Dupont-Guichardet-Wigner quasi-homomorphisms on mapping class groups

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Abstract

We compute a series of quasi-homomorphisms on mapping class groups extending previous work of Barge and Ghys in [3] and of Gambaudo and Ghys in [18]. These quasi-homomorphisms are pull-backs of the Dupont-Guichardet-Wigner quasi-homomorphisms on pseudo-unitary groups by quantum representations. As application we prove that either the images of the mapping class groups by quantum representations are not isomorphic to higher rank lattices or else the kernels have a large number of normal generators.

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1 Introduction and statements

The first purpose of this paper is to compute the quasi-homomorphisms on mapping class groups defined in [14], by extending previous work of Barge and Ghys in [3] and of Gambaudo and Ghys in [18]. Notice that Bestvina and Fujiwara proved in [2] that there are uncountably many distinct quasi-homomorphisms on the mapping class groups of genus $g \geq 2$. Our quasi-homomorphisms are constructed as pull-backs of Dupont-Guichardet-Wigner cocycles by quantum representations of mapping class groups $\mathcal{M}_g$ of oriented surfaces of genus $g \geq 2$ into pseudo-unitary groups. The second purpose is to use quasi-homomorphisms for obtaining information about the images of the mapping class groups.

1.1 Quantum representations

In [1], Blanchet, Habegger, Masbaum and Vogel defined the TQFT functor $\mathcal{V}_p$, for every $p \geq 3$ and a primitive root of unity A of order $2p$. These TQFT should correspond to the so-called $SU(2)$-TQFT, for even p and to the $SO(3)$-TQFT, for odd p (see also [27] for another $SO(3)$-TQFT). As it is known these TQFT determine and are determined by a series of representations of an extension of the mapping class groups $\mathcal{M}_g$.

Definition 1.1. *Let $p \in \mathbb{Z}_+$, $p \geq 3$ and A be a primitive $2p$-th root of unity. The quantum representation $\rho_{p,A}$ is the projective representation of the mapping class group associated to $\mathcal{V}_p$, the TQFT at the root of unity A. We denote therefore by $\widetilde{\rho}_{p,A}$ the linear representation of the central extension $\widetilde{\mathcal{M}_g}$ of the mapping class groups M_g (of the genus g closed orientable surface)* *which resolves the projective ambiguity of $\rho_{p,A}$ (see [19, 29]). Furthermore $N(g,p)$ will denote the dimension of the space of conformal blocks associated by the TQFT $\mathcal{V}_p$ to the closed orientable surface of genus g.*

Remark 1.1. The unitary TQFTs arising usually correspond to the following choices of the root of unity:

$$A_p = \begin{cases} -\exp\left(\frac{2\pi i}{2p}\right), & \text{if } p \equiv 0(\text{mod } 2); \\ -\exp\left(\frac{(p+1)\pi i}{p}\right), & \text{if } p \equiv 1(\text{mod } 2). \end{cases}$$

For $p \geq 5$ an odd prime we denote by $\mathcal{O}_p$ the ring of cyclotomic integers $\mathcal{O}_p = \mathbb{Z}[\zeta_p]$, if $p \equiv -1(\text{mod} 4)$ and $\mathcal{O}_p = \mathbb{Z}[\zeta_{4p}]$, if $p \equiv 1(\text{mod } 4)$ respectively, where ζ_p is a primitive p-th root of unity. The main result of [20] states that, for every odd prime $p \geq 5$, there exists a free $\mathcal{O}_p$-lattice $S_{g,p}$ in the $\mathbb{C}$-vector space of conformal blocks associated by the TQFT $\mathcal{V}_p$ to the genus g closed orientable surface and a non-degenerate Hermitian $\mathcal{O}_p$-valued form on $S_{g,p}$ such that (a central extension of) the mapping class group preserves $S_{g,p}$ and keeps invariant the Hermitian form. Therefore the image of the mapping class group consists of unitary matrices (with respect to the Hermitian form) with entries in $\mathcal{O}_p$. Let $P\mathbb{U}(\mathcal{O}_p)$ be the group of all such matrices, up to scalar multiplication. Throughout this paper we will consider, for the sake of simplicity, that $p \equiv -1(\text{mod } 4)$. Similar results hold for the other p.

It is known that $P\mathbb{U}(\mathcal{O}_p)$ is an irreducible lattice in a semi-simple Lie group $P\mathbb{G}_p$ obtained by the so-called restriction of scalars construction from the totally real cyclotomic field $\mathbb{Q}(\zeta_p + \zeta_p^{-1})$, ζ_p being a p-th root of unity, to $\mathbb{Q}$. Specifically, let us denote by $\mathbb{G}_p$ the product $\prod_{\sigma \in S(p)} S\mathbb{U}^\sigma$. Here $S(p)$ stands for a set of representatives for the classes of complex valuations σ of $\mathcal{O}_p$ modulo complex conjugacy. The factor $S\mathbb{U}^\sigma$ is the special unitary group associated to the Hermitian form conjugated by σ, thus corresponding to some Galois conjugate root of unity. Denote also by $\widetilde{\rho}_p$ and ρ_p the representations $\prod_{\sigma \in S(p)} \widetilde{\rho}_{p,\sigma(A_p)}$ and $\prod_{\sigma \in S(p)} \rho_{p,\sigma(A_p)}$, respectively. When p is an odd prime $p \geq 5$ and $g \geq 3$ then it is known that $\widetilde{\rho}_{p,A_p}$ takes values in $S\mathbb{U}$ (see [?]). Notice that the real Lie group $\mathbb{G}_p$ is a semi-simple algebraic group defined over $\mathbb{Q}$.

In [14] the first author proved that $\widetilde{\rho}_p(\widetilde{\mathcal{M}_g})$ is a discrete Zariski dense subgroup of $\mathbb{G}_p$ whose projections onto the simple factors of $\mathbb{G}_p$ are topologically dense, for $g \geq 3$ and $p \geq 5$ is a prime.

Remark 1.2. A similar result holds for the $SU(2)$-TQFT. Specifically let $p = 2r$ where $r \geq 5$ is prime. According to ([1], 1.5) there is an isomorphism of TQFTs between $\mathcal{V}_{2r}$ and $\mathcal{V}_2' \otimes \mathcal{V}_r$, and hence the projection on the second factor gives us a homomorphism $\pi : \rho_{2r}(M_g) \to \mathbb{G}_r$. Furthermore the image of the TQFT representation associated to $\mathcal{V}_2'$ is finite. Therefore, $\pi \circ \rho_{2r}(M_g)$ is a discrete Zariski dense subgroup of $\mathbb{G}_r$.

1.2 Quasi-homomorphisms

In ([5],Thm.1.3) Burger and Iozzi proved that for any pair of integers $1 \leq m < n$ there is an explicit bounded cohomology class $c_{SU(m,n)} \in H_b^2(SU(m,n); \mathbb{R})$ such that for any discrete group Γ, two Zariski dense representations $\rho : \Gamma \to SU(m,n)$ are non-conjugate if and only if the corresponding cohomology classes $\rho^*(c_{SU(m.n)}) \in H_b^2(\Gamma; \mathbb{R})$ are distinct. Moreover, if distinct, then these classes are $\mathbb{Z}$-linearly independent. The class $c_{SU(m,n)}$ plays therefore the role of a sort of universal character for Zariski dense representations.

An explicit construction of the class $c_{SU(m,n)}$ was provided by Guichardet-Wigner [23] and by Dupont [9]. Both constructions give equal generators of the group $H_b^2(SU(m,n); \mathbb{R}) \cong \mathbb{R}$. Let us outline the construction given by Dupont-Guichardet et Guichardet-Wigner in [23, 10]. Consider a real semi-simple Lie group G with maximal compact subgroup K such that the homogeneous space

G/K is an irreducible Hermitian symmetric space of non-compact type. Then, if ω denotes the Kähler form on G/K, for any point $x_0 \in \mathcal{X}$ we have a 2-cocycle $c_{I(\mathcal{X})} : I(\mathcal{X}) \times I(\mathcal{X}) \to \mathbb{Z}$ given by:

$$c_{I(\mathcal{X})}(g_1, g_2) = \frac{1}{4\pi} \int_{\Delta(g_1(x_0), g_2(x_0), g_1 g_2(x_0))} \omega, \qquad g_1, g_2 \in I(\mathcal{X}),$$

where $\Delta(x, y, z)$ denotes an oriented smooth triangle on $\mathcal{X}$ with geodesic sides. Although the interior of the triangle with geodesic sides is not uniquely defined the value of the cocycle is well-defined because ω is closed. Any two different choices of x_0 give cohomologous cocycles.

Let us now describe how this construction amounts in our case to compute homogeneous quasi-homomorphims on suitable central extensions of the mapping class groups. Let G be a topological group. The ordinary cohomology group $H^2(G, \mathbb{R})$ is usually an extremely large group, for instance for non-compact Lie groups it is typically uncountable (see [30]). In contrast, the *bounded* cohomology group $H_b^2(G; \mathbb{R})$ is often a much more manageable group and contains a large amount of information on the group G. There is a canonical comparison map $H_b^2(G; \mathbb{R}) \to H^2(G; \mathbb{R})$ whose kernel is described by quasi-homomorphisms: a map $\varphi : G \to \mathbb{R}$ is a quasi-homomorphism if $\sup_{a,b \in G} |\partial\varphi(a, b)| < \infty$, where $\partial\varphi(a, b) = \varphi(ab) - \varphi(a) - \varphi(b)$ is the boundary 2-cocycle. The quasi-homomorphism φ is homogeneous if $\varphi(a^n) = n\varphi(a)$, for every $a \in G$ and $n \in \mathbb{Z}$. If G is a uniformly perfect group then any quasi-homomorphism is at bounded distance of a unique homogeneous one which can be computed by an averaging process. Let us denote the vector space of quasi-homomorphisms by $QH(G)$ and its quotient by the subspace generated by the bounded functions and the group homomorphisms by $\widetilde{QH}(G)$. It is known that there is an exact sequence:

$$0 \to \widetilde{QH}(G) \to H_b^2(G; \mathbb{R}) \to H^2(G; \mathbb{R}).$$

Recall now that $\mathcal{M}_g$ is perfect when $g \geq 3$ and that the universal central extension $\widetilde{\mathcal{M}_g}^{\text{univ}}$ of $\mathcal{M}_g$ is a subgroup of finite index 12 in the central extension $\widetilde{\mathcal{M}_g}$ (see [29]). Let $g \geq 3$, $p \geq 5$ be a prime number and and $SU(m, n)$ be the non-compact simple factor of $\mathbb{G}_p$ corresponding to the primitive root of unity ζ. Then the representation $\widetilde{\rho}_{p,\zeta}$ is determined by the bounded cohomology class $\widetilde{\rho}_{p,\zeta}^*(c_{SU(m,n)})$. But as the universal extension of the mapping class groups is both perfect and has no non-trivial extensions, we have an isomorphism $\widetilde{QH}(\widetilde{\mathcal{M}_g}^{\text{univ}}) \simeq H_b^2(\widetilde{\mathcal{M}_g}^{\text{univ}}; \mathbb{R})$. As $\widetilde{\mathcal{M}_g}^{\text{univ}}$ is of finite index in $\widetilde{\mathcal{M}_g}$, the same isomorphism holds for the second group. That is, there exists a quasi-homomorphism, unique up to a bounded quantity, $L_\zeta : \widetilde{\mathcal{M}_g} \to \mathbb{R}$ (respectively one defined on $\widetilde{\mathcal{M}_g}^{\text{univ}}$) verifying

$$\partial L_\zeta = \frac{1}{4\pi(m+n)} \widetilde{\rho}_{p,\zeta}^* c_{SU(m,n)}$$

and one is obtained from the other by restricting. Let $\overline{L}_\zeta$ denote the unique homogeneous quasi-homomorphism in the class of L_ζ. We have the following immediate consequence of the theorem of Burger and Iozzi from [5] and the density result from [14]:

