

# On power subgroups of mapping class groups\*

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## Abstract

In the first part of this paper we prove that the mapping class subgroups generated by the  $D$ -th powers of Dehn twists (with  $D \geq 2$ ) along a sparse collection of simple closed curves on an orientable surface are right angled Artin groups. The second part is devoted to power quotients, i.e. quotients by the normal subgroup generated by the  $D$ -th powers of all elements of the mapping class groups. We show first that for infinitely many  $D$  the power quotient groups are non-trivial. On the other hand, if  $4g + 2$  does not divide  $D$  then the associated power quotient of the mapping class group of the genus  $g \geq 3$  closed surface is trivial. Eventually, an elementary argument shows that in genus 2 there are infinitely many power quotients which are infinite torsion groups.

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## 1 Introduction and statements

The aim of this paper is to give a sample of results concerning power subgroups of mapping class groups. We denote by  $M(S)$  the mapping class group of the orientable surface  $S$ , namely the group of isotopy classes of homeomorphisms that fix point-wise the boundary components. Set  $\Sigma_{g,k}^r$  for the orientable surface of genus  $g$  with  $k$  boundary components and  $r$  punctures. We will omit the indices  $k$  and  $r$  in  $\Sigma_{g,k}^r$  when they are zero.

**Definition 1.1.** *Let  $A$  be a collection of (isotopy classes of) simple closed curves on the surface  $S$ . We denote by  $M(S)(A; D)$  the subgroup generated by  $D$ -th powers of Dehn twists along curves in  $A$ .*

When  $A$  is a set  $SCC(S)$  of representatives for all simple closed curves up to homotopy on the surface  $S$  the group  $M(S)(SCC(S); D)$  will be denoted by  $M(S)[D]$ . We will omit the indices  $k$  and  $r$  in  $M(\Sigma_{g,k}^r)[D]$  and  $M(\Sigma_{g,k}^r)(A; D)$  when they are zero. For simplicity, when we do not need to specify the surface  $\Sigma_{g,k}^r$  we will use the notation  $M_{g,k}^r$  for  $M(\Sigma_{g,k}^r)$  and respectively  $M_{g,k}^r[D]$  for  $M(\Sigma_{g,k}^r)[D]$ , with the same convention concerning the indices  $k$  and  $r$ , which we omit when they are zero.

Observe that  $M_g[D]$  is a normal subgroup of  $M_g$ , whose definition is similar to that of the congruence subgroups of the symplectic groups. In fact, let  $T_a$  denote the Dehn twist along the simple closed curve  $a$ . Then for every  $h \in M_g$  we have  $hT_a^D h^{-1} = T_{h(a)}^D \in M_g[D]$ . As  $M_g[D]$  is generated by the  $T_a^D$ , for  $a$  running over the set of all simple closed curves, it follows that  $M_g[D]$  is a normal subgroup.

The first results on  $M_g[D]$  were obtained by Humphries ([19]) who proved that  $M_g/M_g[2]$ , for each  $g \geq 1$ ,  $M_2/M_2[3]$  and  $M_3/M_3[3]$  are finite, while  $M_2/M_2[D]$  is infinite when  $D \geq 4$ .

On the other hand, using quantum topology techniques we proved in [13] that the groups  $M_g[D]$  are of infinite index in  $M_g$ , if  $g \geq 3$ , and  $D \notin \{1, 2, 3, 4, 6, 8, 12\}$ .

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Mapping class groups have interesting actions on various moduli spaces, for instance on spaces of  $SU(2)$  representations of surface groups. It is known (see [15]) that the whole mapping class group acts ergodically. Actually the same proof extends trivially to show that  $M_g[D]$  still acts ergodically. This yields the first examples of infinite index subgroups of the mapping class group acting ergodically.

Methods from quantum topology also show that:

$$\bigcap_{D \in \mathcal{D}} M_g[D] = 1$$

if  $g \geq 2$  and  $\mathcal{D}$  is any infinite set of positive integers. In fact, the kernel of the  $SO(3)$  quantum representation of level  $k$  of  $M_g$  contains  $M_g[k]$ . Then the asymptotic faithfulness theorem from [2, 12] yields the claim.

However, these results seem to exhaust our present knowledge about the groups  $M_g[D]$ . It is not known, for instance, whether the following holds:

**Conjecture 1.1.** *The group  $H_1(M_g[D])$  is not finitely generated if  $D \geq 3$ ,  $g \geq 4$  or  $D \geq 4$ ,  $g \in \{2, 3\}$ .*

If true, this would imply that  $M_g/M_g[D]$  is infinite for the above values of  $D$  and  $g$ .

*Remark 1.1.* The groups  $M_g[2]$  have finite index in  $M_g$  (see [19]) and hence are finitely generated. However the quantum representations at 4-th roots of unity (see [36, 41]) and 6-th roots of unity (see [42]) have finite image. Thus the quantum method used for large  $D$  cannot decide whether  $M_g[4]$  and  $M_g[6]$  have finite index or not. It is likely that  $M_g[D]$  is of infinite index for every  $D \geq 4$  and  $g \geq 3$ . Notice also that a similar problem for pure braid groups was considered in [20].

A question of Ivanov (see [24], Question 12) is particularly relevant for the structure of the group  $M_g[D]$  by asking about the possible relations between powers of Dehn twists. We formulate it here as a conjecture, under a slight restriction on  $D$ :

**Conjecture 1.2.** *The group  $M_g[D]$  (for  $D \geq 3$ ,  $g \geq 4$  or  $D \geq 4$ ,  $g \in \{2, 3\}$ ) has the following presentation:*

1. Generators  $Z_a$  (standing for  $T_a^D$ ), where  $a$  belongs to the (infinite) set  $SCC(\Sigma_g)$  of simple closed curves on the surface;
2. Relations of braid type:

$$Z_{T_a^D(b)} = Z_a Z_b Z_a^{-1}$$

for each pair  $a, b \in SCC(\Sigma_g)$ .

We denote by  $\mathcal{A}_\Gamma$  the right angled Artin group associated to the graph  $\Gamma$ , which is defined by the following presentation:

1. Generators  $Z_a$ , where  $a$  belongs to the set of vertices of  $\Gamma$ ;
2. Relations

$$Z_a Z_b = Z_b Z_a, \text{ if } a \text{ and } b \text{ are connected by an edge in } \Gamma$$

A related (but much weaker) Conjecture is as follows:

**Conjecture 1.3.** *Let  $C_{\Sigma, D} \subset SCC(\Sigma)$  be a set of representatives of the orbits set  $SCC(\Sigma)/M(\Sigma)[D]$ . Consider the associated intersection graph  $\Gamma(C_{\Sigma, D})$ , whose vertex set is  $C_{\Sigma, D}$  and edges join vertices corresponding to disjoint curves on the surface. Then the homomorphism  $A_{\Gamma(C_{\Sigma, D})} \rightarrow M(\Sigma)[D]$  which sends the generators  $Z_a$  into the elements  $T_a^D$  is an isomorphism on its image for  $D \geq 3$ ,  $g \geq 4$  or  $D \geq 4$ ,  $g \in \{2, 3\}$ . Here  $g$  denotes the genus of the surface  $\Sigma$ . Notice that this homomorphism is not surjective, in general, but the normal closure of its image is  $M(\Sigma)[D]$ .*

*Remark 1.2.* According to Ishida (see [21]) the group generated by two Dehn twists is either free abelian (if the curves are disjoint or coincide) or generate the braid group  $B_3$  in 3 strands (if the curves intersect in one point) or free (if the curves intersect in at least two points). In particular the subgroup generated by two  $D$ -th powers of Dehn twists is either free abelian or free, supporting the Conjecture 1.3. See also [7] or ([18], Thm. 3.5) for the braid case. Relations between multi-twists are also given in [35].

**Proposition 1.1.** *The analogues of Conjecture 1.2 for  $D = 2$  and any closed orientable surface  $\Sigma$  of genus  $g \geq 3$  are false as stated, namely there are additional relations in a presentation of  $M_g[2]$  with the given generators.*

*Proof.* According to Humphries (see [19]) the subgroup  $M_g[2]$  can be identified to the kernel of the homomorphism  $M_g \rightarrow Sp(2g, \mathbb{Z}/2\mathbb{Z})$ . Hain proved in [17] (see also [34]) that any finite index subgroup of  $M_g$  (for  $g \geq 3$ ) containing the Torelli subgroup (i.e. the subgroup of mapping classes acting trivially in homology) has trivial first cohomology. Thus  $H^1(M_g[2]) = 0$ . But the abelianization of the group presented by the relations from Conjecture 1.2 is a free abelian group of rank equal to the cardinal of  $SCC(\Sigma_g)/M_g[2]$ . This contradiction shows that in  $M_g[2]$  there are additional relations.  $\square$

Remark also that the analogue of Conjecture 1.2 cannot hold when  $D = 1$  either. In fact the abelianization of  $M_g$  would be a nontrivial free abelian group, contradicting the fact that  $M_g$  is perfect when  $g \geq 3$  and has torsion abelianization otherwise.

