Braided Houghton groups as mapping class groups*

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To the memory of my friend Gheorghe Ionesei

In [9] we introduced the braided Ptolemy-Thompson group T^* playing the role of a mapping class group for an infinite planar surface and proved that T^* is finitely presented. The group T^* is a somewhat simplified version of the mysterious acyclic extension considered earlier by Greenberg and Sergiescu (see [13]), whose algebraic properties are still poorly understood today.

The aim of this note is to use the method introduced in [9] in order to recover the braided Houghton groups as mapping class groups of infinite surfaces. In particular the braid group on infinitely many strands is realized as the commutator subgroup of an explicit finitely presented group. This has been done previously by Dynnikov who used the so-called three pages representations of braids and links in ([7]). Our groups are slightly different from those considered by Dynnikov and their presentation is of different nature, because it comes from a geometric description in terms of mapping classes. Moreover, we obtain that the word problem of the braided Houghton groups is solvable. A version of our construction was used by Degenhardt, who introduced the braided Houghton groups BH_n in his (unpublished) thesis [6]. Then Kai-Uwe Bux described a conjectural approach to the finiteness properties of these groups in [4].

In order to define the mapping class group of an infinite surface we need to fix the behaviour of homeomorphisms at infinity. The main ingredient used in [9] consists of adjoining rigid structures, as defined below:

Definition 1. *A* rigid structure *d on the surface Σ is a decomposition of Σ into 2-disks with disjoint interiors, called elementary pieces. We suppose that the closures of the elementary pieces are still 2-disks.*

We assume that we are given a family F of compact subsurfaces of Σ such that each member of F is a finite union of elementary pieces, and called the family of admissible subsurfaces of Σ.

Given the data (Σ, d, F) we can associate the asymptotic mapping class group $\mathcal{M}(\Sigma, d, F)$ as follows. We restrict first to those homeomorphisms that act in the simplest possible way at infinity.

Definition 2. *A homeomorphism φ between two surfaces endowed with rigid structures is* rigid *if it sends the rigid structure of one surface onto the rigid structure of the other.*

The homeomorphism $\varphi : \Sigma \to \Sigma$ is said to be asymptotically rigid *if there exists some admissible subsurface $C \subset \Sigma$, called a support for φ, such that $\varphi(C) \subset \Sigma$ is also an admissible subsurface of Σ and the restriction $\varphi|_{\Sigma - C} : \Sigma - C \to \Sigma - \varphi(C)$ is rigid.*

As it is customary when studying mapping class groups we consider now isotopy classes of such homeomorphisms.

Definition 3. *The group $\mathcal{M}(\Sigma, d, F)$ of isotopy classes of asymptotically rigid homeomorphisms is called the* asymptotic mapping class group *of Σ corresponding to the rigid structure d and family of admissible subsurfaces F.*

Remark 1. Two asymptotically rigid homeomorphisms that are isotopic should be isotopic among asymptotically rigid homeomorphisms.

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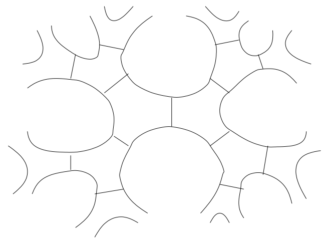
Example 1. One of the simplest nontrivial examples of infinite surface with a large symmetry group is obtained by thickening the planar binary tree in the plane. The ribbon tree $\mathcal{D}$ obtained this way has a natural rigid structure.

The elementary pieces of the rigid structure are hexagons obtained by thickening the tripodes issued from the vertices of the binary tree, each one made of three mid-edges going from the vertex to the midpoints of the adjacent edges, as in the picture below:



The family of admissible subsurfaces consists of those subsurfaces of $\mathcal{D}$ that are unions of finitely many elementary pieces.

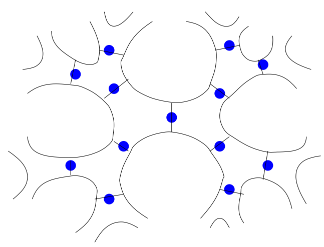
The asymptotic mapping class group of $\mathcal{D}$ with this rigid structure can be identified with the so-called Ptolemy-Thompson group T. Notice that T is a simple finitely presented group which admits an alternative description as the group of piecewise linear homeomorphisms of the circle $S^1 = [0,1]/\{0 \sim 1\}$, mapping images of dyadic into images of dyadic numbers, differentiable outside finitely many images of dyadic numbers, with derivatives powers of 2.

