^{E_\Gamma}$ with a fixed Euclidean space, which is done by choosing a labelling $\sigma : E_\Gamma \to \{1, 2, ...\sharp E_\Gamma\}$ of its edges. Thus we have homeomorphism $\mathbf{t}_{\Gamma,\sigma} : \mathcal{T}_{g,s;or} \to \mathbb{R}^{\sharp E_\Gamma}$ given by $(\mathbf{t}_{\Gamma,\sigma}(S))_k = (\mathbf{t}_\Gamma(S))_{\sigma^{-1}(k)\in E_\Gamma}$. Further we can compare the coordinates $\mathbf{t}_{\Gamma,\sigma}$ and $\mathbf{t}_{F(\Gamma,\sigma)}$, for two labelled fatgraphs which are related by a flip. We can state:

Proposition 4.1.4 *A flip acts on the edge coordinates of a fatgraph as follows:*



where $\phi(z) = \log(1 + e^z)$. Here it is understood that the coordinates associated to the edges not appearing in the picture remain unchanged.

Proof. The flip on the graph corresponds to the following flip of ideal triangulations:



Then the coordinates a, b, c, d, z using the left-hand-side graph are the following cross-ratios: $a = [Q_3, P_\infty, P_0, P_{-1}]$, $b = [Q_4, P_{-1}P_\infty, P_0]$, $c = [Q_1, P_0, P_\infty, P]$, $d = [Q_2, P, P_0, P_\infty]$, $z = [P_{-1}, P_\infty, P, P_0]$. Let a', b', c', d', z' be the coordinates associated to the respective edges from the right-hand-side graph, which can again be expressed as cross-ratios as follows: $a' = [Q_3, P_\infty, P, P_{-1}]$, $b' = [Q_4, P_{-1}, P, P_0]$, $c = [Q_1, P_0, P_{-1}, P]$, $d = [Q_2, P, P_{-1}, P_\infty]$, $z = [P_\infty, P, P_0, P_{-1}]$. One uses for simplifying computations the half-plane model where, up to a Möbius transformation, the points P_{-1}, P_∞, P, P_0 are sent respectively into $-1, \infty, e^z$ and 0. The flip formulas follow immediately. □

Remark. Similar computations hold for Penner's λ-coordinates on the decorated Teichmüller spaces. However the transformations of $\mathbf{R}^{6g-6+2s}$ obtained using λ-coordinates are *rational* functions. □

Let us denote by $Aut^\omega(\mathbb{R}^m)$ the group of real analytic automorphisms of $\mathbb{R}^m$.

Corollary 4.1.1 *1. We have a faithful representation $\rho : \mathcal{M}_{g,s} \to Aut^\omega(\mathbb{R}^{6g-6+3s})$ induced by the $P_{g,s}$ action on the Teichmüller space $\mathcal{T}_{g,s;or}$ if $(g, s) \neq (1, 1)$.*

2. The groupoid $P_g^s \subset P_{g,s}$ leaves invariant the Teichmüller subspace $\mathcal{T}_g^s \subset \mathcal{T}_{g,s;or}$. Therefore the formula given in proposition 4.1.4 above for the flip actually yields a representation of P_g^s into $Aut^\omega(\mathbb{R}^{6g-6+2s})$. The restriction to the mapping class groups is a faithful representation $\rho : \mathcal{M}_g^s \to Aut^\omega(\mathbb{R}^{6g-6+2s})$ if $(g, s) \neq (1, 1)$, and a faithful representation of $\mathrm{PSL}(2, \mathbb{R})$ *when $(g, s) = (1, 1)$.*

Proof. The representation of $\mathcal{M}_{g,s}$ (respectively $\mathcal{M}_g^s$) is injective because the mapping class group acts effectively on the Teichmüller space. Therefore if the class of any (marked) Riemann surface is preserved by a homeomorphism then this homeomorphism is isotopic to the identity.

