ON GROUPS WITH LINEAR SCI GROWTH

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Abstract. We prove that the simple connectivity at infinity growth of sci hyperbolic groups and most non-uniform lattices is linear. Using the fact that the end-depth of finitely presented groups is linear we prove that the linear growth of simple connectivity at infinity is preserved under amalgamated products over finitely generated one-ended groups.

Keywords: simple connectivity at infinity, quasi-isometry, end-depth, lattices in Lie groups, amalgamated products.

MSC Subject: 20 F 32, 57 M 50.

1. Introduction

The metric spaces (X, d_X) and (Y, d_Y) are *quasi-isometric* if there are constants λ, C and continuous maps $f : X \to Y$, $g : Y \to X$ (called (λ, C)-quasi-isometries) such that the following:

$$d_Y(f(x_1), f(x_2)) \leqslant \lambda d_X(x_1, x_2) + C,\ d_X(g(y_1), g(y_2)) \leqslant \lambda d_Y(y_1, y_2) + C,$$

$$d_X(fg(x), x) \leqslant C,\ d_Y(gf(y), y) \leqslant C,$$

hold for all $x, x_1, x_2 \in X, y, y_1, y_2 \in Y$.

Definition 1. *A connected locally compact topological space X with $\pi_1 X = 0$ is simply connected at infinity (abbreviated sci and one writes also $\pi_1^\infty X = 0$) if for each compact $k \subseteq X$ there exists a larger compact $k \subseteq K \subseteq X$ such that any closed loop in $X - K$ is null homotopic in $X - k$.*

The sci is a fundamental tameness condition for non-compact spaces, as it singles out Euclidean spaces among contractible manifolds, following classical results of Stallings and Siebenmann. This notion was extended by Brick and Mihalik (see [4]) to a group-theoretical framework as follows:

Definition 2. *A finitely presented group G is simply connected at infinity (abbreviated sci) if for some (equivalently any) finite complex X_G with $\pi_1 X_G = G$ its universal covering $\widetilde{X_G}$ is sci.*

It is known that not all finitely presented groups are sci, for example M. Davis (see e.g. [9]) constructed word hyperbolic groups G (of virtual cohomological dimension $n \geq 4$ by the results of Bestvina and Mess from [1]) which are not sci.

All groups considered in the sequel will be finitely generated and a system of generators determines a word metric on the group. Although this depends on the chosen generating set the different word metrics are quasi-isometric. In [10] we enhanced the topological sci notion in the case of groups by taking advantage of this metric structure.

Definition 3. *Let X be a sci non-compact metric space. The sci growth $V_X(r)$ (called rate of vanishing of π_1^∞ in [10]) is the infimal $N(r)$ with the property that any loop in the complement of the metric ball $B(N(r))$ of radius $N(r)$ (centered at the identity) bounds a 2-disk outside $B(r)$.*

Remark 1. *It is easy to construct examples of metric spaces with arbitrarily large V_X.*

It is customary to introduce the following (rough) equivalence relation on real valued functions: the real functions f and g are equivalent $f \sim g$ if there exists constants c_i, C_j (with $c_1, c_2 > 0$) such that:

$$c_1 f(c_2 x) + c_3 \leq g(x) \leq C_1 f(C_2 x) + C_3, \text{ for all } x.$$

The authors acknowledge partial support from Research Funding Program Heracleitus II of the University of Athens and the European Union (M.G.), CMIRA Explora Pro 1200613701 (L.F.) and the European Commissions Marie Curie Intra-European Fellowship (D.O.).

It is proved in [10] that the (rough) equivalence class of $V_X(r)$ is a quasi-isometry invariant. In particular, if G is sci, then the (rough) equivalence class of the real function $V_G = V_{\widetilde{X_G}}$ is a quasi-isometry invariant of the group G, where $\widetilde{X_G}$ is the universal covering space of any finite complex X_G, with $\pi_1(X_G) = G$.

If G is sci and V_G is a linear function we will say that G has *linear sci*. In contrast with the abundance of equivalence classes of geometric invariants of finitely presented groups, (like group growth, Dehn functions or isodiametric functions) the metric refinements of topological properties seem highly constrained. We already found in [10] that many cocompact lattices in Lie groups and in particular geometric 3-manifold groups have linear sci. The aim of this paper is to further explore this phenomenon by considerably enlarging the class of groups with linear sci. Our first result is:

Theorem 1. *If G is a sci word hyperbolic group then V_G is linear.*

The sci and its refinement (the sci growth rough equivalence class) are 1-dimensional invariants at infinity for a group G, in the sense that they take care of loops and disks. The 0-dimensional analog of the simple connectivity at infinity is the connectivity at infinity, namely the one-endedness. One could adapt the notion of sci growth to the growth of an end. This was already considered by Cleary and Riley (see [6]). Let us recall that a group G is *one-ended* (or 0-connected at infinity) if for any compact subset L of the Cayley graph X of G, there exists a compact subset $K \supset L$ such that any two points out of K can be joined by a path contained in $X - L$. This leads to the following metric refinement which is the 0-dimensional counterpart of the sci growth:

Definition 4. *Let X be a one-ended metric space. The end-depth $V_0(X)$ of X is the infimal $N(r)$ with the property that any two points which sit outside the ball $B(N(r))$ of radius $N(r)$ can be joined by a path outside $B(r)$.*

If G is a finitely generated one-ended group then the end-depth of G is the (rough) equivalence class of the real function $V_{0,G} = V_0(\widetilde{X})$, where $\widetilde{X}$ is a Cayley graph of G associated to a finite generating set.

One can define in the same way the end-depth of a specific end of a space or finitely generated group which are not necessarily one-ended. In [21] one proved that the (rough) equivalence class of $V_{0,G}$ is a well-defined quasi-isometry invariant of one-ended finitely presented groups. Examples of groups whose Cayley graphs have dead-ends (i.e. end-depth functions strictly larger than $x + c$, for any c) were obtained in [6]. Our second result shows that the (rough) equivalence class of the end-depth is not meaningful:

Theorem 2. *Any finite type one-ended group has a linear end-depth. More precisely we have the inequality:*

$$V_0(X)(r) \leq 2r, \text{ for large enough } r$$

where X denotes the Cayley graph X associated to a finite generating set of the group G.

This is a useful step in establishing:

Theorem 3.
(1) *If G_1 and G_2 are one-ended finitely presented groups with linear sci and the finitely generated subgroup H has one end, then the amalgamated free product $G = G_1 *_H G_2$ has linear sci.*
(2) *If the finitely presented group G_1 has linear sci and the finitely generated subgroup H has one end, then the HNN-extension $G = G_1 *_H$ has linear sci.*

Theorem 3 is similar to (but under stronger restrictions than) the results obtained by Mihalik and Tschantz in [17, 18, 19] in the context of semistability. Notice that the sci is not preserved under amalgamated products over multi-ended subgroups (see [12]).

Previous results of [10] dealt with all cocompact lattices in connected Lie groups but the solvable ones. We now consider non-uniform lattices. Our main result in this direction is:

Theorem 4.
(1) *Let G be a semisimple Lie group for which the associated symmetric space G/K is of dimension $n \geq 4$ and of $\mathbb{R}$-rank greater than or equal to 2. Let Γ be an irreducible, non-uniform lattice in G of $\mathbb{Q}$-rank one. Then Γ is sci with linear sci growth.*
(2) *Every lattice $\Gamma \subset SO(n, 1)$, $n \geq 2$ has a linear sci growth.*

We believe that the last result also holds for all $\mathbb{R}$-rank 1 Lie groups, and for non-uniform lattices of $\mathbb{Q}$-rank > 1. Further any lattice in a simply connected solvable Lie group is uniform and it is a polycyclic group which admits a strongly polycyclic subgroup of finite index. Thus it would suffice to show that strongly polycyclic groups have linear sci in order to obtain the linear sci property for all solvable Lie groups and hence by an easy argument for all uniform lattices in connected Lie groups.