Proposition 1.1. *([14]) The quasi-homomorphism $\overline{L}_\zeta$ is a class function (i.e. invariant on conjugacy classes) on $\widetilde{\mathcal{M}_g}$ which encodes all information about the representation $\widetilde{\rho}_{p,\zeta}$. Namely, if $\rho' : \widetilde{\mathcal{M}_g} \to SU(m, n)$ is some Zariski dense representation and L' is the corresponding homogeneous quasi-homomorphism, then $\overline{L}_\zeta = L'$ if and only if ρ' is conjugate to $\widetilde{\rho}_{p,\zeta}$ Furthermore, the classes of those L_ζ, for which $1 \leq m < n$, are linearly independent over $\mathbb{Q}$ in $\widetilde{QH}(\widetilde{\mathcal{M}_g})$.*

Remark 1.3. Notice that Bestvina and Fujiwara proved in [2] that $\widetilde{QH}(\mathcal{M}_g)$, and hence $\widetilde{QH}(\widetilde{\mathcal{M}_g})$ are infinitely generated.

1.3 Main results

One aim of this paper is to obtain an explicit formula for the quasi-homomorphism $\overline{L}_\zeta$, extending the one obtained by Barge and Ghys in [3] for the Maslov cocycle on the symplectic groups. The first ingredient is the construction of the lift $\widehat{\rho}_{p,\zeta} : \widetilde{M}_g^{\text{univ}} \to \widetilde{SU(m,n)}$. Specifically we will prove:

Proposition 1.2. *There is a unique lift* $\widehat{\rho}_{p,\zeta} : \widetilde{M}_g^{\text{univ}} \to \widetilde{SU(m,n)}$ *of* $\widetilde{\rho}_{p,\zeta}$.

In order to proceed with the computation we have to introduce two more ingredients. The first one is the Dupont-Guichardet-Wigner quasi-homomorphism Φ_{DGW} on the universal covering $\widetilde{SU(m,n)}$ of $SU(m,n)$.

Definition 1.2. *A Dupont-Guichardet-Wigner quasi-homomorphism* $\Phi_{DGW} : \widetilde{SU(m,n)} \to \mathbb{Z}$ *is some primitive of the pull-back of* $c_{SU(m,n)}$ *on* $\widetilde{SU(m,n)}$*. Namely, it is a quasi-homomorphism satisfying:*

$$\Phi_{DGW}(\widetilde{x}\widetilde{y}) - \Phi_{DGW}(\widetilde{x}) - \Phi_{DGW}(\widetilde{y}) = c_{SU(m,n)}(x,y)$$

for all $x, y \in SU(m,n)$ *and their arbitrary lifts* $\widetilde{x}, \widetilde{y} \in \widetilde{SU(m,n)}$*. The quasi-homomorphism is normalized if*

$$\Phi_{DGW}(cz) = \Phi_{DGW}(z) + 1$$

for $z \in \widetilde{SU(m,n)}$*, where* c *denotes the generator of* $\ker(\widetilde{SU(m,n)} \to SU(m,n))$*. All Dupont-Guichardet-Wigner quasi-homomorphisms are at bounded distance and the unique homogeneous normalized Dupont-Guichardet-Wigner quasi-homomorphism is denoted* $\overline{\Phi}_{DGW}$*.*

We can now state our first result:

Theorem 1.1. *Let* $\overline{\Phi}_{DGW} : \widetilde{SU(m,n)} \to \mathbb{R}$ *be the homogeneous normalized Dupont-Guichardet-Wigner quasi-homomorphism. Then for any bounded cocycle in the class of* $c_{SU(m,n)}$ *its associated homogeneous quasi-homomorphism* $\overline{L}_\zeta$ *is given by:*

$$\overline{L}_\zeta(x) = \overline{\Phi}_{DGW}(\widehat{\rho}_{p,\zeta}(x))$$

where $\widehat{\rho}_{p,\zeta}(x)$ *is the lift of* $\widetilde{\rho}_{p,\zeta}(x)$ *to* $\widetilde{SU(m,n)}$*. Moreover, we have:*

$$\overline{L}_\zeta(x) \equiv \frac{1}{2\pi}\left(\sum_{\lambda \in S(\widetilde{\rho}_{p,\zeta}(x))} n^+(\lambda)\arg(\lambda)\right) \in \mathbb{R}/\mathbb{Z}$$

where $S(x)$ *is the set of eigenvalues of* x *and* $n^+(\lambda)$ *is the positivity multiplicity of* λ *(see section 2.3 for details).*

The second part of the present paper is devoted to applications of these methods to the study of the quantum representations. Let us introduce more terminology. Set $s_{p,g}$ for the number of simple non-compact factors of the semi-simple Lie group $\mathbb{G}_p$. We also write $s^*_{p,g}$ for the number of such factors of non-zero signature i.e. of the form $SU(m,n)$ with $1 < m < n$. Each simple factor is associated to a primitive root of unity ζ of order p. Those ζ corresponding to non-compact simple factors or with non-zero signature will be called non-compact roots and respectively non-compact roots of non-zero signature. Denote also by $r_{p,g}$ the minimal number of normal generators of $\ker \widetilde{\rho}_p$ (i.e. of $\ker \widetilde{\rho}_{p,\zeta}$, for any primitive ζ) within $\widetilde{M}_g^{\text{univ}}$, namely the minimum number of relations needed to add in order to obtain the quotient $\widetilde{\rho}_p(\widetilde{M}_g^{\text{univ}})$.

Theorem 1.2. *Let $g \geq 2$, p prime and $(g,p) \neq (2,5)$. Either $\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{univ}})$ is not isomorphic to a higher rank lattice, or else $r_{p,g} \geq s_{p,g}$ and hence $r_{p,g} \geq s_{p,g} \geq \left[\frac{(g-1)}{2g}p\right]$, if $g \geq 3$ and $r_{p,2} \geq s_{p,2} \geq \left[\frac{p}{3}\right]$, respectively.*

Remark 1.4. The content of this theorem is that whenever $\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{univ}})$ is isomorphic to a higher rank lattice the group $\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{univ}})$ should be the quotient of $\widetilde{M_g}^{\mathrm{univ}}$ by a *large* number of relations, growing linearly on p. This would be against the belief that, except for the case $g = 1$, the only relations arising in the level p quantum representations are T_γ^p, where γ runs over simple curves. Specifically, consider the subgroup $M_g[p]$ generated by p-th powers of all Dehn twists. It is known that $M_g[p]$ is contained within the kernel $\ker \rho_{p,\zeta}$ of the *projective* quantum representation. One might conjecture that the two subgroups coincide in genus $g \geq 3$. Now, $M_g[p]$ is *normally* generated by the p-th powers of Dehn twists along a family of curves containing one bounding simple closed curve in each genus and one non-separating one. This gives an upper bound of $1 + \left[\frac{g}{2}\right]$ normal generators which is uniformly bounded independently on p. For instance this conjectured equality of groups holds when the surface is a once-punctured torus (see [15]) or a 4-punctured sphere.

2 Computing the Dupont-Guichardet-Wigner quasi-homomorphism

2.1 Outline of the proof of Theorem 1.1

Proof of Theorem 1.1. The first step in computing $\overline{L}_\zeta$ is to obtain a formula for $\overline{\Phi}_{DGW}$, following the similar computation from [3] of its symplectic counterpart, as follows:

Proposition 2.1. *The Dupont-Guichardet-Wigner homogeneous quasi-homomorphism Φ_{DGW} : $\widetilde{SU(m,n)} \to \mathbb{R}$ is the unique continuous lift of the map $\overline{\phi}_0 : SU(m,n) \to \mathbb{R}/\mathbb{Z}$, defined when g is semi-simple by the formula:*

$$\overline{\phi}_0(g) = \frac{1}{2\pi}\left(\sum_{\lambda \in S(g)} n^+(\lambda)\arg(\lambda)\right) \in \mathbb{R}/\mathbb{Z}$$

where $S(g)$ is the set of eigenvalues of g and $n^+(\lambda)$ their positivity multiplicity (see section 2.3 for details).

The independence of $\overline{L}_\zeta$ on the chosen bounded cocycle is a consequence of the fact that $SU(m,n)$ is uniformly perfect. This will be proved in section 3.

Eventually the continuity of the homogeneous quasi-homomorphism $\overline{\Phi}$ and Proposition 1.2 imply our claim. □

The proof of Proposition 1.2 is given in subsection 2.2. The continuity of $\overline{\Phi}$ is proved in subsection 2.3 and the proof of Proposition 2.1 is postponed to subsection 2.4.

2.2 Proof of Proposition 1.2

The construction of a lift is very general and it can be obtained by purely formal considerations. The proof of Proposition 1.2 is a consequence of the result below for $G = \widetilde{M_g}^{\mathrm{univ}}$ and $H = SU(m,n)$. Since G is already the universal central extension of a M_g ($g \geq 3$) there is a canonical isomorphism $\widetilde{G} \to G$. The explicit description of $\widehat{\rho}_{p,\zeta}$ comes from the constructive proof.