An important step towards a solution to Conjecture 1.3 was taken in the recent preprint [29] of Koberda, where it is proved the following. For any irredundant (see [29] for the definition) collection  $\{f_1, f_2, \dots, f_k\}$  of mapping classes of homeomorphisms, each one being either a Dehn twist or a pseudo-Anosov homeomorphism supported on a single connected subsurface, there exists  $N_0$  such that  $\{f_1^N, f_2^N, \dots, f_k^N\}$  is a right angled generating system for a right angled Artin subgroup of the mapping class group, for any  $N \geq N_0$ .

The first result of this paper supports further evidence for the last two conjectures. Let  $A$  be a finite collection of simple closed curves on a surface  $S$  and denote by  $F(A)$  the regular neighborhood of  $A$  in  $S$ . We assume that curves are isotoped so that for each  $a, b \in A$  the number  $i(a, b)$  of intersection points between  $a$  and  $b$  is minimal. We pick up a base point  $p$  on the surface  $S$  and a set of distinct points  $p_s^0 \in a_s$ , for  $s \in A$ .

**Definition 1.2.** *The collection  $A$  of curves on the surface  $S$  is sparse if it is finite and for some choice of paths  $\gamma_{pp_s^0}$  joining  $p$  to  $p_s^0$  the free subgroup  $O(A) \subset \pi_1(F(A), p)$  generated by the homotopy classes of based loops  $\gamma_s a_s \gamma_s^{-1}$ ,  $s \in A$ , embeds into  $\pi_1(S, p)$  under the map induced by the inclusion  $F(A) \hookrightarrow S$ . The collection  $A$  is nontrivial if the group  $O(A)$  is nontrivial.*

**Theorem 1.1.** *Let  $D \geq 2$  and  $A$  be a nontrivial sparse collection of curves on  $\Sigma_{g,d}$ , where  $d \geq 1$ . Then the subgroup  $M(\Sigma_{g,d})(A; D)$  is a right angled Artin group.*

*Remark 1.3.* One can construct sparse sets  $A$  by considering free subgroups generated by primitive elements in  $\Sigma_{g,d}$ .

*Remark 1.4.* J.Crisp and L.Paris considered before the question of finding presentations of subgroups generated by non-trivial powers of the standard generators in Artin groups. They established in [7] the Tits conjecture, which claimed that these subgroups are right angled Artin groups. M.Lönne proved in [32] similar results in the braid group setting, by showing that the subgroups generated by the powers of band generators are again right angled Artin groups if the powers are at least 3.

*Remark 1.5.* Recently, M.Kapovich proved in [27] (making use of our result above) that all right angled Artin groups associated to finite graphs embed into the group of Hamiltonian symplectomorphisms of any symplectic manifold.

The second part of this article is concerned with power subgroups and quotients. Recall the following:

**Definition 1.3.** *Set  $X_g[D]$  for the  $D$ -th power subgroup of  $M_g$ , namely the subgroup generated by powers  $h^D$  of all elements of  $h \in M_g$ . Then  $X_g[D]$  is a normal subgroup of  $M_g$  whose quotient is a torsion group.*

*Remark 1.6.* Newman ([39]) proved that the  $D$ -th power subgroup of  $PSL(2, \mathbb{Z})$  (and hence of  $SL(2, \mathbb{Z})$ ) is of infinite index when  $D = 6m \geq 48000$ . This was extended by Fine and Spellman (see [11]) to the  $2p$ -th power subgroups of  $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/p\mathbb{Z}$  (for odd prime  $p$ ).

A natural question is whether power quotients of the mapping class group could be non-trivial, or even infinite torsion groups. Our second result gives some answers in particular cases:

**Theorem 1.2.** *1. Choose an ordered basis of  $H_1(\Sigma_g, \mathbb{Z})$  and denote by  $P : M_g \rightarrow Sp(2g, \mathbb{Z})$  the homomorphism which sends a mapping class into the matrix describing its action in homology. Then, for every  $g \geq 2$  there exist infinitely many integers  $D$  for which  $P(X_g[D])$  is a proper subgroup of  $Sp(2g, \mathbb{Z})$ . In particular  $M_g/X_g[D]$  are non-trivial torsion groups, for these values of  $D$ .*

2. If  $4g + 2$  does not divide  $D$  and  $g \geq 3$  then  $M_g = X_g[D]$ .

The question concerning the existence of infinite torsion quotients of  $M_g$  (see the question of Ivanov in [24]) has an elementary solution for genus  $g = 2$ . Using arguments similar to those of Korkmaz in [30] we show that:

**Theorem 1.3.** *The group  $M_2/X_2[360D]$  is an infinite torsion group (of exponent  $360D$ ) for  $D \geq 8000$ .*

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## 2 Subgroups of mapping class groups generated by powers of Dehn twists

### 2.1 Finitely generated subgroups generated by powers in braid groups

The analogs of the groups  $M(\Sigma_g)(A; D)$  in the case of braid groups have been considered long time ago by Coxeter. The braid group  $B_n$  in  $n$  strands has the usual presentation, due to Artin:

$$B_n = \langle \sigma_1, \sigma_2, \dots, \sigma_{n-1} \mid \sigma_i \sigma_j = \sigma_j \sigma_i, \text{ if } |i - j| > 1, \sigma_{i+1} \sigma_i \sigma_{i+1} = \sigma_i \sigma_{i+1} \sigma_i, 1 \leq i \leq n - 2 \rangle$$

It is well-known that the quotient of  $B_n$  by the normal subgroup generated by  $\sigma_i^2$  is the permutation group  $S_n$ . Consider, after Coxeter (see [4]):

**Definition 2.1.** *The subgroup  $B_n\{D\}$  of  $B_n$  is the group generated by the powers  $\sigma_i^D$  of the standard generators  $\sigma_i$ . Let also  $N(B_n\{D\})$  denote the normal closure of  $B_n\{D\}$  in  $B_n$ .*

Coxeter gave in [4] the list of all those quotients  $B_n/N(B_n\{D\})$  which are finite, together with their respective description (see also [5, 6]), as follows:

**Proposition 2.1** (Coxeter). *The group  $N(B_n\{D\})$  is of finite index in  $B_n$  if and only if  $(D - 2)(n - 2) < 4$ . Away from the trivial cases  $D = 2$  or  $n = 2$  we have another five groups:*

1.  $n=3$ 
  - (a) For  $D = 3$  the quotient  $B_3/N(B_3\{3\})$  is isomorphic to  $SL(2, \mathbb{Z}/3\mathbb{Z})$  and has order 24;
  - (b) For  $D = 4$  the quotient  $B_3/N(B_3\{4\})$  is a non-split extension of the symmetric group  $S_4$  on a set of 4 elements by  $\mathbb{Z}/4\mathbb{Z}$  and has order 96;
  - (c) For  $D = 5$  the quotient  $B_3/N(B_3\{5\})$  is isomorphic to  $GL(2, \mathbb{Z}/5\mathbb{Z})$  and has order 600;
2. For  $n = 4, D = 3$  the factor group  $B_4/N(B_4\{3\})$  has order 648 and is the central extension of the Hessian group  $(\mathbb{Z}/3\mathbb{Z})^2 \rtimes PSL(2, \mathbb{Z}/3\mathbb{Z})$  by  $\mathbb{Z}/3\mathbb{Z}$ .
3. For  $n = 5, D = 3$  the factor group  $B_5/N(B_5\{3\})$  has order 155 520 and is the central extension of the simple group of order 25 920 by  $\mathbb{Z}/6\mathbb{Z}$ .

*Remark 2.1.* 1. There is an analogue of the Conjecture 1.3 for the punctured disk  $\Sigma_{0,1}^n$ , where we replace powers of Dehn twists by powers of half-twists (i.e. braids). Notice that  $N(B_n\{2D\})$  is a subgroup of  $M_{0,1}^n[D]$ .

2. J.Tits conjectured that  $B_n\{D\}$  and more generally the subgroups generated by powers of the standard generators in Artin groups are right angled Artin groups. The later conjecture was settled in full generality by Crisp and Paris [7].
3. It seems unknown whether the analogue of the Conjecture 1.3 for  $N(B_n\{D\})$  holds for  $D \geq 3$ . Notice that for  $D = 2$  there exist nontrivial relations among squares of band generators (which are Dehn twists) according to [32].