The results of this paper naturally lead to the question of the existence of sci groups with super-linear sci growth. The next step is to understand whether CAT(0) groups which are sci have linear sci.

Acknowledgements: The authors are indebted to Y. de Cornulier, F. Haglund, P. Papazoglou, V. Poenaru, M. Sapir and A. Valette for useful discussions and advice.

2. Preliminaries on hyperbolic groups

Let (X, d) be a geodesic metric space, which in our case will be the Cayley graph of a finitely generated group G. A geodesic triangle is δ-*slim* if every side is contained in the δ-neighbourhood of the union of its other sides.

Given a geodesic triangle Δ with vertices x, y, z in X, let Δ' be a Euclidean comparison triangle with vertices x', y', z' and sides of the same lengths as those of Δ, and $f : \Delta \to \Delta'$ an identification map. Suppose that the maximal inscribed circle C in Δ' meets its sides at $c'_z \in [x', y']$, $c'_x \in [y', z']$ and $c'_y \in [x', z']$. There is a unique isometry f_Δ of Δ' into a metric tripod T (i.e. a tree with one vertex w of degree 3 and three vertices x'', y'', z'' of degree one), such that:

$$d(w, x'') = d(x', c_z) = d(x', c_y)$$
$$d(w, y'') = d(y', c_z) = d(y', c_x)$$
$$d(w, z'') = d(z', c_x) = d(z', c_y)$$

Let $F = f_\Delta \circ f : \Delta \to T$. The triangle Δ is δ-*thin* if for all $p \in T$, the diameter $diam(F^{-1}(p)) \leqslant \delta$. Moreover, we call the points $c_x = F^{-1}(w) \cap [y, z]$, $c_y = F^{-1}(w) \cap [x, z]$, $c_z = F^{-1}(w) \cap [y, x]$, the *internal points* of Δ.

The group G is δ-*hyperbolic*, for some $\delta \geqslant 0$, if all geodesic triangles in X are δ-thin. Then, all geodesic triangles are also δ-slim and there is $z \in X$ so that the Gromov product based at z is δ-hyperbolic ([5], ch. III-H). The group G is *hyperbolic* if it is δ-hyperbolic for some $\delta \geqslant 0$. We remark that the notion of hyperbolicity for a group is independent of the choice of the presentation. It is clear that if G is δ-hyperbolic, for some $\delta \geqslant 0$, then it is also δ'-hyperbolic, for any $\delta' > \delta$.

Suppose from now on that G is a δ-hyperbolic group and X its Cayley graph associated with a finite presentation $\langle S \mid R \rangle$.

Let $\gamma : [0, \infty) \to X$ be a geodesic path, either finite or infinite. For any x, $y \in \gamma$, we denote by $[x, y]_\gamma$ the subpath of γ that connects x to y.

Bestvina and Mess [1] proved the following:

Proposition 1 ([1])**.** *Let G be a hyperbolic one-ended group. There is $c \geqslant 0$ so that for all $x \in X$ there exists an infinite geodesic ray starting at the identity of G which passes within c of x.*

We say that two geodesic rays are *asymptotic* if their images in X are at finite Hausdorff distance. This defines an equivalence relation on the collection of geodesic rays in X. The *boundary* ∂X of X is the collection of equivalence classes, under this relation, of geodesic rays in X.

Lemma 1 ([5], III.H, Lemma 3.3)**.** *Let γ_1, $\gamma_2 : [0, \infty) \to X$ be two asymptotic, unit speed geodesic rays. Then*

(1) *If $\gamma_1(0) = \gamma_2(0)$, then $d(\gamma_1(t), \gamma_2(t)) \leqslant 2\delta$ for all $t > 0$.*
(2) *In general, there exist t_1, $t_2 \in (0, \infty)$ such that $d(\gamma_1(t_1 + t), \gamma_2(t_2 + t)) \leqslant 5\delta$ for all $t \geqslant 0$.*

Given a class $\gamma(\infty) \in \partial X$ of a geodesic ray γ, there is a unit speed geodesic ray starting from the identity, 1, of G which is asymptotic to γ. Thus, we identify ∂X with the collection of asymptotic classes of unit speed geodesic rays starting at 1 (see [5], [14]). We say that a geodesic ray $\gamma : [0, \infty) \to X$ connects the point $\gamma(0) \in X$ to a point $x \in \partial X$ if x is the equivalence class of γ, i.e. $x = \gamma(\infty)$. Let $\gamma : (-\infty, \infty) \to X$ be a bi-infinite geodesic in X. We denote by γ^- and γ^+ the geodesic rays whose image is equal to γ restricted to $(-\infty, 0]$ and $[0, \infty)$ respectively. Moreover, we say that γ connects two points $x,\, y \in \partial X$ if $\{x, y\} = \{\gamma^-(-\infty), \gamma^+(\infty)\}$. We remark that any two distinct points in ∂X are connected by a bi-infinite geodesic.

All geodesics considered will be assumed to be unit speed geodesics. Let $\gamma_1,\, \gamma_2 : [0, \infty) \to X$ be two geodesic rays starting from the identity. We say that γ_1 and γ_2 *diverge* if they are not asymptotic, i.e. they correspond to different points on ∂X. Moreover, if t_0 is the infimal $t > 0$ such that the distance from $\gamma_1(t)$ to $\gamma_2(t)$ is greater than δ, then we say that γ_1 and γ_2 *diverge at* t_0. The continuous function $d(\gamma_1(t), \gamma_2(t))$ goes from 0 to ∞, therefore $d(\gamma_1(t_0), \gamma_2(t_0)) = \delta$.

Lemma 2 ([5], III.H, Lemma 3.2)**.** *Let $\gamma_1, \gamma_2 : [0, \infty) \to X$ be two divergent geodesic rays in X, issued from the identity that correspond to points $x,\, y \in \partial X$. There is a bi-infinite geodesic γ that joins x to y and is contained in the δ-neighborhood of $\gamma_1 \cup \gamma_2$.*

There is a natural topology on $X \cup \partial X$ making it a compact metrizable space. Let $\alpha > 1$ and $x \in X$. We say that a metric d_α on ∂X is a *visual metric* with base point x and visual parameter α if there is $c > 0$, the constant of the visual metric, so that:

(1) The metric d_α induces the natural boundary topology on ∂X.
(2) For any distinct points $x,\, y \in \partial X$ and any bi-infinite geodesic γ connecting them we have:

$$\frac{1}{c} \cdot \alpha^{-d(\gamma,x)} \leqslant d_\alpha(x, y) \leqslant c \cdot \alpha^{-d(\gamma,x)}$$

Since (X, d) is a proper δ-hyperbolic space, there is $\alpha_0 > 1$, called the *global visual parameter* of X, such that for any base point x_0 and any $\alpha \in (1, \alpha_0)$, the boundary ∂X admits a visual metric d_α with respect to x_0 (see [5], [13]). For the purposes of this paper we will consider a visual metric, d_{2^α}, on ∂X with base point the identity of G and visual parameter $2^\alpha \in (1, \alpha_0)$ for some appropriate $\alpha \in \mathbb{R}$. If c is the constant of this visual metric, let $c_1 \in \mathbb{R}$ be minimal such that $c \leqslant 2^{c_1}$. Then, for all $x, y \in \partial X$ and any bi-infinite geodesic γ that connects x and y, we have:

$$2^{-c_1-\alpha\cdot d(1,\gamma)} \leqslant d_{2^\alpha}(x, y) \leqslant 2^{c_1-\alpha\cdot d(1,\gamma)}$$

We say that α and c_1 are the 2-*visual parameters* of the visual metric d_{2^α}. For sake of simplicity we will use from now on $d_{\partial X}$ for the aforementioned visual metric d_{2^α} on ∂X.

