A crash course in quantum invariants of knots [1]

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CHAPTER 1

Higher dimensional manifolds versus knots

0.1. The fundamental objects we are dealing with in this sequel are unions of disjoint closed loops in the three space, called links. When there is only one component then it is called a knot.

0.2. TOP, PL,DIFF. The essential structures relevant for the manifold topology are TOP, PL and DIFF. Let us state briefly the terms. A separable Hausdorff topological space is an n-dimensional *topological manifold* if every point has an open neighborhood homeomorphic to an open subset of $\mathbf{R}^n_+$. Such open subsets are called charts. The topological manifold is a $\mathcal{C}^r$-manifold if it has an an open covering with charts (i.e. an atlas) for which the associated transition functions between the images of charts in $\mathbf{R}^n_+$ are $\mathcal{C}^r$-maps. Two $\mathcal{C}^r$-structures on a manifold are isotopic if the identity map is isotopic through homeomorphisms to a $\mathcal{C}^r$-diffeomorphism. Although the notion of $\mathcal{C}^r$-manifold exists for any $r = 0, 1, 2, ..., \infty$, only the cases $r = 0$ and $r = \infty$ are interesting. In fact a theorem of Munkres states that for any $1 \leq r \leq k \leq \infty$ a $\mathcal{C}^r$-manifold has a $\mathcal{C}^k$-structure for which the induced $\mathcal{C}^r$-structure is $\mathcal{C}^r$-diffeomorphic to the initial one. This $\mathcal{C}^k$-structure is unique up to isotopy through $\mathcal{C}^r$-diffeomorphisms.

Now TOP is the category of rough topological manifolds (corresponding to $r = 0$) and DIFF is the category of smooth manifolds (corresponding to $r = \infty$). There is an intermediary structure for which the transition function of an atlas are piecewise linear functions, with respect to the linear structure of $\mathbf{R}^n_+$. This is the PL category.

0.3. The Hauptvermutung. The aim of manifold topology is to classify the topological manifolds and to find a way to tell which manifolds carry a PL (or a DIFF) structure and how many different such. For instance a cornerstone for the seventies was the

HAUPTVERMUTUNG PROBLEM 1. *Let M and N be two compact DIFF (PL) n-manifolds and $f : M \to N$ be a homeomorphism (if M has boundary then it is supposed that the restriction to the boundary is DIFF (PL)). Does then exists a diffeomorphism (resp. PL homeomorphism) $g : M \to N$ (possibly isotopic or homotopic to f) ?*

The answer to this problem is negative in dimensions $n \geq 6$ ($n \geq 5$ if there is no boundary) for general M and N. For instance there exists an obstruction $\wr \in H^3(M, \partial M; \mathbf{Z}/2\mathbf{Z})$ which vanishes if and only if f is isotopic to a PL homeomorphism. Moreover if M is simply connected then f is homotopic to a PL homeomorphism if and only if the boundary of the previous obstruction $\delta\wr \in H^4(M, \mathbf{Z}/2\mathbf{Z})$ vanishes. In particular there exist only a finite number of smooth (and PL) structures on a given topological manifold.

The surgery machinery, though sophisticated, yields some quite satisfactory (even if not very explicit) classification results for simply connected n-manifolds ($n \geq 5$). It should be stressed that the simple connectedness is essential: in fact any finitely presented group can be obtained as the fundamental group of a closed smooth n-manifold, for any $n \geq 4$. Since the isomorphism problem for groups is undecidable (after S.Novikov) it follows that the classification of non-simply connected n-manifolds is also undecidable.

Let us mention that PL and DIFF agree for $n \leq 6$.

0.4. Dimension 4. The dimension $n = 4$ remains quite mysterious despite many efforts and a lot of exciting results. M.Freedman obtained the classification of simply connected 4-manifolds, in terms of the intersection pairing and the Kirby-Siebenmann obstruction. Meantime, using Donaldson and the latter Seiberg-Witten invariants it has been shown that the smooth structures are rather special. Only a few positive definite bilinear forms over $\mathbf{Z}$ can be realized as intersection forms of smooth manifolds, namely the diagonal ones. Let us mention that $\mathbf{R}^n$ has an unique smooth structure for $n \neq 4$, but $\mathbf{R}^4$ has uncountable many different exotic structures. It is not known at present whether the sphere S^4 has any exotic smooth structure, but there are a lot of candidates for that.

0.5. Dimension 3. The dimensions $n \leq 3$ do not have such peculiarities. In fact TOP, PL and DIFF are equivalent categories. Any compact 3-manifold is triangulable and has an unique DIFF structure according to E.Moise. The corresponding statement for surfaces is due to F.Rado. Notice that surfaces carry even complex structures (not unique this time). The surfaces are classified by their orientability type, genus and number of boundary circles. The classification problem for compact 3-manifolds is one of most popular problem in topology and it is still unsolved. Most activity was done on various aspects of W.Thurston's Geometrization Conjecture. We recall that the famous Poincare Conjecture, which states that a closed 3-manifold homotopy equivalent to S^3 is homeomorphic to S^3 will be also a byproduct of the Geometrization ideas.

0.6. Wild arcs. The distinction between TOP to DIFF apply also in the case of embeddings, thus for knots. A link L in $\mathbf{R}^3$ is *tame* if it consists of disjoint simple curves made of from a finite number of linear segments. Usually the links are considered in S^3 hence the linear structure is that of the boundary of the 4-simplex.

There exist *wild* knots exhibiting surprising features which belong to the TOP realm. Let us mention that there exist wild arcs (Artin-Fox arcs) in $\mathbf{R}^3$ whose complement are not simply connected. Constructing a tube around one such arc we find a sphere S^2 topologically embedded in S^3 which does not bound a ball on each side, since one complementary component is not simply-connected (the Schoenflies Conjecture in dimension 2). These embeddings are not locally flat at two points. If all points had been locally flat it would follow that the sphere bounds a ball on each side (after E.Brown).

These aspects are intimately related to other pathologies arising for open 3-manifolds. In particular there exist open contractible 3-manifolds which are not homeomorphic to $\mathbf{R}^3$, because they are not simply connected at infinity.

In order to rule out such problems we will restrict ourselves from now on to tame knots and links. The natural equivalence relation on tame links is that induced by PL homeomorphisms.

0.7. Knots and links.

DEFINITION 1.1. *Two links are equivalent if there exists an orientation preserving PL homeomorphism of S^3 transforming one into the other. If the links are oriented we ask that the orientation agree.*

Remark that a tame link is equivalent to a smooth link. We recall that a diffeomorphism is orientation preserving if the jacobian matrix has positive determinant (using local coordinates). A more familiar way to see the equivalence is via isotopy.

DEFINITION 1.2. *Two embeddings $\varphi_0, \varphi_1 : K \to X$ are isotopic if there exist a family of embeddings $\varphi_t : K \to X$ relating them, such that the map $\varphi : K \times [0,1] \to X \times [0,1]$, $\varphi(x,t) = (\varphi_t(x), t)$ is a PL map (actually an embedding).*
The PL homeomorphisms h_0 and h_1 of X are isotopic if there exists a one parameter family $h_t : X \to X$ of homeomorphisms, $t \in [0,1]$, such that the map $h : X \times [0,1] \to X \times [0,1]$, $h(x,t) = (h_t(x), t)$ is a PL map (actually a homeomorphism).

Notice that for compact $K \subset int(X)$ two embeddings φ_0, φ_1 are isotopic if and only if they are ambient isotopic, i.e. there exists a PL homeomorphism h of X isotopic to identity such that $\varphi_1 = h \circ \varphi_0$.

REMARK 1.1. *A PL homeomorphism (diffeomorphism) of S^3 is orientation preserving if and only if it is isotopic to identity.*

Sometimes it is important to consider a slightly larger equivalence relation, by allowing the PL homeomorphism be orientation reversing. Then the links are said to have *the same type*. Thus either the link or its mirror image is equivalent to the other.

0.8. Complements of knots. The complement of the link L is $S^3 - N(L)$ where $N(L)$ is a tubular neighborhood of the link in S^3. Since L is tame $N(L)$ can be chosen a sufficiently small metric neighborhood of L. Each topological invariant associated to the complement becomes then an invariant for the link, in particular the fundamental group $\pi_1(S^3 - N(L))$, which is called the group of L. We have the following result of Gordon and Luecke:

THEOREM 1.1. *Two knots whose complements are homeomorphic (by an orientation preserving homeomorphism) have the same type.*

We should stress that the analogous statement for links with more components is not true, in general.
A knot is *composed* if it can be written as the connected sum of two nontrivial knots and otherwise it is *prime*. Recall that the connected sum of two oriented knots is defined as follows: consider the knots are separated by a 2-plane and choose a thin ribbon standardly embedded in the three space, laying with its extremities on the two knots, such that their orientations are not parallel. Excise the small arcs from the knots and replace them with the arcs from the remaining part of the boundary of the ribbon. A classical result asserts the uniqueness of the prime (connected sum) decomposition for knots. Whitten have proved that the previous theorem implies also that:

THEOREM 1.2. *Two prime knots whose groups are isomorphic have (orientation preserving) homeomorphic complements.*

For a composed knot the group does not determine the complement. However the group of the knot determines the type of each of its prime components. Thus examples of different knots with the same group arise by making connected sum of knots with different orientations for the prime components.

CHAPTER 2

Diagrams and the linking number

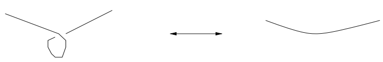
1. The Reidemeister moves

1.1. The generic projections of a link $L \subset \mathbf{R}^3$ give planar graphs, with some extra structure. The Reidemeister theorem relies any two such projections coming from the same link by means of a finite number of *local moves*. This does not means that we have information about the number of such moves or about their precise location in the space.

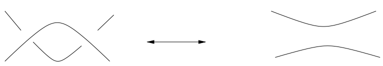
1.2. Diagrams. A good *link projection* is a projection of the link on a plane which is an immersion having only ordinary double points singularities. For a given plane we can move slightly the link by an arbitrary small isotopy such that the projection on the given plane is a link diagram, by genericity. A careful thought shows also that the set of those planes on which the links has a good projection is dense in the set of all planes (identified with the projective space $\mathbf{R}P^3$). When drawing the picture of a link projection we specify at each double point which is the strand which pass over the other one with respect to the natural height. This information completely describes the link, and we call it a *link diagram*. The double points are called *crossings* of the diagram. Notice that a link may have many distinct diagrams.

Theorem 2.1. *Two diagrams of the same link can be obtained one from another by a sequence of planar isotopies and moves of the following type (called Reidemeister moves):*

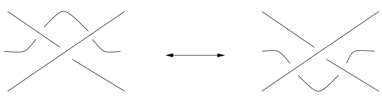
R1



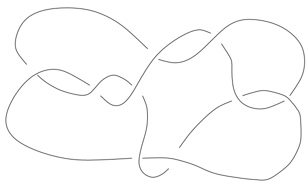
R2



R3

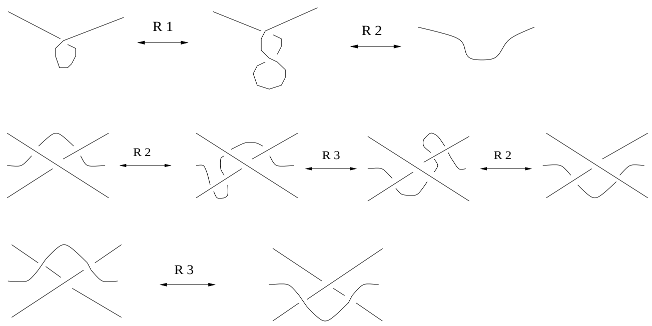


1.3. The theorem above is due to J.W.Alexander, B.G.Briggs and it is commonly attributed to H.Reidemeister. An outstanding question was to find out an explicit upper bound for the number of Reidemeister moves needed to convert a diagram of the unknot having n crossings into the trivial (i.e. no crossings) diagram. Recently J.Hass and J.C.Lagarias obtained th bound 2^{cn} for $c < 10^{11}$. Goeritz found in the thirties that there exist diagrams of the unknot such that any sequence of Reidmeister moves converting it to the trivial diagram should pass through diagrams with more crossings than the initial one. An example is given in the figure below:



1.4.

Proof. There are some additional moves similar to R1, R2 and R3 which are are easy consequences of the formers:



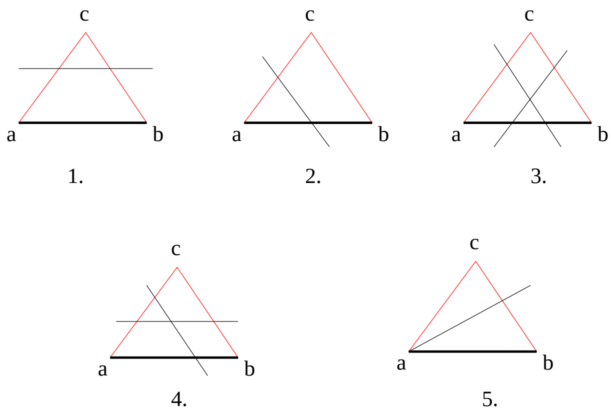
The last one is simply a R3 applied to the strand passing under the two other strands. The latter will be also called Reidemeister moves from now on, of the appropriate type.

The simplest link associated to a planar diagram is obtained by pushing slightly in the normal direction the strands which pass over, around every crossing. Let consider two diagrams whose associated links are isotopic. The isotopy relating them decomposes into a finite sequence of *triangular moves*, when taking sufficiently fine subdivisions of the links, by the PL requirements.

Consider a triangle $T \subset \mathbf{R}^3$ having the interior disjoint from the link K such that $K \cap T$ is an edge of T. Set then K' for the link obtained by excising the common edge and replacing it by the two other edges of T. We say then that K and K' are related by a *triangular move*.

Let us follow the transformation a triangular move induces for the planar projections of links. Set abc for the projection of T, where ab corresponds to the common edge. If $c \in ab$ then nothing changes for the projections, hence we can suppose the triangle abc is nontrivial.

We can partition abc into smaller triangles and express this way the initial move as a composition of triangular moves coming from the latter. It suffices then to consider the case when the triangle abc has the minimal possible intersection with the rest of the diagram. This means that the intersection is connected and it contains either one vertex, or one crossing or it consists in a segment. The possible configurations, up to a symmetry interchanging a and b are pictured below. We didn't figured the configurations with one vertex since they have the same features as the case where only a segment.

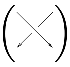
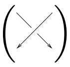


Notice that (any of) the extra strand in the diagram has the same type of crossings with all edges of *abc*, e.g. if once it passes over one edge, then also the second crossing is a overpass. This follows from the fact that the interior of T is disjoint from the knot. Now all the triangular moves in the figures can be expressed in terms of Reidmeister moves: (1.) is R2, (2.) is an isotopy, (3.) is R3, (4.) is the composition of two R2 and one R3, (5.) is an R1.