## 2.2 Proof of Theorem 1.1

Consider the regular neighborhood  $F(A)$  of  $A$  in  $\Sigma_{g,d}$ , which is a subsurface of genus  $g(A)$  with  $k(A)$  boundary components. Then  $g(A) \leq g$ , but the number  $k(A)$  of boundary components of  $F(A)$  depends on the geometry of  $A$  and can be arbitrarily large. Set  $i(A) = \frac{1}{2} \sum_{a,b \in A} i(a,b)$  for the total number of intersection points of curves in  $A$ . Here  $i(a,b)$  is the minimal number of intersection points between curves in the isotopy classes of  $a$  and  $b$ , respectively. We assume that curves in  $A$  are isotoped so that for each  $a, b \in A$  the number of intersection points between  $a$  and  $b$  equals  $i(a,b)$  and there are not triple intersections among curves in  $A$ .

We will adapt the proof of the Tits conjecture given in [7]. In the present situation we will be concerned with an Artin group  $B(A)$  (to be defined later) associated to the collection  $A$  and its representation into the mapping class group of  $F(A)$ .

We can obtain  $F(A)$  as the result of plumbing one annulus neighborhoods  $Ann_a$  for each curve  $a$  in  $A$ . In particular the core curves of the annuli are transverse to each other. Pick-up one base point  $p_a^0$  in the boundary of  $Ann_a$ , for each  $a \in A$ . We can suppose that all  $p_a^0$  belong to  $\partial F(A)$ . Choose one distinguished boundary component  $a^+$  for each annulus  $Ann_a$ . There is no loss of generality in assuming that each  $p_a^0$  belongs to  $a^+$  and a small arc of  $a^+$  centered at  $p_a^0$  is contained within  $\partial F(A)$ .

Give an orientation to every curve  $a \in A$  and a total order  $<$  on  $A$ .

If we travel along  $a^+$  in the direction given by the orientation and starting at  $p_a^0$  we will meet a number of intersection points between  $a^+$  and the other curves  $b^+$ , where  $b \in A$ . We denote them in order  $p_a^1, p_a^2, \dots, p_a^{d(a)}$ . Denote then by  $S = \cup_{a \in A} \cup_{0 \leq j \leq d(a)} \{p_a^j\}$  the set of all these points. It is clear that  $S \subset \partial F(A)$ .

The groupoid  $\pi_1(F(A), S)$  is the fundamental groupoid of  $F(A)$  based at the points of  $S$ . Since  $F(A)$  has boundary it follows that  $\pi_1(F(A), S)$  is a free groupoid (see [8], p.7).

Furthermore the mapping class group  $M(F(A))$  acts by automorphisms on the fundamental groupoid  $\pi_1(F(A), S)$ , because  $S \subset \partial F(A)$  and elements of  $M(F(A))$  are classes of homeomorphisms fixing point-wise the boundary.

Consider the following elements of  $\pi_1(F(A), S)$ :

1. For every  $s \in A$  the elementary loop  $\alpha_s$  is  $s^+$  based at  $p_s^0$ , with its orientation. Thus  $\alpha_s$  is parallel to the central curve  $s$  in the annulus  $Ann_s$ .
2. For every  $s \in A$  and  $i \in \{0, 1, \dots, d(s) - 1\}$  consider the arc  $p_s^i p_s^{i+1}$  of  $s^+$  which joins  $p_s^i$  to  $p_s^{i+1}$ . We call them admissible arcs. Observe that the arc  $p_s^{d(s)} p_s^0$  is not admissible.

The dual intersection graph of  $A$  is defined as the graph whose vertices are the elements of  $A$  and two vertices are connected by an edge if the corresponding curves intersect. Assume henceforth that the dual intersection graph of  $A$  is connected. Then admissible arcs and elementary loops generate the groupoid  $\mathbb{F} = \pi_1(F(A), S)$ .

Let then  $\Gamma_A$  be the subgroup of  $M(F(A))$  generated by the Dehn twists  $T_a$ , for all  $a \in A$ .

Set  $\mathbb{B}$  for the sub-groupoid of  $\mathbb{F}$  generated by the admissible arcs.

We will need some terminology and facts from [7]. Any element of  $\mathbb{F}$  can uniquely be written in the reduced form:

$$w = \mu_0 \alpha_{s_1}^{k_1} \mu_1 \cdots \alpha_{s_m}^{k_m} \mu_m$$

where  $\mu_i \in \mathbb{B}$ ,  $\mu_i$  is non-trivial if  $i \neq 0, m$  and  $k_i \neq 0$ .

We say that  $w$  has a *square* in  $\alpha_s$  if for some  $j$  we have  $s_j = s$  and  $|k_j| \geq 2$ , and is *without squares in  $\alpha_s$* , otherwise. Moreover  $w$  is of type  $(\mu, \alpha_t^p)$  if its reduced form is

$$w = \mu_0 \alpha_t^{k_1 p} \mu_1 \cdots \alpha_t^{k_m p} \mu_m, \quad k_j \in \mathbb{Z} \setminus \{0\}, \quad \text{and } \mu = \mu_0 \mu_1 \cdots \mu_m$$

By language abuse we will speak about  $T_a(w)$ , where  $w$  is a word in  $\mathbb{F}$ , using the action of  $\Gamma_A$  by automorphisms on  $\mathbb{F}$ .

**Lemma 2.1.** *Let  $s \in A$  and  $m \in \mathbb{Z} \setminus \{0\}$ .*

1. If  $\mu \in \mathbb{B}$  then  $T_s^m(\mu)$  is of type  $(\mu, \alpha_s^m)$ .
2. Let  $t \in A$ . If  $s = t$  or  $i(s, t) = 0$  then  $T_s^m(\alpha_t) = \alpha_t$ .
3. If  $i(s, t) \neq 0$  then  $T_t^m(\alpha_s)$  is  $u\alpha_s$ , where  $u$  is an element of type  $(1, \alpha_t^m)$ . Thus, if  $|m| \geq 2$  and  $i(s, t) \neq 0$  then  $T_t^m(\alpha_s)$  has a square in  $\alpha_t$ .

*Proof.* If  $s^+, t^+$  intersect at  $p$  we define  $\varepsilon(s, t; p) \in \{-1, 1\}$  as follows. Assume that we travel along  $s^+$  to meet  $p$ . At  $p$  we use the global orientation of the surface for turning right along  $t^+$  and continue travelling this way. If the direction along  $t^+$  is the orientation of  $t^+$  then we set  $\varepsilon(s, t; p) = 1$  and otherwise  $\varepsilon(s, t; p) = -1$ .

Next, we will identify canonically  $\pi_1(F(A), S)$  with  $\pi_1(F(A), S')$  where  $S'$  is a copy of  $S$ , each point  $p_a^j$  being slightly moved in the positive direction along the arc  $a^+$  to a point  $\tilde{p}_a^j$ .

Denote by  $\alpha_t(p_t^j)$  the element  $\tilde{p}_t^0 \tilde{p}_t^j \alpha_t \tilde{p}_t^j \tilde{p}_t^0$ , where  $\tilde{p}_t^0 \tilde{p}_t^j$  is the unique arc of  $\alpha_t$  joining  $\tilde{p}_t^0$  to  $\tilde{p}_t^j$  and consisting only of admissible sub-arcs. Also  $\alpha_t$  is considered as the loop  $t^+$  based at  $\tilde{p}_t^0$ . Then by direct computation we find:

$$T_{\alpha_t}^m(\tilde{p}_s^i \tilde{p}_s^{i+1}) = \begin{cases} \tilde{p}_s^i \tilde{p}_s^{i+1}, & \text{if } p_s^i p_s^{i+1} \cap \alpha_t = \emptyset, \text{ or } s = t \\ \tilde{p}_s^i \tilde{p}_s^{i+1} \alpha_t^{m\varepsilon(s, t; p_s^{i+1})} (p_s^{i+1}) & \text{if } p_s^{i+1} \in t^+ \\ \tilde{p}_s^i \tilde{p}_s^{i+1} & \text{if } p_s^{i+1} \notin t^+ \end{cases}$$

Notice that when the start-point  $p_s^i$  belongs to  $t^+$  the action is trivial since the base-point  $p_s^i$  is slightly pushed out of  $t^+$  in  $S'$ .