Proposition 2.2. *Let G be a perfect group, $\widetilde{G}$ denote its universal central extension and $\psi : G \to H$ a group homomorphism. Let also $\widetilde{H}$ be some central extension of H. There exists an unique lift $\widehat{\psi} : \widetilde{G} \to \widetilde{H}$ making the following diagram commutative:*

$$\begin{array}{ccc} \widetilde{G} & \stackrel{\widehat{\psi}}{\longrightarrow} & \widetilde{H} \\ \downarrow & & \downarrow \\ G & \stackrel{\psi}{\longrightarrow} & H \end{array}$$

Lemma 2.1. *Let G be a perfect group, $p : \widetilde{G} \to G$ denote its universal central extension, $p_U : U \to \widetilde{G}$ be a central extension of $\widetilde{G}$. Then U is a trivial extension and there exists a canonical section $s : \widetilde{G} \to U$ over G.*

Proof. Let us introduce some terminology. Consider $G = \mathbb{F}/R$ be a group presentation of G, where $\mathbb{F}$ is a free group. It is well-known that $\widetilde{G} = [\mathbb{F}, \mathbb{F}]/[\mathbb{F}, R]$. Set $E = \mathbb{F}/[\mathbb{F}, R]$, which contains $\widetilde{G}$. There is a natural surjection $p_E : E \to G$ induced by the inclusion $[\mathbb{F}, R] \subset R$. Set also $p_{\mathbb{F}} : \mathbb{F} \to E$ for the natural projection.

Since $\mathbb{F}$ is a free group there exists a lift $t : \mathbb{F} \to E \times_G U$ of $p_{\mathbb{F}}$. The core of the proof is contained in the following

Lemma 2.2. *The homomorphism t descends to a well-defined homomorphism $\overline{t} : E \to U$ making the following diagram commutative:*

$$\begin{array}{ccc} \mathbb{F} & \stackrel{t}{\longrightarrow} & E \times_G U \\ p_{\mathbb{F}} \downarrow & & \downarrow p_2 \\ E & \stackrel{\overline{t}}{\longrightarrow} & U \end{array}$$

where $p_2 : E \times_G U \to U$ is the projection on the second factor.

Then the restriction $\overline{t}|_{\widetilde{G}} : \widetilde{G} \to U$ is the desired section s. The fact that s is a section over G comes from the construction. Notice also that s does not depend on the choice of t. This proves Lemma 2.1. ☐

Proof of Lemma 2.2. Since $\ker p_E \subset R/[F, R]$ we have $[E, \ker p_E] \subset [F/[F, R], R/[F, R]] = 1$, so that $p_E : E \to G$ is a central extension.

We claim now that $E \times_G U \to E$ is also a central extension. We have the commutative diagram:

$$\begin{array}{ccc} E \times_G U & \stackrel{q}{\longrightarrow} & E \\ p_2 \downarrow & & \downarrow p_E \\ U & \stackrel{p \circ p_U}{\longrightarrow} & G \end{array}$$

First, we have $p_U([U, U]) = [p_U(U), p_U(U)] = [\widetilde{G}, \widetilde{G}] = \widetilde{G}$, since $\widetilde{G}$ is perfect. This implies that $U = [U, U] \cdot \ker p_U$. Since $p_U([U, \ker p \circ p_U]) \subset [\widetilde{G}, \ker p_U] = 1$ we derive that $[U, \ker p \circ p_U] \subset \ker p_U$. Thus, if $x \in \ker p \circ p_U$ and $u, v \in \ker p_U$ then $[x, u], [x, v] \in \ker p_U$. It follows that $x[u, v]x^{-1} = [[x, u]u, [x, v]v] = [u, v]$, because $\ker p_U$ is central abelian in U. But this means that $[\ker p \circ p_U, [U, U]] =$. Since $\ker p \circ p_U$ also commutes with the central $\ker p_U$ we derive that $\ker p \circ p_U$ commutes with $[U, U] \cdot \ker p_U = U$. This shows that $p \circ p_U : U \to G$ is a central extension. Therefore its pull-back on E is a central extension, as claimed.

From the commutative diagram

$$\begin{array}{ccc} \mathbb{F} & = & \mathbb{F} \\ t\downarrow & & \downarrow p_{\mathbb{F}} \\ E\times_G U & \stackrel{q}{\longrightarrow} & E \\ p_2\downarrow & & \downarrow p_E \\ U & \stackrel{p\circ p_U}{\longrightarrow} & G \end{array}$$

we derive that $p_2 \circ t(R) \subset \ker p \circ p_U$, which is central in U. Therefore $p_2 \circ t([\mathbb{F}, R]) \subset [p_2 \circ t(\mathbb{F}), p_2 \circ t(R)] = 1$, so that $p_2 \circ t$ factorizes to a homomorphism $\overline{t} : \mathbb{F}/[\mathbb{F}, R] \to U$, as claimed. □

End of proof of Proposition 2.2. Now if G is perfect and $E \to G$ is some central extension then there exists an explicit and unique homomorphism $\theta : \widetilde{G} \to E$ lifting the identity on G. It suffices to take $U = \widetilde{G} \times_G E$, which a central extension over $\widetilde{G}$. Then Lemma 2.1 furnishes a canonical section $s : \widetilde{G} \to \widetilde{G} \times_G E$. Composing with the projection on the second factor we get our homomorphism $\theta : \widetilde{G} \to E$.

In particular we take $E = G \otimes_H \widetilde{H}$, which is a central extension of G. The construction above yields a canonical homomorphism $\theta : \widetilde{G} \to G \otimes_H \widetilde{H}$. If $p_{\widetilde{H}} : G \otimes_H \widetilde{H} \to \widetilde{H}$ is the projection on the second factor then we can define

$$\widehat{\psi} = p_{\widetilde{H}} \circ \theta$$

□

2.3 Dupont-Guichardet-Wigner cocycles

Let $x = k(x)a(x)n(x)$ be the Iwasawa decomposition of the element $x \in SU(m,n)$, where $k(x) \in K$, $a(x) \in A$, $n(x) \in N$, corresponding to the Iwasawa decomposition $SU(m,n) = KAN$. Here K is chosen to be the maximal compact subgroup $S(U(m) \times U(n))$, A is the group of unitary diagonal matrices with real entries and N is the group of unitary unipotent matrices.

We recall the following particular case of the construction due to Guichardet and Wigner in ([23], Thm.1) for simple Lie groups of isometries of Hermitian domains of non-compact type:

Proposition 2.3. *Consider a smooth function $v : SU(m,n) \to \mathbb{C}^*$ satisfying the following conditions:*

1. *the restriction of v to the maximal compact K is a non-trivial morphism of K into $U(1) \subset \mathbb{C}^*$;*
2. *the restriction of v to $\exp \mathfrak{p}$ is strictly positive and K-invariant;*
3. *$v(k \cdot \exp p) = v(k)v(\exp p)$, for any $k \in K$ and $p \in \mathfrak{p}$, where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of $SU(m,n)$ and $\mathfrak{k}$ is the Lie algebra of K.*

Then there exists a unique smooth 2-cocycle $f_v : SU(m,n) \times SU(m,n) \to \mathbb{R}$ such that

$$\exp(2\pi\sqrt{-1} f_v(g_1, g_2)) = \arg(v(g_1g_2)^{-1} \cdot v(g_1) \cdot v(g_2)), \text{ and } f(1,1) = 0$$

Moreover, the class of f generates the Borel cohomology group $H^2(SU(m,n), \mathbb{R})$.

We will normalize later the cocycle f to a cocycle $c^{\mathbb{Z}}_{SU(m,n)}$ whose class is the generator of the image of $H^2(SU(m,n), \mathbb{Z})$ in $H^2(SU(m,n), \mathbb{R})$.

We will consider the function $v_0 : K \to U(1)$ given by

$$v_0(x) = \arg\det(x)$$

where $x = \begin{pmatrix} x_+ & 0 \\ 0 & x_- \end{pmatrix} \in S(U(m) \otimes U(n))$ and x_+ is the $U(m)$ component of x. Extend now v_0 to all of $SU(m,n)$ by asking it to verify the conditions stated in Proposition 2.3 and setting $v_0(\exp p) = 1$.

We consider also its continuous lift $\Phi_0 : \widetilde{SU(m,n)} \to \mathbb{R}$ to the universal covering, which is uniquely determined by the conditions $\Phi_0(1) = 0$ and $\Phi_0(c) = 1$, where c is a generator of $\ker(\widetilde{SU(m,n)} \to SU(m,n))$. More generally we say that a quasi-homomorphism Φ is *normalized* if $\Phi(c) = 1$. Notice that $\Phi_0|_{\widetilde{K}} : \widetilde{K} \to \mathbb{R}$ is the obvious lift of v_0 to the universal coverings.

Lemma 2.3. *The pull-back of the 2-cocycle f_{v_0} on $\widetilde{SU(m,n)}$ is the boundary of Φ_0. Moreover, $\Phi_0 : \widetilde{SU(m,n)} \to \mathbb{R}$ is a continuous quasi-homomorphism.*

Proof. A maximal compact of the Lie group $\widetilde{SU(m,n)}$ is the semi-simple subgroup $SU(m) \times SU(n)$. The Van Est isomorphism gives us $H^2_c(\widetilde{SU(m,n)}, \mathbb{R}) \cong H^2(su(m,n), su(m) \otimes su(n); \mathbb{R})$, where the right hand side is the Lie algebra cohomology. However, it is well-known (see [24], chap. VIII, or [4]) that $\dim H^2(su(m,n), su(m) \otimes su(n); \mathbb{R}) = \dim \mathrm{Hom}(su(m) \times su(n), \mathbb{R}) = 0$. This proves that $H^2_c(\widetilde{SU(m,n)}, \mathbb{R}) = 0$.

Then Φ_0 is a primitive of the pull-back of f_{v_0}. Furthermore, f_{v_0} is bounded and hence Φ_0 is a quasi-homomorphism. □

We will denote by $\overline{\Phi}$ the homogenization of the quasi-homomorphism Φ defined by

$$\overline{\Phi}(z) = \lim_{n\to\infty} \frac{1}{n} \Phi(z^n)$$

Lemma 2.4. *The homogeneous quasi-homomorphism $\overline{\Phi}$ is continuous.*

Proof. The homogeneous quasi-homomorphism associated to a continuous quasi-homomorphism is also continuous, by the result of Shtern (see [31], Prop.1). □

Notice that the reduction mod $\Phi_0(c)\mathbb{Z}$ of $\overline{\Phi}_0$ makes sense as a map $\overline{\phi}_0 : SU(m,n) \to \mathbb{R}/\mathbb{Z}$, given by $\overline{\phi}_0(x) = \overline{\Phi}_0(\widetilde{x})$, where $\widetilde{x}$ is an arbitrary lift of x.

Lemma 2.5. *There is a unique homogeneous normalized quasi-homomorphism on $\widetilde{SU(m,n)}$. In particular $\overline{\Phi}_0$ is trivial on the preimage of the Borel subgroup $AN \subset SU(m,n)$ to $\widetilde{SU(m,n)}$.*

Proof. The quasi-homomorphism Φ_0 equals 1 on the preimage of AN to $\widetilde{SU(m,n)}$, since Φ_0 is a continuous lift of v_0 and v_0 is trivial on AN. by construction.