1.5. The erudite reader may notice that the theorem follows also from some transversality arguments (the easy part of Cerf's theory). □

2. The linking number

2.1. Consider now an oriented diagram in the plane. This means that each component of the associated link has been given an orientation. We say that a crossing p is positive (and associate $\varepsilon(p) = +1$) if turning counterclockwise the upper strand by $\frac{\pi}{2}$ we obtain the right orientation of the other strand. Otherwise the crossing is negative and $\varepsilon(p) = -1$, i.e:

$\varepsilon\Big($  $\Big) = -1, \qquad \varepsilon\Big($  $\Big) = +1.$

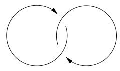
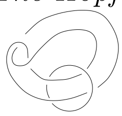
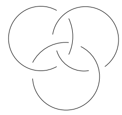
2.2. Let consider now a diagram of a link with two components K and L, an their respective subdiagrams D_K and D_L. We set

$$lk(D_K, D_L) = \frac{1}{2} \sum_{p \in D_K \cap D_L} \varepsilon(p),$$

where $D_K \cap D_L$ denote those crossings where one strand is from D_K and the other one from D_L.

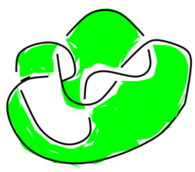
A straightforward check on the diagrams shows that $lk(K, L)$ is invariant at Reidmeister moves. In particular $lk(K, L) = lk(D_K, D_L)$ is independent on the diagram we chose and it is a topological invariant of the link, which is called the *linking number* between K and L. In particular if the linking number is nonzero the knots cannot be separated by an embedded 2-sphere, i.e. the link is not split. Notice that $lk(K, L) = lk(L, K)$ is an integer. In fact any link can be transformed into a trivial link by allowing self-crossings. At the level of diagrams this corresponds to crossing changes. When the sign of a crossing changes the value of the linking number changes by one unit hence the result.

EXAMPLE 2.1.

- *The Hopf link*  *has the linking number +1.*
- *The Whitehead link*  *has linking number 0 but it is not a split link.*
- *The linking numbers of any two components of the Borromean link*  *are zero. If any component is deleted we obtain a split link, but the Borromean link is not split. In fact this link has some "triple linking number" (called triple Massey number) nonzero.*

3. Seifert surfaces for links

3.1. Topological definitions of linking numbers involve the choice of an oriented surface bounding a given knot $K \subset S^3$, which is usually called a *Seifert surface*. The simplest way to find out a bounding surface is the chessboard method. Start with a planar diagram of the link and color the complementary regions in two colors, say green and white in a chessboard fashion. Then each monocolor region forms a surface embedded in S^3 (one surface contains the point at infinity), by imagining the neighborhood of a crossing as a twisted strip.

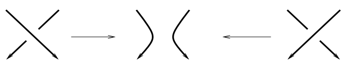


However it can happen that both surfaces obtained this way are unorientable. There is another method to prove that

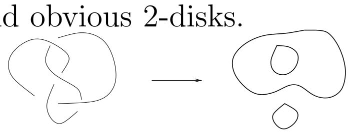
THEOREM 2.2. *Any link in S^3 has a Seifert surface.*

3.2.

PROOF. We transform the crossings of a planar diagram D of the link L in a way compatible with the orientation:



We obtain therefore a number of disjoint circles (called Seifert circles) in the plane. These circles bound obvious 2-disks.



Let us push the disks in the normal direction such that each disk lay at a different height, being therefore disjoint. In order to recover the initial link we have to glue one twisted strip between the disks for each crossing. The surface obtained this way is orientable since the orientations of the Seifert circles match with those induced by the strips. □

4. More about the linking number

4.1. Equivalent definitions of the linking number are provided by the following:

THEOREM 2.3. (1) *Let K be an oriented knot in S^3. Then there is a canonical isomorphism $H_1(S^3 - N(K)) = \mathbf{Z}$, a generator being given by the homology class of a simple loop μ lying in the boundary and bounding a 2-disk in the cylinder $N(K)$ which intersects K in one point (such μ is called a meridian). Then the class of an oriented simple curve $L \subset S^3 - N(K)$ in $H_1(S^3 - N(K))$ is $lk(K, L) lk(K, \mu) \in \mathbf{Z}$.*

(2) *Let F be an oriented surface in S^3 bounded by the knot L. Any other knot K can be slightly moved to become transversal to F. To each intersection point $q \in K \cap F$ we associate a number $\tilde{\varepsilon}(q)$ as follows. The orientation of F together with that of K yield an orientation (frame) for $\mathbf{R}^3$ at q. If this orientation agree with the usual one of $\mathbf{R}^3$ we set $\tilde{\varepsilon}(q) = +1$, otherwise $\tilde{\varepsilon}(q) = -1$. Then*

$$lk(K, L) = \sum_{q \in K \cap F} \tilde{\varepsilon}(q).$$

(3) *Let $\varphi_1, \varphi_2 : S^1 \to \mathbf{R}^3$ be parametrizations of the knots K and L. We have an induced map $\varphi : S^1 \times S^1 \to S^2$ given by*

$$\varphi(x, y) = \frac{\varphi_1(x) - \varphi_2(y)}{\mid \varphi_1(x) - \varphi_2(y) \mid} \in S^2 \subset \mathbf{R}^3,$$

where $\mid \mid$ denotes any norm on $\mathbf{R}^3$. Then we have

$$lk(K, L) = deg\ \varphi = \frac{1}{4\pi} \int_{K \times L} d\,u\, \frac{d\,w \times (u - w)}{\mid u - w \mid^3}.$$

4.2.

PROOF. Let us prove first that $H_1(S^3 - N(K)) \cong \mathbf{Z}$. This follows immediately from the Alexander duality, which states that for each compact ANR $N \subset S^n$ we have an isomorphism $\tilde{H}^{n-i}(N) \cong \tilde{H}_{i-1}(S^n - N)$. Otherwise we can use the Mayer-Vietoris sequence for $N(K)$ and $S^3 - N(K)$:

$$\to H_2(S^3) \cong 0 \to H_1(\partial N(K)) \to H_1(S^3 - N(K)) \oplus H_1(N(K)) \to H_1(S^3) \cong 0 \to$$

Since $\partial N(K)$ is $S^1 \times S^1$ we derive $H_1(S^3 - N(K)) \cong \mathbf{Z}$. The isomorphism is induced by the inclusion of $\partial N(K)$ into $N(K)$ and $S^3 - N(K)$. In particular the meridian is an indivisible (non-zero) element of $H_1(S^3 - N(K))$ and its class in $H_1(N(K))$ is zero, hence the claim. Notice that $H_2(S^3 - N(K)) = H_3(S^3 - N(K)) = 0$ by the same method.

4.3. Let us consider the sum associated to planar diagrams of the knots K and L

$$lk'(D_K, D_L) = \sum_{p \in [D_K \cap D_L]_{K>L}} \varepsilon(p),$$

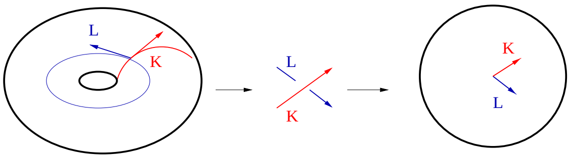
where $[D_K \cap D_L]_{K>L}$ denote those crossings where one strand is from D_K and the other one from D_L and the strand from D_K pass over the strand of D_L. A straightforward check shows that $lk'(K, L)$ is invariant at Reidemeister moves henceforth a topological invariant. If one changes a crossing from D_K or D_L both lk and lk' are preserved. If one changes a crossing between D_K and D_L then both invariants change by the sign of that crossing, henceforth $lk = lk'$.

Consider now the Seifert surface S for the knot L associated to the diagram D_K as constructed before. Imagine the disks bounding the Seifert circles be horizontal circles at various heights. We can isotope K such that K is above S except near the underpasses and intersects S once at each underpass. The number associated to an underpass(i.e. from $[D_K \cap D_L]_{K>L}$) coincides with $\varepsilon(p)$, and thus the point (2) is proved for this particular surface.

4.4. We go back to (1). There exists a natural map $\mathbf{Z} \cong H_1(S^3 - N(K)) \to H_0(S^3 - N(K)) \cong \mathbf{Z}$ constructed as follows. A 1-cycle x in $S^3 - N(K)$ vanishes in $H_1(S^3)$ hence there exists a 2-chain y in S^3 such that $\partial y = x$. Then the intersection $y \cap K$ is a 0-cycle of $S^3 - N(K)$. Then the map from above is $x \to [y \cap K]$. This is an isomorphism because μ is sent in $lk(K, \mu) \in \{-1, +1\}$. On the other hand if L is a simple curve in the complement take a Seifert surface S for L constructed from some diagram of L. Then the image of L by this isomorphism is $[S \cap K] = lk(K, L) \in \mathbf{Z}$, since the signs associated to the intersection points are the same as those associated to the intersection of cycles. Therefore the class of L in $H_1(S^3 - N(K))$ must be $lk(K, L)lk(K, \mu) \in \mathbf{Z}$.

Moreover the isomorphism above does not depend on the particular 2-chain y we chose such that $\partial y = x$. This means that we can choose any Seifert surface S for L, not necessarily that given by a planar diagram. This completes the proof of (2). Another proof of this fact would easily follow from the fact that any two Seifert surface of a knot in S^3 (or any homology sphere) are tube-equivalent (the equivalence relation induced by the attachment of 1-handles away from the boundary).

4.5. Let us denote by $e \in S^2$ the unitary vector corresponding to the normal outward direction to the plane of some projection diagrams of K and L. Then $deg\ \varphi$ is the sum of signs associated to the points of $\varphi^{-1}(e) \in S^1 \times S^1$. We recall that the sign is $+1$ if the orientation agree in the two tangent spaces (at $S^1 \times S^1$ and S^2 respectively) and -1 otherwise. But the points of $\varphi^{-1}(e)$ correspond to the crossings from $[D_K \cap D_L]_{K>L}$. The orientation of the torus is given by the frame consisting in the ordered pair of tangent vectors at K and at L. This frame is sent by the differential of φ into the frame of tangent vectors at the strands involved in a crossing, in the same order, viewed as tangent vectors of the sphere (or the plane of the diagram). Thus the sign to be associated to a crossing is $\varepsilon(p)$.



Eventually observe that the degree of an application can be obtained as the integral of the pull-back of the normalized area form of the target manifold. This proves the second part. □

REMARK 2.1. *All crossing changes involving only strands of K (or only strands of L) do not change the value of the linking number. Thus this is an invariant for pairs (K, L) of curves (not necessarily embedded) up to homotopies which keep them disjoint.*

CHAPTER 3

The Jones polynomial

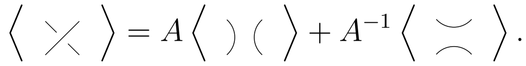
1. The Kauffman bracket

1.1. Definition. The *Kauffman bracket* $<,>$ is a function on the set of unoriented diagrams taking values in the ring $\mathbf{Z}[A, A^{-1}]$ of Laurent polynomials in one variable. It is defined by a *skein relation*, i.e. a relation which takes place in the module generated by all planar diagrams. Specifically we have:

DEFINITION 3.1. *The Kauffman bracket is the unique* $\mathbf{Z}[A, A^{-1}]$*-valued function on diagrams having the properties:*

$$\langle \bigcirc \rangle = 1.$$

$$\langle D \cup \bigcirc \rangle = (-A^2 - A^{-2}) < D > .$$



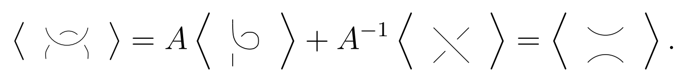
In this definition $\bigcirc$ is the trivial diagram for the unknot, $D \cup \bigcirc$ is the disjoint union of an arbitrary diagram D with the trivial one. The third formula (skein) is a relationship between three diagrams which are identical outside a small disk, and inside this disk they look as the diagrams drawn above.

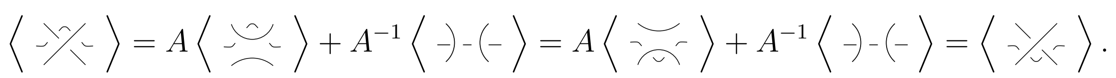
By repeatedly use of the skein relation for all crossings of D we find that $< D >$ is a linear combination of the bracket values of trivial links, which can be computed using the first two formulas. It is easy to check that the order of the crossing is irrelevant hence $<,>$ is well-defined.

PROPOSITION 3.1. *The Kauffman bracket is invariant at the Reidemeister moves R2 and R3 (such a function is called a regular isotopy invariant of links).*

1.2.

PROOF. The verification is by direct calculation on the diagrams. One observes that once the invariance at R2 is achieved the invariance at R3 follows automatically. In other words suppose that the coefficients in the right hand side of the skein formula are arbitrary. Then all the constraints the coefficients should satisfy come from the invariance at R2. This principle holds also for other skein invariants.





We used in the previous calculations the relation

$$<\overline{D}>=<D>^*,$$

where $\overline{D}$ is the diagram obtained from D by changing all crossings and $* : \mathbf{Z}[A, A^{-1}] \to \mathbf{Z}[A, A^{-1}]$ is the algebra involution sending A to A^{-1}. □

1.3. The writhe. Remark that the Kauffman bracket is not invariant at R1. This is the reason to introduce a simple counterterm, called the *writhe* of the diagram. In order to do that we assign an orientation to all components of the link, so that the diagram D is oriented. We indicate this by putting an arrow over the diagram $\vec{D}$. Recall that each crossing of an oriented diagram has been assigned a sign ε. The writhe $w(\vec{D})$ is the sum of $\varepsilon(p)$ over all crossings p of $\vec{D}$.

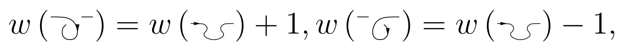
THEOREM 3.1. *Let $\vec{D}$ be a diagram for the oriented link L and D denotes the unoriented adjacent diagram. The function*

$$V_L = (-A)^{-3w(\vec{D})} < D > \in \mathbf{Z}[A, A^{-1}],$$

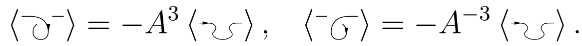
is a link invariant, called the Jones polynomial of L.

1.4.

PROOF. The writhe is invariant at Reidemeister moves R2 and R3 (as the linking number), and it transforms under R1 as follows:



Also for the Kauffman bracket we find



Thus the extra term in V_L cancels the R1 action on Kauffman bracket. □

1.5. Notice that the crossing of a twist involved in R1 is negative or positive independently on the orientation we choose. Thus it makes sense to talk about a negative or a positive twist (self-crossing).