In order to understand better this correspondence we remark that T almost acts on the infinite binary tree. One introduces an *almost* automorphism as an automorphism in the complement of a finite subtree. Then the group T is generated by the following almost automorphisms α, β of the binary tree: β is the order three rotation around a vertex and α is the order four rotation around an edge midpoint. The subgroup $\langle \alpha^2, \beta \rangle \subset T$ is actually the modular group $PSL(2,\mathbb{Z})$ acting on its Bass-Serre tree.

More about Thompson's groups can be found in [5, 11].

Example 2. A more intricate example was considered and studied extensively in [9]. We shift from the ribbon tree $\mathcal{D}$ to the punctured ribbon tree $\mathcal{D}^*$ by adding infinitely many punctures on D located at the midpoints of the edges. Then the ribbon tree $\mathcal{D}^*$ inherits a natural rigid structure from D.

The ribbon tree $\mathcal{D}^*$ inherits a natural rigid structure from D for which the elementary pieces are punctured hexagons, as in the picture below:



The family of admissible subsurfaces consists of those subsurfaces of $\mathcal{D}^*$ that are unions of finitely many elementary pieces.

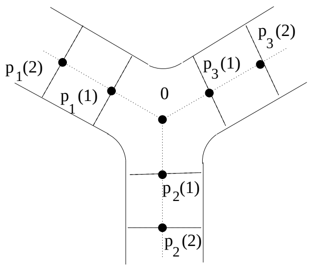
The asymptotic mapping class group of $\mathcal{D}^*$ with this rigid structure is the braided Ptolemy-Thompson group T^* considered in [9]. It is proved there that there exists an exact sequence

$$1 \to B_\infty \to T^* \to T \to 1$$

where B_∞ is the group of braids with compact support in the punctures of $\mathcal{D}^*$. Thus B_∞ is the direct limit of the sequence of braid groups associated to an ascending sequence of punctured admissible subsurfaces of $\mathcal{D}^*$.

We want to turn to simpler examples obtained from thickening trees in the plane and show that interesting groups could be obtained this way. Consider the planar ribbon Y_n, which is a 2-dimensional neigborhood of the wedge of n half-lines (or rays) in the plane that intersect at the origin. Assume that every half-line is endowed with a linear coordinates system in which the origin corresponds to 0 and that the rotation of order n sends them isometrically one into the other.

Let Y_n^* (respectively $Y_n^\sharp$) be the punctured ribbon obtained from Y_n by puncturing it along the set of points of positive (respectively nonnegative) integer coordinates on each half-line. Punctures are therefore identified to nonnegative integers along each ray. The origin has coordinates 0 on all half-lines and does appear only in $Y_n^\sharp$.



There is a family of parallel arcs associated to each ray, by drawing a properly embedded segment orthogonal to the respective half-line and passing through the puncture labelled n, for every $n \in \mathbb{Z}_+ - \{0\}$.

The surface Y_n (respectively Y_n^*, $Y_n^\sharp$) is then divided by these arcs into elementary pieces, which are of two types: one central (respectively punctured for Y_n^*) $2n$-gon containing the origin and infinitely many (respectively punctured) squares which sit along the half-lines. One defines the admissible subsurfaces of Y_n (respectively Y_n^*, $Y_n^\sharp$) to be those (punctured) $2n$-gons which contain the (punctured) central $2n$-gon and are made of finitely many elementary pieces.

Let $\mathcal{M}(Y_n)$ (respectively $\mathcal{M}(Y_n^*)$, $\mathcal{M}(Y_n^\sharp)$) denote the asymptotic mapping class group of Y_n (respectively Y_n^*, $Y_n^\sharp$) with the above rigid structure.

