COHOMOLOGICAL REPRESENTATIONS OF AUTOMORPHISMS GROUPS

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Abstract. We survey several representations of automorphisms groups which arise in a unified manner from a construction due to Long and Moody.

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1. Tangent representations from moduli spaces

1.1. **Mapping class groups as (outer) automorphisms groups.** Set $\Sigma_{g,k}^r$ for the orientable surface of genus g with k boundary components and r marked points. We denote by $\Gamma_{g,k}^r$ the mapping class group of $\Sigma_{g,k}^r$, namely the group of isotopy classes of orientation preserving homeomorphisms that fix pointwise the boundary components and preserve globally the set of marked points. The pure mapping class group $P\Gamma_{g,k}^r$ consists of those classes of homeomorphisms which fix pointwise both the boundary components and each of the marked points.

We set $\pi_{g,k}^r$ for the fundamental group of the closed surface $\Sigma_{g,k}^r$. Recall that, by the Dehn-Nielsen-Baer theorem Γ_g^1 is the group of orientation preserving automorphisms of π_g, namely those which preserve the relator (conjugacy class) instead of reversing it. Further $G_g = \mathrm{Out}^+(\pi_g) = \mathrm{Aut}^+(\pi_g)/\mathrm{Inn}(\pi_g)$, where $\mathrm{Inn}(\pi_g)$ is the subgroup of inner automorphisms of π_g.

Denote by γ_j and δ_j the loops around the punctures and respectively the boundary components and by $[z]$ the conjugacy class in $\pi_{g,k}^r$ of the element z. Let $\mathrm{Aut}^+(\pi_{g,k}^r; C_1, \ldots, C_s)$ denotes the subgroup of automorphisms fixing globally each set of conjugacy classes in $C_1, C_2, \ldots, C_s$.

We denote then by P_r the set of all peripheral conjugacy classes $[\gamma_j]$ and by $\mathbf{P}_r$ the vector consisting of these peripheral conjugacy classes. Similarly, P_k^∂ is the set of all boundary conjugacy classes $[\delta_j]$ and by $\mathbf{P}_k^\partial$ the vector consisting of these peripheral conjugacy classes. We have then the following general statement for the Dehn-Nielsen-Baer theorem:

$$\Gamma_g^r = \mathrm{Out}(\pi_g^r, P_r) = \mathrm{Aut}^+(\pi_{g,k}^r; P_r)/\mathrm{Inn}(\pi_g^r)$$

The notation is intended to specify that each boundary conjugacy class is fixed, while the peripheral conjugacy classes are only globally invariant. If we fix the base point of the fundamental group to be among the marked points, it will be automatically invariant by the pure mapping class group so that:

$$P\Gamma_g^{r+1} = \mathrm{Aut}^+(\pi_g^r; \mathbf{P}_r)$$

Now, let $\Gamma_{g,k}^{r,1} \subset \Gamma_{g,k}^{r+1}$ denote the index $r+1$ subgroup of mapping classes of those homeomorphisms which fix one marked point. Then we have the more general statement:

$$\Gamma_g^{r|1} = \mathrm{Aut}^+(\pi_{g,k}^r; P_r)$$

Here P_r consists of the r conjugacy classes of peripheral loops with the exception of the one around the marked basepoint.

Then we have the following commutative diagram consisting of two exact sequences corresponding to Birman's exact sequence, connected by isomorphisms provided by the Dehn-Nielsen-Baer

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theorem:

$$\begin{array}{ccccccccc} 1 & \to & \pi_g^r/Z(\pi_g^r) & \to & P\Gamma_g^{r+1} & \to & P\Gamma_g^r & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & \pi_g^r/Z(\pi_g^r) & \to & \mathrm{Aut}^+(\pi_g^r;\mathbf{P}_r) & \to & \mathrm{Out}^+(\pi_{g,k}^r;\mathbf{P}_r) & \to & 1 \end{array}$$

We have also a similar commutative diagram in the non pure case:

$$\begin{array}{ccccccccc} 1 & \to & \pi_g^r/Z(\pi_g^r) & \to & \Gamma_g^{r\mid 1} & \to & \Gamma_g^r & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & \pi_g^r/Z(\pi_g^r) & \to & \mathrm{Aut}^+(\pi_{g,k}^r;P_r) & \to & \mathrm{Out}^+(\pi_{g,k}^r,P_r) & \to & 1 \end{array}$$

The group π_g^r is either a free group of rank $2g+r-1$, if $r>0$ or else a surface group. In particular it is centerless when $2g+r-2>0$, which we suppose to be the case from now on.