2. The Jones polynomial

2.1.

REMARK 3.1. *It is immediate that $V_{\bigcirc} = 1$. It is an outstanding conjecture that the converse holds i.e. that $V_L = 1$ if and only if L is the trivial knot.*

REMARK 3.2. *Though the orientation of L was involved in the definition of L the value of V_L does not depend on the (global) orientation of L, since by the remark above the writhe has this property. If $r(L)$ denotes the link with the orientations of all its components reversed then $V_{r(L)} = V_L$. Let $K \subset L$ be a sublink made of several components of L and $L - K$ denotes the result of deletion of the components from K. If $r_K(L)$ denotes the link obtained from L by reversing the orientations of components from K then $V_{r_K(L)} = A^{12lk(K,L-K)} V_L$, where lk is the linking number. This follows by a careful interpretation of the terms of the writhe and linking numbers.*

It is common for the Jones polynomial to make the change of variable $t^{\frac{1}{2}} = A^{-2}$.

2.2. The skein relation.

THEOREM 3.2. *The Jones polynomial is the unique invariant of oriented links satisfying the oriented skein relation*

$$t^{-1}\, V_{L_+} \;-\; t\, V_{L_-} \;=\; (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\, V_{L_0},$$

with the normalization

$$V_{\bigcirc} = 1.$$

In particular $V_L \in \mathbf{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$.

2.3.

PROOF. We have the unoriented skein relations for the Kauffman bracket

$$\langle L_+ \rangle = A \langle\,)(\,\rangle + A^{-1} \langle \asymp \rangle,$$
$$\langle L_- \rangle = A^{-1} \langle\,)(\,\rangle + A \langle \asymp \rangle,$$

from which we derive

$$A \langle L_+ \rangle - A^{-1} \langle L_- \rangle = (A^2 - A^{-2}) \langle\,)(\,\rangle.$$

We introduce appropriate orientations on the strands and use the fact that

$$w(L_+) = w(L_0) + 1, \quad w(L_-) = w(L_0) - 1,$$

in order to derive the oriented skein relation

$$A^4 V_{L_+} \;-\; A^{-4} V_{L_-} \;=\; (A^2 - A^{-2}) V_{L_0}.$$

On the other hand we know that any link can be transformed into the trivial link by means of crossing changes. Thus it suffices to find the values of V for the trivial link with k components. A recurrence argument gives the value $(t^{\frac{1}{2}} - t^{-\frac{1}{2}})^{k-1}$. □

REMARK 3.3. *Before the Jones polynomial was invented the Alexander polynomial was known to satisfy a similar skein relation (when using the so-called Conway normalization). This reads*

$$-\nabla_{L_+} - \nabla_{L_-} = (t - t^{-1}) \nabla_{L_0}.$$

The Alexander invariant has a nice topological interpretation and definition in terms of universal coverings of knot complements.

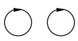
3. Properties of the Jones polynomial

3.1. We should stress that one new property the Jones polynomial has which is not shared by the Alexander polynomial is its ability to distinguish the *chirality* of knots. Let $\overline{L}$ be the mirror image of the link L; the link is called *amphicheiral* if L and $\overline{L}$ are equivalent. If D is a diagram for L then $\overline{D}$ is a diagram for $\overline{L}$ hence the remark after the definition of Jones polynomial yields

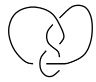
$$V_{\overline{L}}(t) = V_L(t^{-1}).$$

EXAMPLE 3.1. *Let L be the right trefoil knot 3_1*  *. We compute its Jones polynomial by using twice the skein relation*

$$t^{-1}V$$  $$- \; tV$$  $$= (t^{\frac{1}{2}} - t^{-\frac{1}{2}})V$$  ,

$$t^{-1}V$$  $$- \; tV$$  $$= (t^{\frac{1}{2}} - t^{-\frac{1}{2}})V$$  ,

and find $V_L(t) = t + t^3 - t^4$. As this is not invariant at $t \to t^{-1}$ it follows that L is not amphicheiral, and thus not equivalent to the left trefoil knot.

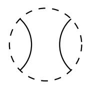
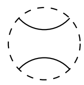
The computations for the amphicheiral eight knot $K =$  *give for instance $V_K(t) = t^{-2} - t^{-1} + 1 - t + t^2$.*

3.2. Mutation. Explicit calculations of Jones polynomial values yield several pairs of different knots having the same polynomial. We can find such pairs in a systematic way as follows.

Let B^3 be a 3-ball whose boundary intersects the link L into 4 points. Choose one axis such that the rotation S by π around it preserves the 4 points. We excise $B^3 \cap L$ and replace it by its image $S(B^3 \cap L)$ to get the link L' (being careful about the orientations inside the ball). The links L and L' are said to be related by a *mutation*.

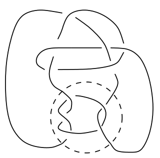
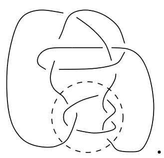
PROPOSITION 3.2. *Two links obtained one from another by mutations have the same Jones polynomials.*

PROOF. By using skein relations we can express the (unoriented) diagram $B^3 \cap L$ as a formal sum of diagrams

$$\alpha$$  $$+ \; \beta$$  .

This expression is invariant by the three rotations in B^3 preserving the 4 boundary points. Notice that the writhe of each diagram in the ball is unchanged by a mutation. The computation of the Kauffman brackets of the L and L' can be done in two steps: first we get rid of crossings inside the ball B^3 by using the skein relation and afterwards we continue with the other crossings. After the first step was completed the formal sums of diagrams contain identical terms, and so $V_L = V_{L'}$. □

3.3. Examples. Specific examples are the Kinoshita-Terasaka and Conway knots which are related by a mutation. They are distinct since their fundamental groups are not isomorphic. An invariant distinguishing them is the number of representations of their group into a finite group.

  .

CHAPTER 4

Alternating links

1. Definitions

DEFINITION 4.1. *A planar diagram is alternating if the crossings we reach when we travel through the components of the link are alternatively overpasses, underpasses, overpasses and so on. A link is alternating if it has some alternating diagram.*

DEFINITION 4.2. *A link in $L \subset S^3$ is split if there exists an embedded 2-sphere $S^2 \subset S^3 - L$ disjoint from L, each complementary ball containing at least one component of L.*

A diagram $D \subset S^2$ is split if there exists a simple closed curve $S^1 \subset S^2 - D$, each complementary disk containing at least one component of D.

DEFINITION 4.3. *A nontrivial link L is prime if every 2-sphere $S^2 \subset S^3$ intersecting L in precisely two points bound on one side a ball intersecting L in a unknotted arc.*

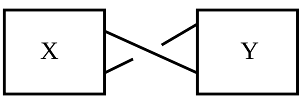
A nontrivial diagram $D \subset S^2$ is prime if every simple closed curve $S^1 \subset S^2$ intersecting D in exactly two points bounds on one side a 2-disk intersecting D in a topologically trivial arc. This means that the arc in the disk represents a diagram for a trivial unknotted arc in the ball. If this diagram is the trivial diagram (without crossings) then we call D strongly prime.

Remark that the problem of recognizing the non-split and prime links from their diagrams is central for the tabulations of knots having a small number of crossings. In the case of alternating links there is a simple solution:

PROPOSITION 4.1. *Let L be a link with an alternating diagram D. Then L is split if and only if D is split; L is prime if and only if D is prime.*

This criterion is effective in testing primeness for knot diagrams having less than 8 crossings since the first non-alternating knot has 8 crossings.

The most spectacular application of the Jones polynomial is the solution to G.Tait conjectures. Let us say that a diagram is reduced if it contains no reducible crossings:



The first two conjectures state that a reduced alternating diagram has the minimal number of crossings a diagram of the respective link can have, and that the writhe of an alternating diagram is a topological invariant. The third Tait conjecture describes a move allowing to pass from a reduced alternating diagram to another, the so-called flip move.

2. The crossing number for alternating links

Let us denote by $b(V)$ the *breadth* of the Laurent polynomial V in one variable, which is the difference between the maximal degree $M(V)$ and the minimal degree $m(V)$ occurring in V.

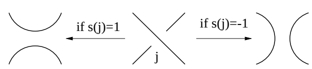
THEOREM 4.1. *Let D be a connected diagram of the link L having n crossings. If $V_L(t) \in \mathbf{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$ denotes the Jones polynomial of L then*

(1) *$B(V_L(t)) \leq n$.*
(2) *If D is alternating and reduced then $B(V_L(t)) = n$.*
(3) *If D is non-alternating and prime then $B(V_L(t)) < n$.*

COROLLARY 4.1. *If a link L has a connected reduced alternating diagram with n crossings then any other diagram of L has at least n-crossings, if alternating, and more than n crossings if non-alternating.*

PROOF. It follows from (2) that $B(V_L) = n$, hence by (1) any diagram of L with m crossings should satisfy $n = B(V_L) \leq m$. If the diagram is non-alternating we use (3). □

In order to proceed to the proof of the theorem we introduce some notations. If D is a diagram having n crossings, we label the crossings by the elements of $\{1, 2, ..., n\}$, and call a *state* of the diagram a function $s : \{1, 2, ..., n\} \to \{-1, 1\}$. To each state we can associate a way to smooth the crossing, following the prescription:



The diagram obtained this way from the state s and the diagram D will be denoted sD. This has no crossings and it consists in a number, say $|sD|$ of disjoint circles. We can write down a state-sum version of the Kaufman bracket value, as follows:

LEMMA 4.1. *We have $< D >= \sum_s < D|s >$, where*

$$< D|s >= \left(A^{\sum_{j=1}^{n}}(-A^2 - A^{-2})^{|sD|-1}\right).$$

PROOF. This follows by recurrence from the skein relation. □

Let s_+ (and respectively s_-) denote the constant state $s_+(j) = +1$ (and respectively $s_-(j) = -1$. Notice that $\sum_{j=1}^{n} s(j) \cong n(mod\ 2)$ and $|\sum_{j=1}^{n} s(j)| = n$ iff $s = s_+$ or $s = s_-$. If the value of $s_{+,-}$ is changed for exactly one crossing then we find a state for which s for which $|\sum_{j=1}^{n} s(j)| = n - 2$.

The diagram D is said to be *plus-adequate* if $|s_+D| > |sD|$ for all states s such that $\sum_{j=1}^{n} s(j) = n - 2$. Also the diagram D is said to be *minus-adequate* if $|s_-D| > |sD|$ for all states s such that $\sum_{j=1}^{n} s(j) = -n + 2$. Eventually a diagram is adequate if it is both plus-adequate and minus-adequate.

LEMMA 4.2. *A reduced alternating diagram is adequate.*

PROOF. Remark first that D is plus-adequate if, for any crossing of D, the two segments of s_+D which replace the crossing are lying in different components of s_+D. Thus it suffices to prove that no component of s_+D does come close to itself around a crossing. Let us color in a chessboard fashion the complementary of D. Since D is alternating the components of s_+D bound monocolor (smoothed) regions (except for a little portion around the crossings). By the same argument the components of s_-D bound monocolor (smoothed) regions of the other color. As there are not reductible crossings we cannot have regions coming close to each other as to be neighbor regions around a former crossing. Thus D is adequate. □

We recall that $< D >\in \mathbf{Z}[A, A^{-1}]$ denotes the Kauffman bracket in the variable A.

Lemma 4.3. *Let D be a diagram with n crossings. Then*

$$M(< D >) \leq n + 2|s_+D| - 2, \text{ with equality if } D \text{ is plus-adequate}$$

$$m(< D >) \geq -n - 2|s_-D| + 2, \text{ with equality if } D \text{ is minus-adequate}$$

In particular for an adequate diagram

$$B(< D >) = 2n + 2|s_+D| + 2|s_-D| - 4.$$

Proof. Choose a sequence of states $s_0 = s_+, s_1, ..., s_k = s$, such that the state s_{r+1} is obtained from s_r by changing the value of the state $s_r(i_r) = +1$ into $s_{r+1}(i_r) = -1$ (the integers $i_0, i_1, ..., i_{k-1}$ are distinct). Then $\sum_{j=1}^{n} s_r(j) = n - 2r$ and the diagrams s_rD and $s_{r+1}D$ are identical except for the smoothing around the crossing labeled i_r. This means that $|s_rD| = |s_{r+1}D|\pm 1$. Therefore $M(< D|s_r >) - M(< D|s_{r+1} >\in \{-4, 0\}$ and hence $M(< D|s_r >\leq M(< D|s_+ >= n + 2|s_+D| - 2$ yielding the claimed inequality.

If D is plus-adequate then $M(< D|s_1 >< M(< D|s_+ >$ because $|s_1D| = |sD| - 1$. Thus $M(< D|s_r >< M(< D|s_+ >$ for $s \neq s_+$. It follows that the term of maximal degree in $< D|s_+ >$ cannot be cancelled by other terms of type $< D|s >$ and this proves that $M(< D >) = n + 2|s_+D| - 2$.

Th same proof applied to $\overline{D}$ gives the second inequality. □

Lemma 4.4. *If D is a connected diagram with n crossings then $|s_+D|+|s_-D| \leq n+2$.*

Proof. We use a recurrence on n. For $n = 0$ it is obvious. Let us consider a diagram D with $n + 1$ crossings. One manner, say the positive one, of smoothing a crossing gives us a connected diagram D'. Then $s_+D = s_+D'$ and $|s_-D| = |s_-D|\pm 1$, hence $|s_+D| + |s_-D| = |s_+D'| + |s_-D'|\pm 1 \leq n + 2\pm 1 \leq n + 3$. □

Lemma 4.5. *If D is a connected alternating diagram with n crossings then $|s_+D| + |s_-D| = n + 2$.*

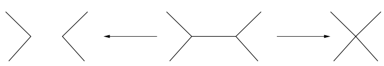
If D is non-alternating and strongly prime then If D is a connected diagram with n crossings then $|s_+D| + |s_-D| < n + 2$.

Proof. If D is alternating we saw earlier that $|s_\pm D|$ is the number of monocolor regions of the respective color, after chessboard coloring the complementary of D (in S^2). Thus $|s_+D| + |s_-D|$ is the total number of such regions, which by an Euler-Poincare argument is exactly $n + 2$.

Let now D be non-alternating and strongly prime. We want to use a recurrence on n. For $n = 2$ is immediate. Let D have $n + 1 \geq 3$ crossings. There exist therefore two consecutive crossings of the same type (overpasses or underpasses). Let c be a third crossing, distinct from the first two.

Since D is strongly prime both ways to smooth the crossing c yield connected diagrams.