Let $x,\, y \in \partial X$ and $t > 0$. A *t-chain* from x to y is a sequence of points $l_1 = x, l_2, \ldots, l_k = y$ in ∂X, for some $k > 1$, such that, for all $i \in \{1, 2, \ldots, k-1\}$, $d_{\partial X}(l_i, l_{i+1}) \leqslant t$. The length of a t-chain is the number of points it consists of.

The crucial point in the proof of Theorem 1, is the following result due to Bonk and Kleiner [2]:

Proposition 2 ([2])**.** *Let G be a one-ended hyperbolic group and $d_{\partial X}$ a visual metric on ∂X. There are constants $c, K > 0$ so that for all $x, y \in \partial X$, $t \in \mathbb{Z}_+$ there is a $\frac{1}{2^t}d(x, y)$-chain of length at most c^t that connects x to y and whose diameter is at most $Kd_{\partial X}(x, y)$.*

We remark that Proposition 2 actually states that ∂X is linearly connected and derives from a result of Bowditch, Svenson and Swarup [3, 23, 24] which states that ∂X has no global cut points.

When X is the Cayley complex of a group G associated with a finite presentation $\mathcal{P} = \langle S \mid R\rangle$, we will only consider geodesics within the Cayley graph, namely the 1-skeleton $X^{(1)}$ of X. Notice that while the Cayley complex may change when adding words equal to the identity to the relators in $\mathcal{P}$, the Cayley graph remains unchanged.

The following will be used in the proof of Theorem 1:

Lemma 3. *Let G be a δ-hyperbolic group, X its Cayley complex associated with a presentation of G that contains as relators all words of length less than 8δ which are equal to the identity in G. Suppose that $n > 0$ and Δ is a geodesic triangle in X outside the ball $B(n + 1.5\delta)$. If one side of Δ has length less than δ, then Δ can be filled outside $B(n)$ in X.*

Proof. Let $\alpha,\, \beta,\, \gamma$ be the sides of the geodesic triangle Δ and $x = \alpha \cap \gamma$, $y = \beta \cap \gamma$, $z = \alpha \cap \beta$ its vertices. We may assume that the lengths of its sides satisfy $\ell(\gamma) \leqslant \ell(\alpha) \leqslant \ell(\beta)$. We remark that, since the triangle Δ is δ-slim and $\ell(\gamma) < \delta$, for any $x' \in \alpha$, $y' \in \beta$ with $d(x', z) = d(y', z)$ we have that $d(x', y') \leqslant 3\delta$. For any $i = 1, \ldots, \ell(\alpha)$ we consider a geodesic segment w_i that connects $\alpha(i)$ to $\beta(i)$ and define the polygon R_i to be one with sides $[\alpha(i-1), \alpha(i)]_\alpha$, w_i, $[\beta(i-1), \beta(i)]_\beta$ and w_{i-1}, where w_0 is trivially the point z. Each R_i is outside $B(n)$ and corresponds to a word in G of length less than 8δ which is equal to the identity. Thus, R_i can be filled by a disc D_i in X outside the ball $B(n)$. If $\ell(\alpha) = \ell(\beta)$, let $R_{\ell(\alpha)+1} = \emptyset$, otherwise let $R_{\ell(\alpha)+1}$ be the remaining triangle with sides $w_{\ell(\alpha)}$, γ, $[\beta(\ell(\alpha)), y]_\beta$. Then, $R_{\ell(\alpha)+1}$ corresponds to a word in G of length less than 8δ which is equal to the identity and thus can be filled by a disc $D_{\ell(\alpha)+1}$ outside the ball $B(n)$. Therefore, Δ can be filled outside the ball $B(n)$ by $D = \bigcup_{i=1}^{\ell(\alpha)+1} D_i$. □

Lemma 4. *Let G be a finitely presented group, $X^{(1)}$ its Cayley graph, $x \in X^{(1)}$ a vertex and β a geodesic ray in $X^{(1)}$ starting from the identity. Let $z \in \beta$ with $d(z, x) = d(\beta, x)$. For any $y \in \beta$ with $d(y, 1) \geq d(z, 1)$, if η is a geodesic from x to y then,*

$$d(1, \eta) \geqslant d(x, 1) - d(x, \beta)$$

Proof. We have $d(y, z) = d(y, 1) - d(z, 1)$, and by the triangle inequality $\ell(\eta) \leqslant d(x, z) + d(y, 1) - d(z, 1)$. Let v be a point on η which is closest to 1. The triangle inequalities applied in the triangles of vertices $1, x, v$ and $1, y, v$ give $\ell(\eta) \geq d(x, 1) + d(y, 1) - 2d(1, v)$. Therefore $2d(\eta, 1) \geq d(x, 1) - d(x, z) + d(z, 1) \geq 2d(x, 1) - 2d(x, z)$. □

Proposition 3. *If γ_1 and γ_2 are two geodesic rays issued from 1 which diverge at t_0 then there is a bi-infinite geodesic γ, that connects $\gamma_1(\infty)$ to $\gamma_2(\infty)$ and:*

$$t_0 - 2.5\delta \leq d(1, \gamma) \leq t_0 + \delta.$$

Proof. From Lemma 2, we have that there is a bi-infinite geodesic, γ, that joins $\gamma_1(\infty)$ to $\gamma_2(\infty)$ and is contained in the δ-neighborhood of γ_1, γ_2. This implies that the ideal triangle, Δ, of vertices $1, \gamma_1(\infty), \gamma_2(\infty)$ is δ-slim. Suppose that $w \in \gamma$ with $d(w, 1) = d(\gamma, 1)$. For any $t < d(w, 1) - \delta$, we obviously have that $d(\gamma_1(t), \gamma)$, $d(\gamma_2(t), \gamma) > \delta$. The ideal triangle Δ being δ-thin, it follows that $d(\gamma_1(t), \gamma_2(t)) \leq \delta$. This yields that $t_0 \geq d(1, w) - \delta$, which establishes the right hand side inequality.

Now, suppose that $d(1, w) < t_0 - \delta$. We set $\gamma^- = [w, \gamma_1(\infty)]_\gamma$ and $\gamma^+ = [w, \gamma_2(\infty)]_\gamma$. There are $w_1 \in \gamma^-$, $w_2 \in \gamma^+$ so that $d(w_1, 1)$, $d(w_2, 1) = t_0$. It follows that $w \in [w_1, w_2]_\gamma$, and therefore

$$d(1, w) \geq t_0 - \frac{d(w_1, w_2)}{2} \tag{1}$$

Since $d(1, w) < t_0 - \delta$, it follows from (1) that $d(w_1, w_2) > \delta$. The fact that Δ is δ-slim further implies that there are $u, z \in \gamma_1 \cup \gamma_2$ such that $d(w_1, u)$, $d(w_2, z) \leq \delta$. By the triangle inequality in the triangles of vertices $1, u, w_1$ and $1, z, w_2$ we derive that $t_0 - \delta \leq d(1, u) \leq t_0 + \delta$ and $t_0 - \delta \leq d(1, z) \leq t_0 + \delta$. We distinguish two cases for u, z, either they belong to the same geodesic ray or to different ones.