Furthermore the difference between any two quasi-homomorphisms of $\widetilde{SU(m,n)}$ is uniformly bounded because $SU(m,n)$ is uniformly perfect. This implies that any two homogeneous quasi-homomorphisms coincide. □

Lemma 2.6. *The homogeneous normalized quasi-homomorphism on $\widetilde{SU(m,n)}$ is the unique continuous normalized lift of the map $\overline{\phi}_0 \circ e : SU(m,n) \to \mathbb{R}/\mathbb{Z}$ where $x = e(x)h(x)u(x)$ is the Jordan decomposition of $x \in SU(m,n)$.*

Proof. In fact $\overline{\Phi}(x) = \overline{\Phi}(e(x))\overline{\Phi}(h(x))\overline{\Phi}(u(x))$. But $h(x)$ is conjugate to some element $a \in A$, $u(x)$ to some element n in N. Since $\overline{\Phi}$ is a class function i.e. conjugacy invariant, we have $\overline{\Phi}(h(x) = \overline{\Phi}(a) = 1$ and similarly $\overline{\Phi}(u(x)) = 1$.

Now also $e(x)$ is elliptic hence it is conjugate to some $k \in K$. Thus there exists g such that $ge(x)g^{-1} = k$. $\overline{\Phi}(x) = \overline{\Phi}(e(x)) = \overline{\Phi}(k) = v(k)$. It is clear that it is conjugacy invariant and homogeneous, and coincides with v on K. It is trivial on AN, and hence it gives a quasi-morphism using ([23], Thm.1). Since it is homogeneous it coincides with $\overline{\Phi}$. $\square$

2.4 Positive eigenvalues of pseudo-unitary operators

Consider now a pseudo-unitary operator $g \in SU(m,n)$. Let $H : V \times V \to \mathbb{C}$ be the indefinite Hermitian form defining the group $SU(m,n)$, where $\dim_{\mathbb{C}} V = m+n$. We will assume henceforth that $1 \le m \le n$.

The spectrum $S(g)$ of T is symmetric with respect to the unit circle, namely if $\lambda \in S(g)$ then $\overline{\lambda}^{-1} \in S(g)$ (see [21], ch.10, section 5). For $\lambda \in S(g)$ we consider the root space $V_\lambda(g) = \ker(g - \lambda I)^{m+n} \subset V$. We have then $V = \oplus_{\lambda \in S(g)} V_\lambda(g)$. Moreover, each $V_\lambda(g)$ splits as $V_\lambda(g) = \oplus_i V_{\lambda,i}(g)$, where each subspace $V_{\lambda,i}(g)$ corresponds to a Jordan block with diagonal λ in the Jordan decomposition of g. The number of such subspaces $V_{\lambda,i}(g)$ (i.e. Jordan blocks) is the geometric multiplicity of λ, namely $\dim\ker(g - \lambda I)$. The collection of dimensions $\dim V_{\lambda,i}$ is the collection of partial multiplicities of λ. Furthermore the collection of partial multiplicities of $\lambda \in S(g)$ agrees with the one for $\overline{\lambda}^{-1}$.

We will use the canonical form of pseudo-unitary operators from ([22], Thm.5.15.1). We will only need a weaker form and state it in a simplest form, though the statement in [22] is more precise:

Proposition 2.4. *Let $g \in SU(m,n)$ have the set of Jordan blocks $J_1, J_2, \ldots, J_{a+2b}$ (where $a+2b \le m+n$) and corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{a+2b}$, not necessarily distinct. We suppose that that $|\lambda_1| = |\lambda_2| = \cdots = |\lambda_a| = 1$, $|\lambda_{a+2i-1}| > 1$ and $\lambda_{a+2i-1} = \overline{\lambda}_{a+2i}^{-1}$, for $1 \le i \le b$. Then there exists a non-singular matrix C such that the following two conditions hold simultaneously:*

$$C^{-1}gC = \bigoplus_{i=1}^{m^+(g)} \lambda_{j_i} K_{j_i} \bigoplus_{i=1}^{m^-(g)} \lambda_{s_i} K_{s_i} \bigoplus_{1 \le i \le b} \begin{pmatrix} \lambda_{a+2i-1} K_{a+2i-1} & 0 \\ 0 & \overline{\lambda}_{a+2i-1}^{-1} K_{a+2i} \end{pmatrix}$$

$$C^* H C = \bigoplus_{i=1}^{m^+(g)} P_{j_i} \bigoplus_{i=1}^{m^-(g)} -P_{s_i} \bigoplus_{1 \le i \le b} \begin{pmatrix} 0 & P_{a+2i-1} \\ P_{a+2i} & 0 \end{pmatrix}$$

where

1. *K_j are unipotent upper triangular matrices (also called Toeplitz blocks), for all $j \le a+2b$;*
2. *Each matrix P_j is a permutation matrix of the form* $\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$ *having the size of the Jordan block J_j, for all $j \le a+2b$;*
3. *The two sets $\{j_1, j_2, \ldots, j_{m_+(g)}\}$ and $\{s_1, s_2, \ldots, s_{m_-(g)}\}$ form a partition of $\{1,2,\ldots,a\}$, so that $m_+(g) + m_-(g) = a$. The sign characteristic $\varepsilon_i \in \{\pm 1\}$, for $1 \le i \le a$ is given by $\varepsilon_i = 1$ iff $i \in \{j_1, j_2, \ldots, j_{m_+(g)}\}$.*
4. *The canonical form is unique, up to a permutation of equal Toeplitz blocks respecting the sign characteristic.*

When g is semi-simple the canonical form is simpler, as follows:

Corollary 2.1. *Let $g \in SU(m,n)$ be a semi-simple element with eigenvalues λ_i, $1 \le i \le m+n$. Let us denote by $\lambda_\alpha, \overline{\lambda}_\alpha^{-1}$, with $\alpha \in N(g) \subset \{1,2,\dots,m+n\}$ those eigenvalues of modulus different from 1, where $|\lambda_\alpha| > 1$. Then there exists a non-singular matrix C such that the following two conditions hold simultaneously:*

$$C^{-1}gC = \oplus_{i=1}^{m^+(g)}(\lambda_{j_i}) \oplus \oplus_{i=1}^{m^-(g)}(\lambda_{s_i}) \oplus \oplus_{\alpha\in N(g)} \begin{pmatrix} \lambda_\alpha & 0 \\ 0 & \overline{\lambda}_\alpha^{-1} \end{pmatrix}$$

$$C^*HC = \oplus_{i=1}^{m^+(g)}(+1) \oplus \oplus_{i=1}^{m^-(g)}(-1) \oplus \oplus_{\alpha\in N(g)} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Here the sets of indices $\{j_1, j_2, \dots, j_{m_+(g)}\}$, $\{s_1, s_2, \dots, s_{m_+(g)}\}$ and $N(g)$ form a partition of $\{1,2,\dots,m+n\}$. The canonical form is unique up to a permutation preserving the eigenvalues and the sign characteristic.

Proof. This result seems to be first stated explicitly by Krein (see [25]) for the symplectic group and by Yakubovich in the present setting (see [34], p.124). □

Definition 2.1. *Let g be a semi-simple element of $SU(m,n)$. The eigenvalues λ_i of g, for $i \in \{j_1, j_2, \dots, j_{m_+(g)}\}$, i.e. those for which $\varepsilon_i = +1$, will be called* positive *(after Gelfand and Lidskii, Krein and Yakubovich) and their* positivity multiplicity *n_i^+ is the multiplicity among positive eigenvalues. By convention, the eigenvalues λ_α with $|\lambda_\alpha| > 1$ are said to be positive and their positivity multiplicity coincide with the usual multiplicity. The remaining eigenvalues will be called negative eigenvalues of g. We will also denote by $n^+(\lambda)$ the positivity multiplicity of the eigenvalue λ (which is 0 for negative ones) of the semi-simple g.*

The positivity seems more subtle when g is not semi-simple. In fact the signature of each block $\varepsilon_j P_j$ equals 0 when its dimension n_j is even and ε_j, when its dimension n_j is odd, respectively. Further, the signature of $\begin{pmatrix} 0 & P_{a+2i-1} \\ P_{a+2i} & 0 \end{pmatrix}$ is always 0. Thus, every eigenvalue involved in a Jordan block is positive with a positivity multiplicity equal to approximatively half of its partial multiplicity.

Lemma 2.7. *Let $g \in SU(m,n)$. Then in a suitable basis of V we can write simultaneously:*

$$e(g) = \bigoplus_{i=1}^{a} \mathrm{diag}(\lambda_i) \bigoplus_{1\le i\le b} \begin{pmatrix} \mathrm{diag}(\frac{\lambda_{a+2i-1}}{|\lambda_{a+2i-1}|}) & 0 \\ 0 & \mathrm{diag}(\frac{\lambda_{a+2i-1}}{|\lambda_{a+2i-1}|}) \end{pmatrix}$$

$$H = \bigoplus_{i=1}^{a} \varepsilon_i X_i \bigoplus_{1\le i\le b} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

where $\mathrm{diag}(\lambda_i)$ *is a diagonal matrix of the size equal to the partial multiplicity n_i of λ_i and X_i is the diagonal matrix of the same size with entries ± 1 of signature $\frac{1}{2}(1-(-1)^{n_i})$.*

Proof. Each Hermitian block $\varepsilon_i P_i$ can be reduced to the form $\varepsilon_i X_i$ by changing the base by some matrix D_i. Then $D_i^{-1}(\lambda_i K_i)D_i = \lambda_i U_i$, where $U_i = D_i^{-1}K_i D_i$ is an unipotent matrix commuting with the scalar λ_i. Thus the semi-simple part of the Jordan decomposition of $D_i^{-1}\lambda_i K_i D_i$ is $\mathrm{diag}(\lambda_i)$.

Further there exists matrices E_i such that the Hermitian block $\begin{pmatrix} 0 & P_{a+2i-1} \\ P_{a+2i} & 0 \end{pmatrix}$ is reduced to $\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$. Then the elliptic part of $E_i^{-1} \begin{pmatrix} \lambda_{a+2i-1}K_{a+2i-1} & 0 \\ 0 & \overline{\lambda}_{a+2i-1}^{-1}K_{a+2i} \end{pmatrix} E_i$ is given by $\begin{pmatrix} \mathrm{diag}(\frac{\lambda_{a+2i-1}}{|\lambda_{a+2i-1}|}) & 0 \\ 0 & \mathrm{diag}(\frac{\lambda_{a+2i-1}}{|\lambda_{a+2i-1}|}) \end{pmatrix}$. □

Furthermore, the elliptic element $e(g)$ is conjugate to some element $\begin{pmatrix} e(g)_+ & 0 \\ 0 & e(g)_- \end{pmatrix}$ of $S(U(m)\times U(n))$, where $e(g)_+ \in U(m)$ corresponds to a maximal invariant positive subspace of V for the Hermitian form H. The previous lemma gives an explicit formula for $e(g)_+$ in the form:

$$e(g)_+ = \bigoplus_{i=1}^{a} \mathrm{diag}_+(\lambda_i) \bigoplus_{1\le i\le b} \mathrm{diag}(\frac{\lambda_{a+2i-1}}{|\lambda_{a+2i-1}|})$$

where $\mathrm{diag}_+(\lambda_i)$ is a diagonal matrix of the size equal to its partial positivity multiplicity, defined as: $n_i^+ = \begin{cases} \frac{n_i}{2}, & \text{even } n_i \\ \frac{n_i+\varepsilon_i}{2}, & \text{odd } n_i \end{cases}$.