Consider the chessboard coloring of the complementary of D. We associate a graph Γ to D, whose vertices are the black regions and the edges connect adjacent regions (facing each other oppositely to some crossing). Since D is strongly prime then no vertex is a cut point of Γ: removing that vertex does not disconnect the graph. The two ways to smooth the crossing c correspond to the following transformation on the graph:



If one manner of smoothing the crossing produces a separating vertex then the other way yields a graph without cut vertices. This means that one of the diagrams, say D' is strongly prime. Then the recurrence hypothesis can be applied and we derive that $|s_+D'| + |s_-D'| \leq n + 2$ which implies as above that $|s_+D| + |s_-D| \leq n + 3$. □

We are able now to complete the proof of the theorem. Aside the transformation $t = A^{-4}$ the Jones polynomial is a multiple of the Kauffman bracket hence $4B(V_L(t)) = B(< D >)$. The lemmas from above imply

$$4B(V_L(t)) \leq 2n + 2|s_+D| + 2|s_-D| - 4 \leq 4n,$$

with equality if D is alternating and reduced.

If D is prime and non-alternating each subdiagram which is the projection of a trivial arc does not contribute to the Jones polynomial (which behaves multiplicatively for connected sum) but only in raising the crossing number. We can therefore suppose that D is strongly prime, and the last lemma ends the proof. □

Remark 4.1. *The knots* $8_{19}, 8_{20}, 8_{21}$ *have Jones polynomials of breadth less than 8. If they had alternating diagrams, these would have less than 8 crossings. We can see that all tabulated knots with at most 7 crossings have Jones polynomials distinct from that of* $8_{19}, 8_{20}, 8_{21}$. *Thus these knots are non-alternating.*

3. The writhe of alternating links

The main result of this section will give a positive answer to the second Tait conjecture.

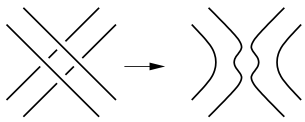
Theorem 4.2. *Let* $\vec{D}$ *and* $\vec{E}$ *be two oriented diagrams of the same link* L*, having* n_D *and respectively* n_E *crossings. If* D *is plus-adequate then*

$$n_D - w(\vec{D}) \leq n_E - w(\vec{E}).$$

In other words the number of negative (resp. positive) crossings of a plus-adequate (resp minus-adequate) diagram is minimal among all diagrams. In particular we obtain that

Corollary 4.2. *The number of crossings in an adequate diagram is minimal. Two adequate diagrams of the same link have the same writhe.*

Proof. If D is a diagram we denote by D^r the diagram where each component is replaced by r parallel copies in the plane. Remark that D is plus/minus-adequate implies that D^r is also plus/minus-adequate. In fact, $s_{\pm}D^r = (s_{\pm}D)^r$. There is no component of $s_{\pm}D^r$ which comes close to itself near a crossing because such a component is parallel to a component of $s_{\pm}D$, and the later are prohibited to have such components. Near a crossing of D the situation is the following:



Set L_j for the various components of L, and D_j, E_j for the corresponding subdiagrams of D and E. Choose some positive integers a_j and b_j verifying $w(D_j) + a_j = w(E_j) + b_j$. We replace now D_j by $\tilde{D}_j$ by adding a_j positive kinks and E_j by $\tilde{E}_j$ by adding b_j positive kinks. Then $\tilde{D}$ remains a plus-adequate diagram and this time $w(\tilde{D}) = w(\tilde{E})$.

On the other hand $\tilde{D}^r$ and $\tilde{E}^r$ are diagrams associated to the same link L^r hence $< \tilde{D}^r > = < \tilde{E}^r >$. The definition of the writhe implies that $w(\tilde{D}^r) = w(\tilde{E}^r)$.

Then the previous lemma 3 gives us the inequalities:

$$M(< \tilde{E}^r >) \leq (n_E + \sum b_j)r^2 + 2(|s_+E| + \sum b_j)r - 2,$$

$$M(< \tilde{D}^r >) = (n_D + \sum a_j)r^2 + 2(|s_+D| + \sum a_j)r - 2.$$

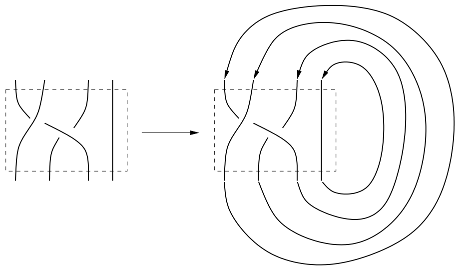
We derive that $n_E + \sum b_j \geq n_D + \sum a_j$, hence $n_D - \sum w(D_j) \leq n_E - \sum w(E_j)$. By adding the sum of all linking numbers between the components of L we obtain the claimed inequality. □

CHAPTER 5

Links as closed braids

1. Alexander's theorem

There exists an intimate relationship between braids and links. Any braid $x \in B_n$ yields an oriented link $\hat{x}$ by closing up the strands of the braid in the most simple way:



Also the up to down orientation of braid strands induces an orientation for the closure. Alexander was the first to observe that this procedure is in fact the most general construction of links.

THEOREM 5.1. *Any oriented link in $\mathbf{R}^3$ is isotopic to a closed braid.*

PROOF. Consider an oriented diagram D in the plane which represents a given link. We suppose that the diagram is polygonal. Fix a point O (the origin in the plane). We can slightly perturb the diagram such that no line determined by the edges of the diagram contain the point O. Let $e = \vec{xy}$ be an edge of D with the orientation inherited from that of the link. If the ray Ox is turning counterclockwise around O in order to arrive in the position Oy, while keeping its other endpoint on $\vec{xy}$ then the edge $\vec{xy}$ is called *positive*, otherwise it will be called *negative*.

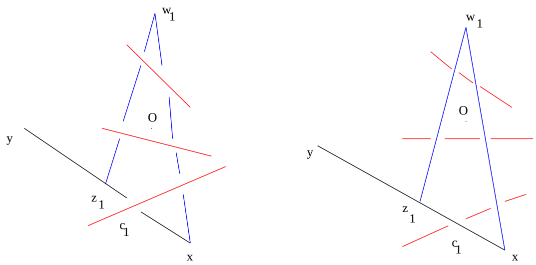
Observe that a diagram having all its edges positives is already the closure of a braid (up to an isotopy). We have therefore to prove that we can change the diagram such that all its edges are positives.

Let xy be a negative edge, and label the crossings on xy by $c_1, c_2, ..., c_p$, starting from x towards y. Choose some intermediary points on xy separating the crossings, say $z_1, z_2, ..., z_{p-1}$. We will us a recurrence on the number of crossings p.

If $p = 0$ we choose some point in the plane w such that O lies in the triangle xwy. Then replace on the diagram the edge $\vec{xy}$ by $\vec{xw} + \vec{wy}$. We have to specify the type of possible crossings the new introduced edges $\vec{xw}$ and $\vec{wy}$ can have with the rest of the diagram. We choose that the curve $\vec{xw} + \vec{wy}$ pass under all the other parts of the diagram. This implies that the associated link is related by a triangular move with the previous link, hence it is isotopic to it. On the other hand the two new edges are positive.

Assume now we have $p+1$ crossings on xy and xz_1 contains only one crossing, namely c_1. Choose as above the point w_1 such that xw_1z_1 contains O in its interior. We will replace the segment $\vec{xz_1}$ by the curve $\vec{xw_1}+\vec{w_1z_1}$. It remains to decide the type of crossings newly introduced.

If c_1 is an underpass of xy we ask that $\vec{xw_1}+\vec{w_1z_1}$ pass under all the other components of the diagram. If c_1 is an overpass of xy we ask that $\vec{xw_1}+\vec{w_1z_1}$ pass over all the other components of the diagram.



In fact in both cases the other strand entering c_1 has to exit the triangle xw_1z_1 by intersecting the new edges in the same type of crossing that c_1. Thus in both cases we used a triangular move not changing the isotopy type of the diagram. This argument permits the inductive reduction of the number of crossings, thereby replacing at the end the edge xy by a sequence of positive ones. We derive the claim. □

2. Vogel's algorithm

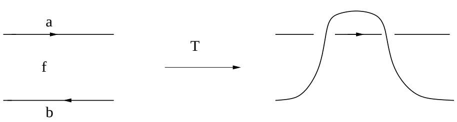
We would like to have a more efficient algorithm to convert a planar diagram into the closure of a braid, in such a way that "a small diagram" has to be associated to a "small braid" from the complexity viewpoint. One such algorithm has recently been proposed by P.Vogel, and we'll outline it below.

Let $D \subset S^2$ be a link diagram. We call *face* any connected component of the complementary of D and *edge* an arc between two consecutive crossings on D.

A triple (f, a, b) consisting of the face f and two edges $a, b \subset \partial f$ is called *admissible* if

- a and b are contained in different Seifert circles.
- f (hence ∂f) inherits an orientation from S^2; then a and b have orientations compatible to that of ∂f.

To such an admissible triple (f, a, b) a R2 move $T = T(f, a, b)$ is associated as follows:



Denote by $\mathcal{D}$ the set of planar diagrams.

THEOREM 5.2. *There exists a function* $\chi : \mathcal{D} \to \mathbf{Z}_+$ *such that*

(1) *If D is a diagram having n Seifert circles then*

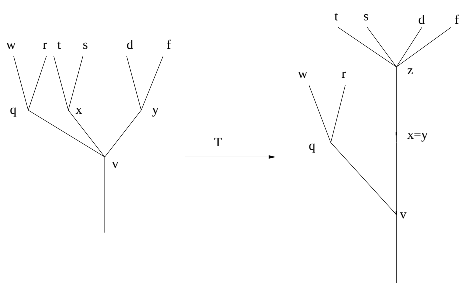
$$n \leq \chi(D) \leq \binom{n+1}{2}.$$

(2) *We have $\chi(D) < \chi(TD)$ if $T = T(f, a, b)$ for some admissible triple (f, a, b).*
(3) *If D is a connected diagram with n Seifert circles such that $\chi(D) \leq \binom{n+1}{2}$ then there exists some admissible triple (f, a, b) in D.*
(4) *The connected diagram D with n Seifert circles is isotopic in S^2 with a closed braid if and only if $\chi(D) = \binom{n+1}{2}$.*

Proof. Let S denote the set of the Seifert circles associated to the diagram $D \subset S^2$. Each oriented circle bounds two regions on the sphere: one which is on the left (when following the orientation of the circle) side and the other one on the right side. As the n Seifert circles are disjoint there are $n+1$ connected components of the complementary of S. We associate a graph $\Gamma(D)$ whose vertices are these connected components. We have an oriented edge from a vertex corresponding to some region to the other if the former is the region on the left side of some Seifert circles whereas the second is the region on the right side of the same circle. Since there are n edges this graph is a tree.

The m-chain c_m is the trivial graph with m edges the partition of a segment into m parts. If $c(\Gamma)$ denotes the number of (oriented) m-chains contained in the graph Γ, for all $m = 1, 2, ...$, then we set $\chi(D) = c(\gamma(D))$. Counting the 1-chains we find that $\chi(D) \geq n$. Notice that $c(c_m) = \binom{m+1}{2}$.

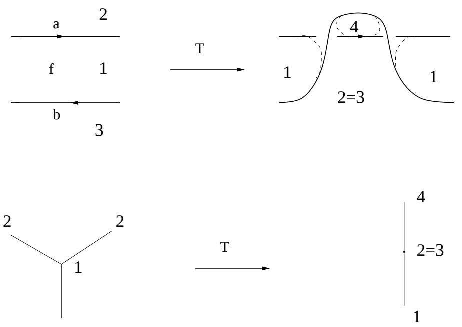
We define a folding transformation T at the level of oriented trees, as follows. Choose some vertex v and two edges leaving v, say vx and vy of the oriented tree Γ. The transformation T associated to this data identifies the two edges (and two vertices), it creates one more vertex z adjacent to the vertex $x = y$, and push all edges which were before adjacent to x or y to be adjacent to z.



We have $c(\Gamma) < c(T\Gamma)$, because all chains can be transported from Γ to $T\Gamma$, possibly raising their length (if they travel through the folding edges), the chains having terminal points x or y give rise to chains having terminal points $x = y$ or z, and there is at least one more chain xz which does not come from Γ.

We can use this transformation as long as the tree Γ is not a chain and in particular $c(\Gamma) \leq c(c_n) = \binom{n+1}{2}$, whenever Γ has n edges.

Notice now that for an admissible triple (f, a, b) the transformation induced by $T(f, a, b)$ at the level of trees $\Gamma(D)$ is the transformation T associated to the vertex f and the two adjacent regions (which are the vertices for the leaving edges).

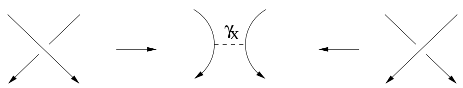


Notice that not every move T at the level of trees can be induced from the transformation associated to an admissible triple.

We have to prove that a diagram D whose associated tree $\Gamma(D)$ is not the chain should contain an admissible triple. If the tree is not a chain there exists a vertex having at least two leaving edges. This means that there exist two Seifert circles containing arcs from a face F, whose orientations agree with that of ∂F.

Let us say that a Seifert circle containing an arc of F is positive (resp. negative) if its orientation agree with that of ∂F (resp. disagrees). Let p (resp. q) be the number of positive Seifert circles (resp negative). Possibly changing the orientation of S^2 we can assume that $p \geq q$. By the assumption $p \geq 2$.

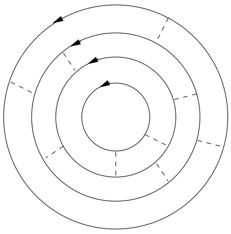
Let us make the following transformation on the diagram. Each crossing is smoothed to obtain a set of Seifert circles, but we keep the trace of the crossing by adding a small arc γ_x between the circles at the place where the twisted bands of the Seifert surface should be attached:



Each such arc joins two Seifert circles having opposite orientations. Set $K = \cup_{\gamma_x \subset F} \gamma_x$. Then cutting open F along the arcs of K we obtain a planar domain $\widehat{F} \subset F$ (with several components), each component of $\widehat{F}$ corresponding to a component (face) of the complementary of D.

Suppose that there is no admissible triple (f, a, b) in D. Then each component f of $\widehat{F}$ touches exactly one positive and one negative circle. Set f_+ for the positive circle. If $f \cap f' \neq \emptyset$ then the Seifert circles associated must be the same i.e. $f_+ = f'_+$. Thus the function $f \to f_+$ is a locally constant function, hence it is a constant. This means that ∂F contains only one positive Seifert circle, contradicting the fact that $p \geq 2$.

Eventually, when $\Gamma(D)$ is a chain, we can use an isotopy on S^2 such that the set S of Seifert circles is the union of concentric circles coherently oriented.