In the first case, without loss of generality we assume that $u, z \in \gamma_1$, so there are t_1, t_2 so that $u = \gamma_1(t_1)$ and $z = \gamma_1(t_2)$. Hence $|t_1 - t_2| \leq 2\delta$ and the triangle inequality shows that $d(w_1, w_2) \leq d(w_1, u) + d(u, z) + d(z, w_2) \leq 4\delta$.

In the second case, without loss of generality we assume that $u \in \gamma_1$, $z \in \gamma_2$, so there are t_1, t_2 so that $u = \gamma_1(t_1)$ and $z = \gamma_2(t_2)$. Hence $|t_0 - t_1|$, $|t_0 - t_2| \leq \delta$ and the triangle inequality shows that $d(w_1, w_2) \leq d(w_1, u) + d(u, \gamma_1(t_0)) + d(\gamma_1(t_0), \gamma_2(t_0)) + d(\gamma_2(t_0), z) + d(z, w_2) \leq 5\delta$.

Hence, in any case, $d(w_1, w_2) \leq 5\delta$. Equation (1) then yields that $d(1, w) \geq t_0 - 2.5\delta$, so our left hand side inequality follows. □

Proposition 4. *Let γ_1 and γ_2 be two geodesic rays issued from 1 which diverge at t_0. If $t \geq 0$ such that $d(\gamma_1(t), \gamma_2(t)) < \delta$, then*

$$t \leq t_0 + 3.5\delta$$

Proof. Assume that $d(\gamma_1(t), \gamma_2(t)) < \delta$ for some $t > t_0 + \delta$, else there is nothing to prove. From Proposition 3, we have that there is a bi-infinite geodesic γ, contained in the δ-neighborhood of $\gamma_1 \cup \gamma_2$ and such that $d(1, \gamma) \leq t_0 + \delta$, so $d(1, \gamma) < t$. As in the previous proof, if $w \in \gamma$ with $d(1, w) = d(1, \gamma)$, then there are $w_1, w_2 \in \gamma$ such that $d(w_1, 1)$, $d(w_2, 1) = t$, $w \in [w_1, w_2]_\gamma$, and therefore

$$d(1, w) \geq t - \frac{d(w_1, w_2)}{2} \tag{2}$$

Again, the fact that the ideal triangle Δ of vertices $1, \gamma_1(\infty), \gamma_2(\infty)$ is δ-slim, further implies that there are $u, z \in \gamma_1 \cup \gamma_2$ such that $d(w_1, u)$, $d(w_2, z) \leq \delta$. By the triangle inequality in the triangles of vertices $1, u, w_1$ and $1, z, w_2$ we derive that $t - \delta \leq d(1, u) \leq t + \delta$ and $t - \delta \leq d(1, z) \leq t + \delta$. We distinguish two cases for u, z, either they belong to the same geodesic ray or to different ones.

In the first case, without loss of generality we assume that $u, z \in \gamma_1$, so there are t_1, t_2 so that $u = \gamma_1(t_1)$ and $z = \gamma_1(t_2)$. Hence $|t_1 - t_2| \leq 2\delta$ and the triangle inequality shows that $d(w_1, w_2) \leq d(w_1, u) + d(u, z) + d(z, w_2) \leq 4\delta$.

In the second case, without loss of generality we assume that $u \in \gamma_1$, $z \in \gamma_2$, so there are t_1, t_2 so that $u = \gamma_1(t_1)$ and $z = \gamma_2(t_2)$. Hence $|t - t_1|$, $|t - t_2| \leq \delta$ and the triangle inequality shows that $d(w_1, w_2) \leq d(w_1, u) + d(u, \gamma_1(t_0)) + d(\gamma_1(t_0), \gamma_2(t_0)) + d(\gamma_2(t_0), z) + d(z, w_2) \leq 5\delta$.

In any case, $d(w_1, w_2) \leq 5\delta$. Equation (2) then yields that $d(1, w) \geq t - 2.5\delta$, so our inequality follows. □

Corollary 1. *Let γ_1 and γ_2 be two geodesic rays issued from 1 which diverge at t_0. If for $p \in \gamma_1$, $q \in \gamma_2$ we have that $d(p, q) < \frac{\delta}{2}$, then $p, q \in B(t_0 + 3.5\delta)$.*

Proof. Let $t_1, t_2 \geq 0$ with $p = \gamma_1(t_1)$ and $q = \gamma_2(t_2)$. Since $d(p, q) < \frac{\delta}{2}$, we get that $|t_1 - t_2| < \frac{\delta}{2}$ and the triangular inequality in the triangle of vertices $p, q, \gamma_2(t_1)$ gives

$$d(\gamma_1(t_1), \gamma_2(t_1)) \leq d(p, q) + d(q, \gamma_2(t_1)) \leq \delta$$

Therefore, Proposition 4 gives that $t_1 \leq t_0 + 3.5\delta$. Similarly, we derive that the same holds for t_2. □

3. Proof of Theorem 1

Lets consider first the case when G is a one-ended, sci hyperbolic group. If c_1 is the constant obtained in Proposition 1 we can assume that X is δ-hyperbolic and $\delta \in \mathbb{N}$ with $\delta > 4c_1 + 2$.

We consider a visual metric on ∂X, denoted again by $d_{\partial X}$, with base point the identity of G and 2-visual parameters α, c, for appropriate α, $c \in \mathbb{R}$. Suppose that c_2, K are the constants of Proposition 2, we can further assume that $\delta > \frac{2c + \log K}{\alpha}$.

Without loss of generality we can assume that the Cayley complex X is associated with a presentation of G that contains as relators all words of length less that 8δ which are equal to the identity in G.

Let $n \in \mathbb{N}$, with $n > 13\delta$. We will show that every loop f outside $B(n + 13\delta)$ is null homotopic outside $B(n)$. Since G is sci, there is $M > 0$ so that every loop outside $B(M)$ is null homotopic outside $B(n)$. Thus, it is enough to consider that f is inside $B(M)$ and consequently that $M > n + 13\delta$.

Let p, q be two vertices on f, with $d(p, q) = 1$. There are unit speed geodesic rays, γ_1, γ_2 issued from the identity which pass within c_1 of p, q, respectively. Denote by x, y the corresponding points on ∂X. Moreover, denote by p' a closest point on γ_1 to p and q' a closest point on γ_2 to q.

Case 1. Suppose that $x \neq y$, and so γ_1, γ_2 diverge.