An immediate consequence is that

$$\det(e(g)_+) = \exp\left(2\pi\sqrt{-1}\left(\sum_{i=1}^{a} n_i^+ \arg(\lambda_i) + \sum_{i=1}^{b} n_{a+2i-1}\arg(\lambda_{a+2i-1})\right)\right)$$

When g is already semi-simple this formula simplifies to

$$\det(e(g)_+) = \exp\left(2\pi\sqrt{-1}\left(\sum_{\lambda\in S(g)} n^+(\lambda)\arg(\lambda)\right)\right)$$

We formulate the result obtained so far in the following:

Lemma 2.8. *For $g \in SU(m,n)$ we have*

$$\overline{\phi}_0(g) = \frac{1}{2\pi}\left(\sum_{i=1}^{a} n_i^+\arg(\lambda_i) + \sum_{i=1}^{b} n_{a+2i-1}\arg(\lambda_{a+2i-1})\right) \in \mathbb{R}/\mathbb{Z}$$

and in particular when g is semi-simple we have

$$\overline{\phi}_0(g) = \frac{1}{2\pi}\left(\sum_{\lambda\in S(g)} n^+(\lambda)\arg(\lambda)\right) \in \mathbb{R}/\mathbb{Z}$$

Proof of Theorem 1.1. Proposition 2.1 shows that the $\overline{\Phi}_{DGW}$ is uniquely determined as a continuous lift of $\overline{\phi}_0$, whose value is given by the previous Lemma.

In our case $\mathbb{G}_p$ is obtained by restriction of scalars from an anisotropic simple $\mathbb{Q}(\zeta)$-group. In particular, the matrices in $\rho_{p,\zeta}(\widetilde{M_g}) \subset SU(m,n)$ are semi-simple. This settles our claim. □

Remark 2.1. The Hermitian form H is given in diagonal form in [1], section 4.

Remark 2.2. When $m = n$ our formula should be compatible with the one given by Barge and Ghys in [3] for $Sp(2m,\mathbb{R})$, since $Sp(2m,\mathbb{R}) \subset U(m,m)$ is the real part of $U(m,m)$. Moreover there exists also a natural embedding $U(m,n) \subset Sp(2(m+n),\mathbb{R})$.

2.5 Comparison with the symplectic quasi-homomorphism

There are two natural homomorphisms, as it will be explained below, $SU(m,n) \hookrightarrow Sp(2(m+n),\mathbb{R})$ and $Sp(2n,\mathbb{R}) \hookrightarrow SU(n,n)$, so that one can wonder whether the quasi-homomorphism Φ_{DGW} in the pseudo-unitary case is given simply by restricting its symplectic counterpart already computed by Barge and Ghys [3], or vice-versa. We will show in this section that the symplectic quasi-homomorphism restricts to the trivial quasi-homomorphism on the pseudo-unitary group. This contrasts with the unitary case, when the pseudo-unitary quasi-homomorphism for $m = n$ restricts to the symplectic quasi-homomorphism from [3].

Now Proposition 2.3 is valid for any real simple Lie group, and as pointed out in ([23], section 3) the Dupont-Guichardet-Wigner cocycles and therefore the associated quasi-homomorphisms are determined by the restriction of the function v to the maximal compact subgroup K. To compare the quasi-homomorphisms $\Phi_{SU(p,q)}$ and $\Phi_{Sp(2g,\mathbb{R})}$, we just need to describe how the functions v in each case restrict from a maximal compact group of one of the groups to a maximal compact subgroup of the another.

2.5.1 From symplectic to special unitary groups

For the symplectic group $Sp(2n, \mathbb{R})$, the maximal compact subgroup is $U(n)$. The embedding $U(n) \hookrightarrow Sp(2n, \mathbb{R})$ sends a matrix $A+iB$ to the matrix $\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$. The function $v_{Sp} : U(g) \to U(1)$ is given on the compact subgroup by $\det(A + iB)$.

The natural inclusions $Sp(2n, \mathbb{R}) \to SU(n, n)$ are determined by the fact that on $\mathbb{C}^{2n}$ the two bilinear forms that in the canonical basis have as matrices

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \text{ and } \Omega = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}$$

are almost isometric. More precisely, there is an invertible map D such that ${}^t\overline{D}\Omega D = -\sqrt{-1}J$ given:

$$D = \frac{1}{\sqrt{2}} \begin{pmatrix} I_n & -\sqrt{-1}I_n \\ -\sqrt{-1}I_n & I_n \end{pmatrix}.$$

As real symplectic matrices preserve the form $-\sqrt{-1}J$, we have an induced injective homomorphism $Sp(2n, \mathbb{R}) \to U(n, n)$ that sends a symplectic matrix S to DSD^{-1}. Symplectic matrices have determinant 1, hence the image lies in $SU(n, n)$. A direct computation shows now that for a matrix in the maximal compact subgroup $U(g)$ one has:

$$D \begin{pmatrix} A & B \\ -B & A \end{pmatrix} D^{-1} = \begin{pmatrix} A + \sqrt{-1}B & 0 \\ & A - \sqrt{-1}B \end{pmatrix}$$

Therefore $\Phi_{SU(n,n}|_{DU(n)D^{-1}} = \Phi_{Sp(2n,\mathbb{R})}$.

For any matrix in $SU(m, m)$, a direct computation shows that its real part is a symplectic matrix, and this gives an embedding $Sp(2m, \mathbb{R}) \to SU(m, m)$ as the real matrices preserving the indefinite Hermitian form. At the level of maximal compact groups this amounts to the map $U(m) \to S(U(m) \times U(m))$ given by $U \to$.

2.5.2 From special unitary to symplectic groups

Denote by $I_{m,n}$ the matrix given by $\begin{pmatrix} I_m & 0 \\ 0 & -I_n \end{pmatrix}$, namely the matrix of the Hermitian form preserved by the groups $SU(m, n)$. As we did for for the unitary group in the previous subsection, we identify a matrix $A + \sqrt{-1}B \in SU(m, n)$ with the matrix with real coefficients $\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$. This gives an embedding $SU(m, n) \hookrightarrow GL_{2(m+n)}(\mathbb{R})$, which is not symplectic. Nevertheless, the image of $SU(p, q)$ preserves the bilinear form given by the matrix $\begin{pmatrix} 0 & I_{m,n} \\ -I_{m,n} & 0 \end{pmatrix}$. This is isometric to the standard symplectic form via the isometry $F = \begin{pmatrix} I_{m,n} & 0 \\ & I_{m+n} \end{pmatrix}$, as a direct computation shows.

As before, we have an induced embedding $SU(m,n) \hookrightarrow Sp(2(m+n),\mathbb{R})$ given by $T \mapsto ZTZ^{-1}$. This map sends the maximal compact subgroup $S(U(m)\times U(n)) \subset SU(m,n)$ into a conjugate of the maximal compact subgroup $U(2(m+n)) \subset Sp(2(m+n),\mathbb{R})$. But on the symplectic groups the function v that induces the Dupont-Guichardet-Wigner cocycles is the determinant and this is invariant under conjugation. Since matrices in $S(U(m)\times U(n))$ have determinant 1, the symplectic Dupont-Guichardet-Wigner cocycle restricts to the trivial cocycle on $SU(m,n)$.

3 Uniform perfectness of $SU(m,n)$

Although the fact that all simple Lie groups are uniformly perfect seems to be folklore, the authors did not find it explicitly in the literature. For all semi-simple Lie groups whose maximal compact is semi-simple any element is the product of 2 commutators (see [7]. However this does not applies precisely to $SU(m,n)$. One also knows that there are elements which are not commutators (from [33]). An explicit bound for the number of reflections needed to write any element in $U(m,n)$ as a product was given in [8] and the number of commutators could be deduced from it. For the sake of completeness we give an explicit bound for $SU(m,n)$ using a similar reasoning. Our bounds are linear in $m+n$, but one might reasonably believe that 3 commutators would suffice.

Let B be an hermitian bilinear $\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$ form on the $\mathbb{C}$-vector space V of signature (m,n). We denote by Q the associated quadratic form $B(v,v) = Q(v)$. A vector $v \in V$ is *isotropic* if $Q(v) = 0$. A subspace of dimension 2 $H \subset V$ is *hyperbolic* if H admits a basis of two isotropic vectors such that $B(u,v) = 1$. Equivalently a subspace of dimension 2 is hyperbolic if and only if the hermitian form restricted to H is non-degenerated and of signature $(1,1)$

Definition 3.1. *Let $a \in \mathbb{C}$ such that $a + \overline{a} = 0$ and $u \in V$ an isotropic vector. Then the map $\tau_{u,a}$ defined by $\tau_{u,a}(v) = v + aB(v,u)u$ is a* transvection*. This is an element in $SU(V)$.*

Definition 3.2. *Let u be an anisotropic vector (i.e. $Q(u) \neq 0$), and $a \in \mathbb{C}$ such that $a\overline{a} = 01$. Then the map $\sigma_{u,a}$ defined by $\sigma_{u,a}(v) = v + (a-1)\frac{B(v,u)}{Q(u)}u$ is a* quasi-reflection *along u.*

Transvections and quasi-reflections enjoy the following properties:

1. Suppose that $u \in V$ is isotropic, that $a, b \in \mathbb{C}^*$ such that $a = -\overline{a}$ and $b = -\overline{b}$. Then:
   (a) $\tau_{u,a}\tau_{u,b} = \tau_{u,a+b}$. In particular $\tau_{u,a}^{-1} = \tau_{u,-a}$
   (b) $\forall c \in \mathbb{C}^*$, $\tau_{cu,a} = \tau_{u,c\overline{c}a}$.
   (c) For all $\sigma \in U(m,n)$, $\sigma\tau_{u,a}\sigma^{-1} = \tau_{\sigma(u),a}$.
   (d) $\forall v \in u^{\perp}$, $\tau_{u,a}(v) = v$.
2. Suppose that $u \in V$ is anisotropic, that $a, b \in \mathbb{C}^*$ such that $a\overline{a} = 1$ and $b\overline{b} = 1$. Then:
   (a) $\sigma_{u,a}\sigma_{u,b} = \sigma_{u,ab}$. In particular $\sigma_{u,a}^{-1} = \sigma_{u,\frac{1}{a}}$.
   (b) $\forall c \in \mathbb{C}$, $\sigma_{cu,a} = \sigma_{u,a}$
   (c) $\forall \tau \in U(p,q)$, $\tau\sigma_{u,a}\tau^{-1} = \sigma_{\tau(u),a}$.
   (d) $\forall v \in u^{\perp}$, $\sigma_{u,a}(v) = v$. In particular $\det \sigma_{u,a} = a$.