Then the arcs γ_x are transversal arcs joining consecutive circles. Replacing these arcs by crossings we find out that D is a closed braid. $\square$

CHAPTER 6

Braid groups, Hecke algebras

1. The symmetric groups

2. Braid groups

3. Hecke algebras

CHAPTER 7

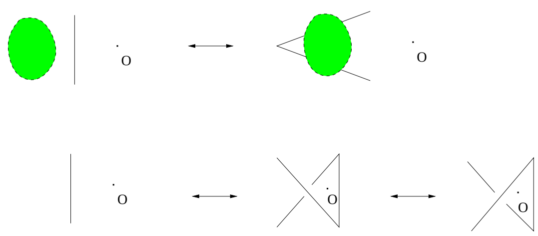
The HOMFLY polynomial

1. Markov's theorem

Any link can be identified with a closed braid in infinitely many ways. As the Reidemeister moves act transitively on the set of diagrams associated to the same link, the theorem of Markov provides the moves relating two braids having the same closure.

THEOREM 7.1. *The closed braids are equivalent links if and only if we can relate them in $\cup_{n\geq 2}B_n$ by a sequence, each term of it being one of the following elementary moves:*
• conjugation $z \in B_n$ *is replaced by* $czc^{-1} \in B_n$.
• stabilization *(or its inverse, namely a* destabilization*) which replace* $z \in B_n$ *by* $zb_n^{\pm 1} \in B_{n+1}$.

A complete proof of this statement is given in the book of J.Birman. The idea is to use in a clever way the triangular moves in order to decompose a sequence of operations between closed braids into the geometric moves (the origin O corresponds to the axis of the braid):



It is useful in applications to have an improved version of Markov's theorem, which is folklore (after J.Birman, H.Morton). The proof we present here is due to S.Kamada. This result provides a rearrangement of the elementary moves, first we can perform only stabilizations and

THEOREM 7.2. *If the closed braids $\widehat{x}$ and $\widehat{y}$ are equivalent then there exists a sequence of elementary moves relating them $x = x_0 \to x_1 \to ... \to x_k \to y_k \to y_{k-1} \to ... \to y_0 = y$, such that, for each j*
• $x_j \to x_{j+1}$ *is either a conjugation or a stabilization.*
• $y_{j+1} \to y_j$ *is either a conjugation or a destabilization (the inverse of a stabilization).*

PROOF. Define a *generalized stabilization* to be the transformation $x \in B_n$ goes to $xw^{-1}b_n^{\pm 1}w \in B_{n+1}$, $w \in B_n$. It is easier to work with them and also these are not far from usual stabilization since we have:

LEMMA 7.1. *The generalized stabilization $x \to x'$ can be decomposed as $x \to y$ a conjugation, $y \to z$ a stabilization, and $z \to x'$ a conjugation.*

PROOF. If $x' = xw^{-1}b_n^{\pm 1}w$ we set $y = wxw^{-1}$ and $z = wxw^{-1}b_n^{\pm 1}$. □

Let us prove that conjugation and generalized stabilization commute each other:

LEMMA 7.2. *Let $x_0 \to x_1$ be a conjugation, $x_1 \to x_2$ be a generalized stabilization. Then there exists some y such that $x_0 \to y$ is a generalized stabilization and $y \to x_2$ is a conjugation.*

PROOF. Let $x_1 = w^{-1}x_0w$, $x_2 = x_1t^{-1}b_n^{\pm 1}t$. We take $y = x_0wt^{-1}b_n^{\pm 1}tw^{-1}$. □

The main argument is the equivalence of a sequence having the shape down-up with another one up-down (with respect to the braid index):

LEMMA 7.3. *Let $x_0 \to x_1$ be a generalized destabilization and $x_1 \to x_2$ be a generalized stabilization. Then there exist y, z such that $x_0 \to y$ is a generalized stabilization, $y \to z$ is a conjugation, and $z \to x_2$ is a generalized destabilization.*

PROOF. We can write $x_0 = x_1w^{-1}b_n^{\varepsilon}w \in B_{n+1}$, $x_2 = x_1t^{-1}b_n^{\delta}t \in B_{n+1}$, where $x_1, w, t \in B_n$, $\varepsilon, \delta \in \{-1, +1\}$. We put then $y = x_0(b_nt)^{-1}b_{n+1}^{\delta}(b_nt) \in B_{n+2}$, $z = b_{n+1}^{-1}yb_{n+1} \in B_{n+2}$. We have therefore

$$\begin{aligned} z &= b_{n+1}^{-1}x_0(b_nt)^{-1}b_{n+1}^{\delta}(b_nt)b_{n+1} = \\ &= b_{n+1}^{-1}x_1w^{-1}b_n^{\varepsilon}wt^{-1}b_n^{-1}b_{n+1}^{\delta}b_ntb_{n+1} = \\ &= b_{n+1}^{-1}x_1w^{-1}b_n^{\varepsilon}wt^{-1}b_{n+1}b_n^{\delta}b_{n+1}^{-1}tb_{m+1} = \\ &= x_1w^{-1}b_{m+1}^{-1}b_m^{\varepsilon}b_{n+1}wt^{-1}b_n^{\delta}t = \\ &= x_1w^{-1}b_nb_{n+1}^{\varepsilon}b_n^{-1}wt^{-1}b_n^{\delta}t = \\ &= x_1t^{-1}b_n^{\delta}t(b_n^{-1}wt^{-1}b_n^{delta}t)^{-1}b_{n+1}^{\varepsilon}(b_n^{-1}wt^{-1}b_n^{\delta}t) \end{aligned}$$

□

A recurrence on the number of consecutive transformations (destabilization, stabilization) in the sequence relating two closed braids yields the wanted rearrangement result. □

Markov's theorem is useful in constructing link invariants from functions on the group algebras of braid groups. The embedding $B_n \hookrightarrow B_{n+1}$ given by $b_j \to b_j$ for $j \le n-1$, induces $\mathbf{C}B_n \hookrightarrow \mathbf{C}B_{n+1}$. Consider a family of linear functionals $t_n : \mathbf{C}B_n \to \mathbf{C}$ fulfilling the following conditions:

$$t_n(xy) = t_n(yx), \forall x, y \in \mathbf{C}B_n,$$
$$t_{n+1}(xb_n) = zt_n(x), \forall x \in \mathbf{C}B_n, \quad , t_{n+1}(xb_n^{-1}) = \tilde{z}t_n(x), \forall x \in \mathbf{C}B_n,$$

for some nonzero $z, \tilde{z} \in \mathbf{C}^*$. Such a family is called a *Markov trace*, and usually we drop the subscript n from t_n.

For an element $x \in B_n$, written in terms of the standard generators as $x = \prod_i b_{j_i}^{\lambda_i}$ we denote by $e(x) = \sum_i \lambda_i$, the *exponent sum* of x. Since the relations in B_n are homogeneous the exponent sum is well-defined (independent on the word chosen in the generators).

Corollary 7.1. *Let $\{t_n\}$ be a Markov trace. The function F which associates to the closed braid $\widehat{x}$, for $x \in B_n$, the value*

$$F(\widehat{x}) = z^{-\frac{e(x)+n}{2}} \tilde{z}^{\frac{e(x)-n}{2}} t_n(x),$$

is a link invariant.

We have to see that this function on x is invariant with respect to conjugation and stabilization which is immediate.

Remark 7.1. *Traces of linear representations of braid groups automatically satisfy the first condition for a Markov trace. Thus such traces could give invariants provided that the representation of different B_n are related in such a way that the second condition holds.*

Remark 7.2. *The general theory of representations of braid groups is highly nontrivial and open subject. The representations in low degree are known to be decompose as sums of Burau and signature representations. It is presently unknown if the braid groups are linear groups. However in a particular case, namely that of representations of Hecke algebras these representations are well-understood (at least for generic values of the parameter).*

2. Markov traces on Hecke algebras

The purpose of this section is to describe the Markov traces which factorize on Hecke algebras. Since they are finite dimensional there are finitely many constraints for a linear functional to be a Markov trace. Moreover these can be explicitly checked out and the result of V.Jones and A.Ocneanu reads:

Theorem 7.3. *For any $z \in \mathbf{C}$ there exists an unique Markov trace on the Hecke algebras $H(q,n)$ thereby verifying:*

$$t(ab) = t(ba), \forall a, b \in H(q,n),$$

$$t(1) = 1,$$

$$t(ab_n) = zt(a), \forall a \in H(q,n).$$

Proof. The quadratic relations can be used to replace any negative exponent of some b_j with a sum of elements with nonnegative powers. Observe first that each element of $H(q,n+1)$ can be written as a linear combinations of words xb_ny, with $x, y \in H(q,n)$ and words from $H(q,n)$. In fact any minimal word has no square powers b_n^2, since the last can be replaced by the linear combination $(q-1)b_n - q$. We use now a recurrence on n. For $n = 2$ it is already clear. Consider now a minimal word from $H(q,n+1)$ in which b_n appears twice. Thus the word is b_nxb_n with $x \in H(q,n)$. According to the recurrence hypothesis $x = yb_{n-1}z$, where $y, z \in H(q,n-1)$, or $x \in H(q,n-1)$. In the first case $b_nxb_n = b_nyb_{n-1}zb_n = yb_{n-1}b_nb_{n-1}z$, in the former there is a factor b_n^2 which can be simplified from the word.

The map $H(q,n) \oplus H(q,n) \otimes_{H(q,n-1)} H(q,n) \to H(q,n+1)$, given by $x \oplus y \otimes z \to x + yb_nz$ is a morphism of $H(q,n) - H(q,n)$ bimodules. The previous discussion proved this map is surjective and a dimension count proves it is an isomorphism.

Assume now we have a Markov trace t defined on $H(q,n)$, which we want to extend over $H(q,n+1)$. Then the formula

$$t(xb_ny) = zt(xy), \forall x, y \in H(q,n).$$

defines a linear functional $t : H(q,n+1) \to \mathbf{C}$. It remains to prove that t is a trace. It is sufficient to verify the trace property $t(xy) = t(yx)$ for all $x \in H(q,n+1)$ and $y \in H(q,n) \cup \{b_n\}$, since the latter is a generating set for $H(q,n+1)$. According to the isomorphism above we have to check that

$$t(xb_nyb_n) = t(b_nxb_ny), \forall x, y \in H(q,n).$$

We have four cases:

- If $x, y \in H(q,n-1)$ then it is immediate since they commute with b_n.
- If $x = ab_nc$, $a, b, y \in H(q,n-1)$ then

$$\begin{aligned} t(b_nab_{n-1}cb_ny) &= t(ab_nb_{n-1}b_ncy) = t(ab_{n-1}b_nb_{n-1}cy) \\ &= zt(ab_{n-1}^2cy) = (z^2(q-1)+zq)t(acy), \end{aligned}$$

$$\begin{aligned} t(ab_{n-1}cb_nyb_n) &= t(ab_{n-1}cb_n^2y) = (q-1)t(ab_{n-1}cb_ny) + qt(ab_{n-1}cy) \\ &= z^2(q-1)+zq)t(acy). \end{aligned}$$

- If $y = ab_nc$, $a, b, x \in H(q,n-1)$ then the roles of x and y are interchanged above.
- If $x = ab_nc$, $y = fb_nd$, $a, b, f, d \in H(q,n-1)$ then

$$\begin{aligned} t(b_nab_{n-1}cb_nfb_{n-1}d) &= t(ab_nb_{n-1}b_ncfb_{n-1}d) \\ &= t((ab_{n-1}b_nb_{n-1}cfb_{n-1}d) \\ &= zt(ab_{n-1}^2cfb_{n-1}d) \\ &= z(q-1)t(ab_{n-1}cfb_{n-1}d) + zqt(acfb_{n-1}d) \\ &= z(q-1)t(ab_{n-1}cfb_{n-1}d) + z^2qt(acfd), \end{aligned}$$

$$\begin{aligned} t(ab_{n-1}cb_nfb_{n-1}db_n) &= t(ab_{n-1}cfdb_nb_{n-1}b_nd) \\ &= t(ab_{n-1}cfdb_{n-1}b_nb_{n-1}d) \\ &= zt(ab_{n-1}cfb_{n-1}^2d) \\ &= z(q-1)t(ab_{n-1}cfb_{n-1}d) + zqt(ab_{n-1}cfd) \\ &= z(q-1)t(ab_{n-1}cfb_{n-1}d) + z^2qt(acfd). \end{aligned}$$

This proves the claim. □

Remark 7.3. *The definition of a Markov trace concerns also the behaviour of the trace with respect to the other stabilization. But the relation* $t(ab_n^{-1}) = \tilde{z}t(a)$ *for some* $\tilde{z} \in \mathbf{C}$ *follows from the first condition. In fact this is a consequence of the quadratic form of the relations in* $H(q,n)$*:* $b_n^{-1} = \frac{1}{q}b_n - \frac{q-1}{q}$*, hence we can take above* $\tilde{z} = \frac{z-q+1}{q}$.

A Markov trace gives rise to a link invariant by the previous corollary, and we derive:

COROLLARY 7.2. *Set* $\lambda = \frac{1-q+z}{qz}$. *Consider the link* L *presented as the closure of the braid* $x \in B_n$. *Then*

$$X_L(q,\lambda) = \left(-\frac{(1-\lambda q)}{\sqrt{\lambda}(1-q)}\right)^{n-1} (\sqrt{\lambda})^{e(x)} t(x),$$

is a link invariant.

This is called the HOMFLY polynomial of the link L. It is common to use some change of variables

$$t = \sqrt{\lambda}\sqrt{q}, \quad , x = \sqrt{q} - \frac{1}{\sqrt{q}},$$

and to denote it

$$P_L(t,x) = X_L(q,\lambda).$$

3. Properties of the HOMFLY polynomial

THEOREM 7.4. $P_L(t,x)$ *is the unique Laurent polynomial in* t *and* x *verifying the skein relation*

$$t^{-1}P_{\times}(t,x) - tP_{\times}(t,x) = xP_{)(}(t,x),$$

and taking the value 1 for the unknot.

PROOF. Using some Markov elementary moves we can write the three links appearing in the skein relation as the closure of the following braids $\times$ as $\widehat{xb_j^2}$, $\times$ as $\widehat{x}$, and $)($ as $\widehat{xb_j}$. We infer $t(xb_j^2) = (q-1)t(xb_j) + qt(x)$ in the definition of X_L and obtain the skein relation.