The invariance of the subspace $\mathcal{T}_g^s \subset \mathcal{T}_{g,s;or}$ by flips is geometrically obvious, but we write it down algebraically for further use. This amounts to check that the linear equations $t_\gamma = 0$, for $\gamma \in F_\Gamma$ are preserved. Let γ be a left-hand-turn path, which intersects the part of the graph shown in the picture, say along the edges labeled a, z, b. Then the flip of γ intersects the new graph along the edges labeled by $a + \phi(z)$ and $b - \phi(-z)$. The claim follows from the equality $z = \phi(z) - \phi(-z)$. The remaining three cases reduces to the same equation. □

Remark. There is a Pt_g^s-action on the Teichmüller space but it is not free, and actually factors through P_g^s. □

Remark. Assume that there exists an element $\mathbf{r} \in \mathcal{T}_g^s$, which is fixed by *some* $\psi \in \mathcal{M}_g^s$, i.e. $\varphi(\psi)(\mathbf{r}) = \mathbf{r}$. Then $\mathbf{r}$ is contained in some codimension two analytic submanifold $Q_g^s \subset \mathcal{T}_g^s$, and for a given $\mathbf{r}$ its isotropy group is finite. This is a reformulation of the fact that $\mathcal{M}_g^s$ acts properly discontinuously on the Teichmüller space with finite isotropy groups corresponding to the Riemann surfaces with non-trivial automorphism groups (biholomorphic). Moreover the locus of Riemann surfaces with automorphisms is a proper complex subvariety of the Teichmüller space, corresponding to the singular locus of the moduli space of curves. □

4.1.3 Deformations of the mapping class group representations

We want to consider deformations of the tautological representation $\rho = \rho_0$ of $\mathcal{M}_g^s$ obtained in the previous section. We first restrict ourselves to deformations $\rho_h : \mathcal{M}_g^s \to Aut^\omega(\mathbb{R}^{6g-6+2s})$ satisfying the following requirements:

1. The deformation ρ_h extends to the Ptolemy groupoid P_g^s. In particular ρ_h is completely determined by $\rho_h(F)$ and $\rho_h(\tau_{(ij)})$.
2. The image of a permutation $\rho_h(\tau_{(ij)})$ is the automorphism of $\mathbb{R}^{6g-6+2s}$ given by the permutation matrix $P_{(ij)}$, which exchanges the i-th and j-th coordinates.
3. The image $F_h = \rho_h(F)$ of a flip has the same form as for $\rho_0(F)$, namely that given in the picture from proposition 4.1.4, but with a deformed function $\phi = \phi_h$, with $\phi_0 = \log(1 + e^z)$.
4. The linear subspace $\mathcal{T}_g^s \subset \mathcal{T}_{g,s;or}$ is invariant by ρ_h.

Proposition 4.1.5 *The real function $\phi : \mathbb{R} \to \mathbb{R}$ yield a deformation of the mapping class groups (respectively the Ptolemy groupoids) if and only if it satisfies the following functional equations:*

$$\phi(x) = \phi(-x) + x. \tag{4.1}$$

$$\phi(x + \phi(y)) = \phi(x + y - \phi(x)) + \phi(x). \tag{4.2}$$

$$\phi\left(\phi\left(x + \phi(y)\right) - y\right) = \phi(-y) + \phi(x). \tag{4.3}$$

Proof. The first equation is equivalent to the invariance of the linear equations defining the cusps. The other two equations follow from the cumbersome but straightforward computation of terms involved in the pentagon equation. □

4.1.4 Belyi Surfaces

Let S be a compact Riemann surface. It is well known that there exists a non-constant meromorphic function on S, $\phi : S \to \mathbb{CP}^1$.