We further consider a surface with one boundary component $\Sigma_{g,1}^r$ and we take the basepoint on the boundary component. Therefore, the basepoint is automatically invariant by the pure mapping class group. It follows that we also have the alternative description:

$$\Gamma_{g,1}^r = \mathrm{Aut}^+(\pi_{g,1}^r;[\partial\Sigma_{g,1}^r],P_r),$$

Notice that homeomorphisms of $\Sigma_{g,1}^r$ automatically preserve the orientation. Denote by

$$\tau:\Gamma_{g,1}^r \to \mathrm{Aut}^+(\pi_{g,1}^r;[\partial\Sigma_{g,1}^r],P_r),$$

the natural isomorphism, which generalizes the usual Artin representation. The following is rather well-known:

Lemma 1.1. *There is an isomorphism between $\Gamma_{g,1}^{r\mid 1}$ and the semi-direct product $\pi_{g,1}^r \rtimes_\tau \Gamma_{g,1}^r$, if and $2g+r-1>0$, which restricts to an isomorphism between the pure mapping class group $P\Gamma_{g,1}^{r+1}$ and semi-direct product $\pi_{g,1}^r \rtimes_\tau P\Gamma_{g,1}^r$.*

Proof. The embedding of $\Sigma_{g,1}^r$ into $\Sigma_{g,1}^{r+1}$ as the complement of a punctured annulus $\Sigma_{0,2}^1$ induces injective homomorphisms $\pi_1(\Sigma_{g,1}^r,*) \to \pi_1(\Sigma_{g,1}^{r+1},*)$ and $\Gamma_{g,1}^r \to \Gamma_{g,1}^{r\mid 1}$. Here $*$ is a basepoint on the boundary component of the punctured annulus. This provides a splitting of the Birman exact sequence above. Moreover, the action of the subgroup $\Gamma_{g,1}^r$ on the subgroup $\pi_1(\Sigma_{g,1}^r,*)$ coincides with τ. Therefore $\Gamma_{g,1}^{r\mid 1}$ is isomorphic to the given semi-direct product. □

It is easy to see that there is a more general version, in which we consider mapping class groups instead of pure ones (see e.g. [5]). The corresponding semi-direct product is now isomorphic to the stabilizer of the last puncture in the mapping class group of the surface with one extra puncture, provided the surface has boundary.

1.2. **Geometric actions of (outer) automorphisms groups on moduli spaces.** Let π be a finitely generated group, G a connected Lie group. We denote by $\mathrm{Hom}(\pi,G)$ the space of repersentations of π. The group $\mathrm{Aut}(\pi)$ acts on $\mathrm{Hom}(\pi,G)$ by right composition:

$$(\varphi\cdot\rho)(x) = \rho(\varphi^{-1}(x)), \text{ for } \varphi\in\mathrm{Aut}(\pi), \rho\in\mathrm{Hom}(\pi,G), x\in\pi$$

This is a real algebraic action. Let now $\mathfrak{M}_{\pi,G}$ be the character variety of representations $\pi\to G$, or the GIT quotient $\mathrm{Hom}(\pi,G)/G$. Then the action

$$\mathrm{Aut}(\pi)\times\mathrm{Hom}(\pi,G)\to\mathrm{Hom}(\pi,G)$$

above passes to a quotient action of

$$\mathrm{Out}(\pi)\times\mathfrak{M}_{\pi,G}\to\mathfrak{M}_{\pi,G}$$

Let F be a finitely generated group. Fix a surjective homomorphism $\rho:\pi\to F$ whose kernel $\ker\rho$ is denoted K and consider its stabilizer, i.e. the subgroup of those elements whose induced action on F via ρ is trivial:

$$\mathrm{Aut}(\pi,\rho) = \{\varphi;\rho(\varphi(x))=\rho(x), \text{for any } x\in\pi\}\subset\mathrm{Aut}(\pi)$$

Note that $\mathrm{Inn}(K) \subset \mathrm{Aut}(\pi, \rho)$. The image of $\mathrm{Aut}(\pi, \rho)$ in $\mathrm{Out}(\pi)$ will be denoted as $\mathrm{Out}(\pi, \rho)$. However $\mathrm{Inn}(\pi)$ does not preserve ρ. In order to fix this probem consider the following quotient:

$$\widetilde{\mathrm{Out}(\pi, \rho)} = \mathrm{Aut}(\pi, \rho)/\mathrm{Inn}(K)$$