Lemma 5. *There exist a bi-infinite geodesic γ that connects x to y and:*

$$d(1, \gamma) > n + 6.75\delta.$$

Proof of Lemma 5. Suppose that γ_1 and γ_2 diverge at t_0. As $d(p, p') \leq c_1$ and $d(q, q') \leq c_1$ we have $d(p', q') \leq 2c_1 + 1 < \frac{\delta}{2}$. Then, according to Corollary 1 we should have $d(p', 1) \leq t_0 + 3.5\delta$. But, $d(1, p') \geq d(1, p) - c_1 > n + 12.75\delta$, so $t_0 > n + 9.25\delta$. Proposition 3 then gives us that there is a bi-infinite geodesic γ joining x and y that verifies the desired inequality. □

Therefore we have

$$d_{\partial X}(x, y) \leqslant 2^{c - \alpha \cdot d(1, \gamma)} < 2^{c - \alpha \cdot (n + 6.75\delta)}$$

Lemma 6. *There are $k > 0$ and a sequence of points $(w_1, \ldots, w_k)$ which are interpolated by the path $W(p, q)$ with the following properties:*

1. *$W(p, q) \subset X \setminus B(M + \delta)$;*
2. *$w_1 \in \gamma_1$, $w_k \in \gamma_2$;*
3. *For all $i \in \{1, \ldots k\}$, $d(w_i, w_{i+1}) < \delta$;*
4. *For all $i \in \{1, \ldots k\}$, if η_i is a geodesic path from p to w_i, then $d(1, \eta_i) > n + 1.75\delta$.*

Proof of Lemma 6. Let

$$T = \min\{t \in \mathbb{Z}_+ ; t > (M - 3.75\delta - n)\alpha + 2c\}$$

We recall that c_2, K are the constants of Proposition 2, and so there is a $\frac{1}{2^T}d_{\partial X}(x,y)$-chain $L = \{l_1 = x, \dots, l_k = y\}$ in ∂X of length $k = {c_2}^T$ that joins x to y and $diam(L) \leq K d_{\partial X}(x,y)$. This means that for all $i = 1, \dots, k-1$ we have:

$$0 < d_{\partial X}(l_i, l_{i+1}) \leq \frac{1}{2^T} d_{\partial X}(x,y) < 2^{c-\alpha\cdot(n+6.75\delta)-T}$$

Moreover, if L_i is a bi-infinite geodesic in X that connects the points l_i and l_{i+1}, then:

$$d_{\partial X}(l_i, l_{i+1}) \geq 2^{-c-\alpha\cdot d(1,L_i)}$$

The last two inequalities and the choice of T imply that:

$$d(1, L_i) > \frac{\alpha(n+6.75\delta)+T-2c}{\alpha} > M + 3\delta \tag{3}$$

Now, for all $i = 1, \dots, k$ let β_i be a geodesic ray joining 1 to l_i and let w_i be a point of β_i at distance $M + 2\delta$ from 1. Without loss of generality we can assume that $\beta_1 = \gamma_1$ and $\beta_k = \gamma_2$, respectively.

We claim that relation (3) implies that β_i and β_{i+1} diverge outside $B(M + 2\delta)$. In fact, if β_i and β_{i+1} diverged at $t_{0,i} \leq M + 2\delta$, then Proposition 3 would provide us a geodesic L_i connecting l_i and l_{i+1} such that $d(1, L_i) \leq M + 3\delta$, therefore contradicting (3). This claim implies that $d(w_i, w_{i+1}) \leq \delta$ and thus we can join w_i and w_{i+1} with a geodesic path $w(i, i+1)$ lying outside $B(M + \delta)$. We then set $W(p, q)$ to be the union of these paths: $W(p,q) = \bigcup_{i=1}^{k-1} w(i, i+1)$.

From Proposition 2, we have that $d_{\partial X}(x, l_i) \leq K d_{\partial X}(x,y)$. Hence, if E_i is a bi-infinite geodesic joining x and l_i, as before, we get:

$$2^{-c-\alpha d(1,E_i)} \leq d_{\partial X}(x, l_i) \leq 2^{\log K + c - \alpha\cdot(n+6.75\delta)}$$

so that, since $\delta > \frac{2c+\log K}{\alpha}$,

$$d(1, E_i) \geq n + 5.75\delta$$

We conclude as before, using Proposition 3, that γ_1 and β_i diverge outside the ball $B(n + 4.75\delta)$. Let then $p'' \in \gamma_1$ and $z_i \in \beta_i$ be points at distance $n + 4.75\delta$ from 1. The divergence condition implies that $d(p'', z_i) \leq \delta$. On the other hand the triangle inequality:

$$d(p, z_i) \leq d(p, p') + d(p', p'') + d(p'', z_i)$$

implies that

$$d(p, \beta_i) \leq d(p, z_i) \leq d(p', 1) - n - 2.75\delta \leq d(1, p) - n - 1.75\delta$$

Suppose that η_i is a geodesic path from p to w_i. Then Lemma 4 yields us

$$d(1, \eta_i) > d(p, 1) - d(p, \beta_i) \geq n + 1.75\delta$$

□

Let P, Q be geodesic paths that join p to p' and q to q' respectively. We set $\Phi(p,q)$ to be the following closed loop:

$$\Phi(p,q) = P \cup [p', w_1]_{\gamma_1} \cup W(p,q) \cup [w_k, q']_{\gamma_2} \cup Q \cup [p,q]_f$$

Lemma 7. *The loop $\Phi(p,q)$ is null homotopic outside $B(n)$.*

Proof of Lemma 7. Let $\Sigma = \{s_1 = q', \dots, s_{k'} = w_k\}$ be a set of points on $[q', w_k]_{\gamma_2}$, such that for all $i = 1, \dots, k'-1$ we have $d(s_i, s_{i+1}) < \delta$. For any $i = 1, \dots, k'-1$, let Δ_i be a geodesic triangle of vertices p, s_i, s_{i+1} and such that one of its sides is $[s_i, s_{i+1}]_{\gamma_2}$. Also, let Δ_0 be a geodesic triangle of vertices p, q and q' and such that two of its sides are $[p,q]_f$ and Q. From Lemma 4 it follows that, if S_i is a geodesic path from p to s_i, then:

$$d(S_i, 1) \geqslant d(p, 1) - d(p, \gamma_2) > n + 12\delta$$

Therefore, $d(\Delta_i, 1) > n + 12\delta$, and so, from Lemma 3 it follows that it can be filled by a disc outside $B(n + 10.5\delta)$.

We proceed similarly to join p to the points w_i on $W(p,q)$. Specifically, from the properties of the path $W(p,q)$ we get that the corresponding triangles are outside the ball $B(n + 1.75\delta)$ and from Lemma 3 we get that these triangles can be filled outside $B(n + 0.25\delta)$.

The union, $D(p,q)$, of all the fillings (Van Kampen diagrams) of the triangles we have considered, fills $\Phi(p,q)$ outside $B(n)$. □

Case 2. Suppose that $x = y$, and so γ_1 and γ_2 are asymptotic.

By Lemma 1 we have that there are t_1, $t_2 > 0$ so that $\gamma_1([t_1, \infty])$ and $\gamma_2([t_2, \infty])$ travel within 5δ of each other. For $i = 1, 2$, there are $w_i \in \gamma_i([t_i, \infty])$ with $d(w_1, 1)$, $d(w_2, 1) > M + 10\delta$ and $d(w_1, w_2) \leq 5\delta$. Let $W(p, q)$ to be a geodesic path that joins them, so then its length is at most 5δ and $W(p, q) \subset X \setminus B(M + 7.5\delta)$. We proceed as in Case 1 to show that the corresponding loop $\Phi(p, q)$ can be filled outside $B(n)$.

In any case, we start with two points p, q on f at distance $d(p, q) = 1$ and create a closed loop $\Phi(p, q)$ one part of which is $[p, q]_f$ and another is a path, $W(p, q)$, that is outside $B(M + \delta)$. The closed loop $\Phi(p, q)$ can be filled by $D(p, q)$ outside $B(n)$. Moreover, the paths $W(p, q)$ can be chosen in a way such that their union, over all points p, q of distance 1 on f, creates a closed loop f_1 outside $B(M + \delta)$:

$$f_1 = \bigcup_{\substack{p,q \in f \\ d(p,q)=1}} W(p, q)$$

The closed loop f_1 can be filled by a disk A_1 outside $B(n)$. On the other hand we have filled the ring between f and f_1 with A_2 outside $B(n)$:

$$A_2 = \bigcup_{\substack{p,q \in f \\ d(p,q)=1}} D(p, q)$$

Thus, f is filled by $A_1 \cup A_2$ outside $B(n)$.