Define the projective isotropic cone $\mathbb{P}C^0 = \{[v] \in \mathbb{P}V \mid Q(v) = 0\}$.

Proposition 3.1. *Let $[u], [v] \in \mathbb{P}C^0$ be two distinct points. Then there exists an element $\sigma \in SU(V)$ which is the product of at most two transvections such that $\sigma([u]) = [v]$.*

Proof. There are two cases, first we consider the case $B(u,v) \neq 0$. Without loss of generality we may assume that $B(u,v) = 1$, so that $< u, v >$ is an hyperbolic plane. Let $a \in \mathbf{C}$ such that $a + \overline{a} = 0$, $a \neq 0$. Then $x = u + av$ is isotropic, and if $b = -\frac{1}{\overline{a}}$, then a direct computation shows that:

$$\begin{aligned} \tau_{x,b}(u) &= u + av - (\frac{1}{\overline{a}})B(u, u + av)(u + av) \\ &= -av. \end{aligned}$$

If $B(u,v) = 0$, then necessarily $p + q \geq 4$ and we may chose $x \in V \setminus (u^\perp \cup v^\perp)$, so that $B(u,x) \neq 0$ and $B(v,x) \neq 0$. By rescaling u and v we may assume that these two products are 1. For any $\lambda \in \mathbb{C}$ we have $Q(x + cu) = Q(x) + \overline{c} + c$, so for a suitable choice of c $Q(x) = 0$, and therefore both $< x, u >$ and $< y, x >$ are hyperbolic planes. As in the first part there exists therefore a transvection τ_1 such that $\tau_1[u] = [x]$ and a transvection τ_2 such that $\tau_2[x] = [v]$. ☐

Proposition 3.2. *If V is an hyperbolic plane and $[u] \neq [v]$ and $[x] \neq [y]$ are four points in $\mathbb{P}C^0$, then there is a product of at most three transvections that sends $([u],[v])$ to $([x],[y])$*

Proof. By the previous proposition there is a product of at most two transpositions that sends $[u]$ to $[x]$. We assume that $[u] = [x]$, and exhibit a transvection τ such that $\tau[v] = [y]$ and $\tau[u] = [u]$. Write $y = au + bv$, since y is isotropic, we have $a\overline{b} + \overline{a}b = 0$, and $b \neq 0$ since $[y] \neq [x] = [u]$. Set $c = -\frac{a}{b}$, note that $c = -\overline{c}$, then a direct computations shows that $\tau_{u,c}$ is the transvection we are looking for. ☐

Proposition 3.3. *Let V be an hyperbolic plane, then any element in $SU(V) = SU(1,1)$ is the product of at most 7 transvections.*

Proof. Let (u,v) be a hyperbolic pair. Then a map $\sigma \in SU(V)$ sends (u,v) to another hyperbolic pair $(\sigma(u), \sigma(v))$. By the preceding proposition, there is $\tau \in SU(V)$ which is a product of at most three transvections such that $\tau\sigma[u] = [u]$ and $\tau\sigma[v] = v$. In particular there exists $\beta \in \mathbb{C}^*$ such that $\tau\sigma \in SU(V)$ has as associated matrix in the basis u, v $\begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}$ Then $B(u,v) = 1 = B(\tau\sigma(u), \tau\sigma(v)) = \beta\overline{\beta}^{-1}$, so $\beta \in \mathbb{R}$.

Choose $a \in \mathbb{C}^*$ such that $\overline{a} = -a$ then a direct computation shows that

$$\tau\sigma = \tau_{v,-a}\tau_{u,a^{-1}(1-b^{-1})}\tau_{v,ab}\tau_{u,a^{-1}(b^{-2}-b^{-1})}.$$

☐

Lemma 3.1. *Let V be a $\mathbb{C}$-vector space of dimension 2, with a sesquilinear form B and an associated quadratic form Q. Assume that Q takes positive and negative values on V. Then V is hyperbolic, i.e. B has signature $(1,1)$ and in particular is non-degenerated.*

Proof. If B is degenerated, then pick a non-zero vector $v \in V \cap V^\perp$ and complete it into a basis $(v, w$ of V. Then in this basis B has matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & Q(w) \end{pmatrix}$$

For any vector $x = av + bw \in V$ a direct computation shows that $Q(x) = b\overline{b}Q(w)$, so that Q can not take values both positive and negative.

We may thus assume that B is non-degenerated, therefore its signature is either $(2,0)$, $(0,2)$ or $(1,1)$. In the first two case the sign of Q is always positive or negative, and are thus excluded so that B is of signature $(1,1)$, hence V is hyperbolic. ☐

We turn now to the usual decomposition theorem for unitary transformations as in [?]. To any unitary map $\sigma \in U(m,n)$ one associates a subspace $W \subset V$, and a non-degenerated sesquilinear form B_σ on W. To any basis $(u_1, \dots, u_r)$of W one associates then a decomposition of $\sigma = \sigma_{u_1,a_1} \cdots \sigma_{u_r,a_r}$ as a product of transvections and quasi-reflections. One gets always at least one quasi-reflection. And σ_{u_i,a_i} is a transvection if u_i is isotropic with respect to B, and a quasi-reflection otherwise. Notice in particular that this shows that *any $\sigma \in U(m,n)$ is a product of at most $dimW \leq m+n$ quasi-reflections or transvections.*

Proposition 3.4. *Let $\sigma \in SU(m,n)$ and assume that $\sigma = \sigma_{u_1,a_1} \cdots \sigma_{u_r,a_r}$ is written as a product of r quasi-reflections. Then σ can be written as a product of at most $14r$ transvections.*

Proof. We prove the assumption by induction on r.

If $r = 1$, then $\sigma = \sigma_{u_1,a_1}$ is a quasi-reflection in $SU(p.q)$. Since $\det \sigma = 1 = a$, $\sigma = \sigma_{u_1,1} = Id$ is a transvection. If $r = 2$, then $\sigma = \sigma_{u_1,a_1}\sigma_{u_2,a_2}$. Notice that then $\det \sigma = 1 = a_1 a_2$. Let $a = a_1$ There are two cases.

1. If $Q(u_1)Q(u_2) < 0$, then $H = \langle u_1, u_2 \rangle$ is a hyperbolic plane, and $\sigma \in SU(H)$. In particular σ is the a product of at most $7 \leq 14 \times 2$ transvections.
2. if $Q(u_1)Q(u_2) > 0$, the because $p, q \geq 1$ there exists v such that $Q(u_i)Q(v) < 0$. Then $\sigma = \sigma_{u_1,a}\sigma_{u_2,\frac{1}{a}} = \sigma_{u_1,a}\sigma_{v,\frac{1}{a}}\sigma_{v,a}\sigma_{u_2,\frac{1}{a}}$. By the preceding case both $\sigma_{u_1,a}\sigma_{v,\frac{1}{a}}$ and $\sigma_{v,a}\sigma_{u_2,\frac{1}{a}}$ can be write as products of at most 7 transvections and σ is a product of at most 14 transvections..

Assume that $r \geq 3$ and that the proposition is true for any $k < r$. Write $\sigma = \sigma_{u_1,a_1} \cdots \sigma_{u_r,a_r}$ where by hypothesis $a_1 \cdots a_r = 1$. Again there are two cases.

1. If Q changes sign on the set $\{u_1, \dots u_r\}$, then there are two consecutive indices with opposite signs, and without loss of generality we may assume that $Q(u_1)Q(u_2) < 0$. Then

$$\begin{aligned} \sigma &= \sigma_{u_1,a_1}\sigma_{u_2,a_2} \cdots \sigma_{u_r,a_r} \\ &= \sigma_{u_1,a_1}\sigma_{u_2,\frac{1}{a_1}}\sigma_{u_2,a_1}\sigma_{u_2,a_2} \cdots \sigma_{u_r,a_r} \\ q &= \sigma_{u_1,a_1}\sigma_{u_2,\frac{1}{a_1}}\sigma_{u_2,a_1a_2} \cdots \sigma_{u_r,a_r} \end{aligned}$$

   Then as before $\sigma_{u_1,a_1}\sigma_{u_2,\frac{1}{a_1}}$ is a product of at most 7 transvections and $\sigma_{u_2,a_1a_2} \cdots \sigma_{u_r,a_r}$ is, by induction, a product of at most $7(r-1)$ transvections.

2. If the sign of Q is constant on $\{u_1, \dots u_r\}$. Since the decomposition as a product of transvections and quasi-reflections explained before is fully determined by the basis in the space W_σ associated to σ, without loss of generality we may assume that the sign of Q is constant on $L = \langle u_1, \dots, u_r \rangle$, for otherwise changing basis we would get a new decomposition as in the preceding point. In particular B restricted to L is then definite and non-degenerated. Then there exists $v \in V$ such that $v \in \langle u_1, \dots, u_r \rangle^\perp$ and such that $Q(u_1)Q(v) < 0$. Then as $B(v, u_i) = 0$ any quasi-reflection supported by v will commute with any quasi-reflection supported by an u_i, and we have the following decomposition: Then

$$\begin{aligned} \sigma &= \sigma_{u_1,a_1}\sigma_{u_2,a_2} \cdots \sigma_{u_r,a_r} \\ &= \sigma_{u_1,a_1}\sigma_{v,1/a_1}\sigma_{u_2,a_2}\sigma v, 1/a_2 \cdots \sigma_{u_r,a_r}\sigma_{v,1/a_r}. \end{aligned}$$

   Applying r times the case 1, we get that σ is a product of at most $14r$ transvections.

□

Proposition 3.5. *Any transvection is a commutator in $SU(m,n)$.*

Proof. Let $\tau_{u,a}$ be a transvection. complete u in a hyperbolic pair (u, v) and compute in the hyperbolic plane generated by them. The matrix of $\tau_{u,a}$ is:

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}.$$

Choose $b \in \mathbb{C}^* \setminus \{\pm 1\}$, and set $c = \frac{a}{b^2-1}$. Then a direct computation shows that:

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}^{-1}$$

□

Corollary 3.1. *The group $SU(m, n)$ is uniformly perfect, more precisely: any element is a product of at most $14(m + n)$ commutators.*

4 Quasi-homomorphisms on mapping class group quotients

4.1 Restriction homomorphisms and proof of Theorem 1.2

Let us the terminology from the Introduction. Set $s_{p,g}$ for the number of simple non-compact factors of the semi-simple Lie group $\mathbb{G}_p$. We also write $s^*_{p,g}$ for the number of such factors of non-zero signature i.e. of the form $SU(m, n)$ with $1 < m < n$. Each simple factor is associated to a primitive root of unity ζ of order p. Those ζ corresponding to non-compact simple factors or with non-zero signature will be called non-compact roots and respectively non-compact roots of non-zero signature. Denote also by $r_{p,g}$ the minimal number of normal generators of $\ker \widetilde{\rho}_p$ (i.e. of $\ker \widetilde{\rho}_{p,\zeta}$, for any primitive ζ) within $\widetilde{M_g}^{\text{univ}}$, namely the minimum number of relations needed to add in order to obtain the quotient $\widetilde{\rho}_p(\widetilde{M_g}^{\text{univ}})$.