Then $\widetilde{\mathrm{Out}(\pi, \rho)}$ has a well-defined action on $\mathfrak{M}_{\pi,G}$ and keeps the class $[\rho]$ invariant. Note that we have an exact sequence:

$$1 \to F \to \widetilde{\mathrm{Out}(\pi_g, \rho)} \to \mathrm{Out}(\pi, \rho) \to 1$$

For any homomorphism $r : F \to G$ the group $\mathrm{Aut}(\pi, r \circ \rho)$ fixes $r \circ \rho \in \mathrm{Hom}(\pi, G)$. Therefore there is an induced action of the level of Zariski tangent spaces. This provides a linear representation of $\mathrm{Aut}(\pi_g, r \circ \rho)$ on the Zariski tangent space $T_\rho \mathrm{Hom}(\pi, G)$ at $r \circ \rho$, which will be called the *tangent representation* at $r \circ \rho$. Recall that Weil identified $T_{r\circ\rho}\mathrm{Hom}(\pi, G)$ with the space of twisted 1-cocycles $Z^1(\pi, \mathfrak{g}_{Ad\ r\circ\rho})$ with coefficients in the Lie algebra $\mathfrak{g}$ twisted by the composition of the adjoint representation Ad of G with $r \circ \rho$. This linear representation

$$\mathrm{Aut}(\pi, r \circ \rho) \to GL(Z^1(\pi, \mathfrak{g}_{Ad\ r\circ\rho}))$$

could be defined directly at the level of twisted cocycles $\psi : \pi \to \mathfrak{g}_{Ad\ r\circ\rho}$, as a right composition.

We explained above that $\widetilde{\mathrm{Out}}(\pi, r \circ \rho)$ acts on $\mathfrak{M}_{\pi,G}$ and stabilizes the class $[r \circ \rho]$ of $r \circ \rho$. We derive then a linear action of $\widetilde{\mathrm{Out}}(\pi, r \circ \rho)$ on the Zariski tangent space $T_{[\rho]}\mathfrak{M}_{\pi,G}$. By Weil, this amounts to a linear representation:

$$\widetilde{\mathrm{Out}}(\pi, r \circ \rho) \to GL(H^1(\pi, \mathfrak{g}_{Ad\ r\circ\rho}))$$

For non-reductive G, for instance when $G = \mathbb{C}^*$, we have to modify slightly this setting, as it will be explained below.

This setting also extends to families of representations using intermediary quotients. Let us consider the map $\iota_F : \mathrm{Hom}(F, G) \to \mathrm{Hom}(\pi, G)$, given by $\iota_F(r) = r \circ \rho$. We denote by $V_F = \iota_F(\mathrm{Hom}(F, G)) \subset \mathrm{Hom}(\pi, G)$ the closed subset consisting of all those ρ with $\rho(\pi_g)$ isomorphic to a quotient of F. For any homomorphism $r : F \to G$ we have $\mathrm{Aut}(\pi_g, r \circ \rho) \subset \mathrm{Aut}(\pi_g, \rho)$. The group action of $\mathrm{Aut}(\pi_g, \rho)$ on $\mathrm{Hom}(\pi_g, G)$ keeps globally invariant the subvariety V_F. Note that V_F is not pointwise invariant. Consider the Gunning sheaf $TV_F = \cup_{\rho\in V_F} T_\rho \mathrm{Hom}(\pi_g, G)$. As an immediate consequence $\mathrm{Aut}(\pi_g, \rho)$ acts both on TV_F and the pull-back $\iota_F^* TV_F$.

$$\mathrm{Aut}(\pi_g, \rho) \times \iota_F^* TV_F \to \iota_F^* TV_F$$

We have a similar action $\iota_F : \mathfrak{M}_{F,G} \to \mathfrak{M}_{\pi,G}$ whose image $\iota_F(\mathfrak{M}_{F,G})$ is endowed with a Gunning sheaf $T\mathfrak{M}_{F,G} = \cup_{\rho\in\mathfrak{M}_{F,G}} T_\rho \mathfrak{M}_{\pi_g,G}$ and a fiber-preserving action:

$$\widetilde{\mathrm{Out}}(\pi_g, \rho) \times \iota_F^* T\mathfrak{M}_{F,G} \to \iota_F^* T\mathfrak{M}_{F,G}$$

We ignored above the fact that dimensions of the fibers could be of non-constant dimension. If we restrict to the non-singular locus of the varieties $\mathfrak{M}_{F,G}$ or V_F, then Gunning sheaves restrict to fiber bundles. On any open contractible (in the usual topology) non-singular subset $U \subset \mathfrak{M}_{F,G}$ or V_F respectively we obtain linear representations

$$U \times \mathrm{Aut}(\pi, \rho) \to GL(Z^1(\pi, \mathfrak{g}_{Ad\ r\circ\rho}))$$

and

$$U \times \widetilde{\mathrm{Out}}(\pi, \rho) \to GL(H^1(\pi, \mathfrak{g}_{Ad\ r\circ\rho}))$$

respectively, parametrized by U.