Let now  $s, t \in A$  be two curves with  $i(s, t) \neq 0$ . Suppose now that starting from  $p_s^0$  and traveling along  $s^+$  we meet the circle  $t^+$  at the points  $p_s^{j_1}, p_s^{j_2}, \dots, p_s^{j_r}$ ,  $r > 0$ . By direct inspection we find that

$$T_{\alpha_t}^m(\alpha_s) = \tilde{p}_s^0 \tilde{p}_s^{j_1} \alpha_t^{m\varepsilon(s, t; p_s^{j_1})} (p_s^{j_1}) \tilde{p}_s^{j_1} \tilde{p}_s^{j_2} \alpha_t^{m\varepsilon(s, t; p_s^{j_2})} (p_s^{j_2}) \dots \alpha_t^{m\varepsilon(s, t; p_s^{j_{r-1}})} (p_s^{j_{r-1}}) (\tilde{p}_s^0 \tilde{p}_s^{j_r})^{-1} \alpha_s$$

It is immediate that  $T_{\alpha_t}^m(\alpha_s) = u\alpha_s$ , where  $u$  is of type  $(1, \alpha_t^m)$ . □

**Lemma 2.2.** *Let  $x \in \mathbb{F}$ ,  $|m| \geq 2$ . If  $x$  is without squares in  $\alpha_t$  and  $T_{\alpha_t}^m(x)$  has a square in  $\alpha_s$  then either  $s = t$  or else  $i(s, t) = 0$  and  $x$  has a square in  $\alpha_s$ .*

*Proof.* Let  $x = \mu_0 \alpha_{s_1}^{k_1} \mu_1 \dots \alpha_{s_r}^{k_r} \mu_r$  in reduced form. The previous lemma shows that:

1. If  $s_i = t$  then  $v_i = T_{\alpha_t}^m(\alpha_{s_i}^{k_i}) = \alpha_{s_i}^{k_i}$ , where  $k_i \in \{-1, 1\}$ , because  $x$  is without squares in  $\alpha_t$ .
2. If  $s_i$  and  $t$  are disjoint then  $T_{\alpha_t}^m(\alpha_{s_i}^{k_i}) = \alpha_{s_i}^{k_i}$ .
3. If  $i(s_j, t) \neq 0$  then

$$T_{\alpha_t}^m(\alpha_{s_j}^{k_j}) = \begin{cases} u_j (\alpha_{s_j} u_j)^{k_j-1} \alpha_{s_j} & \text{if } k_j > 0 \\ \alpha_{s_j}^{-1} (u_j^{-1} \alpha_{s_j}^{-1})^{-k_j-1} u_j^{-1} & \text{if } k_j < 0 \end{cases}$$

where  $u_j$  is a non-constant term of type  $(1, \alpha_t^m)$ .

4.  $T_{\alpha_t}^m(\mu_j)$  has a reduced form  $y_j$  of type  $(\mu, \alpha_t^m)$ , for all  $j \geq 0$ .

Therefore we can write in reduced form  $T_{\alpha_t}^m(x) = x_0 v_1 x_1 v_2 \dots v_r x_r$  as follows:

1. If either  $s_i = t$  or  $s_i$  and  $t$  are disjoint then  $v_i = T_{\alpha_t}^m(\alpha_{s_i}^{k_i}) = \alpha_{s_i}^{k_i}$ .
2. Assume that  $i(s_j, t) \neq 0$ .
  - (a) If  $k_j > 0$  then  $v_j = (\alpha_{s_j} u_j)^{k_j-1} \alpha_{s_j}$ . Absorb the extra factor  $u_j$  into  $x_{j-1}$ .
  - (b) If  $k_j < 0$  then  $v_j = \alpha_{s_j}^{-1} (u_j^{-1} \alpha_{s_j}^{-1})^{-k_j-1}$ . Absorb the extra factor  $u_j^{-1}$  into  $x_j$ .
3. Eventually  $x_j$  are  $T_{\alpha_t}^m(\mu_j)$ , possibly corrected by the absorption of terms coming from  $v_j$  or  $v_{j+1}$ . Thus  $x_j$  are of reduced form of type  $(\mu_j, \alpha_t^m)$ .

In particular, if  $T_{\alpha_t}^m(x)$  has a square in  $\alpha_s$  then either  $s = t$  or there exists  $j$  such that  $s_j = s$  and  $s$  and  $t$  are disjoint. □

To each set of curves  $A \subset \Sigma_{g,d}$  we can associate the Artin group  $B(A)$ , with the following presentation:

$$B(A) = \langle z_a, a \in A \mid z_a z_b = z_b z_a, \text{ if } a \cap b = \emptyset, z_a z_b z_a = z_b z_a z_b, \text{ if } i(a, b) = 1 \rangle$$

There is a natural homomorphism  $\tau : B(A) \rightarrow M(F(A))$  which sends  $z_a$  into the Dehn twist  $T_a$ .

Consider now the right angled Artin group defined by the presentation:

$$H(A) = \langle w_a, a \in A \mid w_a w_b = w_b w_a, \text{ if } i(a, b) = 0 \rangle$$

There is a map  $\iota : H(A) \rightarrow B(A)$  given by  $\iota(w_a) = z_a^D$ . We will suppose that  $D \geq 2$  in the sequel. The word  $W = w_{s_l}^{n_l} w_{s_{l-1}}^{n_{l-1}} \cdots w_{s_2}^{n_2} w_{s_1}^{n_1}$  is called an  $M$ -reduced expression of the element  $w \in H(A)$  (obtained by interpreting letters as the corresponding generators of  $H(A)$ ) if for any  $i < j$  such that  $s_i = s_j$  there exists  $k$  such that  $i < k < j$  and  $i(s_i, s_k) \neq 0$ . Then the  $M$ -reduced expression for  $w$  ends in  $s$  if, up to change the order of commuting generators, we can arrange that  $s_l = s$ .

Recall now that  $\tau(\iota(w))$  is an automorphism of  $\mathbb{F}$ , for each  $w \in H(A)$ . We will write simply  $w(x)$  or  $W(x)$  for  $\tau(\iota(w))(x)$ , where  $w \in H(A)$ ,  $x \in \mathbb{F}$  and  $W$  is an  $M$ -reduced expression for  $w$ .

The following two lemmas are restatements of Propositions 9 and 10 from [7].

**Lemma 2.3.** *Let  $W$  be an  $M$ -reduced expression for  $w \in H(A)$ ,  $x \in \mathbb{F}$  and  $s \in A$ . Suppose that  $x$  is without squares in  $\alpha_t$  for all  $t \in A$ , and that  $w(x)$  has a square in  $\alpha_s$ . Then  $W$  ends in  $s$ .*

*Proof.* Proceed by induction on the length of the  $M$ -reduced expression  $W$ . Let  $W = T_{s_l}^{n_l} W'$ . If  $W'(x)$  had a square in  $\alpha_{s_l}$  then  $W'$  would end in  $s_l$  (by the induction hypothesis) and hence  $W$  would not be an  $M$ -reduced expression. Hence  $W'(x)$  is without squares in  $\alpha_{s_l}$ .

Now  $W(x) = T_{\alpha_{s_l}}^{D n_l}(W'(x))$  has a square in  $\alpha_t$ . By Lemma 2.2 one has:

1. either  $s_l = t$  and so  $W$  ends in  $t$ ;
2. or else  $s_l$  and  $t$  are disjoint and  $W'(x)$  has a square in  $\alpha_t$ . By induction  $W'$  ends in  $t$ . Since  $s_l$  and  $t$  commute we switch the position of the last two generators and find that  $W$  ends in  $t$ .

□

For a fixed  $a \in A$  the fundamental group  $\pi_1(F(A), p_a^0)$  embeds into the groupoid  $\pi_1(F(A), S)$ . It is also clear that  $\pi_1(F(A), p_a^0)$  is kept invariant by the action of an element  $w \in H(A)$ .

**Lemma 2.4.** *Assume that the dual intersection graph of  $A$  (or, equivalently the surface  $F(A)$ ) is connected. If  $w$  has a nontrivial  $M$ -reduced expression then  $w$  acts non-trivially on  $O(A)$ .*

*Proof.* It is known (see e.g.[7] and references there) that an  $M$ -reduced expression representing the identity in  $H(A)$  is trivial. Take then a non-trivial  $M$ -reduced expression  $W$ , as above. Since the dual intersection graph of curves is nontrivial there exists some  $t \in A$  such that  $i(s_l, t) \neq 0$ . We will show that  $W(\alpha_t) \neq \alpha_t$ . Since  $\alpha_t \in O(A) \subset \pi_1(F(A), p_t^0)$  the action of  $W$  is nontrivial on  $O(A)$ .