Proposition 4.1. *We have $s_{p,g} \geq \left[\frac{(g-1)}{2g}p\right]$, if $g \geq 3$ and $s_{p,2} \geq \left[\frac{p}{3}\right]$, respectively.*

Proof. There is an obvious injection of the pure braid group PB_g on g strands into M_g, when $g \geq 3$. When $g = 2$ we should use the homomorphism $PB_3 \to M_2$ from [16], whose kernel is the center of PB_3. The restriction of the representation $\rho_{p,\zeta}$ to PB_g is not irreducible. One summand of the restriction $\rho_{p,\zeta}|_{B_g}$ is the quantum representation in the space $W_{0,g}$ of conformal blocks associated to a sphere with $g + 1$ boundary components whose boundary circles are labeled by the colors $(0, 2, 2, \ldots, 2)$ (for odd p) and $(0, 1, 1, \ldots, 1)$ (for even p), respectively. It is rather well-known (see e.g. [15, 16] for the case $g = 3, 4$) that this summand is equivalent to the Burau representation β_{q_p} of B_g, where q_p is given by $q_p = \zeta^{-4}$ for odd p and $q_p = \zeta^{-8}$, for even p.

Now, for $g \geq 3$ the Burau representation of B_g has an invariant Hermitian form defined by Squier in [32]. Moreover, we have:

Lemma 4.1. *The restriction of the $\widetilde{M_g}$-invariant Hermitian form H to the subspace of conformal blocks $W_{0,g}$ has the same signature as Squier's invariant form.*

Proof. Squier's invariant Hermitian form is obtained as the restriction of the M_g-invariant Hermitian form H on the space of conformal blocks. This follows from the explicit description of both Hermitian forms from [1] and [32], but the proof is rather long and calculatory. Otherwise, a simpler proof of the equivalence of the signatures of the two forms (which is what needed in the sequel) could be obtained as follows. Kuperberg proved in [26] that Burau's representation of B_g at roots of unity of order $p \neq g$ is Zariski dense in the pseudo-unitary group associated to

Squier's Hermitian form. This implies that the restriction of the quantum Hermitian form H to the subspace $W_{0,g}$ of conformal blocks is equivalent to the Squier Hermitian form and hence the two forms have the same signature. ☐

Now, Squier's form is positive (or negative) if and only if $\arg(q_p) \in (-\frac{2\pi}{g}, \frac{2\pi}{g})$ (see [32]). In particular, Squier's Hermitian form is indefinite and thus H is indefinite for all ζ such that $\arg(q_p) \notin (-\frac{2\pi}{g}, \frac{2\pi}{g})$. This proves that there are at least $\frac{(g-1)}{g}p$ non-compact factors in $\mathbb{G}_p$. ☐

Proposition 4.2. *For $g \geq 2$, $p \geq 5$ and $(g,p) \neq (2,5)$ the real rank of $\mathbb{G}_p$ is at least 2. Furthermore, for $g \geq 4$ and $p \geq 5$ each simple factor non-compact factor of $\mathbb{G}_p$ has rank at least 2. Moreover, the real rank of $\mathbb{G}_p$ is at least $\left[\frac{(g-1)}{2g}p\right]\left(\frac{p-1}{2}\right)^{g-3}$, for $g \geq 3$ and odd p.*

Proof. Let $W_g^{\pm}(\zeta)$ be a maximal positive/negative subspace of the space W_g of conformal blocks in genus g for the Hermitian form H_ζ. Consider a separating curve γ on the closed orientable surface Σ_g whose complementary sub-surfaces have genus $g-1$ and 1 respectively. If we label γ by 0 then the spaces of conformal blocks associated to these two sub-surfaces are isometrically identified to the spaces of conformal blocks of the closed surfaces obtained by capping off the boundary components. Therefore we have natural isometric embeddings $W_g \otimes W_1 \hookrightarrow W_g$. It is well-known that $W_1 = W_1^+$ is positive for any ζ. Therefore we obtain the following isometric embeddings: $W_{g-1}^+(\zeta) \otimes W_1 \hookrightarrow W_g^+(\zeta)$ and $W_{g-1}^-(\zeta) \otimes W_1 \hookrightarrow W_g^-(\zeta)$. In particular we have for odd p

$$\dim W_g^+(\zeta) \geq (\dim W_1)^g = \left(\frac{p-1}{2}\right)^g$$

Lemma 4.2. *If $W_3(\zeta) = W_3^+(\zeta)$ is positive, then $W_g(\zeta) = W_g^+(\zeta)$, i.e. the simple factor associated to ζ is compact.*

Proof. The Hermitian form associated to a 3-holed sphere with arbitrary colorings can be obtained as the restriction of the Hermitian form of the genus 3 surface and hence is positive. This implies that the Hermitian form on every space of conformal blocks is positive. ☐

It follows that either $W_g = W_g^+(\zeta)$ is positive or else

$$\dim W_g^-(\zeta) \geq (\dim W_1)^{g-3} \dim W_3^-(\zeta) \geq \left(\frac{p-1}{2}\right)^{g-3}$$

The two formulas above show that the rank of each simple non-compact factor of $\mathbb{G}_p$ is at least $\left(\frac{p-1}{2}\right)^{g-3}$.

On the other hand if $(g,p) \neq (2,5)$ then by direct calculation one obtains that the Hermitian form associated to the 1-holed torus with the boundary circle colored by 1 (or 2) is not positive definite for all values of the root of unity. The argument above implies that the real rank of $\mathbb{G}_p$ is at least 2. ☐

Remark 4.1. When $g = 2$ and $p = 5$ the group $\mathbb{G}_p$ is the rank one pseudo-unitary group $SU(1,4)$. The image of $\widetilde{\rho}_p(\widetilde{M_g})$ lies naturally into $SU(1,4) \times SU(5)$ and projects onto a lattice in $SU(1,4)$. In fact H_ζ has signature ± 5 or ± 3.

Proposition 4.3. *We have* $\dim H^2(\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{univ}}), \mathbb{R}) \leq r_{p,g}$, *if $g \geq 3$.*

The following construction is the key ingredient in the proof of this proposition. For any primitive ζ of order p we obtain a bounded class $\rho_{p,\zeta}^* K_{SU(m(\zeta),n(\zeta)}$ in $H_b^2(\widetilde{M_g}^{\text{univ}}, \mathbb{R})$. If ζ runs over the non-compact primitive roots of non-zero signature then the result of Burger and Iozzi from [5] tells us (see [14]) that the classes $\rho_{p,\zeta}^* K_{SU(m(\zeta),n(\zeta)}$ are independent over $\mathbb{Q}$, and in particular they are non-vanishing. Notice that they obviously vanish in the usual cohomology as $H^2(\widetilde{M_g}^{\text{univ}}) = 0$.

Moreover, consider the following map

$$\bar{l}_\zeta = \overline{L}_\zeta|_{\ker \rho_p} : \ker \rho_p \to \mathbb{R}$$

Lemma 4.3. *We have $\bar{l}_\zeta \in \mathrm{Hom}(\ker \rho_p, \mathbb{R})^{\widetilde{M_g}^{\text{univ}}}$, namely $\bar{l}_\zeta$ is a group homomorphism invariant by the conjugacy action of $\widetilde{M_g}^{\text{univ}}$.*

Proof. The boundary of $\overline{L}_\zeta$ is $\widetilde{\rho}_p^*(c_{SU(m(\zeta),n(\zeta))})$ which obviously vanishes on $\ker \widetilde{\rho}_p$, namely

$$\widetilde{\rho}_p^*(c_{SU(m(\zeta),n(\zeta))})(x, y) = 0, \text{ if either } x \text{ or } y \in \ker \widetilde{\rho}_p$$

This implies that $\bar{l}_\zeta$ is a homomorphism.

Eventually recall that $\overline{L}_\zeta$ is a homogeneous quasi-homomorphism and thus it is a class function. This implies that $\bar{l}_\zeta$ is also a class function. □

Proof of Proposition 4.3. The 5-term exact sequence in cohomology associated to the exact sequence

$$1 \to \ker \widetilde{\rho}_p \to \widetilde{M_g}^{\text{univ}} \to \widetilde{\rho}_p(\widetilde{M_g}^{\text{univ}}) \to 1$$

gives us:

$$0 = H^1(\widetilde{M_g}^{\text{univ}}, \mathbb{R}) \to \mathrm{Hom}(\ker \widetilde{\rho}_p, \mathbb{R})^{\widetilde{M_g}^{\text{univ}}} \xrightarrow{\iota} H^2(\widetilde{\rho}_p(\widetilde{M_g}^{\text{univ}}), \mathbb{R}) \to H^2(\widetilde{M_g}^{\text{univ}}, \mathbb{R}) = 0$$

It follows that $\iota(\bar{l}_\zeta) \in H^2(\widetilde{\rho}_p(\widetilde{M_g}^{\text{univ}}); \mathbb{R})$.

By the exactness of the sequence above ι is an isomorphism and hence identifies $\mathrm{Hom}(\ker \widetilde{\rho}_p, \mathbb{R})^{\widetilde{M_g}^{\text{univ}}}$ with $H^2(\widetilde{\rho}_p(\widetilde{M_g}^{\text{univ}}), \mathbb{R})$. The next lemma shows that $\dim \mathrm{Hom}(\ker \widetilde{\rho}_p, \mathbb{R})^{\widetilde{M_g}^{\text{univ}}} \leq r_{p,g}$ and the claim follows. □

Lemma 4.4. *Let $\{a_1, a_2, \ldots, a_{r_{p,g}}\}$ be a minimal system of normal generators for $\ker \widetilde{\rho}_p$ within $\widetilde{M_g}^{\text{univ}}$. Then the evaluation homomorphism $E : \mathrm{Hom}(\ker \widetilde{\rho}_p, \mathbb{R})^{\widetilde{M_g}^{\text{univ}}} \to \mathbb{R}^{r_{p,g}}$, given by $E(f) = (f(a_1), f(a_2), \ldots, f(a_n))$ is injective.*

Proof. Any element $x \in \ker \widetilde{\rho}_p$ is a product $x = \prod_i g_i a_i g_i^{-1}$, for some $g_i \in \widetilde{M_g}^{\text{univ}}$. Since $f \in \mathrm{Hom}(\ker \widetilde{\rho}_p, \mathbb{R})^{\widetilde{M_g}^{\text{univ}}}$ is conjugacy invariant we have: $f(x) = \sum_i f(g_i a_i g_i^{-1}) = \sum_i f(a_i)$ and the claim follows. □

Proposition 4.4. *If $s_{p,g} > r_{p,g}$ then $\widetilde{\rho}_p(\widetilde{M_g}^{\text{univ}})$ is not a lattice in $\mathbb{G}_p$.*

Proof. Recall from [14] that $\mathbb{G}_p$ is a real semi-simple linear algebraic semi-simple group defined over $\mathbb{Q}$. Since $\mathbb{G}_p$ is obtained by restriction of scalars from the anisotropic unitary group it follows that all elements of $\mathbb{G}_p(\mathbb{Z})$ are semi-simple, as being obtained as Galois conjugates of unitary and hence diagonalizable matrices. Therefore, by Borel theorem $\mathbb{G}_p(\mathbb{Z})$ is a cocompact lattice in $\mathbb{G}_p$. This was also noticed in [28].