1.3. **Finite representations.** The group $\mathrm{Aut}^+(\pi_g)$ of orientation preserving automorphisms of π_g acts on $\mathrm{Hom}(\pi_g, G)$ by right composition and this action passes to a quotient action of Γ_g on $\mathfrak{M}_{g,G}$.

Let now F be a finite quotient of $\rho : \pi_g \to F$. The subgroup $\mathrm{Aut}^+(\pi_g, \rho)$ is of finite index in $\mathrm{Aut}^+(\pi_g)$. If we fix an embedding $F \subset G$ then $\mathrm{Aut}^+(\pi_g, \rho)$ is the stabilizer of ρ on $\mathrm{Hom}(\pi_g, G)$. Its image $\Gamma_g(\rho)$ in Γ_g is also the stabilizer of the class $[\rho]$ of ρ in $\mathfrak{M}_{g,G}$.

Consider the exact sequence associated to ρ:

$$1 \to K \to \pi_g \to F \to 1$$

where F is finite. We are given a representation $r : F \to GL(V)$ which induces the structure of π_g-module on V. Without loss of generality we can suppose that V is from now on an *irreducible* F-module. For the sake of simplicity we consider first that V is a complex vector space.

Following [8] we call ρ *redundant* if it factors through $\pi_g \to \mathbb{F}_g$ and the kernel of the homomorphism $\mathbb{F}_g \to F$ contains a free generator. Here $\mathbb{F}_g$ is the free group of g generators and the homomorphism $\pi_g \to \mathbb{F}_g$ can be taken as the one induced by the inclusion of the surface Σ_g as the boundary of a handlebody with g handles.

Furthermore $F \subset G$ is *adjoint* if the composition $F \to GL(\mathfrak{g})$ by the adjoint representation $Ad : G \to GL(\mathfrak{g})$ is an irreducible representation.

Theorem 1.1. *Suppose that ρ is a finite adjoint redundant representation of π_g. Then the tangent action at $T_{[\rho]}\mathfrak{M}_{g,G}$ is an arithmetic group of symplectic/orthogonal or linear type.*

This is a consequence of the main result of [8]. Specifically, one decomposes the semisimple algebra $\mathbb{Q}[F]$ into simple algebras:

$$\mathbb{Q}[F] = \mathbb{Q} \oplus \bigoplus_{i=1}^{p} A_i$$

where A_i are ring of matrices $m_i \times m_i$ over a division algebra D_i and having center a number field L_i. Each A_i corresponds to a nontrivial irreducible $\mathbb{Q}$-representation of F. Then the authors of [8] constructed representations of (a finite index subgroup of) $\Gamma_g(\rho)$ into the algebraic group of V_i-automorphisms $\mathrm{Aut}_{A_i}(A_i^{2g-2}, \langle -, - \rangle)$ of A_i^{2g-2} endowed with a skew-Hermitian sesquilinear A_i-valued form. Then the image of this representation is a finite index subgroup of the arithmetic group $\mathrm{Aut}_{\mathfrak{D}_i}(\mathfrak{D}_i^{2g-2})$, where $\mathfrak{D}_i \subset A_i$ is the image of $\mathbb{Z}[F]$ by the projection onto A_i and is an order in A_i.

Proposition 1.1. *Assume that V is nontrivial F-module. Then we have an isomorphism*

$$H^1(\pi_g, V) \to \mathrm{Hom}_{\mathbb{C}[F]}(V, V)^{(2g-2)\dim V}$$

Proof. The five-term exact sequence reads:

$$H^1(F, V^K) \to H^1(\pi_g, V) \to H^1(K, V)^F \to H^2(F, V^K)$$

We use now the following classical fact (see Prop. 2.1 of [2]): If F is a finite group and V is a F-module which is also a $\mathbb{K}$-vector space for a field $\mathbb{K}$ whose characteristic does not divide the order of F then $H^j(F, M) = 0$, when $j > 0$. In particular, this is true in characteristic zero. This implies that the restriction homomorphism $H^1(\pi_g, V) \to H^1(K, V)^F$ is an isomorphism.