Suppose  $W(\alpha_t) = \alpha_t$  and write  $W = T_{s_l}^{n_l} W'$ . Then

$$W'(\alpha_t) = w_{s_l}^{-n_l}(\alpha_t) = T_{s_l}^{-D n_l}(\alpha_t)$$

Lemma 2.1 shows that  $T_{s_l}^{-D n_l}(\alpha_t)$  has a square in  $\alpha_{s_l}$  and further from Lemma 2.2  $W'$  ends in  $s_l$ . But then  $W$  is not  $M$ -reduced, contradiction. This proves the claim. □

**Proposition 2.2.** *Assume that the dual intersection graph of the finite collection  $A$  is connected,  $A$  has at least two elements and  $D \geq 2$ . Then the group  $M(F(A))(A; D)$  is a right angled Artin groups of presentation:*

$$M(F(A))(A; D) = \langle T_a^D, a \in A \mid T_a^D T_b^D = T_b^D T_a^D, \text{ if } a \cap b = \emptyset \rangle$$

*Proof.* Lemma 2.4 shows that the map  $\tau \circ \iota : H(A) \rightarrow M(F(A))$  is injective, since  $M(F(A))$  is a subgroup of the group of automorphisms of  $\mathbb{F}$ . Therefore  $M(F(A))(A; D)$  is isomorphic to  $H(A)$ , as claimed. □

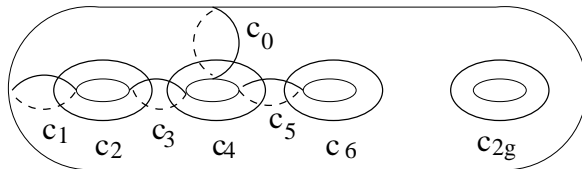
**Corollary 2.1.** *If  $A$  is nontrivial and  $\Sigma_{g,d} \setminus F(A)$  has neither disks nor cylinder components joining two distinct boundary components of  $F(A)$  and  $D \geq 2$  then  $M(\Sigma_{g,d})(A, D)$  is a right angled Artin group of presentation:*

$$M(\Sigma_{g,d})(A; D) = \langle T_a^D, a \in A \mid T_a^D T_b^D = T_b^D T_a^D, \text{ if } a \cap b = \emptyset \rangle$$

*Proof.* The embedding  $F(A) \subset \Sigma_{g,d}$ , with  $F(A)$  different from a disk or an inessential annulus induces a group embedding  $M(F(A)) \hookrightarrow M(\Sigma_{g,d})$  according to ([40], Corollary 4.2) if and only if  $\Sigma_{g,d} \setminus F(A)$  has neither disk nor cylinder components joining two distinct boundary components of  $F(A)$ . Now, since  $A$  is nontrivial  $F(A)$  is neither a disk nor an inessential annulus.  $\square$

*End of proof of Theorem 1.1.* It suffices to consider the case when the dual intersection graph of  $A$  is connected. The mapping class group  $M(\Sigma_{g,d})$  embeds into  $\text{Aut}(\pi_1(\Sigma_{g,d}, p))$ , where the base point  $p$  is chosen on the boundary  $\partial\Sigma_{g,d}$ . By Lemma 2.4 for every nontrivial element  $w \in H(A)$  there is some  $z \in O(A)$  such that  $\iota(w)(z) \cdot z^{-1} \neq 1$ . Since the homomorphism  $j : O(A) \rightarrow \pi_1(\Sigma_{g,d})$  was assumed to be injective it follows that  $\tau(\iota(w))(j(z)) \cdot j(z)^{-1} = j(\iota(w)(z) \cdot z^{-1}) \neq 1$ . Therefore  $\tau(\iota(w))$  acts nontrivially on  $\pi_1(\Sigma_{g,d}, p)$  and thus  $\tau(\iota(w))$  is not identity. This means that  $\tau \circ \iota$  is injective and hence the homomorphism of  $H(A)$  onto  $M(\Sigma_{g,d})(A; D)$  is an isomorphism. This settles Theorem 1.1.

**Corollary 2.2.** *Let  $B = \{c_1, c_2, \dots, c_{2g+1}\}$  and  $C = \{c_1, c_2, \dots, c_{2g}\}$ , where  $c_j$  are the curves from the figure below, which furnish the Dehn-Lickorish-Humphries generators of  $M_g$  and  $D \geq 2$ .*



Let  $\Sigma_{g,2}$  and  $\Sigma_{g,1}$  be the regular neighborhoods in  $\Sigma_g$  of the union of curves from  $B$  and respectively  $C$ . Then the groups  $M(\Sigma_{g,2})(B; D)$  and  $M(\Sigma_{g,1})(C; D)$  are right angled Artin groups with the presentations:

$$M(\Sigma_{g,2})(B; D) = \langle T_{c_j}^D, j = 0, \dots, 2g + 1; T_{c_j}^D T_{c_k}^D = T_{c_k}^D T_{c_j}^D, \text{ if } j < k, k \neq j + 1, (j, k) \neq (0, 4) \rangle$$

and respectively:

$$M(\Sigma_{g,1})(C; D) = \langle T_{c_j}^D, j = 0, \dots, 2g; T_{c_j}^D T_{c_k}^D = T_{c_k}^D T_{c_j}^D, \text{ if } j < k, k \neq j + 1, (j, k) \neq (0, 4) \rangle$$

*Proof.* Here is a direct simpler proof which uses the proof given in [7] for small Artin groups. Let  $E_{2g}$  be the Artin group associated to the Dynkin graph of type  $E_{2g}$ , which is the tree whose vertices are in one-to-one correspondence with the curves  $c_0, c_1, \dots, c_{2g}$  from the figure above and whose edges join two vertices only if the respective curves have one intersection point. Observe that  $A_{2g}$  is the Dynkin subgraph associated to the curves  $c_1, c_2, \dots, c_{2g}$ .

Let now  $E_{2g}[D]$  denote the subgroup of  $E_{2g}$  generated by  $T_{c_j}^D, j = 0, 1, \dots, 2g$ . Crisp and Paris proved in [7] that the subgroup  $E_{2g}[D]$  has the following right angled Artin group presentation:

$$E_{2g}[D] = \langle T_{c_j}^D, j = 0, \dots, 2g \mid T_{c_j}^D T_{c_k}^D = T_{c_k}^D T_{c_j}^D, \text{ if } j < k, k \neq j + 1, (j, k) \neq (0, 4) \rangle$$

The regular neighborhoods  $F(B)$  and  $F(C)$  are homeomorphic to  $\Sigma_{g,2}$  and  $\Sigma_{g,1}$ , respectively.

An essential ingredient of the proof in [7] is the natural representation of the Artin group  $E_{2g}$  into the mapping class group  $M(F(B))$ . Consequently  $E_{2g}$  acts by automorphisms on the fundamental groupoid  $\pi_1(F(B); S)$ , where  $S = \{s_0, \dots, s_{2g}\}$  is a set of boundary base points, one base point for each annulus. Set  $\tau : E_{2g} \rightarrow \text{Aut}(\pi_1(F(B); S))$  for this representation.

Let then  $H(B)$  and  $H(C)$  be the right angled Artin group

$$H(B) = \langle a_j, j = 0, \dots, 2g \mid a_j a_k = a_k a_j, \text{ if } j < k, k \neq j + 1, (j, k) \neq (0, 4) \rangle$$

$$H(C) = \langle a_j, j = 1, \dots, 2g \mid a_j a_k = a_k a_j, \text{ if } j < k, k \neq j + 1, (j, k) \neq (0, 4) \rangle$$

There is a homomorphism  $\iota : H(B) \rightarrow E_{2g}$  that sends each  $a_j$  into  $T_{c_j}^D$ .

The key point of the proof from [7] is that, given any non-trivial element  $w \in H(B)$ , the automorphism  $\tau(\iota(w))$  acts non-trivially on some element of  $\pi_1(F(B), S)$  and hence  $\tau(\iota(w)) \neq 1$ . This shows that  $\iota$  injects  $H(B)$  into  $E_{2g}$ .

However this proof also shows that the right angled Artin group  $H(B)$  injects into the mapping class group  $M(F(B))$ . The corresponding map sends  $a_j$  into the Dehn twist  $T_{c_j}^D$ . As  $M(F(B))$  is actually  $M(\Sigma_{g,2})(B; D)$  the claim follows.

The same proof works for the sub-family  $C$ . □

We can slightly generalize the previous results to subgroups generated by not necessarily equal powers of Dehn twists.

**Proposition 2.3.** *The subgroup of  $M(F(A))$  generated by  $T_a^{D(a)}$ , where  $|D(a)| \geq 2$ ,  $a \in A$  is a right angled Artin group.*

*Proof.* The proof from above applies with only minor modifications. □

This generalization makes sense also in the case where we consider the subgroup generated by suitable powers of (all) Dehn twists. However, if we want to stay among normal subgroups we are restricted to the following class of normal subgroups. Let  $\mu : SCC(S)/M(S) \rightarrow \mathbb{Z}$  be a weight. Notice that if  $S$  is  $\Sigma_g$  (respectively  $\Sigma_{g,1}$ ) then  $SCC(S)/M(S)$  is  $\{0, 1, 2, \dots, \lfloor \frac{g}{2} \rfloor\}$ , and respectively  $\{0, 1, 2, \dots, g\}$ . We associate to each curve the minimal genus of a subsurface it bounds it and respectively 0, if the curve is non-separating.