Definition 4.1.1 *The Riemann surface S is a* Belyi surface *if there exists a ramified covering $\phi : S \to \mathbb{CP}^1$, branched over 0, 1 and ∞.*

A surprising theorem of Belyi ([3]) states that:

Theorem 4.1.1 *S is a Belyi surface if and only if it is defined over $\overline{\mathbb{Q}}$ i.e. as a curve in $\mathbb{CP}^2$ its minimal polynomial lies over some number field.*

Following [24, 22] we can characterize Belyi surfaces in terms of fat graphs as follows:

Theorem 4.1.2 *A Riemann surface S can be constructed as $S(\Gamma) = S(\Gamma, \mathbf{0})$ for some trivalent fatgraph Γ if and only if S is a Belyi surface.*

Proof.
We prove first:

Lemma 4.1.3 *Let $G \subset \mathrm{PSL}(2,\mathbb{Z})$ be a finite index torsion-free subgroup. Then $\mathbb{H}^2/G = S(\Gamma)$ for some trivalent fatgraph Γ.*

Proof. Remark that $A = \{z \in \mathbb{H}^2; 0 < \Re(z) < 1, |z| > 1, |z-1| > 1\}$, is a fundamental domain for $\mathrm{PSL}(2,\mathbb{Z})$, with the property that three copies of it around $\omega + 1$ fit together to give the ideal marked triangle. These three copies are equivalent by means of an order three elliptic element γ of $\mathrm{PSL}(2,\mathbb{Z})$.

A fundamental domain for G is composed of copies of A, and since G is torsion free the three copies A, $\gamma(A)$ and $\gamma^2(A)$ are not equivalent under G, thus they can all be included in the fundamental domain for G. In particular it exists a fundamental domain B for G which is made of copies of the ideal triangle I and hence it is naturally triangulated. Consider the graph Γ dual to this triangulation, which takes into account the boundary pairings, and which inherits an orientation from $\mathbb{H}^2/G$. Then $\mathbb{H}^2/G = S(\Gamma)$. □

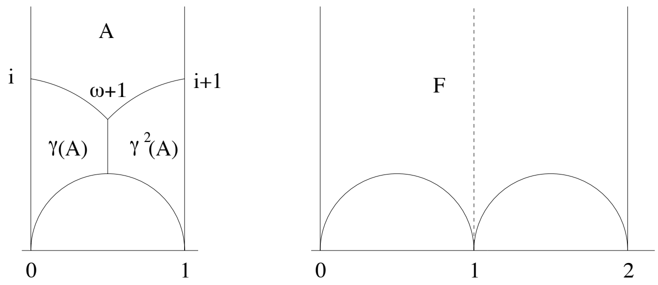


Figure 4.3: Fundamental domains for $\mathrm{PSL}(2,\mathbb{Z})$ and $\Gamma(2)$

Lemma 4.1.4 *S is a Belyi surface if and only if we can find finitely many points on S, $\{p_1,\dots,p_k\}$, such that $S-\{p_1,\dots,p_k\}$ is isomorphic to $\mathbb{H}^2/G$, where G is a finite index torsion free subgroup of* $\mathrm{PSL}(2,\mathbb{Z})$.

Proof. Set $\Gamma(2)=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \mathrm{PSL}(2,\mathbb{Z}) \mid \begin{pmatrix} a & b\\ c & d\end{pmatrix}\equiv\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix} \pmod 2\right\}$. Then $F=\{z\in\mathbb{H}^2; 0<\Re(z)<2, |z-1/2|>\frac{1}{2}, |z-3/2|>\frac{1}{2}\}$ is a fundamental domain for $\Gamma(2)$ composed of 2 ideal triangles glued along a common edge. Thus the 3-punctured sphere $\mathbb{CP}^1-\{0,1,\infty\}$ is $\mathbb{H}^2/\Gamma(2)$. Moreover each ideal triangle is composed of three copies of the fundamental domain of $\mathrm{PSL}(2,\mathbb{Z})$. Therefore, the 3-punctured sphere is a six-fold branched covering of $\mathbb{H}^2/\mathrm{PSL}(2,\mathbb{Z})$.

Let S be a compact Riemann surface. If S is a Belyi surface then $S-\{p_1,\dots,p_k\}$ is a regular smooth finite degree covering of $\mathbb{H}^2/\Gamma(2)$ and thus $S-\{p_1,\dots,p_k\}=\mathbb{H}^2/G$, where G is a finite-index subgroup of $\Gamma(2)$ (and hence of $\mathrm{PSL}(2,\mathbb{Z})$).