A classical result from [3] gives a description of the F-module $H_1(K; \mathbb{Q})$. Another proof is given in [8]. In the case when π_g were replaced by a free group this was a classical result by Gaschútz. Specifically, for every $g \geq 2$ we have an isomorphism of F-modules:

$$H_1(K; \mathbb{Q}) \to \mathbb{Q}^2 \oplus \mathbb{Q}[F]^{2g-2}$$

Some remarks are in order to understand the action of F on the module $H^1(K, V)$. Indeed F acts on K by conjugacy and on V through ρ. Classes in $H^1(K, V)$ are represented by homomorphisms $f : K \to V$, since V is a trivial K-module, and for $\gamma \in F$, $x \in K$ we have:

$$\gamma \cdot f(x) = \rho(\gamma) f(\tilde{\gamma}^{-1} x \tilde{\gamma})$$

where $\tilde{\gamma} \in \pi_g$ is an arbitrary lift of γ. In particular the class of f is F-invariant if for any $\gamma \in F$ and $x \in K$ we have:

$$f(\tilde{\gamma}x\tilde{\gamma}^{-1}) = \rho(\gamma)f(x)$$

By the previous description of the F-action on $H^1(K, V)$ and the Chevalley-Weil description of $H^1(K;\mathbb{C})$ we derive an isomorphism

$$H^1(\pi_g, V) \to \mathrm{Hom}_{\mathbb{C}[F]}(\mathbb{C}[F]^{2g-2} \oplus \mathbb{C}^2, V)$$

On the other hand, for simple $\mathbb{C}[F]$-modules V and W we have $\mathrm{Hom}_{\mathbb{C}[F]}(W, V) = 0$, unless V and W are isomorphic, from Schur's lemma. As a consequence of Maschke's theorem $\mathbb{C}[F] = \mathbb{C} \oplus \bigoplus_{i=1}^m V_i^{\dim(V_i)}$, where V_i are all irreducible $C[F]$-modules. It follows that

$$\mathrm{Hom}_{\mathbb{C}[F]}(\mathbb{C}[F]^{2g-2} \oplus \mathbb{C}^2, V) = \mathrm{Hom}_{\mathbb{C}[F]}(V, V)^{(2g-2)\dim V}$$

□

Now we have an action of $\mathrm{Aut}^+(\pi_g, \rho)$ on $H^1(\pi_g, V)$ induced by the left composition, which we denote by $\phi : \mathrm{Aut}^+(\pi_g, \rho) \to GL(H^1(\pi_g, V))$. Notice however that inner automorphisms do not necessarily act trivially. First, not all inner automorphisms are in $\mathrm{Aut}^+(\pi_g, \rho)$. Second, if the conjugacy ι_α by $\alpha \in \pi_g$ does belong to $\mathrm{Aut}^+(\pi_g, \rho)$, then its image is the automorphism:

$$\phi(\iota_\alpha) = r\rho(\alpha)$$

Since elements in $\mathrm{Aut}^+(\pi_g, \rho)$ which project onto the same element of $\Gamma_g(\rho)$ differ by an inner automorphism from $\mathrm{Aut}^+(\pi_g, \rho)$, it follows that we have an induced representation into a quotient group:

$$\Phi : \Gamma_g(\rho) \to GL(H^1(\pi_g, V))/r(F)$$

This is particularly simple when F is abelian, since $r(F)$ must be a group of scalar matrices and so we obtain a projective representation. In the case considered by [8] the authors rather considered punctured surfaces in order to work directly with the mapping class group $\Gamma_g^1 \subset \mathrm{Aut}^+(\pi_\mathrm{g})$. We have an exact sequence

$$1 \to \pi_g \to \Gamma_g^1 \to \Gamma_g \to 1$$

and the representation Φ lifts to

$$\Phi : \Gamma_g^1(\rho) \to GL(H^1(\pi_g, V))$$

The argument from ([8], section 8.2) shows that its restriction to a suitable finite index subgroup of $\Gamma_g^1(\rho)$ factors through Γ_g, so that Φ lifts to a genuine representation after restriction to a finite index subgroup of $\Gamma_g(\rho)$.

The case when F is an abelian group and V a 1-dimensional irreducible representation of it has been considered by Looijenga in [10] where the associated representations are called Prym representations. This has to be connected with previous construction by Gunning (see [9] in genus 2 and later extended by Chueshev (see [4]) to all genera, which is based on Prym differentials.