The group $\mathcal{M}(Y_n)$ has a particularly simple form. In fact any element of $\mathcal{M}(Y_n)$ corresponds to a triple $((P,Q),r)$, where P and Q are admissible $2n$-gons and r is an order n rotation that gives the recipe for identifying the boundary arcs of P and Q. Moreover, an admissible $2n$-gon $P \subset Y_n$ is completely determined by the vector $v_P \in (\mathbb{Z}_+ - \{0\})^n$ recording the coordinates of those punctures that lie on the boundary arcs of P, one coordinate for each ray. The cyclic group of rotations $\mathbb{Z}/n\mathbb{Z}$ acts on $\mathbb{Z}^n$ by permuting the coordinates and preserves the subgroup $\mathbb{Z}^{n-1} \subset \mathbb{Z}^n$ of the vectors having the sum of their coordinates zero. The map that sends the pair $((P,Q),r)$ into $(v_Q - r(v_P), r) \in \mathbb{Z}^n \rtimes \mathbb{Z}/n\mathbb{Z}$ induces an isomorphism of $\mathcal{M}(Y_n)$ onto the subgroup $\mathbb{Z}^{n-1} \rtimes \mathbb{Z}/n\mathbb{Z}$.

Following the example 2 one expects $\mathcal{M}(Y_n^*)$ and $\mathcal{M}(Y_n^\sharp)$ to be extensions of $\mathcal{M}(Y_n)$ by an infinite braid group B_∞. If $\mathcal{M}(Y_n)$ were abelian then the infinite braid group B_∞ would be the commutator of the extension group. However the semi-direct product $\mathbb{Z}^{n-1} \rtimes \mathbb{Z}/n\mathbb{Z}$ is not direct for $n \geq 3$, and hence it is convenient to restrict to those mapping classes in the groups above coming from end preserving homeomorphisms.

Consider therefore the subgroups $\mathcal{M}_\partial(Y_n)$ (respectively $\mathcal{M}_\partial(Y_n^*)$, $\mathcal{M}_\partial(Y_n^\sharp)$) generated by those homeomorphisms which are end preserving i.e. inducing a trivial automorphism of the ends of Y_n. Alternatively, the homeomorphisms should send each ray into itself, at least outside of a large enough compact.

It follows from above that $\mathcal{M}_\partial(Y_n)$ is isomorphic to $\mathbb{Z}^{n-1}$.

The groups $\mathcal{M}_\partial(Y_n^*)$ are isomorphic to the braided Houghton groups considered by Degenhardt and Dynnikov. It is known that these are finitely presented groups for all $n \geq 3$. The same result holds for the larger related groups $\mathcal{M}_\partial(Y_n^\sharp)$:

Theorem 1. *The groups $\mathcal{M}(Y_n^\sharp)$ and $\mathcal{M}_\partial(Y_n^\sharp)$ are finitely presented for $n \geq 3$. The commutator subgroup of $\mathcal{M}_\partial(Y_n^\sharp)$ is the infinite braid group B_∞ in the punctures of $Y_n^\sharp$.*

Proof. Consider the case of $\mathcal{M}_\partial(Y_n^\sharp)$, the other one being similar. We can express the groups as an extension by the infinite braid group, as follows:

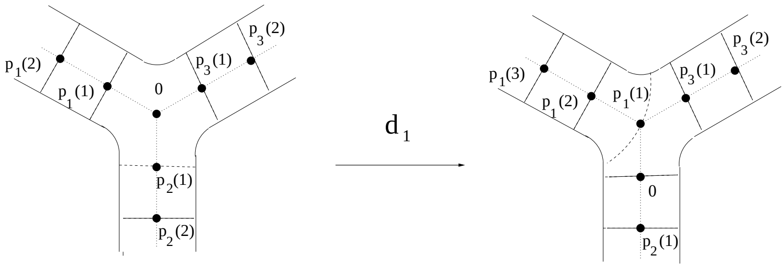
Proposition 1. *There is an exact sequence*

$$1 \to B_\infty \to \mathcal{M}_\partial(Y_n^\sharp) \to \mathbb{Z}^{n-1} \to 1$$

where $B_\infty = \lim_{k\to\infty} B_{kn+1}$ *is the limit of the braid groups of an exhausting sequence of admissible subsurfaces of* $Y_n^\sharp$*.*

Proof. A mapping class $\varphi \in \mathcal{M}_\partial(Y_n^\sharp)$ sends a support $2n$-gon into another support $2n$-gon, by translating the arc on the half-line l_j of k_j units towards the centre. Since the support hexagons should contain the same number of punctures we have $k_1 + k_2 + \cdots + k_n = 0$. The map sending φ to $(k_1, k_2, \ldots, k_n)$ is a surjection onto $\mathbb{Z}^{n-1}$. The rest of the proof is immediate. □