Notice that an inductive use of the skein relation associates a Laurent polynomial P_L in t and x for any diagram of L. But provided we can find $\sqrt{\lambda}$ and $\sqrt{q}$ such that $t = \sqrt{\lambda}\sqrt{q}$, $x = \sqrt{q} - \frac{1}{\sqrt{q}}$, and for which the associated z and $\tilde{z}$ make sense and are non-zero, the value of $P_L(t,x)$ can only depend on L (since equal to that of $X_L(q,\lambda)$. Since there is an open set of values of such t and x the Laurent polynomial P_L depends only on L and not on the particular diagram. □

REMARK 7.4. *Specializing* P_L *at the values of* t *and* x *for which corresponding* q *and* λ *are prohibited (since the associated* z *and* $\tilde{z}$ *are not defined) gives precisely the Alexander-Conway polynomial* $\nabla_L(t) = X_L(t, \frac{1}{t}) = P_L(1, \sqrt{t} - \frac{1}{\sqrt{t}})$. *Notice that we cannot calculate the value of this polynomial by using the trace, since its value does not make sense.*

On the other hand the specialization $X_L(t,t) = P_L\left(\frac{\sqrt{-1}}{t}, \sqrt{-1}\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)\right)$ *is the same as the Jones polynomial* V_L, *because they verify the same skein relation.*

EXAMPLE 7.1. *The right trefoil knot* 3_1 *is the closure of* $b_1^3 \in B_2$. *Since* $b_1^3 = (q^2 - q + 1)b_1 + q(q-1) \in H(q,2)$ *it follows that* $X_L(q,\lambda) = \lambda(1 + q^2 - \lambda q^2) = \lambda q((\sqrt{q} - \frac{1}{\sqrt{q}})^2 + 2 - \lambda q) = 2t^2 - t^4 + t^2x^2 = P_L(t,x)$.

If L *is the trivial link with* k *components, then it is the closure of* $1 \in B_k$, *hence* $X_L(q,\lambda) = \left(-\frac{(1-\lambda q)}{\sqrt{\lambda}(1-q)}\right)^{k-1} = \left(\frac{(t-t^{-1})}{x}\right)^{k-1} = P_L(t,x)$

Let us denote by $\sharp$ the operation of connected sum between two components of some links. This operation it is not uniquely defined but for knots. However we can state:

PROPOSITION 7.1. *We have* $X_{L_1\sharp L_2}(q,\lambda) = X_{L_1}(q,\lambda)X_{L_2}(q,\lambda)$ *regardless for the chosen connected sum.*

PROOF. Assume we choose closed braids $x \in B_n$ and $y \in B_k$ representing L_1 and respectively B_2. Possibly conjugating them we can suppose the last strand of x and the first strand of y belong to the components which are connected. Let $s : B_m \to B_{m+1}$ be the shift map $s(b_j) = b_{j+1}$ and $i : B_m \to B_{m+1}$ be the inclusion $i(b_j) = b_j$. Then the link $L_1\sharp L_2$ is the closure of $i^{k-1}(x)s^{n-1}(y) \in B_{n+k-1}$.

Furthermore if w is a word in $b_1, b_2, ..., b_{n-1}$ and w' is a word in $b_n, b_{n+1}, ..., b_{n+k-2}$ then $t(ww') = t(w)t(w')$ (either by recurrence on the length of w' or just by noticing that $x \to \frac{1}{t(w')}t(xw')$ is a Markov trace on $H_n(q)$, and conclude by the uniqueness). □

PROPOSITION 7.2. *The HOMFLY polynomial is invariant by the reversal of the orientation of all components of the link.*

PROOF. Reversing the arrows in the skein relation of P_L we get the same relation for the link with opposite orientation. □

PROPOSITION 7.3. *If* $\overline{L}$ *is the mirror image of* L *then*

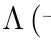
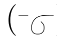
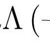
$$X_{\overline{L}}(q,\lambda) = X_L(\frac{1}{q}, \frac{1}{\lambda}), \quad P_{\overline{L}}(t,x) = P_L(\frac{1}{t}, -x).$$

PROOF. The skein relation for $P_L(t,x)$ with crossings changed yield the skein relation associated to $P_L(\frac{1}{t}, -x)$. □

CHAPTER 8

The Kauffman polynomial

0.1. Definition. The Jones polynomial is obtained as the normalization of an invariant of unoriented links, namely the Kauffman bracket. A similar procedure yields the Kauffman polynomial (not to be confused with the Kauffman bracket !). The unoriented regular isotopy invariant is constructed as follows:

THEOREM 8.1. *There exists an unique function* Λ *on unoriented diagrams of* S^2 *taking values in* $\mathbf{Z}[a^{\pm 1}, z^{\pm 1}]$ *satisfying the following conditions:*
- $\Lambda(\bigcirc) = 1$.
- Λ *is invariant at R2 and R3 (it is a regular isotopy invariant).*
- Λ () $= a\Lambda$ (), Λ () $= a\Lambda$ ().
- *It verifies the skein relation:*

Λ () $+ \Lambda$ () $= z\big(\Lambda$ () $+ \Lambda$ ()$\big)$.

COROLLARY 8.1. *Let* $\vec{D}$ *be an oriented diagram for the link* L *and* $w(\vec{D})$ *denote the writhe. Then* $F_L(a,z) = a^{w(\vec{D})}\Lambda(D) \in \mathbf{Z}[a^{\pm 1}, z^{\pm 1}]$ *is a link invariant.*

This link invariant is called the *Kauffman polynomial.*

0.2. Height functions. Before to proceed with the proof of the theorem we need to introduce some notations and definitions. An oriented diagram D is *ordered* if an order is chosen on the set of connected components of the link associated to D. Furtheremore D is pointed if each component is given a base point. By abuse of notation when speaking about a component of the diagram we mean the projection of a connected component of the associated link.

To an ordered pointed (oriented) diagram D we can associate an *ascending diagram* αD as follows. αD is obtained from D by changing some crossings in such a way that, when travelling around the components, the circuits starting at the base points in the sense given by the orientation, and the components being traversed in the given order, then each crossing is reached *the first time* as an underpass . In particular αD is a diagram of a trivial link (with given number of components). A slight generalization of ascending diagrams is provided by the *height* functions. A *height* (or *untying function*) is a real valued function on the link, inducing thereby a function h on the diagram D which is two-valued at crossings (since there is a value for each strand entering the crossing) such that:
- if the components c and d verify $c < d$, then $h(x) < h(y)$ for all $x \in c$, $y \in d$.
- on each component h is strictly increasing from a base point b to a top point t in both directions around the circle.

• at each crossing the value of h on the overpass strand is strictly bigger than that of the underpass.

0.3.

Proof. Let us observe first that there is no canonical way to associate to an unoriented crossing in the plane one of the diagrams $D_+ =$  or $D_- =$ . And in a similar vein their respective smoothings $D_0 =) \ ($ and $D_\infty =$  are also indistinguishable. However their reespective roles in the skein diagram is symmetric, so that it does not matter how we made this assignement.

Set $\delta = \frac{a+a^{-1}}{z} - 1$. If we set $(\Lambda(D_+), \Lambda(D_-), \Lambda(D_0), \Lambda(D_\infty)) = (ax, a^{-1}x, x, \delta x)$ then the skein relation is automatically satisfied.

0.4. The recurrence hypothesis. Let $\mathcal{D}_n$ be the set of unoriented diagrams having no more than n crossings. We want to prove by recurrence that there exists a function $\Lambda : \mathcal{D}_n \to \mathbf{Z}[a^{\pm 1}, z^{\pm 1}]$ such that:
• the skein relation is verified for any four diagrams from $\mathcal{D}_n$.
• the behaviour under R1 is that claimed for diagrams in $\mathcal{D}_n$.
• $\Lambda(D)$ is invariant at those moves R2, R3, R4 which take place in $\mathcal{D}_n$.
• If $D \in \mathcal{D}_n$ has a height then

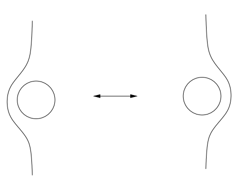
$$\Lambda(D) = a^{\overline{w}(D)} \delta^{\sharp D - 1},$$

where $\sharp D$ denotes the number of components and $\overline{w}(D)$ is the *self-writhe.* The *self-writhe* is the sum of signs ε of those crossings where both strands belong to the same component. Denote by L_j the components of the link L associated to D, and choose an arbitrary orientation of them (and the diagram). We derive therefore:

$$\overline{w}(D) = w(\vec{D}) - \sum_{i,k} lk(L_i, L_k).$$

Notice that $\overline{w}(D)$ does not depend on the orientations we choose.

The Reidemeister move R4 is a composition of two R2, one of them increasing the number of crossing:



Notice that for $n = 0$ the value for the trivial diagrams are specified by the last condition.

0.5. For the induction step consider $D \in \mathcal{D}_n$. We choose an orientation on D, an order on its components and base points. Let αD be the ascending diagram associated to the ordered pointed oriented diagram D. We set then

$$\Lambda(\alpha D) = a^{\overline{w}(D)} \delta^{\sharp D - 1}.$$

Notice that $\sharp D = \sharp \alpha D$.

The diagram αD is obtained from D by switching some crossings. If $\Lambda \mid_{\mathcal{D}_n}$ exists, then according to the skein relation the value of $\Lambda(D_+)$ after switching a crossing is $-\Lambda(D_-) + z(\Lambda(D_0) + \Lambda(D_\infty))$ and $D_0, D_\infty \in \mathcal{D}_{n-1}$, so by the recurrence hypothese their values can be computed. Inductive use of this relation will provide an explicit formula of $\Lambda(D)$ as $\pm\Lambda(\alpha D) + \sum_j \Lambda(D_j)$, where D_j are diagrams having at most $n-1$ crossings. We define therefore

$$\Lambda(D) = \pm\Lambda(\alpha D) + \sum_j \Lambda(D_j).$$

We have to prove now the independence of this definition on the various choices we made, and that the conditions above are satisfied.

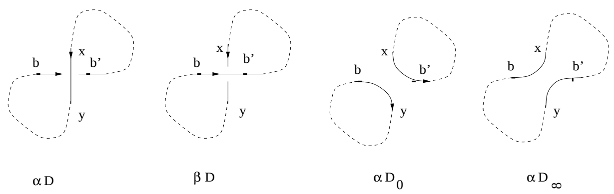
First the order in which the crossings are switched to get αD from D doesn't matter, since the values obtained are the same.

0.6.

LEMMA 8.1. $\Lambda(D)$ *does not depend on the choice of base points.*

PROOF. It suffices to see what happens when the base point b on some component is moved (in the direction of the orientation) to b' which sits just after the first crossing encountered. Let βD denote the ascending diagram associated to the new bas point b'.

If the strand crossed this way lies in another component then $\alpha D = \beta D$. Otherwise the two diagrams are identical outside this crossing where αD is an underpass and βD is an overpass. Let then αD_0 be the diagram obtained from αD by smoothing the crossing in a way compatible with the orientation and αD_∞ be the other smoothing.



Now the diagram αD_0 is an ascending diagram since the dotted arcs $b'x$ and yb are ascending. On the other hand αD_∞ has an untying function, namely the height function inherited from αD (slightly modified on the small arcs bx and $b'y$) for the same reason. The recurrence hypothesis implies that

$$\Lambda(\alpha D_0) = a^{\overline{w}(\alpha D_0)}\delta^{\sharp\alpha D_0 - 1}, \ \ \Lambda(\alpha D_\infty) = a^{\overline{w}(\alpha D_\infty)}\delta^{\sharp\alpha D_\infty - 1},$$

and obviously we have $\overline{w}(\alpha D_0) = \overline{w}(\alpha D_\infty)$, $\sharp\alpha D_\infty = \sharp D$, $\sharp\alpha D_0 = \sharp D + 1$. Furthermore we have the equality $\overline{w}(\alpha D) = \overline{w}(D) + \varepsilon$ $\overline{w}(\beta D) = \overline{w}(D) - \varepsilon$ where ε is the sign of the crossing.

The value of $\Lambda(\beta D)$ computed with the basepoint b' is $\Lambda(\beta D) = a^{\overline{w}(\beta D)}\delta^{\sharp D - 1}$.

The value of $\Lambda(\beta D)$ computed with the basepoint b stems from the ascending diagram associated to βD, which is αD. Since βD is obtained by one switching from αD, $\Lambda(\beta D)$ is determined by the skein relation to that crossing. In fact we know the value of $\Lambda(\alpha D) =$

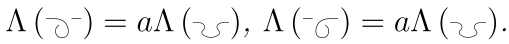
$a^{\overline{w}(\alpha D)}\delta^{\sharp D-1}$, and therefore infering the previous value for $\Lambda(\beta D)$ in the skein relation we get:

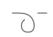
$$a^{\overline{w}(\alpha D_0)}(a\delta^{\sharp D-1}+a^{-1}\delta^{\sharp D-1}) = za^{\overline{w}(\alpha D_0)}(\delta^{\sharp D-1}+\delta^{\sharp D}),$$

which trivially holds. It follows that the value of $\Lambda(D)$ for the base point b, (computed from $\Lambda(\alpha D)$) coincides as that for the base point b' (computed from $\Lambda(\beta D)$). □

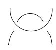
LEMMA 8.2. *The skein relation is verified in* $\mathcal{D}_n$.

PROOF. Let consider $D = D_+$, then choose the base point b just before the crossing (as above) and set αD for its ascending diagram. In αD the crossing was changed, hence the first step (in the inductive definition) in computing $\Lambda(D)$ from $\Lambda(\alpha D)$ is precisely the skein relation $\Lambda(D_+) + \Lambda(D_-) = z(\Lambda(D_0) + \Lambda(D_\infty))$. □

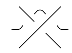
LEMMA 8.3. 

PROOF. If D is the diagram containing the a kink, say the positive one  , and D' is the diagram with the kink removed then choose the base point just before the crossing. We may suppose the crossing is an underpass. Then αD is obtained from $\alpha D'$ by adding a kink, and the same holds at all steps towards the computation of $\Lambda(D)$. Also $\overline{w}(\alpha D) - \overline{w}(\alpha D')$ is the sign of the kink. Now the claim follows from the recurrence hypothesis. □

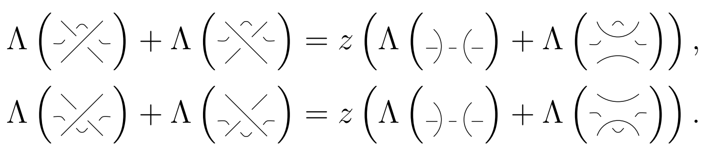
LEMMA 8.4. Λ *is invariant at those moves R2, R3 and R4 which take place in* $\mathcal{D}_n$.

PROOF. Let us consider the diagrams D and D' which are identical outside a small ball where they look like  and  respectively. If we choose the base point just before the crossings, on the strand going underneath then these subdiagrams are not touched in the process of changing crossings towards αD (respectively $\alpha D'$. In particular αD and $\alpha D'$ (as well as all other pairs of corresponding diagrams in the next steps of the computation) are related by the same R2. We have then $\Lambda(\alpha D) = \Lambda(\alpha D')$ and the recurrence hypothesis implies that $\Lambda(D) = \Lambda(D')$.