1.4. **Magnus representations for free groups.** In the case when $\pi = \mathbb{F}_n$ is a free group, there exists a simple description of these representations at the level of the F. Specifically, we first consider $V = \mathbb{Z}[\mathbb{F}_n]$ as a left $\mathbb{F}_n$-module. Then

$$H^1(\mathbb{F}_n, \mathbb{Z}[\mathbb{F}_n]) = I(\mathbb{F}_n) = \ker(\mathbb{Z}[\mathbb{F}_n] \to \mathbb{Z})$$

On the other hand we have an isomorphism

$$I(F_n) \to (\mathbb{Z}[\mathbb{F}_n])^n$$

given by the Fox derivatives. Specifically, if z_i form a free basis of $\mathbb{F}_n$ then we send $x \in F_n$ into $(\frac{\partial x^{-1}}{\partial x_i})_{i=1,n}$, where the Fox derivatives $\frac{\partial}{\partial x_i} : \mathbb{F}_n \to \mathbb{Z}[\mathbb{F}_n]$ form a basis of the space of 1-cocycles and they are determined by

$$\frac{\partial x_j}{\partial x_i} = \delta_{ij}$$

Now any automorphism φ of $\mathbb{F}_n$ induces an automorphism of $I(\mathbb{F}_n)$; under the previous isomorphism this automorphism is described as an element of $GL(n, \mathbb{Z}[\mathbb{F}_n]) \subset GL(V^{\oplus n})$ and is given by the matrix

$$\overline{\left(\frac{\partial \varphi(x_i)}{\partial x_i}\right)} \in GL(n, \mathbb{Z}[\mathbb{F}_n])$$

where $\overline{A}$ is the involution of $\mathbb{Z}[F_n]$ sending each $x \in \mathbb{F}_n$ into x^{-1}.

In particular, given an exact sequence $\rho : \mathbb{F}_n \to F$ we have a representation

$$\mathrm{Aut}(\mathbb{F}_n, \rho) \to GL(H^1(\mathbb{F}_n, \mathbb{Z}[F]))$$

which is obtained from the Magnus representation in $GL(n, \mathbb{Z}[\mathbb{F}_n])$ by evaluating each entry via $\rho : \mathbb{Z}[\mathbb{F}_n] \to \mathbb{Z}[F]$. A similar description holds when we choose a family V_F of representations $r : F \to GL(V)$, in which case the tangent representation

$$\mathrm{Aut}(\mathbb{F}_n, \rho) \to GL(H^1(\mathbb{F}_n, V_{r\circ\rho}))$$

is obtained by evaluating the Magnus representation entries at points of V_F.

2. Long-Moody twisted cohomological induction

2.1. **The construction.** Long and Moody considered in [13] a very general recipee for constructing braid group representations. We generalize his construction here to general automorphisms groups.

Data. Let π be a group, in our case it will be a closed surface group or a free group. Let now B be a group related to the automorphisms group $\mathrm{Aut}(\pi)$, in the sense that it is endowed with a homomorphism $\tau : B \to \mathrm{Aut}(\pi)$.

Our data consists of a (finite dimensional)*B-equivariant linear representation*, namely $\rho : \pi \to GL(V)$ coming along with a linear representation $\beta : B \to GL(V)$ such that ρ is equivariant with respect to the source and target actions τ and β:

$$\beta(b)\rho(f) = \rho(\tau(b)f)\beta(b), \text{for any } b \in B, f \in \pi$$

(Equivariant) twisted cohomological induction. To every B-equivariant representation:

$$(\rho : \pi \to GL(V), \beta : B \to GL(V), \tau : B \to \mathrm{Aut}(\pi))$$

we can associate a new representation

$$\beta^+ : B \to GL(V^+), \text{ where } V^+ = H^1_\rho(\pi, V)$$

by the explicit formula:

$$(\beta^+(b)\psi)(f) = \beta(b)\left(\psi(\tau^{-1}(b)(f))\right)$$

for every $\psi \in Z^1_\rho(\pi, V), f \in \pi, b \in B$.

Proposition 2.1. *The twisted cohomological induction is well-defined.*

Proof. We first have to verify that $\beta^+(b)\psi \in Z^1_\rho(\pi, V)$:

$$\begin{aligned}(\beta^+(b)\psi)(fg) &= \beta(b)\left(\psi(\tau^{-1}(b)(fg))\right) = \beta(b)\left(\psi(\tau^{-1}(b)(f) \cdot \tau^{-1}(b)(g))\right) = \\ &= \beta(b)\left(\psi(\tau^{-1}(b)(f) + \rho(\tau^{-1}(b)f)\psi(\tau^{-1}(b)(g)\right) = \\ &= \beta^+(b)\psi(f) + \beta(b)\rho(\tau^{-1}(b)f)\psi(\tau^{-1}(b)(g) = \\ &= \beta^+(b)\psi(f) + \rho(f)\beta(b)\psi(\tau^{-1}(b)(g) = \beta^+(b)\psi(f) + \beta^+(b)\psi(g)\end{aligned}$$