In conclusion, for all $n > 12\delta$, any loop outside $B(n + 13\delta)$ is null homotopic outside $B(n)$, and therefore G has linear sci.

If G is not one-ended, we work on the connected components of $X \setminus B(n + 13\delta)$.

4. The proof of Theorem 2

The first step is the following lemma:

Lemma 8. *In a homogenous locally finite one-ended graph, through any point p passes a discrete geodesic, i.e. an isometrically embedded copy of the integers.*

Proof. Since the graph is unbounded, for any $n \in \mathbb{N}$ there exist two vertices at distance $2n$, joined by a geodesic segment $u_n, u_{n-1}, ..., u_{-n}$. By homogeneity, we can choose as u_0 a fixed base point $u_0 = x_0$. Now, this is true for any natural $n \in \mathbb{N}$, and since the graph is locally finite, there exists, by a compacity argument (e.g. diagonal extraction), the desired geodesic. □

Now, Theorem 2 follows from the following proposition.

Proposition 5. *Let X be a graph as before. Let $r \in \mathbb{N}$ be a natural number and K be a finite subset of X whose diameter is at most $2r$. Denote by C a connected component of $X - K$. Then for any point x in C, we have the following alternative:*

- *either x belongs to a geodesic ray (i.e. an embedded copy of the natural numbers) of X within C (and this in particular implies that C is infinite),*
- *or else the distance from x to K is at most r and C is bounded.*

Proof. Let x be a point of C. Then, by Lemma 8, there exists a discrete geodesic (u_n) with $n \in \mathbb{Z}$ such that $u_0 = x$. If x does not belong to any geodesic ray contained in C, then one can find n and $m > 0$ (both minimal) such that u_n and u_{-m} belong to K. Since the diameter of K is by hypothesis $\leq 2r$, then one has $m + n \leq 2r$. This means that the distance $d(x, K)$ from x to K is $\min\{m, n\} \leq \frac{m+n}{2}$. Hence x is within r from K, and this ends the proof of the proposition. □

End of proof of Theorem 2. Whenever K is a ball $B(r)$ of radius r centered at the neutral element of the Cayley graph of the group G, then the previous proposition implies that any bounded connected component of $X - B(r)$ is included in the ball $B(2r)$ having the same center and radius $2r$. In particular one has that $V_0(r) \leq 2r$. □

An alternative proof of Theorem 2. Suppose that there is a positive integer $r \geqslant 2$, such that $V_0(r) > 2r$. Then, there is a bounded connected component A of $X \setminus B(r)$ and $a \in A$ such that $d(a, B(r)) > r$. As G is one-ended, there is an unbounded connected component C of $X \setminus B(r)$. Consider the action of a on X by multiplication. Since $d(a, B(r)) > r$, clearly $aB(r) = B(a, r) \subset A$ and there are $x \in A$, $y \in C$ so that $ax, ay \in B(r)$. Here $B(a, r)$ denotes the metric ball of radius r

centered at a. Therefore, there is a path γ in $B(r)$ that joins ax to ay. Then $a^{-1}\gamma$ is a path that joins an element of A to an element of C, so it must pass through $B(r)$. Thus, there is w on γ so that $a^{-1}w \in B(r)$. This however implies that $w \in B(r) \cap aB(r)$ which is a contradiction. This proves that $V_0(r) \leqslant 2r$ and hence the end depth of G is linear. □

5. Proof of Theorem 3

Consider the amalgamated product $G = G_1 *_H G_2$. Let X_1 and X_2 be the standard 2-complexes associated to some finite presentations of G_1 and G_2, respectively. Let S_H be a finite set of generators of H which are represented by a wedge of loops Y in both X_1 and X_2. The space X obtained by attaching X_1 and X_2 along Y has fundamental group G. Let C_H be the Cayley graph of H corresponding to the generators S_H. The image of $\widetilde{Y}$ in $\widetilde{X}_i$ is then homeomorphic to C_H. Furthermore the universal covering $\widetilde{X}$ is constructed from coset copies of the universal coverings $\widetilde{X}_1$ and $\widetilde{X}_2$ which are attached along copies of C_H.

We consider a metric ball $B(r)$ of radius r in $\widetilde{X}$ centered at a fixed point. By compactness $B(r)$ intersects only finitely many copies of $\widetilde{X}_1$ and $\widetilde{X}_2$. Since $\widetilde{X}_1$ and $\widetilde{X}_2$ have linear sci there exists a constant c_1 such that any loop lying in one copy of either $\widetilde{X}_1$ or $\widetilde{X}_2$ which is outside $B(c_1 r)$ is contractible by a nullhomotopy outside $B(r)$.

Since the one-ended group H has linear end-depth by Theorem 2, one can find a constant c_2 such that any two points of a copy of C_H lying outside $B(c_2 r)$ can be connected by a path within that copy C_H not intersecting $B(c_1 r)$.

The proof that any loop of $\widetilde{X}$ which lies outside $B(c_2 r)$ bounds a disk outside $B(r)$ is now standard following [12]. Any edge loop L starting at $g \in G$ can be written as a word $g a_1 a_2 a_3 \cdots a_n$, with $a_i \in G_1$, when i is odd and $a_i \in G_2$, when i is even, such that the equality $a_1 a_2 \cdots a_n = 1$ holds in G. The structure theorem for amalgamated products implies that there exists some i so that $a_i \in H$ (see [16]). Thus the edge subpath l corresponding to the element $a_i \in H$ starts and ends in the same copy of C_H.

We will show that l can be homotoped in $\widetilde{X}$ rel end points into this copy of C_H. As L lies outside $B(c_2 r)$, the end points of l are outside $B(c_2 r)$ and by the above argument they can be connected by some path p lying within the same copy of C_H and which does not intersect $B(c_1 r)$. The resulting loop $l \cup p$ obtained by gluing together l and p at their common end points is therefore contained in one copy of either $\widetilde{X}_1$ or else of $\widetilde{X}_2$. Moreover, $l \cup p$ lies in the complement of $B(c_1 r)$. By hypothesis G_i have linear sci and thus $l \cup p$ can be contracted out of $B(r)$. This establishes the claim. The word associated to the path p belongs to H and it can be absorbed into a_{i-1}. Thus we obtain a free homotopy of L outside $B(r)$ to a loop L' starting at g which corresponds to a word strictly shorter than that of L. Then by induction on n we can decrease the length n until the resulting loop has $n = 1$. This proves the first part of Theorem 3.