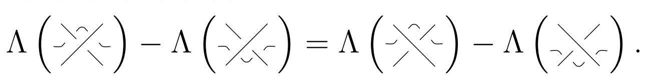
The same idea works for the move R3.

Consider for instance the diagram  . Assume the base points are chosen outside this configuration and label the three strands as follows. The strand 1 is the strand which has to be traversed first, then the strand 2 and 3 is the last one. Therefore in the ascending diagram the strand labeled 1 goes underneath the others and the strand labeled 3 goes over the two others. Thus in general we can't keep the initial configuration unaltered in the ascending diagram.

We derive from the skein relation that:



Since Λ is R2 invariant it follows that



The pair of diagrams on the righthand side is still related by a R3 move, and has one crossing switched with respect to the lefthand side pair. The crossing allowed to be switched is the one opposite to the moving strand. But a R3 move can be viewed as the result of isotopying the strand going underneath and also as the result of moving the strand going over. Thus we are allowed to switch another crossing. The transformations induced by the allowed crossing switches act then transitively on the set of configurations of three strands. Thus one can replace the initial configuration by that arising in the ascending diagram, using crossing changes. We reduced ourselves to the case when the diagram D and its ascending diagram αD are identical on the three strands configuration. Since the values Λ takes on two ascending diagrams related by a move R3 coincide, the recurrence hypothesis shows that Λ is R3 invariant.

Notice also that R4 has no influence on the computation of Λ. □

LEMMA 8.5. *If the diagram $D \in \mathcal{D}_n$ has a height function, then*

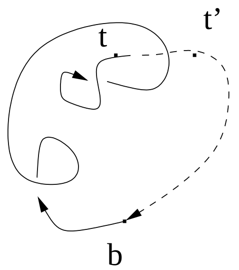
$$\Lambda(D) = a^{\overline{w}(D)}\delta^{\sharp D-1}$$

holds.

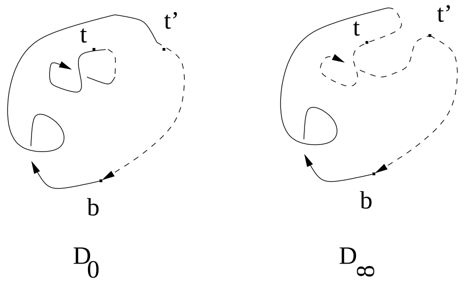
PROOF. If the top points are just before the base points then the diagram is ascending and the formula is verified. We use a recurrence on the (total) number of crossings separating the top points from the base points, when following the orientation direction.

Consider a top point t sitting before a crossing and choose a point t' just after the crossing. If the crossing is an overpass then changing slightly the height function we obtain another untying function whose top point is now t'. If the other strand entering the crossing is from another component the same argument holds. Thus by recurrence we can conclude.

The remaining case is when we have an underpass which is a self-crossing. Let us repersent by a thick line the arc on which the height is increasing and by a dotted line the arc on which the height is decreasing (following the orientation). Then the situation is schematically as in the picture below:



This means that the other strand entering the crossing cannot be dotted from the monotonicity assumption. Consider the diagram D' where this crossing has been changed. A slight perturabtion of our height function provides a height for D'. Moreover we have now an overpass and the previous discussion implies that $\Lambda(D') = a^{\overline{w}(D')}\delta^{\sharp D-1}$. One applies further the skein relation to this crossing. Both diagrams D_0 and D_∞ obtained by annuling the crossing have untying functions suggested by the pictures:



Thus $\Lambda(D_0)$ and $\Lambda(D_\infty)$ are known (by the recurrence hypothesis) and the skein relation yileds our claim. □

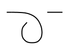
LEMMA 8.6. *Λ is independent on the orientation of the components.*

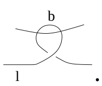
PROOF. Let D and αD as above. After changing the orientation of some component of D one associates the ascending diagram βD. Moreover βD has an untying function with respect to the initial orientation, hence $\Lambda(\beta D) = a^{\overline{w}(\beta D)} \delta^{\sharp D - 1}$, hence it has the same value as it would have been computed with the new orientation. This implies the result. □

LEMMA 8.7. *Λ is independent on the order of the components.*

PROOF. Let D and αD as above. We choose another order on the components and associate the ascending diagram βD with the new order. It suffices to show that, with respect to the initial order, $\Lambda(\beta D) = a^{\overline{w}(\beta D)} \delta^{\sharp D - 1}$.

In order to prove that we will define a simplification procedure which eventually reduces the number of crossings while preserving the value of Λ, and so by recurrence we conclude.

Let consider a loop l (a subarc of the diagram which starts and stops at the same crossing))  from the diagram, possibly cut by other arcs. If l contains no crossings and bounds a 2-disk then an R4 move moves it far away from the diagram. If the loop has exactly one crossing then we can use R1 to get rid of it. This move does not preserve Λ but it modifies the self-writhe accordingly. The simplified diagram is still ascending (with respect to the new order).

Otherwise there exist some other arcs cutting the loop l. Choose then such a loop l which is minimal with respect to the inclusion (between the disks bounded by the immersed curve l in the plane). The minimality implies that each arc (of D) cutting l should be simple (in the interior of the disk bounded by l), thereby determining 2-gones 

. Further choose an innermost such 2-gon b; again the arcs cutting b should be simple and must cut both edges of b. The set of these arcs partitions the 2-gon into polygons. A simple argument yields the existence of a minimal triangle having one edge on the boundary of b. Choosing the base points outside this triangle it follows that the triangle has cross-overs at its three vertices that are appropriate for a R3 move, because the diagram βD is ascending. Then change the diagram by moving the part of the boundary over the interior vertex of the triangle by a R3 move.

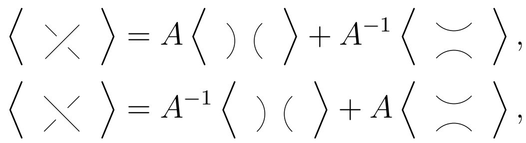
This procedure can be repeated, since the new obtained diagrams are still ascending, until no more arcs cut the 2-gon, in which case a R2 move removes it and diminishes the number of crossings. □

The sequnce of lemmas completes the induction hypothesis and the theorem follows. □

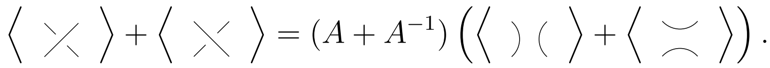
PROPOSITION 8.1. *We have*

$$F_L(-t^{\frac{3}{4}}, t^{-\frac{1}{4}} + t^{\frac{1}{4}}) = V_L(t).$$

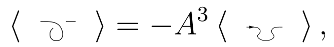
PROOF. We know that



hence



Since



we derive that

$$< D >= \Lambda(D)(-A^3, A + A^{-1}).$$

□

REMARK 8.1. $F_{\overline{L}}(a, z) = F_L(a^{-1}, z)$, *beacuse when the crossings are changed the skein relation is preserved, but the behaviour on the R1 move forces* $a \to a^{-1}$. *Also if one reverses the orientation of all components then the Kauffman polynomial is inchanged.*

CHAPTER 9

The Yang-Baxter equation

1. Enhanced Yang-Baxter operators

1.1. Yang-Baxter operators. Let V be a free $\mathbf{k}$-module of finite type over a commutative ring $\mathbf{k}$. Every endomorphisme $R : V \otimes V \to V \otimes V$ induces endomorphisms

$$R_j : V^{\otimes n} \to V^{\otimes n}, \quad R_j = \underbrace{1 \otimes 1 \otimes ... \otimes 1}_{j-1} \otimes R \otimes 1 \otimes ... \otimes 1.$$

Then R is called a *Yang-Baxter* operator if it satisfies the equation

$$R_1 R_2 R_1 = R_2 R_1 R_2$$

in $End(V^{\otimes 3}$. From now on we will consider that R is an automorphism. In that case a Yang-Baxter operator defines a representation of the braid groups by setting $\rho : B_n \to GL(V^{\otimes n})$, $\rho(b_j) = R_j$.

1.2. Enhanced Yang-Baxter operators. An enhancement of a Yang-Baxter operator is a linear operator $\mu : V \to V$ represented in some basis $\{e_1, e_2, ..., e_m\}$ by a diagonal matrix $\mu = diag(\mu_1, \mu_2, ..., \mu_m)$ (m being the rang of V) such that

$$R(\mu \otimes \mu) = (\mu \otimes \mu)R,$$

$$\sum_j R_{ij}^{kj} \mu_j = \alpha\beta\delta_i^k, \quad \sum_j (R^{-1})_{ij}^{kj} \mu_j = \alpha^{-1}\beta\delta_i^k,$$

where α, β are units in $\mathbf{k}$, and $Re_i \otimes e_j = \sum_{pq} R_{ij}^{pq} e_p \otimes e_q$.

PROPOSITION 9.1. *If (R, μ, α, β) is an enhanced Yang-Baxter operator then the function associating to a closed braid $\widehat{x}$, with $x \in B_n$, the value*

$$F(x) = \alpha^{-e(x)} \beta^{-n} tr(\rho(x)\mu^{\otimes n}),$$

is a link invariant, where tr is the usual trace on the endomorphisms of $V^{\otimes n}$..

The proof is the by now standard check of the invariance at Markov moves. □

2. The series A_n

Let $\mathbf{k} = \mathbf{Z}[q^{\pm 1}]$, $m \geq 1$. Reshetikhin associated to the fundamental representation of the Lie algebra of type A_{m-1} the following Yang-Baxter operator:

$$R = -q \sum_{i=1}^{m} E_{ii} \otimes E_{ii} + \sum_{i \neq j}^{m} E_{ij} \otimes E_{ji} + (q^{-1} - q) \sum_{i<j}^{m} E_{ii} \otimes E_{jj},$$

where $\dim V = m$, and E_{ij} is the endomorphism of V given by $E_{ij}e_k = \delta_{ik}e_j$. It follows that

$$R^{-1} = -q^{-1}\sum_{i=1}^{m} E_{ii}\otimes E_{ii} + \sum_{i\neq j}^{m} E_{ij}\otimes E_{ji} + (-q^{-1}+q)\sum_{i<j}^{m} E_{ii}\otimes E_{jj}.$$

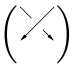
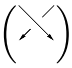
It can be directely verified that this is a Yang-Baxter operator, but it is rather cumbersome. The enhancement associated to R is given by $\mu = diag(\mu_1, ...\mu_m)$, $\mu_j = q^{2j-m-1}$, $\alpha = -q^m$, $,\beta = 1$.

The proposition above says that a link invariant Q is associated to this enhanced Yang-Baxter operator.

Let us observe that R satisfies the equation

$$R - R^{-1} = (q^{-1} - q)\mathbf{1}_{V\otimes V}.$$

This implies that the invariant Q verifies the following skein relation:

$q^m Q$  $- q^{-m}Q$  $= (q - q^{-1})Q$  .

In particular we recover the HOMFLY polynomial $P_L(t, x)$, since we have:

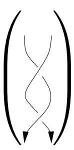
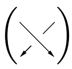
$$Q(L)(q) = P_L(q^m, q - q^{-1}).$$

Another solution found by Cherednik, Drinfeld, Faddeev, Jimbo, Kulish, Reshetikhin and Skyanin is

$$R = \sum_{k=0}^{m-1}\sum r, s = 1m - kq^{rs-\frac{1}{2}(k(r-s)+(m+1)(r+s))}[q^{-m+r}]_k[q^r]_k[q]_k^{-1}E_{s+k,r+k}\otimes E_{r,s},$$

where $[a]_k = (1-a)(1-aq)...(1-aq^{k-1})$. The enhancement associated to it is $\mu = diag(q^{m-1}, ..., q^{m-2}, ..., q, 1)$.

Remark that for $m = 3$ the invariant X_m associated to this operator verifies the cubic relation:

X_3  $(-q^{-1} + q^{-3} - q^{-4})X_3$  $(-q^{-4} + q^{-5} - q^{-7})X_3$  $q^{-8}X_3$  $= 0.$

However this cubical skein relation is not sufficient to compute all values of X_3 from that of trivial links.

REMARK 9.1. *If (R, μ, α, β) is an enhanced Yang-Baxter operator satisfying $\sum_j a_j R^j = 0$ then the associated invariant F verifies the skein relation $\sum_j a_j\alpha^j F_{L_j}) = 0$, where L_j is the diagram of an iterated j right twist.*

3. The series B_n, C_n, D_n

Let $\nu \in \{-1, +1\}$ such that $\nu = -1$ whenever m is odd. We put $i' = m + 1 - i$, for $i = 1, ..., m$, and

$$\bar{i} = \begin{cases} i - \frac{\nu}{2} & \text{if } 1 \leq i < \frac{m+1}{2}, \\ i & \text{if } i = \frac{m+1}{2}, \text{ odd } m, \\ i + \frac{\nu}{2} & \text{if } \frac{m+1}{2} < i \leq m. \end{cases}$$

$$\varepsilon(i) = \begin{cases} 1 & \text{if } 1 \leq i \leq \frac{m+1}{2}, \\ -\mu & \text{if } \frac{m+1}{2} \leq i \leq m. \end{cases}$$

The values of $(m\nu)$ for the series B_n, C_n, D_n, A_n^2 are $(2n+1,-1), (2n,1), (2n,-1), (n+1,-1)$. We take $\mathbf{k} = \mathbf{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ for odd m and $\mathbf{k} = \mathbf{Z}[q,q^{-1}]$ for even m.