Moreover this representation on $Z^1_\rho(\pi, V)$ descends to $H^1_\rho(\pi, V)$. Indeed, if $\psi \in B^1_\rho(\pi, V)$, say $\psi(g) = \rho(g)v - v$, for any $g \in \pi$ for some $v \in V$, then

$$\begin{aligned}(\beta^+(b)\psi)(g) &= \beta(b)\left(\psi(\tau^{-1}(b)(g))\right) = \beta(b)(\rho(\tau^{-1}(b)g)v - v) = \\ &= \beta(b)\rho(\tau^{-1}(b)(g))v - \beta(b)v = \rho(g)\beta(b)v - \beta(b)v \in B^1_\rho(\pi, V)\end{aligned}$$

□

Lemma 2.1. *A couple* $(\rho : \pi \to GL(V),\ \beta : B \to GL(V))$ *satisfying the* B*-equivariance is equivalent to a representation* $\boldsymbol{\beta} : \pi \rtimes_\tau B \to GL(V)$ *of the semi-direct product group* $\pi \rtimes_\tau B$ *obtained by using the action of* B *on* π *by means of* τ*.*

Proof. Indeed $\boldsymbol{\beta}|_\pi = \rho$, while $\boldsymbol{\beta}|_{s(B)} = \beta$, where $s : B \to \pi \rtimes_\tau B$ is a section of the split extension. □

2.2. **Examples.** *Braid group representations.* Long and Moody used this method to define from a series of representations $\rho_n : B_n \to GL(V_n)$ of the braid groups B_n a new series of linear representations $\rho^+_{n+1} : B_n \to GL(V^n_{n+1})$ (see [13], Thm.2.1). Note the shift in the subscript. We identify B_n and the mapping class group of the 2-disk with n punctures. The stabilizer of the (first) puncture is isomorphic to the semi-direct product $F_n \rtimes_\tau B_n \subset B_{n+1}$, where τ denotes the Artin representation $\tau : B_n \to \mathrm{Aut}(F_n)$. Then twisted cohomological induction yields a representation $\rho^+_{n+1} : B_n \to GL(H^1_{\rho_{n+1}}(\pi, V_{n+1}))$. As π is the free group on n generators, the standard free resolution reads (see [2], I.4.4, IV.2, ex.3):

$$0 \to \mathbb{Z}[\pi]^n \to \mathbb{Z}[\pi] \to \mathbb{Z} \to 0$$

Therefore $H^1_\rho(\pi, V)$ is isomorphic to $V^{\oplus n}$. With this identification at hand one could write explicitly β^+ in terms of generators and he values of β (see [13], Thm.2.2).

It is already noticed that there are several embeddings of some semi-direct product $\pi \rtimes B_n$ within B_{n+1}. Above we considered the pure braid local system in which π is freely generated by $g_1 = \sigma_1^2$, $g_2 = \sigma_2\sigma_1^2\sigma_2^{-1}$, $g_3 = \sigma_3\sigma_2\sigma_1^2\sigma_2^{-1}\sigma_3^{-1}$, …, $g_n = \sigma_n\sigma_{n-1}\cdots\sigma_2\sigma_1^2\sigma_2^{-1}\cdots\sigma_{n-1}^{-1}\sigma_n^{-1}$. The action of B_n, which is generated by $\sigma_2, \sigma_3, \ldots, \sigma_n$ normalizes the subgroup π, and the conjugacy action is identified to the action of B_n on the fundamental group π of the punctured disk.

If we set $g_1 = (\sigma_2\sigma_3\cdots\sigma_n)^n$ and then inductively $g_{i+1} = \sigma_i g_i \sigma_i^{-1}$ then the subgroup π generated by $g_1, g_2, \ldots, g_n$ is also free of rank n and the subgroup B_n generated by $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ also normalizes π. This provides the inner automorphism local system $\pi \rtimes B_n$. Moreover, as we have an obvious map $p : \pi \rtimes B_n \to \mathbb{Z} \rtimes B_n$, we can use an arbitrary representation $\beta_n : B_n \to GL(V_n)$ and consider $(\beta_n \circ p)^+ : B_n \to GL(V_n^{\oplus n})$.

Mapping class group representations. According to Lemma 1.1, $\Gamma^{r|1}_{g,1} = \pi^r_{g,1} \rtimes \Gamma^r_{g,1}$. The Long-Moody twisted cohomological induction machinery provides then for any representation $\beta : \Gamma^{r+1}_{g,1} \to GL(V)$ another linear representation

$$\beta^+ : \Gamma^r_{g,1} \to GL(V^{\oplus 2g+r})$$

Finite index subgroups of mapping class groups. Consider a homomorphism $\rho : \pi \to GL(V)$ and $B = \mathrm{Aut}^+(\pi, \rho)$, with the usual action on π and the trivial action β on V. Then β^+ is the tangent action of B on $\mathrm{Hom}(\pi, GL(V))$ at ρ.