Let the line l_j be punctured along the points $p_j(i)$ at distance i from the origin. Consider the mapping class of the homeomorphism d_j which translates all punctures of the line $l_j \cup l_{j+1}$ one unit in the counterclockwise direction, as in the figure below:



We use the convention that the groups acts on the right: thus the composition ab denotes a followed by b. Moreover, the set of indices corresponding to the rays is $\{1, 2, \ldots, n\}$, which is naturally identified to $\mathbb{Z}/n\mathbb{Z}$; let then $<$ denote the cyclic order on $\mathbb{Z}/n\mathbb{Z}$.

Proposition 2. *Set* $u_i = d_i d_{i+1} d_i^{-1} d_{i+1}^{-1}$*. Then the group* $\mathcal{M}_\partial(Y_n^\sharp)$ *is generated by the* $d_1, d_2, \ldots, d_n$ *and admits the presentation:*

$$d_1 d_2 d_3 \cdots d_n = 1$$

$$u_{i_1} u_{i_2} u_{i_3} u_{i_1} = u_{i_2} u_{i_3} u_{i_1} u_{i_2} = u_{i_3} u_{i_1} u_{i_2} u_{i_3}, \text{ if } i_1 < i_2 < i_3$$

$$d_{i-1}^{-1} u_i d_{i-1} = d_i u_i d_i^{-1} \text{ for all } i$$

$$u_i u_j u_i = u_j u_i u_j, \text{ for all } i, j$$

$$d_{i-1}^{-1} u_i d_{i-1} u_i d_{i-1}^{-1} u_i d_{i-1} = u_i d_{i-1}^{-1} u_i d_{i-1} u_i, \text{ for all } i$$

$$[d_i u_i d_i^{-1}, u_j] = 1, \text{ for all } i \neq j$$

$$[d_i u_i d_i^{-1}, d_j] = 1, \text{ for all } i < j < i-1$$

$$[d_j u_i d_j^{-1} = u_i u_j u_j^{-1}, \text{ for all } i < j < i-1$$

Proof. Let a, b be adjacent punctures. We denote by σ_{ab} the standard braid that interchanges a and b moving them counterclockwise. Remark that $u_i = \sigma_{0p_i(1)}$ is the braid twisting 0 and $p_i(1)$. Moreover, one proves by recurrence that $\sigma_{p_i(k)p_i(k+1)} = d_{i-1}^{-k} u_i d_{i-1}^{k}$. Thus $\mathcal{M}_\delta(Y_n^\sharp)$ is generated by the d_js since their images generate both the quotient $\mathbb{Z}^{n-1}$ and the kernel B_∞. The group B_∞ is identified with the braid group associated to the infinite tree given by the reunion of the n half-lines in the plane, whose vertices are located at punctures. We can use therefore the presentation for B_∞ given by Sergiescu (see [16, 1]). In particular B_∞ is generated by the (infinite) set of braids associated to the edges of the tree, namely the braids $\sigma_{p_i(k)p_i(k+1)}$. By translating the relations between the braids into relations between the d_i and u_i we obtain the following set of vertex and edge relations:

$$u_{i_1} u_{i_2} u_{i_3} u_{i_1} = u_{i_2} u_{i_3} u_{i_1} u_{i_2} = u_{i_3} u_{i_1} u_{i_2} u_{i_3}, \text{ if } i_1 < i_2 < i_3$$

$$u_i u_j u_i = u_j u_i u_j, \text{ for all } i, j$$

$$d_i u_i d_i^{-1} u_i d_i u_i d_i^{-1} = u_i d_i u_i d_i^{-1} u_i, \text{ for all } i$$

and an additional set of commuting relations between twists with disjoint support edges, reading:

$$[\sigma_{p_i(k)p_i(k+1)}, \sigma_{p_j(m)p_j(m+1)}] = 1, \text{when their support edges are disjoint}$$

The twists associated to edges at distance one yield the relations

$$[d_{i-1}^{-1} u_i d_{i-1}, u_j] = 1, \text{ for all } i \neq j$$

Further d_i commutes with $\sigma_{p_i(1)p_i(2)}$ since they have disjoint supports, which lead us to the last relations above. The interesting phenomenon is that these relations actually are sufficient to imply all commutativity relations (between arbitrary far away disjoint twists associated to the edges). We skip the proof, which is a direct calculation.