Then we define  $M(S)[\mu]$  as the subgroup generated by  $T_a^{\mu(a)}$ , where  $a \in SCC(S)$ . As above,  $M(S)[\mu]$  is a normal subgroup of  $M(S)$ .

*Remark 2.2.* If  $\mu(0) = D$  and  $\mu(j) = 1$ , for  $j > 0$  the subgroup  $M(\Sigma_g)[\mu]$  is the level  $D$  subgroup of the mapping class group of  $\Sigma_g$ , namely the kernel of  $M(\Sigma_g) \rightarrow Sp(2g, \mathbb{Z}/D\mathbb{Z})$ . This is proved by McCarthy in ([34], Theorem 2.8). In particular, in this case the subgroup is of finite index.

## 3 Power subgroups of the mapping class group

### 3.1 $M_g[D]$ and symplectic groups

We fix once for all a symplectic basis  $\{a_i, b_i\}_{i=1, \dots, g}$  in homology consisting of classes of simple loops and denote by  $P : M_g \rightarrow Sp(2g, \mathbb{Z})$  the natural homomorphism.

**Proposition 3.1.** *If  $g \geq 2$  then  $P$  sends  $M_g[D]$  onto the special congruence subgroup*

$$Sp(2g, \mathbb{Z})[D] = \ker(Sp(2g, \mathbb{Z}) \rightarrow Sp(2g, \mathbb{Z}/D\mathbb{Z}))$$

*Proof.* The action of the Dehn twist  $T_b$  in homology is given by

$$T_b^D a = a \pm D \langle a, b \rangle b$$

where  $\langle a, b \rangle$  is the algebraic intersection number on  $\Sigma_g$  and the sign  $\pm$  depends on the orientation of  $b$ . Therefore  $T_b^D(a) - a$  belongs to the submodule  $DH_1(S_g, \mathbb{Z})$  of  $H_1(S_g, \mathbb{Z})$ , for any  $b \in H_1(S_g, \mathbb{Z})$ . This implies that  $P(T_b^D) \in Sp(2g, \mathbb{Z})[D]$  and hence  $P(M_g[D])$  is a normal subgroup of  $Sp(2g, \mathbb{Z})[D]$ .

Recall that  $Sp(2g, \mathbb{Z})$  is the group of matrices  $A$  with integer entries which satisfy  $AJA^T = J$ , where the almost complex structure matrix  $J$  is the direct sum of  $g$  blocks  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ .

Consider the elementary matrices

$$\begin{aligned} SE_{i\tau(i)}[D] &= I_{2g} + DE_{i\tau(i)} \\ SE_{ij} &= I_{2g} + DE_{ij} - (-1)^{i+j} DE_{\tau(j)\tau(i)} \end{aligned}$$

where  $\tau$  is the permutation  $\tau(2j) = 2j - 1, \tau(2j - 1) = 2j$ , for  $1 \leq j \leq g$  and  $E_{ij}$  denotes the matrix having a single non-zero unit entry at position  $(ij)$ . By direct computation we find that:

$$\begin{aligned} SE_{12}[D] &= P(T_{a_1}^{-D}) \\ SE_{13}[D] &= P(T_{b_2}^{-D} T_{a_1}^{-D} T_c^D) \\ SE_{14}[D] &= P(T_{a_2}^D T_{a_1}^D T_f^{-D}) \end{aligned}$$

where  $c$  and  $f$  are simple closed curves whose homology class is  $a_1 + b_2$  and  $a_1 + a_2$  respectively.

Therefore the elementary congruence subgroup of level  $D$ , which is defined as the matrix group generated by the matrices  $SE_{ij}[D]$ , is contained in  $P(M_g[D])$ . Now, a deep result of Mennicke (see [37, 38, 3]) says that the elementary congruence subgroup coincides with the congruence subgroup  $Sp(2g, \mathbb{Z})[D]$ , if  $g \geq 2$ . Therefore  $P(M_g[D]) = Sp(2g, \mathbb{Z})[D]$ , as claimed.  $\square$

*Remark 3.1.* If  $g = 1$  then  $M_g[D]$  might be of infinite index in  $SL(2, \mathbb{Z})$  (see [39]).

**Corollary 3.1.** *The group  $M_g[D]$  is torsion-free and consists of pure mapping classes when  $D \geq 3$  and  $g \geq 2$ .*

*Proof.* Serre's Lemma tells us that torsion elements in the mapping class group act non-trivially on the homology with  $\mathbb{Z}/D\mathbb{Z}$  coefficients for any  $D \geq 3$ .

The second claim is a simple consequence of Ivanov's results (see [22, 23]) concerning pure classes. Recall that a mapping class  $h$  is pure if  $h^n(\gamma) = \gamma$  implies that  $h(\gamma) = \gamma$ , for each isotopy class of a simple closed curve  $\gamma$ .  $\square$

## 3.2 Power subgroups and symplectic groups

We start by analyzing the images of the power subgroups in the symplectic group. This amounts to find the power subgroups of the symplectic group. Let  $g \geq 2$  and recall that  $P$  denotes the homomorphism  $P : M_g \rightarrow Sp(2g, \mathbb{Z})$  induced by a homology basis. We already saw in section 3.1 that  $P(M_g[D]) = Sp(2g, \mathbb{Z})[D]$ . Moreover since  $P$  is surjective  $P(X_g(D))$  is a normal subgroup of  $Sp(2g, \mathbb{Z})$  containing  $Sp(2g, \mathbb{Z})[D]$ . We have then an obvious surjective homomorphism:

$$L : Sp(2g, \mathbb{Z}/D\mathbb{Z}) = Sp(2g, \mathbb{Z})/Sp(2g, \mathbb{Z})[D] \rightarrow Sp(2g, \mathbb{Z})/P(X_g(D))$$

Our first technical result is the following:

**Lemma 3.1.** *For any integer  $D \not\equiv 0 \pmod{6}$  and any ideal  $J \subset \mathbb{Z}/D\mathbb{Z}$  there exists an element in the kernel of  $L$  which is not central after reduction mod  $J$ .*

*Proof.* It suffices to find a matrix in  $C \in Sp(2g, \mathbb{Z}/D\mathbb{Z})$  whose power  $C^D$  is neither the identity  $\mathbf{1}$  nor  $-\mathbf{1}$  modulo the ideal  $J$ , since the center of  $Sp(2g, \mathbb{Z}/D\mathbb{Z})$  consists of  $\{\mathbf{1}, -\mathbf{1}\}$  (see [28], Prop.2.1). Since  $C^D$  belongs to  $\ker L$  this will prove the lemma.

We look after  $C$  of the form  $A \oplus A \oplus \cdots \oplus A$  where  $A$  is a 2-by-2 matrix. We take a lift of  $A$  with integer entries. Then  $C^D$  has the form  $A^D \oplus A^D \oplus \cdots \oplus A^D$ . Since  $A \in SL(2, \mathbb{Z})$  we have

$$A^2 = tA - \mathbf{1}$$

where  $t$  is the trace of  $A$ . It follows that

$$A^D = Q_{D-1}(t)A - Q_{D-2}(t)\mathbf{1}$$

where  $Q_k(t) \in \mathbb{Z}[t]$  are polynomials in the variable  $t$  determined by the recurrence relation:

$$Q_{n+1}(t) = tQ_n(t) - Q_{n-1}(t)$$

with initial values  $Q_0 = 1, Q_1(t) = t$ .

We obtain therefore, by induction on  $D$ , the following formulas:

$$Q_{D-1}(0) = \begin{cases} (-1)^{\frac{D-1}{2}}, & \text{if } D \equiv 1 \pmod{2}, \\ 0, & \text{if } D \equiv 0 \pmod{2}. \end{cases}$$

$$Q_{D-1}(-1) = \begin{cases} 1, & \text{if } D \equiv 2 \pmod{3}, \\ -1, & \text{if } D \equiv 0 \pmod{3}, \\ 0, & \text{if } D \equiv 1 \pmod{3}. \end{cases}$$

If the reduction mod  $J$  of  $C^D$  is trivial for all  $C$  as above then  $Q_{d-1}(t) \equiv 0 \pmod{J}$  for all  $t$ , since there exist matrices  $A$  of given trace  $t$  having some entry off-diagonal which is congruent to 1 mod  $D$ . Now, either  $Q_{D-1}(-1)$  or  $Q_{D-1}(0)$  is  $\pm 1 \pmod{D}$ , hence  $J$  is trivial. This proves the claim.  $\square$

*Remark 3.2.* The conclusion of Lemma 3.1 does not hold when  $D \equiv 0 \pmod{6}$ . For instance  $Q_5(t) = t(t-1)(t+1)(t^2-3)$  and thus  $Q_5(t) \equiv 0 \pmod{6}$  for every integer  $t$ . More generally  $Q_{6k-1}(t) \equiv 0 \pmod{6}$ , for every integer  $k$ . It suffices to observe that:

$$Q_{D-1}(1) = \begin{cases} 1, & \text{if } D \equiv 1 \pmod{6}, \text{ or } D \equiv 2 \pmod{6}, \\ -1, & \text{if } D \equiv 4 \pmod{6}, \text{ or } D \equiv 5 \pmod{6}, \\ 0, & \text{if } D \equiv 3 \pmod{6}, \text{ or } D \equiv 6 \pmod{6}. \end{cases}$$

and use the previous computations for  $Q_{D-1}(0)$  and  $Q_{D-1}(1)$ .