Surface braid groups. We can consider the braid group $B(\Sigma_{g,1}, r) = \ker(\Gamma^r_{g,1} \to \Gamma_{g,1})$ on the surface $\Sigma_{g,1}$ on r strands. The isomorphism from Lemma 1.1 provides an isomorphism between the stabilizer of the last strand in $B(\Sigma_{g,1}, r+1)$ and the semi-direct product $\pi^r_{g,1} \rtimes B(\Sigma^r_{g,1})$.

Magnus representations of $\mathrm{Aut}(\mathbb{F}_n)$ *and of Torelli groups* In the case when $\pi = \mathbb{F}_n$ and $\rho : \pi \to F$ has a characteristic kernel K, Magnus constructed a crossed-homomorphism $\mathrm{Aut}(\mathbb{F}_n) \to GL(n, \mathbb{Z}[F])$ whose restriction

$$\mathrm{Aut}(\mathbb{F}_n, \rho) \to GL(n, \mathbb{Z}[F])$$

is a homomorphism (see [14]). Note that Magnus' homomorphism coincides with the morphism β^+ provided by the construction above to the data ρ and β being the left action of $\mathrm{Aut}(\mathbb{F}_n)$ on $V = \mathbb{Z}[F]$, after identifying $GL(n, \mathbb{Z}[F])$ with a subgroup of $GL(V^{\oplus n})$ According to ([14], Prop. 3.4)

$$\ker \beta^+ = \ker\left(\mathrm{Aut}(\mathbb{F}_n) \to \mathrm{Aut}\left(\frac{\mathbb{F}_n}{[K, K]}\right)\right)$$

By choosing F to be the derived quotients series, the groups $\mathrm{Aut}(\mathbb{F}_n, \rho)$ form the filtration IA_k of $\mathrm{Aut}(\mathbb{F}_n)$, each one being the kernel of the Magnus representation of the former group. By choosing F to be the lower central quotients series we obtain the Andreadakis filtration $L^k\mathrm{Aut}(\mathbb{F}_n)$.

We can restrict this construction to the subgroup $\Gamma_{g,1} = \mathrm{Aut}(\pi_{g,1}, [\partial\Sigma_{g,1}])$. If F belongs to the central series quotients we obtain the Johnson filtration $J^k\Gamma_{g,1}$, starting with the Torelli group $T_{g,1}$ (for $k = 2$). Note that we have the exact sequences:

$$1 \to \mathrm{Hom}\left(H_1(\pi_{g,1}), \frac{\gamma_k(\pi_{g,1})}{\gamma_{k+1}(\pi_{g,1})}\right) \to \mathrm{Aut}\left(\frac{\pi_{g,1}}{\gamma_{k+1}(\pi_{g,1}}\right) \to \mathrm{Aut}\left(\frac{\pi_{g,1}}{\gamma_{k+1}(\pi_{g,1})}\right) \to 1$$

from which we obtain the Johnon homomorphism:

$$\tau_{k+1} : J^{k+1}\Gamma_{g,1} \to \mathrm{Hom}\left(H_1(\pi_{g,1}), \frac{\gamma_k(\pi_{g,1})}{\gamma_{k+1}(\pi_{g,1})}\right)$$

Note that it has the property that:

$$\ker \tau_{k+1} = J^{k+2}\Gamma_{g,1}$$

It seems still unknown whether the Magnus homomorphisms

$$J^k\Gamma_{g,1} \to GL\left(2g, \mathbb{Z}\left[\frac{\pi_{g,1}}{\gamma_{k+1}(\pi_{g,1})}\right]\right)$$

is faithful or not when $k \geq 3$ (see [14] for details).

2.3. Properties.

Proposition 2.2. *If π is either a free group or a surface group and β is unitary then β^+ is unitary.*

Proof. See [13]. □

2.4. Open questions. A linear representation is cohomological if it can be obtained by iterated Long-Moody induction from the trivial representation.

Problem 2.1. It is true that any quantum representation of the mapping class group $\Gamma_{g,1}$, $g \geq 3$, is a factor (subrepresentation) of a cohomological representation?

Here by quantum representation we mean a representation obtained from a modular tensor category with zero anomaly, e.g. obtained from the Turaev-Viro construction.

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