It follows that subgroup generated by the u_is and their conjugates by the d_js is B_∞. Further B_∞ is a normal subgroup inside the group given by the presentation above, as the conjugates by d_j of the u_is can be expressed in terms of the braid generators $\sigma_{p_i(k)p_i(k+1)}$. Then we can provide an infinite presentation of $\mathcal{M}\partial(Y_n^\sharp)$ by means of Hall's lemma which puts together the Sergiescu presentation of B_∞ and that of $\mathbb{Z}^{n-1}$. Then all relations of this infinite presentation are consequences of those from the statement. The claim follows. □

It follows from above that $B_\infty \subset [\mathcal{M}\partial(Y_n^\sharp), \mathcal{M}\partial(Y_n^\sharp)]$ because $u_i = [d_i, d_{i+1}]$ and thus each braid generators $\sigma_{p_i(k)p_i(k+1)}$ is conjugate to a commutator. Moreover the quotient $\mathcal{M}\partial(Y_n^\sharp)/B_\infty$ is the abelian group $\mathbb{Z}^{n-1}$ and hence B_∞ is the commutator subgroup of $\mathcal{M}\partial(Y_n^\sharp)$. Notice that the same holds for the braided Houghton group $\mathcal{M}\partial(Y_n^*)$. □

The two versions $\mathcal{M}\partial(Y_n^\sharp)$ and $\mathcal{M}\partial(Y_n^*)$ of braided Houghton groups are very closed to each other. Let $\mathcal{M}_\partial^*(Y_n^\sharp)$ denote the subgroup of those mapping classes fixing the central puncture 0. Therefore

Proposition 3. *There is an exact sequence*

$$1 \to \mathbb{F} \to \mathcal{M}_\partial^*(Y_n^\sharp) \to \mathcal{M}\partial(Y_n^*) \to 1$$

where $\mathbb{F}$ is a free group on infinitely many generators that is normally generated in $\mathcal{M}\partial(Y_n^\sharp)$ by one element.

Proof. The projection homomorphism is that induced by the inclusion $Y_n^\sharp \to Y_n^*$ and corresponds to forgetting the central puncture. It is classical that the kernel $\mathbb{F}$ of this map is isomorphic to $\pi_1(Y_n^*, 0)$ which is free on infinitely many generators. The element $u_1^2 \in \mathcal{M}\partial(Y_n^\sharp)$ is represented by the closed loop based at the origin encircling the puncture $p_1(1)$. However $\mathcal{M}\partial(Y_n^\sharp)$ acts transitively by conjugacy on the set of loops that encircle punctures of Y_n^* and thus the element u_1^2 normally generates the kernel. □

Theorem 2. *The groups $\mathcal{M}\partial(Y_n^\sharp)$ (and its versions) have solvable word problem.*

Proof. For any word w in the generators d_i we have that there exists a support of w made of elementary pieces not farther than $|w| + 1$ units from the central $2n$-gon. Then the proof given in [9] can be adapted to our situation. Onserve that we actually use the fact that the word problem is solvable in braid groups. □

Remark 2. Let S_∞ denote the infinite permutation group of punctures of Y_n^* obtained as direct limit of finite permutation groups of punctures in an ascending sequence of admissible subsurfaces.

The Houghton groups H_n considered by Brown ([3]) are quotients of $\mathcal{M}\partial(Y_n^*)$ induced from the obvious homomorphism $B_\infty \to S_\infty$ sending braids into the associated permutations. This means that we have natural exact sequences

$$\begin{array}{ccccccccc} 1 \to & B_\infty & \to & \mathcal{M}\partial(Y_n^\sharp) & \to & \mathbb{Z}^{n-1} & \to 1 \\ & \downarrow & & \downarrow & & \downarrow & \\ 1 \to & S_\infty & \to & H_n & \to & \mathbb{Z}^{n-1} & \to 1 \end{array}$$

Remark 3. The group $\mathcal{M}\partial(Y_2^\sharp)$ (and its variants) is generated by two elements, namely $d = d_1 = d_2^{-1}$ and $u_1 = \sigma_{0p_1(1)}$. However, $\mathcal{M}\partial(Y_2^\sharp)$ is not finitely presented since the commutativity relations coming from the braid group are independent, namely we have infinitely many relations $[d^k u d^{-k}, d^m u d^{-m}] = 1$, for all integers m, k with $|m - k| \geq 1$. Also $\mathcal{M}\partial(Y_2^\sharp)$ surjects onto the Houghton group H_2 which is known to be infinitely presented. In some sense $\mathcal{M}\partial(Y_2^\sharp)$ is similar to the lamplighter groups.