In order to prove the second part let us recall the HNN construction. If H is a finitely generated subgroup of the finitely presented group G_1 and $f : H \to G_1$ is a monomorphism from H into G_1 we set $K = f(H)$. Suppose that H is generated by $a_1, \ldots, a_n$ and denote by c_i the generators $f(a_1), f(a_2), \ldots f(a_n)$ of K. Let

$$\langle b_1, \ldots, b_m, a_1, \ldots, a_n, c_1, \ldots, c_n \quad | \quad p_1 = 1, \ldots, p_k = 1 \rangle$$

be a presentation for G_1. Then the the HNN-extension $G = G_1 *_H$ of G_1 by f has the presentation

$$\langle b_1, \ldots, b_m, a_1, \ldots, a_n, c_1, \ldots, c_n, t \quad | \quad p_1 = 1, \ldots, p_k = 1, c_1 = t^{-1} a_1 t, \ldots, c_n = t^{-1} a_n t \rangle.$$

Consider the 2-complex X_1 associated to the given presentation of G_1. It contains two wedges of circles Y_H, Y_K associated to finite set of generators of H and K. Consider the space X obtained from a copy of X_1 and a copy of $Y_H \times [0, 1]$ where $Y_H \times \{0\}$ is identified with the copy of Y_H in X_1 and $Y_H \times \{1\}$ is identified with the copy of Y_K in X_1 by means of f. The universal covering space $\widetilde{X}$ of X can be constructed from coset copies of $\widetilde{X}_1$ and $C_H \times [0, 1]$, where C_H denotes the Cayley graph of H. As above C_H is the image of $\widetilde{Y}_H$ inside $\widetilde{X}_1$.

As in the case of an amalgamated product above the key tool is Britton's lemma giving the structure of an HNN extension which we state as follows. If we have the equality $g_0 t^{i_1} g_1 t^{i} \cdots t^{i_n} g_n = 1$ in G, where $g_i \in G_1$, then for some k, either $i_k > 0$, $i_{k+1} < 0$, and g_k is in K or else $i_k < 0$, $i_{k+1} > 0$, and g_k is in H.

We denote by $B(r)$ the metric ball at the identity in $\widetilde{X}$. Since each copy of $\widetilde{X}_1$ has linear sci there is c such that any loop in $\widetilde{X}_1$ outside the metric ball $B(cr)$ contained in one copy of $\widetilde{X}_1$ bounds a disk not intersecting $B(r)$. The metric ball $B(cr)$ intersects only finitely many copies of $\widetilde{Y}_H \times [0,1]$. By Theorem 2 since H is one-ended, one can choose c large enough such that any two points of one copy of $\widetilde{Y}_H \times [0,1]$ which lie outside $B(cr)$ can be joined by a path within this copy, not intersecting $B(r)$.

Let L be an edge loop in $\widetilde{X} - B(c^2 r)$. This loop can be represented by a word $g_0 t^{i_1} g_1 t^{i_2} \cdots t^{i_n} g_n$, where $g_i \in G_1$ and which is equal to 1 in G. If $\sum_{j=1}^{n} |i_j| = 0$ then the loop is contained in one copy of $\widetilde{X}_1$ and thus is contractible out of $B(r)$, by hypothesis. When $\sum_{j=1}^{n} |i_j| > 0$, let k be the one provided by Britton's lemma in the form stated above. Then the edge path corresponding to the word $t^{\mathrm{sgn}(i_k)} g_k t^{\mathrm{sgn}(i_{k+1})}$ can be closed in either C_H (or C_K) by means of a path with the same end points which does not intersect $B(cr)$. Here sgn(i) denotes the sign of the non-zero i. We obtain a loop lying in a copy of $\widetilde{X}_1$ outside of $B(cr)$ which can therefore be contracted outside $B(r)$. Thus the loop L is homotopic outside $B(r)$ to a new loop for which the quantity $\sum_{j=1}^{n} |i_j|$ dropped-off by two units. The claim follows by induction.

Remark 2. *If G_i are one-ended sci and H is finitely generated multi-ended then $G_1 *_H G_2$ is one-ended but* not *sci according to Jackson (see* [12]*).*

6. Proof of Theorem 4

Let G be a connected, semisimple Lie group with trivial center and without compact factors. Unlike uniform lattices, nonuniform lattices Γ in G are not quasi-isometric to the symmetric space $X = G/K$ since they do not act co-compactly on X. But one can consider the following construction: chop off every cusp of the quotient X/Γ and look at the lifts of each cusp to X, giving a Γ-equivariant union of horoballs in X. These horoballs are not disjoint in general; these can be made disjoint by cutting the cusps far enough out precisely when Γ has $\mathbb{Q}$-rank one. The resulting space is called the *neutered space* X_0 associated to Γ, and Γ acts co-compactly on it. The natural metric on X_0 is the path metric induced from X, given by the infimal length in X of paths contained in X_0 that join the two points. Then Γ endowed with the word metric is quasi-isometric to X_0 endowed with the path metric. However, sometimes the path metric on X_0 might be distorted with respect to the original metric on X. In order to circumvent this difficulty we consider first only higher rank groups.

Proof of Theorem 4. Since G has higher rank a result due to Lubotzky, Mozes and Raghunathan (see [15]) states that the embedding of Γ endowed with the word metric into G endowed with a left invariant metric is Lipschitz and hence a quasi-isometric embedding. The projection $G \to G/K$ is a quasi-isometry and hence Γ is quasi-isometric to an orbit $\Gamma \cdot x_0 \subset X$ endowed with the restriction of the Riemannian metric d_X on X. Finally the embedding of an orbit of Γ into the neutered space X_0 is a quasi-isometry when we consider the metric $d_X|_{X_0}$ on X_0.

By the quasi-isometry invariance of the sci growth, it will be sufficient to prove that X_0 endowed with the metric $d_X|_{X_0}$ has a linear V_{X_0}. The metric balls $B_{(X_0, d_X|_{X_0})}(x_0, r)$ of radius r centered at $x_0 \in X_0$ for this non-geodesic metric are easy to describe, namely:

$$B_{(X_0, d_X|_{X_0})}(x_0, r) = B_{(X, d_X)}(x_0, r) \cap X_0$$

in terms of the Riemannian metric balls $B_{(X, d_X)}(x_0, r)$.

Now, the neutered space X_0 is obtained from X by removing a collection of disjoint horoballs, as the $\mathbb{Q}$-rank of Γ is at least 2. Then any ball $B_{(X, d_X)}(x_0, r)$ of X intersects only finitely many such horoballs.

This implies that the metric sphere $S_{(X_0, d_X|_{X_0})}(x_0, r) \subset \partial B_{(X_0, d_X|_{X_0})}(x_0, r)$ is obtained from the usual metric sphere $\partial B_{(X, d_X)}(x_0, r)$ in X by removing from it the intersection with a disjoint union of finitely many horoballs.

We need now a lemma which explains the geometry of such intersections:

Lemma 9. *Let X be a proper CAT(0) manifold, H be a horoball, and B be a sphere of X. If the center c of B does not belong to H, then $B \cap H$ is convex (i.e. topologically a ball).*

Proof. Let $f_c(x) = d(x, c)$ be the distance function to a fixed point $c \notin H$. Then f_c restricted to H has only a critical point in H, namely the projection $p(c)$ of c on H, where it achieves a nondegenerate minimum. Since f_c is proper, the level sets on H retract onto $p(c)$. □

From Lemma 9 we derive that the metric spheres in $(X_0, d_X|_{X_0})$ are obtained from S^{n-1} by removing finitely many disjoint disks D^{n-1}. This means that, whenever the dimension n of X is $n \geq 4$, the metric spheres in X_0 are simply connected. This implies that $V_{(X_0,d_X|_{X_0})}(r) = r$ is linear and hence Γ has linear sci.