*Remark 3.3.* Observe that  $Q_n$  is the  $n$ -th Chebyshev polynomial of the second kind

$$Q_n(t) = \frac{\sin(n+1)\arccos(t/2)}{\sin \arccos(t/2)}$$

which can be given by the explicit formula

$$Q_n(t) = \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} (-1)^k \frac{(n-k)!}{k!(n-2k)!} t^{n-2k}$$

Notice that the usual definition for the Chebyshev polynomial uses the variable  $x$ , where  $t = 2x$ .

**Proposition 3.2.** *Suppose that  $g \geq 2$ ,  $D$  is of the form  $p^m$  for a prime  $p$ ,  $m \in \mathbb{Z}_+$  and additionally  $g \geq 3$ ,  $m \geq 2$  when  $p \in \{2, 3\}$ . Then  $P(X_g[D])$  is all of  $Sp(2g, \mathbb{Z})$ .*

*Proof.* The normal subgroups of symplectic groups over local rings were described by Klingenberg and Jehne, in the case when  $D = p^m$ ,  $p$  prime and  $p \notin \{2, 3\}$ . The most general statement can be found in ([16], Thm. 9.1.7, p.517) where one also considered  $p \in \{2, 3\}$  but  $g \geq 3$ . The above cited result is that under these conditions all normal subgroups of  $Sp(2g, \mathbb{Z}/D\mathbb{Z})$  (where  $D = p^m$ , such that  $\mathbb{Z}/D\mathbb{Z}$  is a local ring) are congruence subgroups, namely they contain the kernel of the homomorphism  $Sp(2g, \mathbb{Z}/D\mathbb{Z}) \rightarrow Sp(2g, (\mathbb{Z}/D\mathbb{Z})/J)$ , for some ideal  $J$ . The previous lemma tells us that the smallest possible ideal  $J$  (called the order ring of the respective element) for which the kernel contains the element  $C^D$  is the ring  $\mathbb{Z}/D\mathbb{Z}$  itself, if  $p \notin \{2, 3\}$  or  $m \geq 2$ . Then Klingenberg's result from ([28], Lemma 3.2, see also [26]) shows that the normal subgroup generated by  $C^D$  is all of  $Sp(2g, \mathbb{Z}/D\mathbb{Z})$ .  $\square$

*Remark 3.4.* The projective symplectic group  $PSp(2g, \mathbb{Z}/D\mathbb{Z})$  is simple when  $D$  is prime, except when  $g = 1, D \in \{2, 3\}$  (where it coincides with the permutation group  $S_3$  and respectively the alternating group  $A_4$ ) and  $g = 2, D = 2$  (when it coincides with the permutation group  $S_6$ ).

*Remark 3.5.* When  $g = 2$  and  $D = 2$  the image of  $P(X_2(2))$  is of index 2 in  $Sp(4, \mathbb{Z}/2\mathbb{Z})$ . The subgroup generated by squares of elements in  $S_6$  is the index 2 alternating subgroup  $A_6$ . In fact any square has even signature and  $A_6$  is also the commutator subgroup of  $S_6$ . Observe that  $[a, b] = (ab)^2$ , if  $a^2 = b^2 = 1$  and commutators of transpositions generate  $A_6$ . Finally we have the exact sequence:

$$1 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow P(X_2(2)) \rightarrow A_6 \rightarrow 1$$

to be compared with:

$$1 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow Sp(4, \mathbb{Z}/2\mathbb{Z}) \rightarrow S_6 \rightarrow 1$$

In the general case when  $D$  is not a power of a prime the image of  $X_g(D)$  might be strictly smaller than  $Sp(2g, \mathbb{Z}/D\mathbb{Z})$ . This is clear when  $D \equiv 0 \pmod{6}$ , since Remark 3.2 shows that the image of  $P(X_g(D)) \subset Sp(2g, \mathbb{Z}/D\mathbb{Z})$  into  $Sp(2g, \mathbb{Z}/6\mathbb{Z})$  must be central. A similar result holds more generally. Let us set:

$$o_c(D) = \min\{d; A^d \in Z(Sp(2g, \mathbb{Z}/D\mathbb{Z})), \text{ for any } A \in Sp(2g, \mathbb{Z}/D\mathbb{Z})\}$$

where  $Z(G)$  stands for the center of the group  $G$ . Write  $D$  as  $D = q_1 q_2 \cdots q_m D'$ , where  $q_j$  are powers of distinct primes and  $D' \in \mathbb{Z}$ . Set  $V = \{j; o_c(q_j) \text{ divides } D\} \subset \{1, 2, \dots, m\}$  and  $\nu(D) = \prod_{j \in V} q_j$ . Consider also the general congruence subgroup  $GSp(2g, \mathbb{Z}/D\mathbb{Z})[F]$  which is the preimage of  $Z(Sp(2g, \mathbb{Z}/F\mathbb{Z}))$  under the reduction mod  $F$  homomorphism  $Sp(2g, \mathbb{Z}/D\mathbb{Z}) \rightarrow Sp(2g, \mathbb{Z}/F\mathbb{Z})$ .

**Proposition 3.3.** *The image  $P(X_g(D))$  is contained in the general congruence subgroup  $GSp(2g, \mathbb{Z})[\nu(D)]$ .*

*Proof.* Consider the homomorphism  $p_j : Sp(2g, \mathbb{Z}/D\mathbb{Z}) \rightarrow Sp(2g, \mathbb{Z}/q_j\mathbb{Z})$  which reduces entries modulo  $q_j$ . If  $A \in Sp(2g, \mathbb{Z}/D\mathbb{Z})$  then  $p_j(A^D)$  is central for any  $A \in Sp(2g, \mathbb{Z}/D\mathbb{Z})$  if  $o_c(q_j)$  divides  $D$ . Therefore the  $D$ -th power subgroup of  $Sp(2g, \mathbb{Z}/D\mathbb{Z})$  is contained into  $\bigcap_{j \in V} GSp(2g, \mathbb{Z}/D\mathbb{Z})[q_j]$ , which can be identified with  $GSp(2g, \mathbb{Z})[\nu(D)]$ .  $\square$

### 3.3 Proof of Theorem 1.2

Theorem 1.2 (1) can be restated as follows:

**Proposition 3.4.** *There exist infinitely many integers  $D$  for which  $P(X_g[D])$  is a proper subgroup of  $Sp(2g, \mathbb{Z})$ , for  $g \geq 2$ . In particular  $M_g/X_g[D]$  are non-trivial torsion groups, for these values of  $D$ .*

*Proof.* It is clear that  $o_c(q)$  is a divisor of the order of  $Sp(2g, \mathbb{Z}/q\mathbb{Z})$ , although this upper bound is far from being optimal. Let  $D = \text{l.c.m.}(o_c(q), q)$ . Thus we can write  $D = qD'$  for some integer  $D'$ , and we know that  $o_c(q)$  divides  $D$ . Therefore  $\nu(D)$  is divisible by  $q$ . Henceforth there exist infinitely many integers  $D$  for which  $P(X_g(D))$  is a proper subgroup of  $Sp(2g, \mathbb{Z})$ , by Proposition 3.3. In particular  $M_g/X_g(D)$  is a non-trivial torsion group.  $\square$

Notice however that  $P(X_g(D))$  is always of finite index since it contains the congruence subgroup  $P(M_g[D])$ . The second step in the study of  $X_g(D)$  is to understand the interactions with the torsion subgroup of  $M_g$ . We restate here Theorem 1.2 (2) for the sake of completeness.

**Proposition 3.5.** *We have  $X_g(D) = M_g$ , for  $g \geq 3$ , if  $4g + 2$  does not divide  $D$ .*

*Proof.* The chain relation (see e.g. [10], 4.4) shows that whenever  $c_1, c_2, \dots, c_k$  are simple closed curves forming a chain i.e. consecutive  $c_j$  have a common point and are otherwise disjoint, then:

1. if  $k$  is even we have:

$$(T_{c_1} T_{c_2} \cdots T_{c_k})^{2k+2} = T_d$$

and also:

$$(T_{c_1}^2 T_{c_2} \cdots T_{c_k})^{2k} = T_d$$

where  $d$  is the boundary of the regular neighborhood of the union of the  $c_j$ .