We know as part of the Matsushima vanishing theorem that for cocompact lattices Γ in semi-simple Lie groups $\mathbb{G}$ the restriction homomorphism $H^j(\mathbb{G},\mathbb{R}) \to H^j(\Gamma,\mathbb{R})$ is an isomorphism as long as $j \leq \mathrm{rk}_{\mathbb{R}}\mathbb{G} - 1$ (see [4], ch.7, Thm.3.4). Then Proposition 4.2 shows that $H^2(\mathbb{G}_p,\mathbb{R}) \to H^2(\Gamma,\mathbb{R})$ is an isomorphism for any odd $p \geq 5$.

Recall now that $\mathbb{G}_p$ is a product of $s_{p,g}$ pseudo-unitary groups of type $SU(m,n)$, each factor being a simple group of isometries of some irreducible Hermitian space. Then by [23] we have $H^2(\mathbb{G}_p,\mathbb{R}) = \mathbb{R}^{s_{p,g}}$ is the vector space generated by the set of Dupont-Guichardet-Wigner classes of the simple factors. In particular, if $s_{p,g} > r_{p,g}$ then the restriction map could not be an isomorphism $H^2(\mathbb{G}_p,\mathbb{R}) \to H^2(\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{univ}}),\mathbb{R})$ cannot be an isomorphism by dimension reasons and so $\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{univ}})$ would not be a lattice in $\mathbb{G}_p$. □

Proof of Theorem 1.2. Since $\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{univ}})$ is a Zariski dense discrete subgroup (as it is contained in $\mathbb{G}_p(\mathbb{Z})$) in $\mathbb{G}_p$ then if it would be isomorphic to a higher rank irreducible lattice then it must have a finite index subgroup which is a lattice in a product of factors of $\mathbb{G}_p$. This follows from Margulis super-rigidity and the arithmeticity of lattices in higher rank Lie groups. But then the Zariski closure of $\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{univ}})$ will be contained in this product of factors. On the other hand, one proved in [14] that $\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{univ}})$ is Zariski closed in $\mathbb{G}_p$ and hence it is a lattice within $\mathbb{G}_p$. Now Proposition 4.4 and Proposition 4.1 settle the claim. □

Proposition 4.5. *If $s^*_{p,g} > r_{p,g}$ then $\widetilde{QH}(\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{univ}}))$ cannot be trivial.*

Proof. Let also denote by $i_{p,\zeta} : \widetilde{\rho}_{p,\zeta}(\widetilde{M_g}^{\mathrm{univ}}) \to PU(m(\zeta),n(\zeta))$ the obvious inclusion. The discussion above shows that $i^*_{p,\zeta}K_{SU(m(\zeta),n(\zeta)}$ is represented by $\iota(\overline{l}_\zeta)$. Thus there exists a linear combinations of the classes $i^*_{p,\zeta}K_{SU(m(\zeta),n(\zeta)}$ which vanishes in $H^2(\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{univ}}),\mathbb{R})$ if and only if the corresponding linear combination in the homomorphisms $\overline{l}_\zeta$ vanishes identically.

If ζ runs over the non-compact primitive roots of non-zero signature then the result of Burger and Iozzi from [5] tells us (see [14]) that the classes $\rho^*_{p,\zeta}K_{SU(m(\zeta),n(\zeta)}$ are independent over $\mathbb{Q}$. If $\widetilde{QH}(\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{univ}}))$ were trivial, then the cohomology classes $\rho^*_{p,\zeta}K_{SU(m(\zeta),n(\zeta)}$ in $H^2(\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{univ}}),\mathbb{R})$ would be independent over $\mathbb{Q}$. But these are integral classes (i.e. lying within the image of $H^2(\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{univ}}),\mathbb{Z})$ in the corresponding real cohomology group), since they are pull-backs of integral classes on $H^2(\mathbb{G}_p,\mathbb{R})$. Therefore they would be linearly independent over $\mathbb{R}$. On the other hand there are $s^*_{p,g}$ such classes living within the vector space $\mathrm{Hom}(\ker\widetilde{\rho}_p,\mathbb{R})^{\widetilde{M_g}^{\mathrm{univ}}}$ which is of dimension at most $r_{p,g}$. This contradiction proves the claim. □

Remark 4.2. If $\widetilde{QH}(\rho_p(M_g))$ were infinite dimensional then $\rho_p(M_g)$ would not be boundedly generated.

4.2 Computations of quasi-homomorphisms

The aim of this section is to compute explicit values of $\overline{L}_\zeta$. Denote by $h^+_g(\zeta)$ the dimension of the maximal positive subspace of the Hermitian form H_ζ.

Remark 4.3. Let c denote the generator of the center of $\widetilde{M_g}^{\mathrm{univ}}$. Since c is central we have: $\overline{L}_\zeta(cx) = \overline{L}_\zeta(x) + \overline{L}_\zeta(c)$. Consider the following map:

$$\delta\overline{L}_\zeta(x,y) = \overline{L}_\zeta(\widetilde{x}\widetilde{y}) - \overline{L}_\zeta(\widetilde{x}) - \overline{L}_\zeta(\widetilde{y})$$

where $x, y \in M_g$ and $\widetilde{x}, \widetilde{y} \in \widetilde{M_g}^{\mathrm{univ}}$ are arbitrary lifts of x, y. Then $\delta\overline{L}_\zeta$ is well-defined and is a 2-cocycle on M_g. It follows then that the class of $\delta\overline{L}_\zeta$ is $\overline{L}_\zeta(c)$ times a generator of $H^2(M_g)$.

Therefore, if $\overline{L}_\zeta(c) = \overline{L}_{\zeta'}(c)$, then $\delta\overline{L}_\zeta - \delta\overline{L}_{\zeta'}$ is a boundary, namely the boundary of $L_\zeta - L_{\zeta'}$, which descends to M_g.

Proposition 4.6. *Suppose that $h_g^+(\zeta)$ is not divisible by the prime $p \geq 5$. Then $\iota(\overline{l}_\zeta) \neq 0 \in H^2(\widetilde{\rho}_p(\widetilde{M_g}^{\text{univ}}); \mathbb{R})$.*

Proof. It is enough to compute $\overline{L}_\zeta(c)$:

Lemma 4.5. *We have*

$$\overline{L}_\zeta(c) \equiv -6h_g^+(\zeta)\arg(\zeta) \pmod{2\pi\mathbb{Z}}$$

Proof. We know that $\widetilde{\rho}_{p,\zeta}(c) = \zeta^{-6}$. The formula follows then from Theorem 1.1. □

If $\overline{L}_\zeta(c) \not\equiv 0 \in \mathbb{R}/2\pi\mathbb{Z}$ then $\overline{L}_\zeta(c) \neq 0$. This means in particular that $\overline{L}_\zeta(c^n) \neq 0$ for any $n \neq 0$. However, we know that $c^p \in \ker \widetilde{\rho}_{p,\zeta}$. Thus $\overline{l}_\zeta(c^p) \neq 0$ so that $\overline{l}_\zeta$ is not identically zero. □

Remark 4.4. Explicit computations of $h_2^+(\zeta)$ seem to be rather complicated. We know that $h_2^+(\exp(\frac{4\pi}{5})) = 1$ and hence for $p = 5$ and $g = 2$ the assumptions of the previous proposition are satisfied.

Remark 4.5. We want now to express $\overline{L}_\zeta(T_\gamma)$. These are class functions so we can compute them using convenient basis in the space of conformal blocks. When the same holds true for $\overline{L}_\zeta(T_\gamma^p)$ the previous method will provide non-trivial elements of $\text{Hom}(\ker \widetilde{\rho}_p, \mathbb{R})^{\widetilde{M_g}^{\text{univ}}}$. Changing ζ might result in families of linearly independent elements and thus to lower bounds for the rank of $H^2(\widetilde{\rho}_p(\widetilde{M_g}^{\text{univ}}), \mathbb{R})$.

Remark 4.6.

1. The representation $\widetilde{\rho}_{p,\zeta}$ is defined on the bigger group $\widetilde{M_g}$ which contains $\widetilde{M_g}^{\text{univ}}$ as a subgroup of index 12. What is the class of $\widetilde{\rho}_{p,\zeta}^* c_{SU(m,n)} \in H^2(\widetilde{M_g})$? We have $H^2(\widetilde{M_g}) = \mathbb{Z}/12\mathbb{Z}$.
2. The same could be done using $c_{PU(m,n)}$. Is it true that the class of $\rho_{p,\zeta}^* c_{PU(m,n)}$ is $\frac{mn}{m+n}\overline{L}_\zeta(c)$ times a generator of $H^2(M_g)$?

Remark 4.7. It is not clear whether we can define L_ζ directly on $\widetilde{\rho}_{p,\zeta}(\widetilde{M_g}^{\text{univ}})$, namely if L_ζ descends to this quotient. If it does, then it also defines a class in $H^2(\rho_{p,\zeta}(M_g))$. Furthermore we could find the action of the Galois conjugacy σ which sends ζ to ζ' at the level of quasi-homomorphisms. In particular we can find whether the difference of the boundary of the quasi-homomorphisms L_ζ and $\sigma^* L_\zeta$ is the boundary of a quasi-homomorphism X on $\rho_{p,\zeta}(M_g))$ or not. When pulled-back on $\widetilde{M_g}^{\text{univ}}$ the quasi-homomorphism X is not bounded because its bounded cohomology classes is non-vanishing according to Burger-Iozzi (see [5] and [14]). If we were able to show that there is at least one non-trivial quasi-homomorphism on $\rho_{p,\zeta}(M_g))$ then it would follow that this group cannot be an irreducible higher rank lattice in a semi-simple Lie group, according to the result of Burger and Monod from [6].

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