For the second part of the Theorem 4 consider Γ a non-uniform lattice in $SO(n, 1)$. A non-uniform lattice Γ acts properly and co-compactly by isometries on $X_0 = \mathbb{H}^n - \mathcal{F}$ where $\mathcal{F}$ is a finite union of disjoint open horoballs. The result does not follow from Lemma 9, as the metric on this truncated hyperbolic metric space is the path metric, which is exponentially distorted. Nevertheless this space is CAT(0) (by [5], Cor. 11.28 p.362 and [22]). Metric balls are therefore homeomorphic to balls and their boundaries are spheres. In order to understand the topology of the metric spheres it suffices to consider a neighborhood of one horoball H. Given $c \in X_0$ consider the cone in $\mathbb{H}^n$ with vertex c which is tangent to the horoball H along an equidistant $(n-1)$-sphere $S^{n-1}(c) \subset \partial H$. If p belongs to the visible n-disk bounded by S^{n-1} on ∂H the geodesics segments joining p and c for the hyperbolic metric $d_{\mathbb{H}^n}$ and the path metric on X_0 coincide. When $p \in \partial H$ is outside the visible disk a geodesic segment in the path metric consists of a spherical segment pq joining p to $q \in S^{n-1}(c)$ followed by a geodesic segment qc. It follows that metric spheres in the path metric are obtained from a sphere by deleting a number of disjoint disks corresponding to visible disks at distance smaller than the radius. For $n \geq 4$ these are simply connected and this shows that X_0 with its path metric has linear sci. □

7. Other classes of groups with linear sci

Recall now that a *Coxeter group* is a group W with presentation of the following form:

$$\langle s_1, s_2, \ldots, s_n | s_i^2 = 1 \text{ for } i \in \{1, 2, \ldots, n\}, (s_i s_j)^{m_{ij}} = 1\rangle$$

where $i < j$ ranges over some subset of $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$ and $m_{ij} \geq 2$. Let W be a Coxeter group and C be its Cayley graph.

Proposition 6. *Coxeter groups which are sci have linear sci.*

Proof. The Davis complex (see [7]) D_W of a finitely-generated Coxeter group W is a CAT(0) cell complex D_W on which W acts on cellularly, properly, and with finite quotient. The links of vertices of D_W are all isomorphic to a fixed finite simplicial complex L, where L can be described combinatorially in term of subsets of the generating set of W.

It has been proved in [8] that a Coxeter group is sci if and only if its *nerve* L and all its *punctured links* $L - \sigma$ are simply connected (where σ is any simplex of L). The boundary of a metric ball in D_W is a connected sum of various punctured links $L - \sigma$, and hence it is simply connected.

Now any loop outside the metric ball of radius r can be contracted onto the boundary of the metric ball and there contracted to a point. This implies that $V_{D_W}(r) = r$.

The action of the Coxeter group on the Davis complex is not free but has finite stabilizers. Moreover there exists a finite index subgroup which acts freely on the Davis complex. This finite index subgroup is still sci and quasi-isometric to D_W and hence by the previous arguments it has linear sci. This implies that the W has linear sci. □

Remark 3. *The same proof works for the right-angled Artin groups, namely a sci right-angled Artin group has a linear sci growth.*

Remark 4. *The connectivity of the punctured links determines the connectivity at infinity of W. However in* [9] *the authors constructed a CAT(0) cell complex acted properly and cocompactly by W whose nerve and punctured links are* not *simply connected, though as W is sci. In some sense the geometric property of groups which is closest to the simple connectivity of large spheres is the linear sci.*

Proposition 7. *If $1 \to H \to G \to K \to 1$ is an exact sequence of finitely presented infinite groups where either H or K has one end then G has a linear sci growth.*

Proof. The proof in [12] that G is sci actually furnishes the estimate $V_G(n) \leq n + C$ for some constant c. □

8. Semistability and simple connectivity at infinity

The aim of this section is to put the sci growth into a more general context. Recall that a *ray* in a noncompact topological space X is a proper map $\gamma : [0, \infty) \to X$. Two rays γ_1 and γ_2 converge to the same end of X if for any compact $C \subset X$ there exists R such that $\gamma_1([R, \infty))$ and $\gamma_2([R, \infty))$ lie in the same component of $X - C$. The set of rays under this equivalence relation is the same as the set of ends of X.

Definition 5. *An end of X is* semistable *if any two rays of X converging to this end are properly homotopic. This is equivalent (see e.g.* [20]*) to the following: for any ray γ converging to the end and for any n there exists a $N \geq n$ such that any loop based on a point of γ with image outside the metric ball $B(N)$ of radius N and fixed center can be pushed (rel γ) to infinity by a homotopy in $X - B(n)$.*

A topological space is semistable if all its ends are semistable. This definition was extended to groups, as follows:

Definition 6. *A finitely presented group G is* semistable *if for some (equivalently any) finite complex X_G with $\pi_1 X_G = G$ its universal covering $\widetilde{X_G}$ is semistable.*

There are many classes of groups known to be semistable (see e.g. [20, 17, 18] and also [3, 23, 24] for the case of hyperbolic groups). There are presently not known examples of finitely presented groups which are *not* semistable. There is a well-defined notion of topological fundamental group at infinity associated to a semistable end of a group (see [11]). Now, following [10] we consider the following metric refinement of the semistability:

Definition 7. *Let X be a non-compact metric space, e an end of X and γ a ray converging to e. The* semistability growth *function $S_e(r)$ is the infimal N with the following property: for any $R \geq N$ and any loop l based on γ which lies in $X - B(N)$ there exists a homotopy rel γ supported in $X - B(n)$ which moves l to a loop in $X - B(R)$.*

Set S_G for $S_{\widetilde{X_G}}$, where X_G is a finite complex with fundamental group G, whenever this is defined. It is not difficult to see that the equivalence class of S_G is a well-defined quasi-isometry invariant of the finitely presented group G.

The principal result of this section is the following connection between sci growth and semistability growth:

Proposition 8. *Assume that G is finitely presented sci group. Then $V_G = S_G$.*

Proof. For given r as the space $\widetilde{X_G}$ is sci there exists some large enough $N(r)$ so that any loop within $\widetilde{X_G} - B(N(r))$ bounds a disk outside $B(r)$. Let l be a loop not intersecting $B(S_G(r))$. By the semistability assumption one can homotope l in $\widetilde{X_G} - B(r)$ to a loop l' lying within $X - B(N(r))$. But l' bounds a disk outside $B(r)$ and hence l bounds a disk outside $B(r)$, as claimed. This proves that $V_G(r) \leq S_G(r)$.

For the reverse inequality let l be a loop based at $\gamma(V_G(r) + \varepsilon)$ (for arbitrarily small ε) outside $B(V_G(r))$, where γ is a given ray. Then l bounds a disk outside $B(r)$, which yields a nullhomotopy of the based loop l to the base point p. We push then p along γ as far as we want. This proves that $S_G(r) \leq V_G(r)$. □

Remark 5. *Some of the present results could be extended to cover the semistability growth as well. For instance, word hyperbolic groups have linear semistability by minor modification of our proof.*

Remark 6. *Mihalik and Tschantz have proved (see* [17, 18, 19]*) that amalgamated products and HNN extensions of semistable groups over arbitrary finitely generated subgroups are semistable. Our theorem 3 is similar in spirit (but less general). We don't know whether Theorem 3 can be extended to multi-ended subgroups and linear semistability. This construction might be a source of groups with nonlinear semistability. On the opposite if this were true in the same generality as Mihalik-Tschantz theorem cited above then finitely generated one relator group would also have linear semistability by the argument used in* [18] *and Coxeter groups would have linear semistability. Notice that the proof from* [20] *yields the linearity of the semistability growth only in the case of irreducible Coxeter groups. The general case depends on the behavior of semistability growth under amalgamated products over multi-ended subgroups.*

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