2. if  $k$  is odd we have:

$$(T_{c_1} T_{c_2} \cdots T_{c_k})^{k+1} = T_{d_1} T_{d_2}$$

and respectively:

$$(T_{c_1}^2 T_{c_2} \cdots T_{c_k})^k = T_{d_1} T_{d_2}$$

where  $d_1, d_2$  are the boundary curves of the regular neighborhood of the union of the  $c_j$ .

As a consequence the element  $a = T_{c_1} T_{c_2} \cdots T_{c_{2g}}$  is of order  $4g + 2$  and the element  $b = T_{c_1}^2 T_{c_2} \cdots T_{c_{2g}}$  is of order  $4g$ , where  $c_1, c_2, \dots, c_{2g}$  are the curves from the first figure.

**Lemma 3.2.** *The normal subgroup generated by  $a^k$  is  $M_g$  when  $k \leq 2g$  and  $g \geq 3$  and of index 2 when  $g = 2$ .*

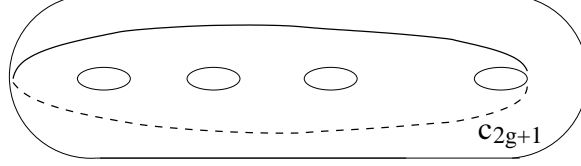
*Proof.* See ([31], Theorem 4).  $\square$

Let  $\pi : M_g \rightarrow M_g/X_g(D)$  be the projection. We have then  $a^{4g+2} = 1$ . Set  $k = \gcd(4g+2, D) < 4g+2$ . In the quotient  $M_g/X_g(D)$  we have also  $\pi(a^D) = 1$  and hence  $\pi(a^k) = 1$ . We have either  $k \leq 2g$  or else  $k = 2g+1$ .

If  $k < 2g+1$  Lemma 3.2 shows that the quotient  $M_g/X_g(D)$  is trivial.

If  $k = 2g+1$  recall that we have also  $b^{4g} = 1$  and hence  $\pi(b) = 1$ . This implies that  $\pi(a) = \pi(T_{c_1}T_{c_2} \cdots T_{c_g}) = \pi(T_{c_1}^{-1})$ .

By recurrence on  $k$  we can show that  $a^k(c_1) = c_{k+1}$ , if  $k \leq 2g$ , where  $c_{2g+1}$  is the curve from the figure below:



Thus

$$T_{c_1}^{-1}a^kT_{c_1}a^{-k} = T_{c_1}^{-1}T_{a^k(c_1)} = T_{c_1}^{-1}T_{c_{k+1}}$$

Therefore

$$\pi(T_{c_1}^{-1}T_{c_{k+1}}) = \pi(T_{c_1}^{-1}a^kT_{c_1}a^{-k}) = 1$$

so that

$$\pi(T_{c_1}) = \pi(T_{c_2}) = \cdots = \pi(T_{c_{2g}})$$

The braid relations in  $M_g$  read

$$T_{c_0}T_{c_4}T_{c_0} = T_{c_4}T_{c_0}T_{c_4}$$

and

$$T_{c_1}T_{c_0} = T_{c_0}T_{c_1}$$

from which one can find

$$\pi(T_{c_0}) = \pi(T_{c_1})$$

Thus the images by  $\pi$  of all standard  $2g+1$  generators of  $M_g$  coincide and since  $H_1(M_g) = 1$ , for  $g \geq 3$ , we obtain:

$$\pi(T_{c_i}) = 1, \text{ for all } i = 0, 1, \dots, 2g$$

Thus the quotient group is trivial.  $\square$

*Remark 3.6.* One knows that  $M_g/M_g[2]$  is finite (see [19]), when  $g \geq 2$ , and  $M_g/X_g(2)$  is the further quotient obtained by adjoining all squares as relations. Thus the quotient is a finite commutative 2-torsion group. But  $M_g$  is perfect (when  $g \geq 3$ ) and hence it does not have surjective morphisms into nontrivial abelian groups. Thus  $M_g/X_g(2)$  should be trivial, for  $g \geq 3$ .

*Remark 3.7.* For every non-separating curve  $d$  we can find a chain  $c_1, c_2, \dots, c_{2g-1}$  whose boundary is made of two curves isotopic to  $d$  and hence

$$(T_{c_1}^2T_{c_2} \cdots T_{c_{2g-1}})^{2g-1} = T_d^2$$

Since  $T_d$  and  $T_{c_i}$  commute we have

$$((T_{c_1}^2T_{c_2} \cdots T_{c_{2g-1}})^{1-g}T_d)^{2g-1} = T_d$$

Thus every Dehn twist along a non-separating curve is a  $(2g-1)$ -power. Since these Dehn twists generate  $M_g$  it follows that  $X_g(2g-1) = M_g$ , for  $g \geq 3$ .

**Corollary 3.2.** *The index of a normal subgroup of  $M_g$  is a multiple of  $4g+2$ , when  $g \geq 3$ .*

*Proof.* In fact  $X_g[N]$  is contained in a normal subgroup of index  $N$ . Proposition 3.5 implies the claim.  $\square$

### 3.4 Proof of Theorem 1.3

For a group  $G$  denote by  $Q(G)[D]$  the quotient of  $G$  by its  $D$ -th power subgroup  $X(G)[D]$ . The key ingredient we shall use is the deep result of Adian and Novikov (see [1]), Lysënok ([33]) and Sergei Ivanov (see [25]) that the free Burnside group  $Q(\mathbb{F}_2)[D]$  is infinite for large  $D$  (e.g.  $D \geq 8000$ ).

**Lemma 3.3.** *If  $G \rightarrow H$  is surjective then  $Q(G)[D] \rightarrow Q(H)[D]$  is also surjective.*

*Proof.* It suffices to see that it is well-defined and thus surjective. □

**Lemma 3.4.** *If  $G \subset H$  is a subgroup of index  $n$  and  $Q(G)[D]$  is infinite then  $Q(H)[n!D]$  is infinite. When  $G$  is a normal subgroup then  $Q(H)[nD]$  is infinite.*

*Proof.* If  $G$  is a normal subgroup of  $H$  then for every  $a \in H$  we have  $a^n \in G$ . If  $G$  is not necessarily normal then we claim that for every  $a \in H$  we have  $a^{n!} \in G$ . In fact, by our assumption there are only  $n$  distinct left cosets of  $G$  in  $H$ . Thus the following  $(n+1)$  left cosets  $G, aG, a^2G, \dots, a^nG$  cannot be distinct. This means that there exists some non-zero integer  $p \leq n$  such that  $a^p \in G$ . Since  $p$  divides  $n!$  it follows that  $a^{n!} \in G$ , as claimed.

Therefore  $X(H)[nD] \subset X(G)[D] \subset G \subset H$  if  $G$  is normal and  $X(H)[n!D] \subset X(G)[D] \subset G \subset H$ , otherwise. The Lemma follows from this. □

**Lemma 3.5.** *We have  $Q(M_0^n)[n(n-1)(n-2)(n-3)D]$  is infinite if  $n \geq 4$  and large  $D \geq 8000$ .*

*Proof.* Observe that  $M_0^n$  contains the index  $n(n-1)(n-2)(n-3)$  normal subgroup  $U$  which preserve point-wise four punctures. Let  $PM_0^n$  denote the subgroup of pure mapping classes in  $M_0^n$  which preserve point-wise all punctures. Then  $U$  surjects onto  $PM_0^4$ , by forgetting all but the four fixed punctures. But  $PM_0^4$  is isomorphic to the free group  $\mathbb{F}_2$ . Thus Lemmas 3.3 and 3.4 settle the claim. □

The proof of Theorem 1.3 follows now from the following exact sequence:

$$1 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow M_2 \rightarrow M_0^6 \rightarrow 1$$

and Lemmas 3.3 and 3.5.

*Remark 3.8.* The same proof shows that the group  $Q(C_{M_g}(j)((2g+2)!D))$  associated to the centralizer  $C_{M_g}(j)$  of the hyper-elliptic involution  $j$  is infinite as soon as  $D$  is large enough.

*Remark 3.9.* One might speculate that for large values of  $D$  the subgroup  $X_g[g!(4g+2)D]$  is of infinite index in  $M_g$  and the quotient is a finitely generated torsion group of exponent  $g!(4g+2)D$ . Moreover, in this case it would exist  $N(g)$ , which divides  $g!(4g+2)$ , such that  $Q(M_g)[N(g)D]$  is infinite for large enough  $D$ , while  $Q(M_g)[D]$  is finite for every  $D$  not divisible by  $N(g)$ . This would follow if it were exist a finite index subgroup of  $M_g$  which surjects onto a free non-abelian group.

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