Conversely, if $S-\{p_1,\dots,p_k\}=\mathbb{H}^2/G$, where G is a finite index torsion free subgroup of $\mathrm{PSL}(2,\mathbb{Z})$, then $S-\{p_1,\dots,p_k\}$ is a finite-degree branched covering of $\mathbb{H}^2/\mathrm{PSL}(2,\mathbb{Z})$, which is a sphere with one cusp and 2 ramification points. Therefore, if we remove the 2 ramification points and their pre-images, we get that $S-\{p_1,\dots,p_k,\dots,p_n\}$ is a regular smooth finite-degree covering of the 3-punctured sphere, i.e. a Belyi surface. □

These lemmas show that any Belyi surface can be constructed out of some fatgraph.

Conversely the fundamental polygon constructed for $G(\Gamma)$ is composed of copies of the ideal triangle. By decomposing each ideal triangle into three copies of the fundamental domain for $\mathrm{PSL}(2,\mathbb{Z})$, we see that $G(\Gamma)$ can be embedded as a finite-index torsion free subgroup of $\mathrm{PSL}(2,\mathbb{Z})$. □

4.2 The geometry of the Teichmüller space

4.2.1 Symplectic structures for the Teichmüller space of punctured surfaces

The Teichmüller space $\mathcal{T}_g^s$ has a natural structure of complex manifold. Let us recall some of its features. Suppose that the Riemann surface S is uniformized by the Fuchsian group $G\subset \mathrm{PSL}(2,\mathbb{R})$.

One considers first the vector space $Q(S)=Q(G)$ of integrable holomorphic quadratic differentials on S. An element $\varphi\in Q(S)$ is a holomorphic function $\varphi(z)$ on $\mathbb{H}^2$ satisfying $\varphi(\gamma(z))\gamma'(z)^2=\varphi(z)$ for all $\gamma\in G$, and $\int_F|\varphi|$ is finite, where F is a fundamental domain for G. Then φ induces a symmetric tensor of type $(2,0)$ on S.

Let then $M(S)$ be the space of G-invariant Beltrami differentials. These are measurable, essentially bounded functions $\mu:\mathbb{H}^2\to\mathbb{C}$ satisfying $\mu(\gamma(z))\frac{\overline{\gamma'(z)}}{\gamma'(z)}=\mu(z)$ for all $\gamma\in G$, and hence define a $(-1,1)$ tensor on S.

There is a natural pairing $(,):M(S)\times Q(S)\to\mathbb{C}$ given by $(\mu,\varphi)=\int_F\mu\varphi$, with null space $N(S)\subset M(S)$ which induces a duality isomorphism between $M(S)/N(S)$ and $Q(S)$.

The holomorphic cotangent space at the point $[S]\in\mathcal{T}_g^s$ is identified with $Q(S)$ and thus the tangent space is naturally isomorphic to $M(S)/N(S)$. Weil introduced a hermitian product on $Q(S)$ defined in terms of the

Petersson product for automorphic forms. This yields the Weil-Petersson (co)metric on $\mathcal{T}_g^s$:

$$< \varphi, \psi >= \frac{1}{2}\mathsf{Re}\int_{\mathbb{H}^2/G} \varphi\overline{\psi}\,(\mathsf{Im}z)^{-2}\,, \quad \text{for } \varphi, \psi \in Q(S).$$

Remark. The Weil-Petersson metric is Kähler, it has negative holomorphic sectional curvature and is invariant under the action of the mapping class group. □

The Kähler form of the Weil-Petersson metric is a symplectic form ω_{WP}. In the case of closed surfaces Wolpert ([34]) derived a convenient expression for ω_{WP} in terms of Fenchel-Nielsen coordinates:

$$\omega_{WP} = -\sum_j d\,\tau_j \wedge d\,l_j.$$

Recall that a pair of pants $\Sigma_{0,3}$ has a hyperbolic structure with geodesic boundary. The lengths $l_j \in \mathbb{R}_+$ of the boundary circles can be arbitrarily prescribed. To each decomposition of S into pairs of pants $P_1, ..., P_{2g-2}$ we have therefore associated the lengths of their boundary geodesics $l_1, ..., l_{3g-3}$. In fact given pairs of pants, not necessarily distinct, P_1 and P_2 with boundary circles c_1 on P_1 and c_2 on P_2, of the same length we can glue the pants by identifying c_1 with c_2 by an isometry. The hyperbolic metric extends over the connected sum. Therefore we can glue together the pants $P_1, ..., P_{2g-2}$ to obtain the Riemann surface S. If a length $l = 0$ then this corresponds to the situation where the surface has a cusps. We can therefore extend this description to punctured surfaces Σ_g^n with cusps at punctures. The pants decomposition is specified therefore by $3g - 3 + n$ geodesics on S. Each boundary circle c belongs to two pairs of pants P_j and P_k. The geodesics joining the circles of P_j to the circles of P_k intersect c into two points. The parameter τ_j is the (signed) hyperbolic distance between these two points. The parameters (τ_j, l_j) are the Fenchel Nielsen coordinates on $\mathcal{T}_g^s$.

Fricke and Klein established that, if one carefully choose the curves $\gamma_1, ..., \gamma_{6g-6+2n}$ then the associated lengths l_j can also give local coordinates on $\mathcal{T}_g^s$. A typical example is to pick up first the curves $\gamma_1, ..., \gamma_{3g-3+n}$ arising from a pants decomposition, and then a dual pants decomposition obtained as follows. Consider the pieces of geodesics which yield the canonical points on the circles, and then identify combinatorially the canonical points. We obtained this way a family of closed loops $\gamma_{3g-3+n+1}, ..., \gamma_{6g-6+2n}$. Wolpert ([35],Lemma 4.2, 4.5) expressed the Kähler form in these coordinates:

Lemma 4.2.1 *Assume that $l_1, ..., l_{6g-6+2n}$ provide local coordinates on $\mathcal{T}_g^s$ and denote:*

$$\alpha^{jk} = \sum_{p\in\gamma_j\cap\gamma_k} \cos\theta_p,$$

where θ_p is the angle between the geodesic γ_j and γ_k at the point p. Let $W = (w_{jk})_{j,k}$ be the inverse of the matrix $A = (\alpha^{jk})_{j,k}$. Then the Weil-Petersson form is:

$$\omega_{WP} = -\sum_{j<k} w_{jk}\,d\,l_j \wedge d\,l_k.$$

4.2.2 Poisson structure for the Teichmüller space of surfaces with boundary

Let G be a connected Lie group, which will be most of the time $\mathrm{PSL}(2,\mathbb{R})$ in this section. Set $M(\Sigma, G) = \mathrm{Hom}(\pi_1(\Sigma), G)/G$ for the moduli space of representations of the fundamental groups.

Goldman ([10]) proved that $M(\Sigma, G)$ is endowed with a natural symplectic structure, whenever Σ is a closed oriented surface. Moreover Fock and Rosly ([8]) was able to show more generally that there is a Poisson structure on $M(\Sigma, G)$, even in the case when Σ is a surface with boundary. Furthermore the symplectic leaves of this structures are precisely the singular submanifolds $M(\Sigma, G)_{\lambda_1,...,\lambda_s}$, where λ_j is the conjugacy class of the holonomy around the j-th boundary component.

Notice also that Zocca have shown that $M(\Sigma, G)$ has a pre-symplectic structure, whose restriction to the symplectic leaves is the symplectic form.

4.2.3 Penner's decorated Teichmüller space

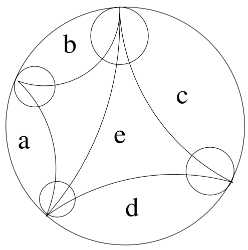
Penner ([24, 25] considered the space $\widetilde{\mathcal{T}}_g^s$ of cusped Riemann surfaces endowed with a horocycle around each puncture, and called it the decorated Teichmüller space. There is a natural family of coordinates (called lambda lengths), associated to the edges of an ideal triangulation $\Delta = \Delta(\Gamma)$ of the surface. For each such edge e one puts $\lambda_e = \sqrt{2exp(\delta)}$, where δ is the signed hyperbolic distance between the two horocycles centered at the two endpoints of the edge e. The sign convention is that $\lambda_e > 0$ if the horocycles are disjoint. It is not difficult to see that these coordinates give a homeomorphism $\widetilde{\mathcal{T}}_g^s \to \mathbb{R}^{6g-6+3s}$. The map which forgets the horocycles $\pi : \widetilde{\mathcal{T}}_g^s \to \mathcal{T}_g^s$ is a fibration having $\mathbb{R}_+^s$ as fibers. Moreover:

Lemma 4.2.2 *The projection π is expressed in terms of Penner and Fock coordinates as follows:*

$$\pi\left((\lambda_e)_{e\in\Delta(\Gamma)}\right) = \left(\log\frac{\lambda_a\lambda_c}{\lambda_b\lambda_d}\right)_{e\in\Delta(\Gamma)},$$

where, for each edge e we considered the quadrilateral of edges a, b, c, d, uniquely determined by the following properties:

- *the cyclic order a, b, c, d is consistent with the orientation of Σ_g^s.*
- *e is the diagonal separating a, b from c, d (see the figure 4.2.2).*
- *each triangle of Δ has an orientation inherited from Σ_g^s, in particular the edge e is naturally oriented. We ask that a (and d) be adjacent to the startpoint of e, while b and c is adjacent to the endpoint of e.*



Proof. The proof is a mere calculation. □

Proposition 4.2.1 *The pull-back $\pi^*\omega_{WP}$ of the Weil-Petersson form on the decorated Teichmüller space $\widetilde{\mathcal{T}}_g^s$ is given in Penner's coordinates as:*

$$\pi^*\omega_{WP} = -2\sum_{T\subset\Delta} d\log\lambda_a \wedge d\log\lambda_b + d\log\lambda_b \wedge d\log\lambda_c + d\log\lambda_c \wedge d\log\lambda_a,$$

where the sum is over all triangles T in Δ whose edges have lambda lengths a, b, c in the cyclic order determined by the orientation of Σ_g^s.

Proof. See [26], Appendix A. □
Remark. For dimensional reasons the pre-symplectic form $\pi^*\omega_{WP}$ is degenerate. □

Proposition 4.2.2 *The Poisson structure on $\mathcal{T}_{g,s;or}$ is given by the following formula in the Fock coordinates (t_e):*

$$P_{WP} = \sum_{T\subset\Delta} dt_a \wedge dt_b + dt_b \wedge dt_c + dt_c \wedge dt_a,$$

where the sum is over all triangles T in Δ whose edges are a, b, c in the cyclic order determined by the orientation of Σ_g^s. This Poisson structure is degenerate. Moreover $\mathcal{T}_g^s \subset \mathcal{T}_{g,s;or}$ is a symplectic leaf and hence the restriction of P_{WP} is the Poisson structure dual to the Weil-Petersson symplectic form ω_{WP}.

4.3 Teichmüller spaces after Thurston

1. Denote by $\mathcal{S}(\Sigma)$ *the set of isotopy classes of simple closed curves on* Σ. Consider the geometric intersection number $\iota(\alpha,\beta)$ for $\alpha,\beta \in \mathcal{S}(\Sigma)$ which is the minimum number of intersection points of two curves in the respective isotopy classes.

Proposition 4.3.1 *1. For every $\alpha \in \mathcal{S}$ there exists $\beta \in \mathcal{S}$ such that $\iota(\alpha,\beta) \neq 0$.*

2. If $\alpha_1 \neq \alpha_2 \in \mathcal{S}$ then there exists $\beta \in \mathcal{S}$ such that $\iota(\alpha_1,\beta) \neq \iota(\alpha_2,\beta)$.