The Yang-Baxter operator associated to these series is

$$R = q \sum_{i;i\neq i'} E_{ii} \otimes E_{ii} + \sum_{i;i=i'} E_{ii} \otimes E_{ii} + \sum_{i,j;i\neq j,j'} E_{ij} \otimes E_{ji} + q^{-1} \sum_{i;i\neq i'} E_{ii'} \otimes E_{i'i} +$$
$$(q-q^{-1}) \sum_{i<j} E_{ii} \otimes E_{jj} + (q^{-1}-q) \sum_{i<j} \varepsilon(i)\varepsilon(j) q^{\bar{i}-\bar{j}} E_{ij'} \otimes E_{i'j}.$$

Following Jimbo its inverse is

$$R = q^{-1} \sum_{i;i\neq i'} E_{ii} \otimes E_{ii} + \sum_{i;i=i'} E_{ii} \otimes E_{ii} + \sum_{i,j;i\neq j,j'} E_{ij} \otimes E_{ji} + q \sum_{i;i\neq i'} E_{ii'} \otimes E_{i'i} +$$
$$(-q+q^{-1}) \sum_{i<j} E_{ii} \otimes E_{jj} + (-q^{-1}+q) \sum_{i<j} \varepsilon(i)\varepsilon(j) q^{\bar{i}-\bar{j}} E_{ij'} \otimes E_{i'j}.$$

This operator admits an enhancement given by $\mu_j = q^{2\bar{j}-m-1}$, $\alpha = q^{m+\nu}$, $\beta = 1$. If $Q_{m,\nu}$ is the invariant of links derived from the enhanced YBO then

$$Q_{m,\nu}(\bigcirc) = -\nu + \frac{(q^{m+\nu} - q^{-m-\nu})}{q-q^{-1}}$$

PROPOSITION 9.2. *For an oriented link diagram D of the link L the function*

$$\tilde{Q}_{m,\nu}(D) = (\sqrt{-1\nu} q^{m+\nu})^{w(D)} Q_{m,\nu}(L)$$

does not depend on the orientation of L. Moreover it verifies the skein relation

$$\tilde{Q}_{m,\nu}\left(\diagup\!\!\!\!\diagdown\right) + \nu \tilde{Q}_{m,\nu}\left(\diagdown\!\!\!\!\diagup\right) = \sqrt{-\nu}(q-q^{-1}) \left(\tilde{Q}_{m,\nu}\left(\,)\,(\,\right) + \tilde{Q}_{m,\nu}\left(\asymp\right)\right)$$

In particular there exists an unique polynomial $Q_\nu \in \mathbf{Z}[a, a^{-1}, x, x^{-1}]$ which is 1 for the trivial knot, $\tilde{Q}_\nu(D) = a^{w(D)} Q_\nu(L)$ is independent of the orientation ad it verifies the skein relation:

$$\tilde{Q}_\nu\left(\diagup\!\!\!\!\diagdown\right) + \nu \tilde{Q}_\nu\left(\diagdown\!\!\!\!\diagup\right) = x \left(\tilde{Q}_\nu\left(\,)\,(\,\right) + \tilde{Q}_\nu\left(\asymp\right)\right).$$

It follows that Q_1 is the Kauffman polynomial and Q_{-1} is essentially equivalent because

$$Q_{-1}(L)(a,x) = (-1)^{\sharp L-1} Q_1(L)(\sqrt{-1}a, -\sqrt{-1}x).$$

It is clear that

$$Q_{m,\nu}(L) = \left(-\nu + \frac{(q^{m+\nu} - q^{-m-\nu})}{q-q^{-1}}\right) Q_\nu(L)(\sqrt{-\nu} q^{m+\nu}, \sqrt{-\nu}(q-q^{-1})).$$

Notice that the product $a^{-\overline{w}(D)} \tilde{Q}_1(D)$ is an isotopy invariant for unoriented links. Since R satisfies

$$(R_\nu + \nu q^{-m-\nu} \mathbf{1})(R_\nu + q^{-1}\mathbf{1})(R_\nu - q\mathbf{1}) = 0,$$

the polynomial $(a^2T-1)(aT^2 - xT + \nu a^{-1})$ (in right twists) annihilates Q_ν.

CHAPTER 10

The G_2-invariant

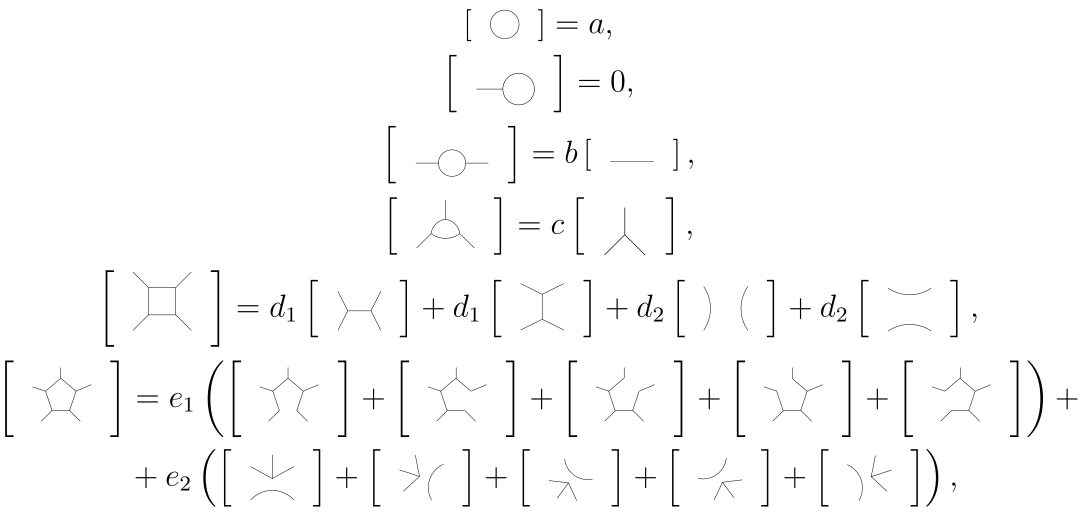
1. The skein relation

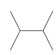
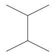
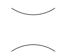
The main result in this section is the explicit construction of the link invariant associated by Reshetikhin and Turaev to the exceptional Lie algebra G_2. The most convenient way is to define it as a function on diagrams of trivalent graphs instead of link diagrams. These are generalizations of the good projections to more complicated graphs either on a plane or on S^2. We allow them to have edges which go in a circle and have no vertex, or have multiple edges and vertices connected to themselves. The Reidemeister theorem has an immediate counterpart for diagrams of graphs, if one introduces some additional moves. Thus the Reidemeister moves for projections of graphs consists in the moves we already encountered together with the moves in which a strand is slided over a vertex:

A function on the diagrams is a regular isotopy invariant for trivalent graphs if it is invariant at all Reidemeister moves but the R1. We will consider in the sequel only trivalent graphs.

Theorem 10.1. *There exists an unique regular homotopy invariant of links and trivalent graphs given by the skein (recurrent) relations:*

$$[\,\emptyset\,] = 1,$$



[] $= f_1$ [] $+ g_1$ [] $+ f_2$ [ ) ( ] $+ g_2$ [],

and the parameters are given by

$$a = q^5 + q^4 + q + 1 + q^{-1} + q^{-4} + q^{-5},$$
$$b = -q^3 - q^2 - q - q^{-1} - q^{-2} - q^{-3},$$
$$c = q^2 + 1 + q^{-2},$$
$$d_1 = -q - q^{-1},$$
$$d_2 = q + 1 + q^{-1},$$
$$e_1 = 1, e_2 = -1,$$
$$f_1 = \frac{1}{1+q^{-1}},\ f_2 = \frac{q}{1+q^{-1}},\ g_1 = \frac{1}{1+q},\ g_2 = \frac{q^{-1}}{1+q},$$

where q is an indeterminate.

2. The existence

The invariant defined above is best understood as linearly recurrent invariant, following G.Kuperberg.

Let I be an invariant of graphs. A planar graph with boundary is the intersection of a planar graph with a 2-disk which is transverse to the boundary. The intersection of edges with the boundary circle are some degree 1 vertices called endpoints. There is a natural way to extend I to an invariant for graphs with boundary b, which takes values in a vector space V_b depending only on the boundary b. In fact take for V_b the set of all $\mathbf{C}$-valued functions on the set of all trivalent graphs (or links) with boundary $-b$ (having the opposite orientation). To a given graph G having $\partial G = b$ we associate the function $I_G \in V_b$ given by $I_G(G') = I(G \cup G')$, where $G \cup G'$ is the union of the graphs compatible with the boundary identification. So far this extension is not very interesting since the space V_b is very large. However it can happen that only a small portion of V_b is needed to understand the invariant I. In fact the spaces V_b and $V{-}b$ are canonically isomorphic (think of the two complementary disks making up S^2) and the linear function I_G extends by linearity to a bilinear form

$$I : V_b \times V_b \to \mathbf{C}.$$

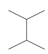
It is reasonably to look at the quotient $W_b = V_b / \ker I$ by the left kernel of the bilinear map and the non-degenerate bilinear form

$$I : W_b \times W_b \to \mathbf{C}.$$

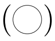
This map contains all information to recover I. Alternatively set $\tilde{W}_b$ for the span of all I_G, for $\partial G = b$. Then $\tilde{W}_b$ is isomorphic to W_b. The invariant I is called linearly recurrent if W_b is finite dimensional for all b.

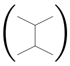
If this dimension is small then it is often possible to define the invariant recursively by describing it for some graphs G with $\partial G = b$ and knowing a specific base for W_b.

If no other structure is given then the boundary b is specified by the cardinal of its endpoints.

For instance the Kauffman bracket is uniquely determined by the requirements that the 4-point spanning space W_4 is 2-dimensional and that the elements  and

 form a basis of it, and the 0-point spanning space W_0 is 1-dimensional. In fact any diagram invariant having this property should satisfy the liner dependence conditions:

$I\,($  $) = a,$

$I_G\,($  $) = bI_G\,($  $) + cI_G\,($  $),$

where a, b and c are parameters. Imposing the invariance at R2 (and R3 will be automatically satisfied) we find that $bc = 1$, $abc + b^2 + c^2 = 0$, yielding the usual values for parameters in the Kauffman bracket.

The existence of the G_2-invariant follows the same way by imposing a basis for the spaces W_k (associated to [ ]) for $k < 6$. We assume that *all acyclic crossingless trivalent graphs with $k < 6$ endpoints are linearly independent vectors in W_k and that the bracket of a crossingless diagram which consists of a face with k sides is a linear combination of these acyclic ones (for $k < 6$).* These acyclic diagrams are exactly the ones in the right hand side of the recurrent formulas stated in the theorem. Thus we assume the existence of the coefficients $a, b, c, d_1, d_2, e_1, e_2$ as given in the statement (but we do not assume yet the formulas given for these coefficients). Notice that there can be only two distinct coefficients for the relation of a pentagon or a square, by the symmetry of the configuration, for otherwise the bracket of a pentagon or a square would satisfy more than one linear equation, which would violate the linear independence assumption. It remains to prove that, under this assumption, there is only one invariant, with the coefficients having the stated values.

Let us show first that these coefficients $a, b, c, d_1, d_2, e_1, e_2$ completely determine the bracket for the crossingless graphs:

LEMMA 10.1. *A crossingless planar graph has at least one simply-connected face with five or fewer sides, which does not share an edge with itself.*

PROOF. If one face shares an edge with itself then there is a circle joining the middle point of this edge with itself through the face. Consider then an innermost such circle. This circle bounds a region whose Euler characteristic is 1 and the argument below will work. If we have a partition of a disk into polygons then there the sum of Euler characteristic for the simply-connected ones is at least 1 because the non-simply connected ones contribute by a nonpositive integer. Since each vertex has at least three adjacent edges and each edge is shared by two faces, we derive that simply connected faces with less than 6 edges must exist. ☐

We have to derive now the values for the coefficients which are consistent for the values of the bracket on all crossingless graphs. In fact if A and B are faces with five or fewer sides in a graph, we have to see that reducing first A and then B yields the same result as reducing first B and afterwards B. This is obvious when the faces are non adjacent, but we have to check out explicitly the reduction for adjacent faces. Since such a configuration is expressed in terms of acyclic crossingless graphs with $k < 6$ edges the coefficients of the basis vectors should agree for the two reduction procedures.

Doing that for a pentagon adjacent to a triangle we derive:

$$cd_1 = e_1 d_1 + e_1 c + e_1 b$$

$$cd_2 = e_1d_2 + e_2b$$

We can perform this procedure for all possible pairs of k-gon, n-gon but for $k = n = 5$. We obtain the equations:

$$b^2 = bd_1 + d_2 + ad_2$$
$$c^2 = bd_1 + cd_1 + d_2$$
$$= 2e_1b + 2e_2 + ae_2 + e_1c$$
$$cd_1 = d_1e_1 + be_1 + ce_1 + e_2$$
$$cd_2 = d_2e_1 + be_2$$
$$d_1e_1 + d_2 = e_1^2$$
$$d_1e_1 + d_1^2 = e_1^2 + ce_1 + d_1e_1$$
$$d_1e_1 = e_1^2 + d_1e_2? + e_2$$
$$d_1e_2 + d_2c = e_1e_2 + be_2 + 2d_2e_1$$
$$d_1e_2 = e_1e_2 + d_2e_1$$
$$d_1e_2 + d_1d_2 = e_1e_2 + e_2c$$

In the case $n = k = 5$ we cannot impose the coefficients agree, since the graphs involved have six endpoints. However, if the coefficients agree, then the two reductions give the same element. The equations resulting this way are:

$$e_2d_1 = e_1d_2 + e_1e_2$$
$$e_2 = e_1^2$$

The case when two faces share more than one edge is inconsequential.

The case $e_1 = 0$ produces only trivial solutions. We normalize the solutions by $e_1 = 1$, and we are left a second free parameter which is $d_1 = -q - q^{-1}$.

To extend the invariant to links and graphs with crossings we further assume that a crossing is a linear combination of the four acyclic crossingless graphs in W_4. To find the coefficients f_j, g_j thus introduced we must investigate the behaviour of the bracket under the Reidemeister moves.

The sliding moves R2 and sliding over a vertex give a set of equation having the unique solution stated in the theorem:

$$f_1g_1d_1 + f_1^2c + g_1^2c + f_1g_1b + g_1g_2 + f_1f_2 = 0$$
$$f_1g_1d_1 + f_1g_2 + f_2g_1 = 0$$
$$f_1g_1d_2 + f_1f_2b + f_2^2 + g_1g_2b + g_2^2 + f_2g_2a = 0$$
$$f_1g_1d_2 + f_2g_2 = 1$$
$$f_1^2d_1 + f_1g_1c + g_1f_1e_1g_1^2d_1 = 0$$
$$g_1f_1e_1 + g_1f_2 = f_1$$
$$g_1f_1e_1 + g_2f_1 = g_1$$
$$g_1f_1e_1 + g_1^2d_1 + g_2g_1 = 0$$
$$f_1^2d_1 + f_1f_2 + g_1f_1g_1? = 0$$
$$g_1f_1e_2 + g_2f_2 = 0$$
$$f_1^2d_2 + f_1g_2b + g_1f_1e_2 + g_1g_2c + g_2^2 = 0$$

$$g_1 f_1 e_2 + g_1^2 d_2 + f_2 g_1 b + f_2 f_1 c + f_2^2 = 0$$
$$f_1^2 d_2 + g_1 f_1 e_2 = f_2$$
$$g_1 f_1 e_2 + g_1^2 d_2 = g_2$$

When checking the invariance at R3 again we cannot impose the coefficients agree since the graphs involved have six endpoints, but we can see that they indeed agree. This ends the proof of the theorem. □