Remark 4. Since all generators of B_∞ are conjugate the abelianization of B_∞ is $\mathbb{Z}$. The abelianization homomorphism $B_\infty \to \mathbb{Z}$ induces an extension $\mathcal{M}\partial(Y_n^\sharp)^{ab}$ as follows:

$$\begin{array}{ccccccc} 1\to & B_\infty & \to & \mathcal{M}\partial(Y_n^*) & \to & \mathbb{Z}^{n-1} & \to 1 \\ & \downarrow & & \downarrow & & \downarrow & \\ 1\to & \mathbb{Z} & \to & \mathcal{M}\partial(Y_n^*)^{ab} & \to & \mathbb{Z}^{n-1} & \to 1 \end{array}$$

For $n = 2$ it follows that $\mathcal{M}\partial(Y_2^*)^{ab}$ is abelian, generated by the images of d and u. In particular, we obtain that $\mathcal{M}\partial(Y_2^*)^{ab} \cong H_1(\mathcal{M}\partial(Y_2^*)) = \mathbb{Z}^2$.

For $n \geq 3$ the group $\mathcal{M}\partial(Y_n^*)^{ab}$ is a nontrivial (non-abelian) extension of $\mathbb{Z}^{n-1}$ by $\mathbb{Z}$.

Remark 5. Degenhardt ([6]) proved that the braided Houghton groups are F_{n-1} but not F_n for $n \leq 3$ and conjectured that this holds for all n. This would be a parallel to the results obtained by Brown (see [3]) for the usual Houghton groups H_n. Progress towards the settlement of this conjecture was made by Kai-Uwe Bux in [4]. It seems that the groups $\mathcal{M}\partial(Y_n^\sharp)$ are also F_{n-1} but not F_n.

This behavior is in contrast with the case of the Thompson group T (which is FP_∞) and its braided version T^* (which is at least FP_3 (see [10]) and expected to be FP_∞). It is therefore likely that $\mathcal{M}\partial(Y_n^\sharp)$ are not combable (hence not automatic) although the result of [10] would suggest that they might be asynchronously combable with quadratic Dehn function. If the similarity with the braid groups is pushed one step farther then the braided Houghton groups should have solvable conjugacy problem, as well.

Remark 6. Given three rays in the binary tree we can associate an embedding of Y_3^* into D^* that induces injective compatible homomorphisms $\mathbb{Z}^2 \rtimes \mathbb{Z}/3\mathbb{Z} \to T$ and $\mathcal{M}(Y_3^*) \to T^*$.

Remark 7. One does not know which other planar graphs yield finitely presented group asymptotic mapping class groups. One may enlarge the category of graphs to that of coloured graphs, in which automorphisms and almost automorphisms are required to preserve the colouring.

An interesting class of coloured planar trees comes from universal coverings of ribbon graphs, associated to punctured surfaces and 2-dimensional orbifolds. The ribbon structure of the graph is a cyclic order around each vertex. There is a natural colouring of vertices and edges of the graph by using different colours, and this induces a colouring on the universal covering tree. Moreover, the tree has a natural embedding in the plane which uses the induced cyclic order around the vertices.

However P.Greenberg ([12]) showed that asymptotic mapping class groups of universal coverings of (coloured) ribbon graphs (called projective Thompson groups) have infinitely many generators, as soon as the genus of the surface is positive. Moreover, if the genus is zero Laget ([15] proved that the asymptotic mapping class groups are finitely presented groups. It seems that the finite presentability holds more generally for all the groups obtained from the 2-orbifolds of genus zero. The basic example in this respect is the Thompson group T which arises from the 2-orbifold associated to the group $PSL(2, \mathbb{Z})$, namely the sphere with a cusp, one singular point of order 2 and another one of order 3.

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