The map $\iota : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ induces $\iota_* : \mathcal{S} \to \mathbb{R}^{\mathcal{S}}$; let

$$P : \mathbb{R}^{\mathcal{S}} - \{0\} \longrightarrow \mathbb{P}(\mathbb{R}^{\mathcal{S}})$$

be the quotient map defining the projectivisation. Since $\iota_*(\mathcal{S}) \subset \mathbb{R}^{\mathcal{S}} \setminus \{0\}$ it makes sense to define:

$$\mathcal{PS}(\Sigma) = \text{ closure of } (P(\iota_*(\mathcal{S}))) \subset \mathbb{P}(\mathbb{R}^{\mathcal{S}})$$

Proposition 4.3.2 *$\mathcal{PS}(\Sigma)$ is homeomorphic to a sphere $S^{6g+2b-7}$, where Σ is of genus g with b boundary components, provided that $\chi(\Sigma) < 0$.*

Remark.
If $\mathcal{S}'(\Sigma)$ denotes the set of isotopy classes of non-empty unions of disjoint simple loops in Σ (none bounding a disk or isotopic to γM) then the same construction yields $\mathcal{PS}'(\Sigma)$. Moreover $\mathcal{PS}'(\Sigma) = \mathcal{PS}(\Sigma)$. □

2. We identify points of $\mathcal{T}(\Sigma)$ with marked hyperbolic structures on Σ. Define the map

$$\iota : \mathcal{T}(\Sigma) \times \mathcal{S}(\Sigma) \to \mathbb{R}_+$$

where $\iota(\mathcal{T},\alpha)$ states for the infimum length of a simple closed curve $a \subset \Sigma$ where a represents α and Σ has metric τ. It is known that a is the unique closed geodesic representing α. Moreover two hyperbolic metrics τ_1, τ_2 are isotopic if

$$\iota(\tau_1,\alpha) = \iota(\tau_2,\alpha) \text{ for all } \alpha \in \mathcal{S}\ .$$

We obtain then a map $\iota_* : \mathcal{T}(\Sigma) \to \mathbb{R}^{\mathcal{S}} \setminus \{0\}$ and by projecting onto $P(\mathbb{R}^{\mathcal{S}} \setminus \{0\})$ we set

$$\overline{\mathcal{T}}(\Sigma) = \text{ closure of } (P \circ \iota_*(\mathcal{T}(\Sigma))) \subset \mathbb{P}(\mathbb{R}^{\mathcal{S}})\ .$$

Proposition 4.3.3 (W.Thurston) $\overline{\mathcal{T}}(\Sigma) = P \circ \iota_*(\mathcal{T}(\Sigma)) \cup \mathcal{PS}(\Sigma)$, *and with this topology* $\overline{\mathcal{T}}(\Sigma) \cong D^{6g-6+2b}$, *such that $\mathcal{PS}(\Sigma)$ corresponds to $S^{6g-7+2b}$. Thus $\overline{\mathcal{T}}(\Sigma)$ is a compactification of $\mathcal{T}(\Sigma)$.*

Remark. M.F. Vignéras proved that there are non isometric Riemann surfaces with the same length spectrum. If one adds the marking by means of the conjugacy classes in $\pi_1(\Sigma)$ corresponding to the length of closed geodesics then the two negatively curved surfaces should be isometric (as shown by J.-P. Otal). □

Corollary 4.3.1 *The action of* $\mathrm{Mod}(\Sigma)$ extends continuously to $\overline{\mathcal{T}}(\Sigma)$. *In particular, if ϕ is a diffeomorphism of Σ then either*

1. *ϕ fixes an element of the Teichmüller space, or*
2. *ϕ fixes a projective class of a measurable foliation.*

Remarks.

1. $\mathcal{T}(\Sigma)$ has a smooth structure but $\mathcal{PS}(\Sigma)$ does not have (unless $g = 1$) a natural smooth structure. However $\mathcal{PS}(\Sigma)$ has a projective piecewise integral structure and $\mathrm{Mod}(\Sigma)$ acts by piecewise integral projective homeomorphisms on $\mathcal{PS}(\Sigma)$.
2. There exists an invariant measure on $\mathcal{PS}(\Sigma)$. However the $\mathrm{Mod}(\Sigma)$-action on $\mathcal{PS}(\Sigma)$ is *not* properly discontinuous, but an *ergodic* action, as proved by H.Masur.

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